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NORMALITY AND NON NORMALITY OF CERTAIN SEMIGROUPS AND ORBIT CLOSURES

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Abstract. Given a representation $\rho : G \to \mathrm{GL}(N)$ of a semisimple group G, we discuss the normality or non normality of the cone over $\rho(G)$ using the wonderful compactification of the adjoint quotient of G and its projective normality [K]. These methods are then used to discuss the normality or non normality of certain other orbit closures including determinantal varieties.

1 Introduction

Given a finite dimensional representation $\rho : G \to \mathrm{GL}(V)$, where we assume that G is an algebraic group defined over an algebraically closed field of characteristic 0 and that ρ is rational, a natural object to consider is the cone Z_V over $\rho(G)$, i.e., the closure of the linear transformations in $\mathrm{End}(V)$ which are multiples of some $\rho(g)$ for $g \in G$. It is easy to see that Z_V is a semigroup and one may ask about geometric properties of this semigroup. In this paper we are going to analyze the normality of Z_V under the assumption that G is semisimple and that V is an highest weight module. In particular, we are going to perform this discussion in the case in which V is irreducible.

It turns out that in this case, Z_V is almost never normal. Indeed, we prove that it is normal if and only if V is a minusculrepresentation (for the definition see Section 3). On the other hand we are able in all cases to exhibit the normalization of Z_V, which turns out to be of the form Z_W for "the largest" G-module having the same highest weight as V, that is W is the sum of all irreducible modules whose highest weight is less than or equal to that of V in the dominant ordering. After this is achieved, we make the observation that our methods can be used to analyze a much larger class of orbit closures which include, among others, various varieties of matrices satisfying rank conditions, in particular determinantal varieties. Thus we slightly generalize our methods to treat this case as well.

We now briefly explain how our results are obtained. The main observation is that the coordinate ring of Z_V is also the homogeneous coordinate ring of the associated projective variety. This projective variety turns out to be the image under a suitable morphism of a certain completion of a homogeneous space (in the case of the semigroups over G, this is the wonderful compactification of G introduced and studied in [DCP]). Thus results about such compactifications can be used. In particular we crucially use their projective normality recently proved by Kannan in [K], not only directly, but also as the main idea behind our arguments.

To finish I would like to stress that many of our results are contained or implicit in the work of others. Affine semigroups for semisimple or reductive groups have been considered by various authors (see for example [Re1], [Re2], [RP], [Ri], [V]). Some of the questions considered below have been addressed in [F] and results closely related to those of this paper have been obtained in [T]. In particular a great deal of the content of our Theorem 3.1 can be obtained using [V] together with [K]. Under the normality assumption, the property of having rational singularities has been proved in similar terms in [Ri]. Nonetheless, we have decided to give complete proofs, including those of such known facts as the projective normality result of Kannan.

Our paper is divided as follows. After a section giving our notations, we discuss the normality of semigroups in Section 3. In Section 4 the generalization to other orbit closures is explained. In order to obtain this result, we need to slightly generalize the theory of wonderful completions of an adjoint group. Since the proofs of many assertions are essentially identical to those of the corresponding assertions in the case of group compactifications, they are sometimes omitted. Finally in the last section we discuss various examples.

2 Notation

Let k denote an algebraically closed field of characteristic 0. Let G be a semisimple simply connected algebraic group over k. Choose a maximal torus T and a Borel subgroup $B \supset T$ in G, and let $W = N(T)/T$ be the Weyl group of (G, T).

Consider the character group $P := X(T)$ of T, and let $\Delta \subset P$ denote the roots of (G, T) with Δ_+ the positive roots relative to B. Similarly consider the dual lattice $\check{P}$, and the set of coroots $\check{\Delta}$ with positive coroots $\check{\Delta}_+$. Set P_+ equal to the semigroup of dominant weights, the weights λ such that $\langle \lambda, \check{\alpha} \rangle \geq 0$ for each positive coroot $\check{\alpha}$. Here $\langle \ , \ \rangle$ is the canonical W-invariant inner product on $P \otimes \mathbb{R}$. P_+ is a fundamental domain for the action of the Weyl group W on P.

Let us order P by setting $\lambda \geq \mu$ if $\lambda - \mu$ is a positive linear combination of positive roots. This order is called the dominant order.

Recall that the set of isomorphism classes of finite dimensional irreducible representations of G is in bijection with P_+, the bijection being defined as follows. Any such representation contains a unique line preserved by B called the highest weight line. Given a non zero vector v in such a line, then T acts on

v by multiplication by a character $\lambda \in P_+$. The "unique" (up to isomorphism) irreducible module corresponding to λ will be denoted by V_λ.

P can be identified with the Picard group of G/B. We fix this identification in such a way that, if $\lambda \in P_+$, the line bundle $\mathcal{L}_\lambda$ corresponding to λ is such that

$$H^0(G/B, \mathcal{L}_\lambda) \equiv V_\lambda.$$

Let us recall a few facts about the weight structure of V_λ. We say that a weight $\mu \in P$ appears in a representation V if there is a non zero T eigenvector $v \in V$ of weight μ. One has:

(1) μ appears in V if and only if $w\mu$ does for every $w \in W$.
(2) If μ appears in V_λ, then $\mu \leq \lambda$.
(3) μ appears in V_λ if and only if the unique dominant weight of the form $\nu = w\mu$ satisfies $\nu \leq \lambda$.

We shall say that a finite dimensional G-module U has highest weight $\lambda \in P_+$, if it has a non zero vector of weight λ and each weight μ which appears in U satisfies $\mu \leq \lambda$.

Recall that a dominant weight λ is called minuscule if it satisfies one of the following

(1) $\langle \lambda, \check{\alpha} \rangle \leq 1$ for all positive coroots $\check{\alpha}$.
(2) If μ is dominant and $\mu \leq \lambda$, then $\mu = \lambda$.
(3) A weight appears in V_λ if and only if it is in the W-orbit of λ.

Given now $\lambda \in P_+$, we define its saturation as the set

$$\Sigma(\lambda) = \{\mu \in P_+ \mid \mu \leq \lambda\}.$$

It is well known that λ is minuscule if and only if $\Sigma(\lambda) = \{\lambda\}$.

3 Some Semigroups

Given a G-module V, denote by $I \in \mathrm{End}(V)$ the identity map. Consider the morphism $\gamma : G \times G_m \to \mathrm{End}(V)$ defined by $\gamma((g, z)) = gzI = zgI$. The image of this morphism is the set of G translates of the homotheties.

Notice that clearly the image of γ lies in $\mathrm{GL}(V)$ and as a morphism

$$\gamma : G \times G_m \to \mathrm{GL}(V),$$

γ is a group homomorphism. Furthermore $\gamma(G \times G_m)$ is stable under the action of $G \times G$ by left and right multiplication.

We set Z_V equal to the closure of $\gamma(G \times G_m)$ in $\mathrm{End}(V)$. Note that, by continuity, Z_V is closed under composition, i.e., it is a semigroup, and it is stable under the action of $G \times G$ by left and right multiplication. The main goal of this paper will be to discuss the normality of these semigroups for certain representations V of G. First some notation. Fix a dominant weight $\lambda \in P_+$ and let $\Sigma(\lambda)$ be its saturation. Define $W_\lambda = \oplus_{\mu \in \Sigma_\lambda} V_\mu$.

We set

$$Z_\lambda := Z_{V_\lambda} \quad \text{and} \quad \mathcal{Z}_\lambda := Z_{W_\lambda}.$$

We can now state:

Theorem 3.1. 1) *$\mathcal{Z}_\lambda$ is a normal variety with rational singularities.*

2) *If V is a G-module of highest weight λ, then $\mathcal{Z}_\lambda$ is the normalization of Z_V, and it is equal to Z_V if and only if W_λ is a subrepresentation of V.*
In particular $\mathcal{Z}_\lambda$ is the normalization of Z_λ and it is equal to Z_λ if and only if λ is minuscule.

The proof is an application of some of the results in [DCP], so before giving it, let us recall these facts.

In [DCP] one studies the following variety. Consider $\omega_1, \dots, \omega_n$, the fundamental weights for (G,T). Let $p_i \in \mathbb{P}(\mathrm{End}(V_{\omega_i}))$ be the point representing the line spanned by the identity. Define X to be the closure in $\mathbb{P}(\mathrm{End}(V_{\omega_1})) \times \cdots \times \mathbb{P}(\mathrm{End}(V_{\omega_n}))$ of the orbit $G(p_1, \dots, p_n)$. X is called the wonderful compactification of the adjoint group $\overline{G} = G/Z(G)$, and it has a number of very nice properties. Here we shall need some of them.

First of all X is a smooth $G \times G$-variety with open orbit $G(p_1, \dots, p_n)$, the complement of which is a divisor with normal crossings and smooth irreducible components $D_1, \dots, D_n$. Given a subset $I \subset \{1, \dots, n\}$, the smooth subvariety $D_I := \cap_{i \in I} D_i$ is the closure of the $G \times G$-orbit $\mathcal{O}_I = D_I - \cup_{J \supsetneq I} D_J$ ($D_\emptyset = X$) and each $G \times G$-orbit equals one of the $\mathcal{O}_I$. In particular X contains a unique closed $G \times G$-orbit which can be seen to be isomorphic to $G/B \times G/B$.

One knows that the Picard group of $G/B \times G/B$ can be identified with $P \times P$, and one has that the homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(G/B \times G/B)$ is an injection whose image is the lattice consisting of pairs of the form $(\lambda, -w_0(\lambda))$, w_0 being the longest element in W. Thus $\mathrm{Pic}(X)$ can be identified with P. Under this identification, the classes of the $\mathcal{O}(D_i)$ correspond to the simple roots α_i.

Furthermore take $\lambda \in P_+$ and U a G-module of highest weight λ.

Consider the point $p \in \mathbb{P}(\mathrm{End}(U))$ representing the line spanned by the identity. Set $X(U) = \overline{Gp}$. Then the obvious map $G(p_1, \dots, p_n) \to X(U)$ given by $g(p_1, \dots, p_n) \to gp$ extends to a morphism

$$\phi : X \to X(U) \to \mathbb{P}(\mathrm{End}(U))$$

(in fact if λ is regular ϕ gives an isomorphism of X onto $X(U)$).

Furthermore under the identification of $\mathrm{Pic}(X)$ with P, λ corresponds to the class of $\phi^*(\mathcal{O}(1))$, $\mathcal{O}(1)$ being the tautological line bundle on $\mathbb{P}(\mathrm{End}(U))$.

The above discussion obviously applies to both V_λ and W_λ. Furthermore it is clear that Z_λ is nothing else than the affine cone over $X(V_\lambda)$, while $\mathcal{Z}_\lambda$ is the affine cone over $X(W_\lambda)$. Thus the coordinate rings of Z_λ and $\mathcal{Z}_\lambda$ can be identified with graded $G \times G$ stable subrings of the ring

$$A := \oplus_n H^0(X, L_{n\lambda}),$$

L_μ being the line bundle on X corresponding to μ.

Let us take now for each simple root α_i the unique (up to a scalar) $G\times G$ invariant section of $H^0(X, L_{\alpha_i})$, whose set of zeroes is the divisor D_i. We can filter the ring

$$R := \oplus_{\mu\in P} H^0(X, L_\mu)$$

by the order of vanishing on the $D_i's$. We get a filtration $R = R_0 \supset R_1 \supset \cdots \supset R_m \supset \cdots$, where

$$R_i = \sum_{h_1+\cdots+h_n=i} s_1^{h_1}\cdots s_n^{h_n} R.$$

One then easily gets from [DCP]

Proposition 3.2. *The associated graded ring $GrR = \oplus_i R_i/R_{i+1}$, is isomorphic to the polynomial ring $C[x_1,\dots,x_n]$, where x_i is the class of s_i in R_1/R_2 and*

$$C = \oplus_{\lambda\in P} H^0(G/B\times G/B, \mathcal{L}_{(\lambda,-w_0(\lambda))}),$$

$\mathcal{L}_{(\lambda,-w_0(\lambda))}$ being the line bundle on $G/B\times G/B$ corresponding to $(\lambda,-w_0(\lambda))\in P\times P$.

Proof. Fix $\lambda\in P$. We have already seen that if we restrict the line bundle L_λ to $G/B\times G/B$, we get the line bundle $L_{(\lambda,-w_0(\lambda))}$. Also if we consider the restriction map

$$H^0(X, L_\lambda)\to H^0(G/B\times G/B, \mathcal{L}_{(\lambda,-w_0(\lambda))}),$$

we have that this is surjective with kernel $R_1(\lambda) = R_1\cap H^0(X, L_\lambda)$.

Given two sequences $\underline{h} = \{h_1,\dots,h_n\}$ and $\underline{k} = \{k_1,\dots,k_n\}$, we shall say that $\underline{k}\geq\underline{h}$ if $k_i\geq h_i$ for each $i = 1,\dots,n$. If we now fix such a sequence $\underline{h}$, we set $R_{\underline{h}}(\lambda)$ equal to the image of the map

$$H^0(X, L_{\lambda-\sum h_i\alpha_i})\to H^0(X, L_\lambda)$$

given by multiplication by $s_1^{h_1}\cdots s_n^{h_n}$. Then by [DCP], we know that this map induces an isomorphism of $G\times G$-modules,

$$\psi_{\underline{h}}(\lambda) : R_{\underline{h}}(\lambda)/\sum_{\underline{k}>\underline{h}} R_{\underline{k}}(\lambda)\to H^0(X, L_{\lambda-\sum h_i\alpha_i})/R_1(\lambda-\sum h_i\alpha_i)\cong$$

$$\cong H^0(G/B\times G/B, \mathcal{L}_{(\lambda-\sum h_i\alpha_i, -w_0(\lambda-\sum h_i\alpha_i))}).$$

Since clearly

$$\mathrm{Gr}R = \oplus_{\lambda\in P,\underline{h}} R_{\underline{h}}(\lambda)/\sum_{\underline{k}>\underline{h}} R_{\underline{k}}(\lambda),$$

we get the required isomorphism

$$\psi : \mathrm{Gr}R\to C[x_1,\dots,x_n]$$

by setting $\psi(a) = \psi_{\underline{h}}(\lambda)(a)x_1^{h_1}\cdots x_n^{h_n}$ for $a\in R_{\underline{h}}(\lambda)/\sum_{\underline{k}>\underline{h}} R_{\underline{k}}(\lambda)$. $\square$

As an application of this proposition, we have

Proposition 3.3. *Let $\lambda \in P_+$. Then the ring*

$$A := \oplus_n H^0(X, L_{n\lambda})$$

is normal with rational singularities.

Proof. By a result of Kempf and Ramanathan [KR, Theorem 2], the ring

$$\oplus_{(\lambda,\mu)\in P\times P} H^0(G/B \times G/B, \mathcal{L}_{(\lambda,\mu)})$$

is normal with rational singularities. Now if we let $T \times T$ act on this ring by $(t_1, t_2)s = \lambda(t_1)\mu(t_2)s$, for $s \in H^0(G/B \times G/B, \mathcal{L}_{(\lambda,\mu)})$, then the ring

$$C = \oplus_{\lambda\in P} H^0(G/B \times G/B, \mathcal{L}_{(\lambda,-w_0(\lambda))}),$$

is the subring of invariants under the action of the subgroup $\Gamma \subset T\times T$ defined as the intersection of the kernels of the characters $(\lambda, -w_0(\lambda))$ as λ varies in P. Thus by [B], it is also normal with rational singularities. It follows that also the polynomial ring $\mathrm{Gr}R = C[x_1, \dots, x_n]$ is normal with rational singularities. Using a result of Elkik [E, Theorem 4], we then deduce that R itself is normal with rational singularities.

At this point, let us act on R with T by $ts = \lambda(t)s$ if $s \in H^0(X, L_\lambda)$. Then A is the subring of invariants under the action of the subgroup $\mathrm{Ker}(\lambda)$. Thus again by [B], it is normal and with rational singularities. □

Let us now observe the following easy Lemma. Consider the coordinate ring $k[G]$ of G. One knows that as a $G \times G$-module, $k[G] = \oplus_{\lambda\in P_+}\mathrm{End}(V_\lambda)$. We have

Lemma 3.4. *Given $\lambda, \mu \in P_+$, we set $C_{\lambda,\mu} = \{\nu \in P_+ \mid (V_\lambda \otimes V_\mu \otimes V_\nu^*)^G \neq 0\}$. Then, in $k[G]$,*

$$\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu) = \oplus_{\nu\in C_{\lambda,\mu}}\mathrm{End}(V_\nu).$$

Proof. If we consider $k[G]$ as a G-module with respect to the G action induced by left multiplication, then $\mathrm{End}(V_\lambda)$ is the isotypic component of the irreducible module V_λ. It follows that if we consider $\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu)$ as a G-module, each of its irreducible components has to be isomorphic to an irreducible module V_ν with $\nu \in C_{\lambda,\mu}$.

The decomposition $k[G] = \oplus_{\lambda\in P_+}\mathrm{End}(V_\lambda)$ is a decomposition into distinct pairwise non isomorphic irreducible $G \times G$-modules and $\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu)$ is stable under the action of $G \times G$. From this we deduce that there is a subset $C'_{\lambda,\mu}$ of $C_{\lambda,\mu}$ such that

$$\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu) = \oplus_{\nu\in C'_{\lambda,\mu}}\mathrm{End}(V_\nu).$$

In particular

$$\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu) \subseteq \oplus_{\nu \in C_{\lambda,\mu}} \mathrm{End}(V_\nu).$$

From these facts we see that in order to prove equality, it suffices to see that, as a G-module, $\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu)$ contains a copy of V_ν for every $\nu \in C_{\lambda,\mu}$.

To see this, consider the open G-orbit in $G/B \times G/B$, i.e., the orbit of (B, B^-), B^- being the opposite Borel subgroup to B with respect to T. Let $\pi : G \to G/B \times G/B$ be the map defined by $\pi(g) = g(B, B^-) = (gBg^{-1}, gB^-g^{-1})$. Notice that π can be written as the composition of the diagonal embedding $G \to G \times G$ and of the $G \times G$ action. Take the line bundle $L_{(\lambda,\mu)}$ on $G/B \times G/B$ with the property that $H^0(G/B \times G/B, L_{(\lambda,\mu)}) = V_\lambda \otimes V_\mu$ as a G-module. Since G is simply connected, its Picard group is zero and we get an embedding π^* of $H^0(G/B \times G/B, L_{(\lambda,\mu)})$ into $k[G]$. Since π is G-equivariant, and factors through the diagonal $G \to G \times G$, we easily deduce that the image of π^* is a copy of $V_\lambda \otimes V_\mu$ contained in $\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu)$. In particular, it follows from the definition of $C_{\lambda,\mu}$, that we can find a copy of V_ν in $\mathrm{End}(V_\lambda)\mathrm{End}(V_\mu)$ for every $\nu \in C_{\lambda,\mu}$, as desired. □

We now recall the PRV conjecture proved by Kumar [Ku] and Mathieu [Ma].

Theorem 3.5. *Given two dominant weights λ, μ, if a dominant weight ν is of the form $w\lambda + w'\mu$, for some $w, w' \in W$, then $\nu \in C_{\lambda,\mu}$.*

From this we deduce

Theorem 3.6. [K] *Let $\lambda, \mu \in P$. Then the multiplication map*

$$H^0(X, L_\lambda) \otimes H^0(X, L_\mu) \to H^0(X, L_{\lambda+\mu})$$

is surjective.

Proof. As in the above Lemma 3.4, we can restrict to the open G-orbit and embed $H^0(X, L_\lambda)$ into $k[G]$ as the $G \times G$-submodule $\oplus_{\gamma \in \Sigma(\lambda)} \mathrm{End}(V_\gamma)$. Thus, following all our identifications and Lemma 3.4, we are reduced to prove that, given $\nu \in P_+$ with $\nu \leq \lambda + \mu$ there exist $\lambda' \leq \lambda$ and $\mu' \leq \mu$ both dominant, such that $\nu \in C_{\lambda',\mu'}$.

To see this, recall that the natural map $V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$ is surjective (since it is non zero, G-equivariant and $V_{\lambda+\mu}$ is irreducible). So each weight appearing in $V_{\lambda+\mu}$ can be written as a sum of a weight appearing in V_λ and of a weight appearing in V_μ. Now notice that since $\nu \leq \lambda + \mu$ and ν is dominant, for every $w \in W$, $w\nu \leq \nu \leq \lambda + \mu$. We deduce that ν appears in $V_{\lambda+\mu}$. By the above remark write

$$\nu = \lambda'' + \mu''$$

with λ'' appearing in V_λ and μ'' in V_μ. We now know that there exist w, w' such that $\lambda' = w\lambda''$ and $\mu' = w'\mu''$ are dominant and appear in V_λ and in V_μ respectively. The PRV conjecture then implies that $\nu \in C_{\lambda',\mu'}$ and we are done. □

Remark 3.7. Although the variety X is defined in arbitrary characteristic and both the computation of the Picard group of X and Proposition 3.2 hold (see [S1]), Theorem 3.6 is false in positive characteristic, contrary to the claim contained in the Appendix of [K].

The easiest example is the following. Let $\mathbb{F}$ be a field of characteristic 2. Let $G = \mathrm{SL}(4)$ and let $\lambda = \omega_2$ be the second fundamental weight.

Consider a matrix $Y = (y_{i,j})$, $1 \leq i, j \leq 4$, of indeterminates. Given two sequences $1 \leq i_1 < \cdots < i_h \leq 4$ and $1 \leq j_1 < \cdots < j_h \leq 4$ (of course $h \leq 4$), we denote by $[i_1, \ldots, i_h | j_1, \ldots, j_h]$ the determinant of the minor of Y formed by the rows $i_1, \ldots, i_h$ and the columns $j_1, \ldots, j_h$. We can then identify $H^0(X, L_{\omega_2})$ with the subspace of $\mathbb{F}[z_{i,j}]$ spanned by 2×2 minors of Y and $H^0(X, L_{2\omega_2})$ with the span of the polynomials $[s,t|s',t'][u,v|u',v']$ and $[s,t,u|s',t',u'][v|v']$ with $1 \leq s,t,u,v,s',t',u',v' \leq 4$. The image of $H^0(X, L_{\omega_2}) \otimes H^0(X, L_{\omega_2})$ in $H^0(X, L_{2\omega_2})$ is the the span of the polynomials $[s,t|s',t'][u,v|u'v']$. A direct computation (see [Br] section 4 and 5), shows that the polynomial $[2,3,4|2,3,4][1|1]$ does not belong to this image.

Notice that if we set $V = \bigwedge^2 \mathbb{F}^4$, then the coordinate ring of Z_{ω_2} is the subring of $\mathbb{F}[y_{i,j}]$ generated by $H^0(X, L_{\omega_2})$. In [Br] it is also shown that this ring is neither normal nor Cohen–Macaulay. So we get that also Theorem 3.1 does not hold in this case.

Using Theorem 3.6 we can now prove the first part of our Theorem 3.1, namely that $\mathcal{Z}_\lambda$ is a normal variety with rational singularities.

To see this let us see that $k[\mathcal{Z}_\lambda] = A$. Since both rings are graded and generated in degree 1, ($k[\mathcal{Z}_\lambda]$ by definition and A by the Theorem 3.6) and since $k[\mathcal{Z}_\lambda] \subseteq A$, it suffices to see that the two rings coincide in degree 1. Now $A_1 = H^0(X, L_\lambda) = \oplus_{\mu \in \Sigma(\lambda)} \mathrm{End}(V_\mu)$. On the other hand, the degree one part $k[\mathcal{Z}_\lambda]_1$ of $k[\mathcal{Z}_\lambda]$, is the $G \times G$-module spanned by the identity in $\mathrm{End}(W_\lambda)$. Since $W_\lambda = \oplus_{\mu \in \Sigma(\lambda)} V_\mu$, it is immediate that $k[\mathcal{Z}_\lambda]_1 = \oplus_{\mu \in \Sigma(\lambda)} \mathrm{End}(V_\mu)$, proving our claim.

It remains to prove the second part of Theorem 3.1. By our assumption on V, we have that there is a subset $\Omega \subset \Sigma(\lambda)$ containing λ and positive integers n_μ for $\mu \in \Omega$, such that

$$V \simeq \oplus_{\mu \in \Omega} V_\mu^{\oplus n_\mu}.$$

If we take the identity map $I \in \mathrm{End}(V)$, we then deduce that the $G \times G$ span of I is isomorphic to $\oplus_{\mu \in \Omega} \mathrm{End}(V_\mu)$. It follows that we can assume, without loss of generality, $n_\mu = 1$ for all $\mu \in \Omega$. Also we deduce that $Z_V \subset \oplus_{\mu \in \Omega} \mathrm{End}(V_\mu)$. In particular, as we have already seen above, we get that $\mathcal{Z}_\lambda \subset \oplus_{\mu \in \Sigma(\lambda)} \mathrm{End}(V_\mu)$. We thus have that the obvious $G \times G$-equivariant projections

$$\oplus_{\mu \in \Sigma(\lambda)} \mathrm{End}(V_\mu) \to \oplus_{\mu \in \Omega} \mathrm{End}(V_\mu) \to \mathrm{End}(V_\lambda)$$

restrict to dominant morphisms

$$\mathcal{Z}_\lambda \to Z_V \to Z_\lambda.$$

Thus we get inclusions $k[Z_\lambda] \subset k[Z_V] \subset k[\mathcal{Z}_\lambda]$.

Notice that $k[Z_V] = k[\mathcal{Z}_\lambda]$ if and only if $\Omega = \Sigma(\lambda)$, since only in this case do the two rings coincide in degree 1. Also using the above inclusions, it is clear that in order to show our claims, it suffices to see that $k[\mathcal{Z}_\lambda]$ and $k[Z_\lambda]$ have the same quotient field and that $k[\mathcal{Z}_\lambda]$ is integral over $k[Z_\lambda]$.

Let us see that $k[\mathcal{Z}_\lambda]$ and $k[Z_\lambda]$ have the same quotient field. Indeed both the representation of $G \times G_m$ in $Gl(V_\lambda)$ and of $G \times G_m$ in $Gl(W_\lambda)$, have as kernel the subgroup $S \subset Z(G) \times G_m$, $Z(G)$ being the center of G, consisting of those pairs $(z, t) \in Z(G) \times G_m$ for which $\lambda(z)t = 1$. This clearly implies that $\mathcal{Z}_\lambda$ and Z_λ contain the dense open set $G \times G_m/S$, hence they are birational.

It remains to see that $k[\mathcal{Z}_\lambda]$ is integral on $k[Z_\lambda]$. For this, it is clearly sufficient to show that the degree one part of $k[\mathcal{Z}_\lambda]$ is integral over $k[Z_\lambda]$.

Restricting to the open $G \times G$-orbit, we can identify, for each m, the degree m part $k[Z_\lambda]_m$ of $k[Z_\lambda]$ with the subspace of $k[G]$ spanned by the products $f_1 f_2 \cdots f_m$, with $f_i \in \mathrm{End}(V_\lambda)$. Using Lemma 3.4 we deduce that as a $G \times G$-module,

$$k[Z_\lambda]_m = \bigoplus_{\{\nu |\ \mathrm{Hom}_G(V_\nu, V_\lambda^{\otimes m}) \neq 0\}} \mathrm{End}(V_\nu).$$

From this we see that our claim will immediately follow from the following statement.

Proposition 3.8. (Cf. also [T] Lemma 1) *Let μ, λ be dominant weights. Assume that $\mu \leq \lambda$. Then there exists a positive integer H such that $V_{H\mu}$ is an irreducible component of $V_\lambda^{\otimes H}$.*

Proof. We know that μ is a convex linear combination of the weights $w\lambda$, as w varies in the Weyl group W. So let us write

$$\mu = a_1 w_1 \lambda + \cdots + a_m w_m \lambda$$

with $a_1 + a_2 + \cdots + a_m = 1$, $a_i > 0$, a_i rational. Let us make induction on m. If $m = 1$ there is nothing to prove, since the fact that λ is the unique dominant weight in its W-orbit clearly implies that $\mu = \lambda$. Assume $m \geq 2$. Write

$$\mu = a_1 w_1 \lambda + (1 - a_1)\Big(\frac{a_2}{1 - a_1} w_2 \lambda + \cdots + \frac{a_m}{1 - a_1} w_m \lambda\Big).$$

Now notice that there is a positive integer N such that

$$\begin{aligned}\tilde{\mu} &= N\Big(\frac{a_2}{1 - a_1} w_2 \lambda + \cdots + \frac{a_m}{1 - a_1} w_m \lambda\Big) \\ &= \frac{a_2}{1 - a_1} w_2 N\lambda + \cdots + \frac{a_m}{1 - a_1} w_m N\lambda \in P,\end{aligned}$$

is a convex combination of $m - 1$ W-translates of $N\lambda$. Thus also every W-translate of $\tilde{\mu}$ has the same properties and, by the inductive hypothesis, if we denote by $\overline{\mu}$ the unique W-translate of $\tilde{\mu}$ which lies in P_+, we obtain that there exists a M such that $V_{M\overline{\mu}}$ is an irreducible component of $V_{N\lambda}^{\otimes M}$. But we know that $V_{N\lambda}^{\otimes M}$ is a summand of $V_{\lambda}^{\otimes MN}$, so that $V_{M\overline{\mu}}$ is an irreducible component of $V_{\lambda}^{\otimes MN}$. Let us now consider

$$NM\mu = a_1 w_1 NM\lambda + (1-a_1)M\tilde{\mu}.$$

Since $0 < a_1 < 1$, we can find two positive integers s and r with $r < s$ and $a_1 = r/s$. Clearing denominators, we obtain that

$$sNM\mu = w_1 rNM\lambda + (s-r)M\tilde{\mu}.$$

Now $(s-r)M\tilde{\mu}$ is a W-translate of $(s-r)M\overline{\mu}$, so that applying the PRV conjecture, we deduce that $V_{sNM\mu}$ is an irreducible component of $V_{rNM\lambda} \otimes V_{(s-r)M\overline{\mu}}$. On the other hand $V_{rNM\lambda}$ is an irreducible component of $V_{\lambda}^{\otimes rNM}$ while $V_{(s-r)M\overline{\mu}}$ is an irreducible component of $V_{M\overline{\mu}}^{\otimes s-r}$ which in turn is a direct summand of $V_{\lambda}^{\otimes (s-r)NM}$. Thus $V_{sNM\mu}$ is an irreducible component of

$$V_{\lambda}^{\otimes rNM} \otimes V_{\lambda}^{\otimes (s-r)NM} = V_{\lambda}^{\otimes sNM}.$$

Setting $H = sNM$ we obtain our claim. □

Remark 3.9. In [DCP, section 4], it is shown that whenever λ is regular, then both $X(V_\lambda)$ and $X(W_\lambda)$ are isomorphic to X. This can be used to deduce directly that $k[\mathcal{Z}_\lambda]$ is integral on $k[Z_\lambda]$. Also we have that Z_λ contains the origin as its unique singular point and the natural projection

$$f : \mathcal{Z}_\lambda \to Z_\lambda$$

is bijective. But as we have proved in Theorem 3.1, it is an isomorphism only for $G = \mathrm{SL}(2)^n$ and $\lambda = (\omega_1, \dots, \omega_1)$, in which case Z_λ is the cone over the Segre embedding of $(\mathbb{P}^3)^n$.

4 A Generalization

As in the previous section, G will denote a semisimple simply connected algebraic group over k, $T \subset G$ a maximal torus and $B \subset G$ a Borel subgroup containing T. We shall continue to denote by $\overline{G}$ the adjoint quotient of G.

We now take another semisimple simply connected group $\mathcal{G}$, a maximal torus $\mathcal{T}$ in $\mathcal{G}$ and a Borel subgroup $\mathcal{B} \supset \mathcal{T}$ in $\mathcal{G}$. We shall set $\tilde{P} = X(\mathcal{T})$, the character group of $\mathcal{T}$. Let $\tilde{\Delta} \subset \tilde{P}$ denote the set of roots, and let $\tilde{\Delta}_+ \subset \Delta$ be the positive roots relative to $\mathcal{B}$. Finally, let $\tilde{P}_+$ be the semigroup of dominant weights.

We now assume that $\mathcal{G}$ contains a parabolic subgroup $\mathcal{P} \supset \mathcal{B}$ having the following property. If $S \subset \mathcal{P}$ denotes the solvable radical of $\mathcal{P}$, we have a surjective homomorphism $\pi : \mathcal{P}/S \to \overline{G} \times \overline{G}$ with finitekernel. Equivalently we assume that the semisimple Levi factor of $\mathcal{P}$ is isogenous to $G \times G$. Composing

π with the quotient homomorphism $\mathcal{P} \to \mathcal{P}/S$, we get a surjection $\pi' : \mathcal{P} \to \overline{G} \times \overline{G}$.

We set K equal to the preimage under π' of the diagonal subgroup in $\overline{G} \times \overline{G}$.

Notice that we get an action of $G \times G$ on S and a surjective homomorphism $\gamma : S \rtimes (G \times G) \to \mathcal{P}$ with finite kernel. Using γ, we can consider any $\mathcal{P}$-module and hence any $\mathcal{G}$-module as a $S \rtimes (G \times G)$-module .

Let us consider now the wonderful compactification X of $\overline{G}$ and define

$$\mathcal{Y} = \mathcal{G} \times_{\mathcal{P}} X.$$

We want to make a study of some of the properties of $\mathcal{Y}$. This study is in fact essentially identical to that of X, so that we shall only sketch the proofs of the various assertions. First of all notice that, since we have an obvious $\mathcal{G}$-equivariant fibration

$$p : \mathcal{Y} \to \mathcal{G}/\mathcal{P},$$

with fiber X, we immediately deduce that all $\mathcal{G}$-orbits in $\mathcal{Y}$ are of the form $\mathcal{G} \times_{\mathcal{P}} \mathcal{O}$, $\mathcal{O}$ being a $G \times G$-orbit in X. This gives a codimension preserving bijection between $\mathcal{G}$-orbits in $\mathcal{Y}$ and $G \times G$-orbits in X with the property that since $\mathcal{G}/\mathcal{P}$ is projective, if $\mathcal{O}$ is any $G \times G$-orbit in X, then $\overline{\mathcal{G} \times_{\mathcal{P}} \mathcal{O}} = \mathcal{G} \times_{\mathcal{P}} \overline{\mathcal{O}}$. In particular each orbit closure in $\mathcal{Y}$ is smooth.

Recall that the complement of the open orbit, which is isomorphic to $\mathcal{G}/K$, is a divisor $\mathcal{D}$ with normal crossings and smooth irreducible components $\mathcal{D}_i$, $i = 1, \ldots, n$, each of which is the closure of a $\mathcal{G}$-orbit. Furthermore, each orbit closure in $\mathcal{Y}$ is the transversal intersection of those among the $\mathcal{D}_i$'s which contain it. Finally $\mathcal{Y}$ contains a unique closed orbit $\cap_{i=1}^{n} \mathcal{D}_i$ which is isomorphic to $\mathcal{G} \times_{\mathcal{P}} (G/B \times G/B) \simeq \mathcal{G}/\mathcal{B}$.

We are now going to determine the Picard group of $\mathcal{Y}$. Recall that in the previous section we have seen that the homomorphism $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G/B \times G/B)$ induced by the inclusion $i : G/B \times G/B \to X$ as the closed orbit is injective and has as image the lattice consisting of pairs of the form $(\lambda, -w_0(\lambda))$, w_0 being the longest element in the Weyl group W. Consider now the inclusions $j : \mathcal{G}/\mathcal{B} \to \mathcal{Y}$ as the closed orbit and the inclusion $h : G/B \times G/B \to \mathcal{G}/\mathcal{B}$ as the fiber over $[\mathcal{P}]$ of the fibration $\mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}$. We have

Proposition 4.1. *The homomorphism* $j^* : \mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(\mathcal{G}/\mathcal{B})$ *is injective and has as image the lattice* $(h^*)^{-1} i^*(\mathrm{Pic}(X))$.

Proof. We have a commutative diagram of inclusions

$$\begin{tikzcd} G/B \times G/B \arrow[r, "h"] \arrow[d, "i"] & \mathcal{G}/\mathcal{B} \arrow[d, "j"] \\ X \arrow[r, "\tilde{h}"] & \mathcal{Y} \end{tikzcd}$$

where $\tilde{h}$ is the inclusion of X as the fiber over $[\mathcal{P}]$ of the fibration $\mathcal{Y} \to \mathcal{G}/\mathcal{P}$.

Now observe that since $\mathcal{Y}$ has a finite number of orbits, it contains a finite number of fixed points under the action of $\mathcal{T}$. Thus applying [BB], we deduce that $\mathcal{Y}$ has a paving by affine spaces. The same is true both for $\mathcal{G}/\mathcal{P}$ and for X. Using this fact and the previous diagram, we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(\mathcal{G}/\mathcal{P}) \arrow[r, "(pj)^*"] & \mathrm{Pic}(\mathcal{G}/\mathcal{B}) \arrow[r, "h^*"] & \mathrm{Pic}(G/B \times G/B) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Pic}(\mathcal{G}/\mathcal{P}) \arrow[r, "p^*"] \arrow[u, "\mathrm{id}"] & \mathrm{Pic}(\mathcal{Y}) \arrow[r, "\tilde{h}^*"] \arrow[u, "j^*"] & \mathrm{Pic}(X) \arrow[r] \arrow[u, "i^*"] & 0 \end{tikzcd}$$

whose horizontal sequences are exact. Now recall that i^* is injective. This clearly implies that also j^* is injective and has as image $(h^*)^{-1}i^*(\mathrm{Pic}(X))$ as desired. $\square$

Now that we have computed the Picard group of $\mathcal{Y}$, we can proceed to analyze the space of sections of a line bundle on $\mathcal{Y}$. We can clearly assume that the homomorphism $\pi' : \mathcal{P} \to \overline{G} \times \overline{G}$ takes the Borel subgroup $\mathcal{B}$ to the Borel subgroup $\overline{B} \times \overline{B}$, which is the image in $\overline{G} \times \overline{G}$ of $B \times B$ and also takes the maximal torus $\mathcal{T}$ to the maximal torus $\overline{T} \times \overline{T}$, which is the image in $\overline{G} \times \overline{G}$ of $T \times T$. We can also assume that under the homomorphism $\gamma : S \rtimes (G \times G) \to \mathcal{P}$, $T \times T$ is mapped to $\mathcal{T}$. Set Q equal to the root lattice $X(\overline{T})$. Our various maps induce homomorphisms

$$Q \oplus Q \xrightarrow{\pi'^*} \tilde{P} \xrightarrow{\gamma^*} P \oplus P \ ,$$

with the composition being the inclusion of the root lattice into the weight lattice for the maximal torus $T \times T$ in $G \times G$. Clearly we can identify $P \oplus P$ with $\mathrm{Pic}(G/B \times G/B)$, $\tilde{P}$ with $\mathrm{Pic}(\mathcal{G}/\mathcal{B})$ and γ^* with h^*. So $\mathrm{Pic}(\mathcal{Y})$ gets identified with the sublattice of $\tilde{P}$ consisting of those elements λ such that $\gamma^*(\lambda) = (\lambda', -w_0(\lambda'))$ for a suitable $\lambda' \in P$. In particular $\mathrm{Pic}(\mathcal{Y})$ contains a copy of the root lattice Q. This lattice consists of the elements $\tilde{\tau} = \pi'^*((\tau, -w_0\tau))$, $\tau \in Q$ (we want to stress that in all this discussion w_0 denotes the longest element in the Weyl group of G). Let $\{\alpha_1, \ldots, \alpha_n\} \subset Q$ denote the set of simple roots with respect to the Borel subgroup $B \subset G$ and $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ the corresponding subset of $\tilde{P}$.

Before we proceed, let us prove a well known and easy Lemma. Let $\mathcal{U} \subset \mathcal{P}$ denote the unipotent radical in $\mathcal{P}$. Given a $\mathcal{G}$-module M, set $M^{\mathcal{U}}$ equal to the subspace of vectors which are invariant under the action of $\mathcal{U}$. $M^{\mathcal{U}}$ is clearly a $G \times G$-module.

Lemma 4.2. *The $\mathcal{G}$-module M is irreducible if and only if the $G \times G$-module $M^{\mathcal{U}}$ is irreducible.*

Proof. M (resp. $M^{\mathcal{U}}$) is irreducible if and only if contains a unique line stable under $\mathcal{B}$ (resp. $B \times B$). But a line stable under $\mathcal{B}$ is automatically contained in $M^{\mathcal{U}}$ and a $B \times B$ stable line in $M^{\mathcal{U}}$ is automatically $\mathcal{B}$ stable, so that M is irreducible if and only if $M^{\mathcal{U}}$ is irreducible. $\square$

We now have

Proposition 4.3. *Let $\lambda \in \mathrm{Pic}(\mathcal{Y})$ and let L_λ be the corresponding line bundle on $\mathcal{Y}$, $\mathcal{L}_\lambda$ its restriction to $\mathcal{G}/\mathcal{B}$. The the restriction map*

$$H^0(\mathcal{Y}, L_\lambda) \to H^0(\mathcal{G}/\mathcal{B}, \mathcal{L}_\lambda)$$

is surjective.

Proof. We can clearly assume that λ is dominant. Otherwise $H^0(\mathcal{G}/\mathcal{B}, \mathcal{L}_\lambda) = 0$ and there is nothing to prove. If λ is dominant, then $H^0(\mathcal{G}/\mathcal{B}, \mathcal{L}_\lambda) = M_\lambda^*$, M_λ being the irreducible module of highest weight λ. We get an associated morphism $\psi : \mathcal{G}/\mathcal{B} \to \mathbb{P}(M_\lambda)$. We are going to extend this morphism to $\mathcal{Y}$. This will clearly imply our claim.

As we have seen in the previous Lemma, the subspace $M_\lambda^{\mathcal{U}}$ is an irreducible $G \times G$-module whose highest weight is of the form $(\lambda', -w_0(\lambda'))$, since $\lambda \in \mathrm{Pic}(\mathcal{Y})$ for a suitable dominant weight $\lambda' \in P_+$. It follows that, as a $G \times G$-module, $M_\lambda^{\mathcal{U}}$ is isomorphic to $\mathrm{End}(V_{\lambda'})$.

Consider the identity map in $I \in \mathrm{End}(V_{\lambda'}) = M_\lambda^{\mathcal{U}} \subset M_\lambda$. The line spanned by I is clearly stable under the action of $\mathcal{P}$ and so is the corresponding point $[I] \in \mathbb{P}(M_\lambda^{\mathcal{U}})$. In the previous section, we have seen that we have a $G \times G$-equivariant morphism from X onto the closure of the $G \times G$-orbit of $[I]$. It is clear, from the above considerations, that this morphism is indeed $\mathcal{P}$-equivariant.

Using the $\mathcal{G}$ action and the inclusion $\mathbb{P}(M_\lambda^{\mathcal{U}}) \to \mathbb{P}(M_\lambda)$, we then obtain a $\mathcal{G}$-equivariant morphism

$$\tilde{\psi} : \mathcal{Y} = \mathcal{G} \times_{\mathcal{P}} X \to \mathbb{P}(M_\lambda)$$

which clearly extends ψ as desired. $\square$

Once the above result has been established, most of the results which follow are proven exactly as in [DCP, section 8] or as in section 3, so we leave their proof to the reader or only sketch them. The first is the following.

Proposition 4.4. *We can order the divisors $\mathcal{D}_i$, $i = 1, \ldots, n$ in such a way that, under the above identifications, the class in* $\mathrm{Pic}(\mathcal{Y})$ *of $\mathcal{O}(D_i)$ is $\tilde{\alpha}_i$.*

Let us now choose for each $i = 1, \ldots, n$, a non zero section $t_i \in H^0(\mathcal{Y}, L_{\tilde{\alpha}_i})$ whose set of zeros is $\mathcal{D}_i$.

Consider the ring

$$\mathcal{R} = \bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{Y})} H^0(\mathcal{Y}, L_\lambda).$$

As in section 3, given sequences $\underline{h} = \{h_1, \ldots, h_n\}$ and $\underline{k} = \{k_1, \ldots, k_n\}$ of non negative integers, we shall say that $\underline{k} \geq \underline{h}$ if $k_i \geq h_i$ for each $i = 1, \ldots, n$ and set $|\underline{h}| = h_1 + \ldots + h_n$. If we now fix such a sequence $\underline{h}$, we set $\mathcal{R}_{\underline{h}}(\lambda)$ equal to the image of the map

$$H^0(\mathcal{Y}, L_{\lambda - \sum h_i \tilde{\alpha}_i}) \to H^0(\mathcal{Y}, L_\lambda)$$

given by multiplication by $t_1^{h_1}\cdots t_n^{h_n}$. Clearly $\mathcal{R}_{\underline{k}}(\lambda)\subset\mathcal{R}_{\underline{h}}(\lambda)$ if and only if $\underline{k}\geq\underline{h}$ and $\oplus_{\lambda\in\mathrm{Pic}(\mathcal{Y})}\mathcal{R}_{\underline{h}}(\lambda)$ is the ideal generated by $t_1^{h_1}\cdots t_n^{h_n}$.

Theorem 4.5. (1) *For each $\lambda\in\mathrm{Pic}(\mathcal{Y})$,*

$$\mathcal{R}_{\underline{h}}(\lambda)/\sum_{\underline{k}>\underline{h}}\mathcal{R}_{\underline{k}}(\lambda)$$

is isomorphic as a $\mathcal{G}$-module to $H^0(\mathcal{G}/\mathcal{B},\mathcal{L}_{\lambda-\sum_i h_i\tilde{\alpha}_i})$. In particular as a $\mathcal{G}$-module, we have an isomorphism

$$H^0(\mathcal{Y},L_\lambda)\simeq\bigoplus_{(h_1,\dots,h_n)}H^0(\mathcal{G}/\mathcal{B},\mathcal{L}_{\lambda-\sum_i h_i\tilde{\alpha}_i}).$$

(2) *If we set*

$$\mathcal{C}=\bigoplus_{\lambda\in\mathrm{Pic}(\mathcal{Y})}H^0(\mathcal{G}/\mathcal{B},\mathcal{L}_\lambda),$$

and

$$\mathcal{R}_i=\bigoplus_{|\underline{h}|=i,\lambda\in\mathrm{Pic}(\mathcal{Y})}\mathcal{R}_{\overline{h}}(\lambda)=\sum_{|\underline{h}|=i}t_1^{h_1}\cdots t_n^{h_n}\mathcal{R},$$

then associated graded ring

$$\mathrm{Gr}\mathcal{R}=\oplus_{i\geq 0}\mathcal{R}_i/\mathcal{R}_{i+1}$$

is isomorphic to the polynomial ring $\mathcal{C}[x_1,\dots,x_n]$, where for $j=1,\dots,n$, x_j is the image of the t_j in $\mathcal{R}_1/\mathcal{R}_2$.

(3) *Let $\lambda\in\mathrm{Pic}(\mathcal{Y})$ be a dominant weight. Then the ring*

$$\mathcal{R}_\lambda=\bigoplus_{n\geq 0}H^0(\mathcal{Y},L_{n\lambda})$$

is normal with rational singularities.

Proof. The proof of (1) is identical to that given in [DCP] for the case of X. (2) then follows repeating the proof of Proposition 3.2 and (3) the one of Proposition 3.3. □

Before we proceed, let us remark that, by Lemma 4.2, the restriction of the map

$$h^*:M=H^0(\mathcal{G}/\mathcal{B},\mathcal{L}_\lambda)\to H^0(G/B\times G/B,\mathcal{L}_{\lambda|G/B\times G/B})$$

to $M^{\mathcal{U}}$ is an isomorphism for all dominant $\lambda\in\tilde{P}_+$. Using this we obtain

Proposition 4.6. *Let $\lambda,\mu\in$ Pic $(\mathcal{Y})\cap\tilde{P}_+$, then the multiplication map*

$$m:H^0(\mathcal{Y},L_\lambda)\otimes H^0(\mathcal{Y},L_\mu)\to H^0(\mathcal{Y},L_{\lambda+\mu})$$

is surjective.

Proof. Since the map m is $\mathcal{G}$-equivariant and $H^0(\mathcal{Y}, L_{\lambda+\mu})^{\mathcal{U}}$ generates $H^0(\mathcal{Y}, L_{\lambda+\mu})$ as a $\mathcal{G}$-module, it is sufficient to show that the restriction of m to the $\mathcal{U}$-invariants is surjective.

Using the notations of Section 3, notice that, by the definition of the divisors $\mathcal{D}_i$, the restriction of the section t_i to X is, up to scalar, the section s_i vanishing on the boundary divisor D_i of X. This implies that we have, for each $\underline{h} = (h_1, \ldots, h_n)$ and for any $\lambda \in \mathrm{Pic}(\mathcal{Y})$, a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{Y}, L_{\lambda - \sum h_i\tilde{\alpha}_i}) & \longrightarrow & H^0(\mathcal{Y}, L_\lambda) \\ \Big\downarrow{\scriptstyle \tilde{h}^*} & & \Big\downarrow{\scriptstyle \tilde{h}^*} \\ H^0(X, L_{\lambda - \sum h_i\tilde{\alpha}_i|X}) & \longrightarrow & H^0(X, L_{\lambda|X}) \end{array} \quad .$$

Setting $h^*(\lambda) = (\lambda', -w_0(\lambda'))$ and, by abuse of notation, still denoting by $\tilde{h}^*$ the map induced by $\tilde{h}^*$ on a subquotient, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_{\underline{h}}(\lambda)/\sum_{\underline{k}>\underline{h}} \mathcal{R}_{\underline{k}}(\lambda) & \longrightarrow & H^0(\mathcal{G}/\mathcal{B}, \mathcal{L}_{\lambda - \sum_i h_i\tilde{\alpha}_i}) \\ \Big\downarrow{\scriptstyle \tilde{h}^*} & & \Big\downarrow{\scriptstyle h^*} \\ R_{\underline{h}}(\lambda')/\sum_{\underline{k}>\underline{h}} R_{\underline{k}}(\lambda') & \longrightarrow & H^0(G/B \times G/B, \mathcal{L}_{(\lambda' - \sum_i h_i\alpha_i, -w_0(\lambda' - \sum_i h_i\alpha_i))}) \end{array}$$

where the horizontal arrows are isomorphisms. Now assume that λ is dominant. Then it is easy to see, $\lambda' - \sum_i h_i\alpha_i$ is dominant if and only if $\lambda - \sum_i h_i\tilde{\alpha}_i$ is dominant. This implies, by the remark made before our Proposition, that for all $\underline{h}$,

$$h^* : H^0(\mathcal{G}/\mathcal{B}, \mathcal{L}_{\lambda - \sum_i h_i\tilde{\alpha}_i})^{\mathcal{U}} \to H^0(G/B \times G/B, \mathcal{L}_{(\lambda' - \sum_i h_i\alpha_i, -w_0(\lambda' - \sum_i h_i\alpha_i))})$$

is an isomorphism, so that also the restriction of $\tilde{h}^*$ to the space of $\mathcal{U}$ invariants is an isomorphism.

Take $\lambda, \mu \in \tilde{P}_+$. We get a diagram

$$\begin{array}{ccc} H^0(\mathcal{Y}, L_\lambda)^{\mathcal{U}} \otimes H^0(\mathcal{Y}, L_\mu)^{\mathcal{U}} & \xrightarrow{m} & H^0(\mathcal{Y}, L_{\lambda+\mu})^{\mathcal{U}} \\ \Big\downarrow{\scriptstyle \tilde{h}^* \otimes \tilde{h}*} & & \Big\downarrow{\scriptstyle \tilde{h}^*} \\ H^0(X, L_{\lambda|X}) \otimes H^0(X, L_{\mu|X}) & \xrightarrow{m} & H^0(X, L_{\lambda+\mu|X}) \end{array}$$

where the vertical arrows are isomorphisms.

By Theorem 3.6, we have that the map

$$H^0(X, L_{\lambda|X}) \otimes H^0(X, L_{\mu|X}) \xrightarrow{m} H^0(X, L_{\lambda+\mu|X})$$

is surjective and this, together with our previous considerations, implies our claim. $\square$

We are now going to use the properties of $\mathcal{Y}$ to study certain orbit closures.

Let us take a representation M of $\mathcal{G}$ and a non zero vector $v \in M$ which, as we can suppose without loss of generality, spans M as a $\mathcal{G}$-module. The assumptions we are going to make on v are

Assumptions 4.7 1) There is a character $\chi : K \to k^*$ such that $kv = \chi(k)v$, for all $k \in K$.

2) Let $W \subset M$ be the $G \times G$-module spanned by v. Then W is a highest weight module.

Let us make some considerations. By assumption 1) the diagonal subgroup in $G \times G$ fixes v, so that the orbit map $G \times G \to (G \times G)v$ factors through the map $f : G \times G \to G$ given by $f((g_1, g_2)) = g_1 g_2^{-1}$, for all $g_1, g_2 \in G$. Thus, we get an $G \times G$-equivariant inclusion of the vector space W into the coordinate ring $k[G]$. In particular, using assumption 2), we deduce that there is a dominant $\lambda' \in P_+$ and a subset $\Omega' \subset \Sigma(\lambda')$ containing λ', such that, as a $G \times G$-module,

$$W \simeq \oplus_{\mu' \in \Omega'} \mathrm{End}(V_{\mu'}).$$

Also, by assumption 1), we have that $\mathcal{P}$ preserves the line spanned by v, so that W is stable under the action of $\mathcal{P}$ and $W \subset M^{\mathcal{U}}$. Since v spans M as a $\mathcal{G}$-module, we deduce that indeed $W = M^{\mathcal{U}}$. Using Lemma 4.2 and our description of $\mathrm{Pic}(\mathcal{Y})$, we then deduce that there is a subset $\Omega \subset \mathrm{Pic}(\mathcal{Y})$ mapped bijectively onto Ω' by $\tilde{h}^* : \mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(X) = P$ such that, as a $\mathcal{G}$-module,

$$M \simeq \oplus_{\mu \in \Omega} M_\mu.$$

Definition 4.8. The variety $Y(v, \Omega)$ is the cone over the orbit $\mathcal{G}v$, i.e., if we let G_m act on M by homotheties,

$$Y(v, \Omega) = \overline{(\mathcal{G} \times G_m)v}.$$

In order to simplify notations we denote by U the G-module $\oplus_{\mu' \in \Omega'}(V_{\mu'})$. We have

Lemma 4.9. *Set* $Z = \overline{(\mathcal{P} \times G_m)v} \subset W$. *Then*

(1) Z *is isomorphic as a* $G \times G$ *variety to* Z_U.

(2) $Y(v, \Omega) = \mathcal{G}Z$.

Proof. (1) Notice that up to rescaling, we can assume that the isomorphism

$$\psi : W \to \oplus_{\mu' \in \Sigma} \mathrm{End}(V_{\mu'})$$

has the property that $\psi(v) = \sum_{\mu \in \Sigma} \mathrm{Id}_{V_\mu}$. On the other hand, we have a $G \times G$-equivariant inclusion of $\oplus_{\mu \in \Sigma} \mathrm{End}(V_\mu)$ into $\mathrm{End}(U)$ taking $\sum_{\mu \in \Sigma} \mathrm{Id}_{V_\mu}$ to the identity. Composing, we get a $G \times G$-equivariant inclusion $\tilde{\psi} : W \to \mathrm{End}(U)$ with $\tilde{\psi}(v) = \mathrm{Id}_U$. Clearly the restriction of $\tilde{\psi}$ to Z gives the required isomorphism with Z_U.

(2) Since both $Y(v,\Omega)$ and Z are cones, it suffices to show that $\mathcal{G}\tilde{Z} = \tilde{Y}(v,\Omega)$ for the projective varieties $\tilde{Z} = (Z - \{0\})/G_m$ and $\tilde{Y}(v,\Omega) = (Y(v,\Omega) - \{0\})/G_m$. Notice that $\tilde{Z}$ is $\mathcal{P}$-stable, so that the morphism $\mathcal{G}\times\tilde{Z} \to \tilde{Y}(v,\Omega)$ factors through $\mathcal{G}\times_P \tilde{Z}$. Since $\mathcal{G}/\mathcal{P}$ and $\tilde{Z}$ are complete, also $\mathcal{G}\times_{\mathcal{P}}\tilde{Z}$ and hence its image $\mathcal{G}\tilde{Z}$ are complete, proving that $\mathcal{G}\tilde{Z}$ coincides with $\tilde{Y}(v,\Omega)$. □

Notice that if $\mathcal{G} = \mathcal{P} = G\times G$, then the variety $Y(v,\Omega)$ coincides with the variety Z_U considered in Section 3.

The following Lemma strongly restricts our choice of Ω. Let $\lambda\in\Omega$ be the unique element such that $\tilde{h}^*(\lambda) = \lambda'$. Given $\mu' = \lambda' - \sum h_i\alpha_i \in \Sigma(\lambda')$, we set $\rho_\lambda(\mu') = \lambda - \sum h_i\tilde{\alpha}_i$. Notice that the set $\Omega(\lambda) := \rho_\lambda(\Sigma(\lambda'))$ coincides with the set of highest weights of irreducible components of the $\mathcal{G}$-module $H^0(\mathcal{Y}, L_\lambda)$.

Lemma 4.10. $\Omega = \rho_\lambda(\Omega')$. *In particular* $\Omega\subset\Omega(\lambda)$.

Proof. By what we have seen in Section 3, we have a $G\times G$-equivariant morphism

$$\psi : X \to \tilde{Z}.$$

So we deduce from part 2) of the previous Lemma that we have a surjective $\mathcal{G}$-equivariant morphism

$$\phi : \mathcal{Y} \to \tilde{Y}(v,\Omega)$$

whose restriction to X equals ψ. The line bundle on $\mathcal{Y}$ which is the pull back of $\mathcal{O}(1)$ on $\tilde{Y}(v,\Omega)$ is equal to L_λ for some $\lambda\in \mathrm{Pic}\ (\mathcal{Y})$ with the property that $\tilde{h}^*(\lambda) = \lambda'$. It follows that the $\mathcal{G}$-module M is a direct summand in $H^0(\mathcal{Y}, L_\lambda)^*$. Thus Proposition 4.6 implies our claim. □

We are ready to show:

Theorem 4.11. 1) *The variety* $Y(v,\Omega)$ *is normal with rational singularities if and only if* $\Omega = \Omega(\lambda)$.

2) *For a general* $\Omega\subset\Omega(\lambda)$, $Y(v,\Omega(\lambda))$ *is the normalization of* $Y(v,\Omega)$. *In particular* $Y(v,\{\lambda\})$ *is normal if and only if* λ' *is minuscule.*

Proof. The argument given in the above lemma implies that we can identify the coordinate ring $k[Y(v,\Omega)]$ of $Y(v,\Omega)$ with the subring of the ring $\mathcal{R}_\lambda = \oplus_{n\geq 0} H^0(\mathcal{Y}, L_\lambda^n)$ generated by the $\mathcal{G}$ submodule $M^*\subset H^0(\mathcal{Y}, L_\lambda^n)$. In particular, notice that, up to isomorphism, $Y(v,\Omega)$ depends, as a $\mathcal{G}$-variety, only on Ω and not on the choice of a specific vector v (of course provided that the Assumptions 4.7 are satisfied).

We can now prove Theorem 4.11 exactly as we have shown Theorem 3.1. If $\Omega = \Omega(\lambda)$, we get that $k[Y(v,\Omega(\lambda))]$ and $\mathcal{R}_\lambda$ coincide in degree one by Theorem 4.5. Since by Proposition 4.6 they are both generated by their degree one components, we deduce that $k[Y(v,\Omega(\lambda))] = \mathcal{R}_\lambda$. In particular $k[Y(v,\Omega(\lambda))]$ is normal with rational singularities.

It remains to see that, if $\Omega\subsetneq\Omega(\lambda)$, then $k[Y(v,\Omega)]$ is not normal. This follows in a way completely analogous to the corresponding statement for Z_V, which has been seen in Section 3, so we leave the details to the reader. □

Theorem 4.11 can be extended as follows. Suppose $G = G_1 \times \cdots \times G_s$. Let $M_1, \dots, M_s$ be $\mathcal{G}$-modules and $v_1, \dots, v_s$ be vectors with $v_i \in M_i$ each satisfying Assumptions 4.7. Assume furthermore that for each $i = 1, \dots, s$, and $j \neq i$, $G_j \times G_j$ fixes v_i. By what we have already seen, for each i, M_i is a highest weight module of highest weight λ_i and we get a subset $\lambda_i \in \Omega_i \subset \mathrm{Pic}(\mathcal{Y})$, such that $M_i \simeq \oplus_{\mu \in \Omega_i} M_\mu$. Also, we have the subsets $\Omega_i' = \tilde{h}^*(\Omega_i) \subset P_+$. Denote by S the subspace in $M = M_1 \oplus \cdots \oplus M_s$ spanned by the vectors $v_1, \dots, v_s$.

We then define $Y(v_1, \dots, v_s; \Omega_1, \dots, \Omega_s)$ as the closure of $\mathcal{G}S \subset M$. One obtains

Theorem 4.12. 1) *The variety $Y(v_1, \dots, v_s; \Omega_1, \dots, \Omega_s)$ is normal with rational singularities if and only if $\Omega_i = \Omega(\lambda_i)$ for each $i = 1, \dots, s$.*

2) *For a general sequence $\Omega_1, \dots, \Omega_s$, with $\Omega_i \subset \Omega(\lambda_i)$, the normalization of $Y(v_1, \dots, v_s; \Omega_1, \dots, \Omega_s)$ is given by $Y(v_1, \dots, v_s, \Omega_1(\lambda_1), \dots, \Omega_s(\lambda_s))$. In particular, $Y(v_1, \dots, v_s; \{\lambda_1\}, \dots, \{\lambda_s\})$ is normal if and only if λ_i' is minuscule for each $i = 1, \dots, s$.*

Proof. Let $\Gamma \subset \mathcal{T}$ be the intersection of the kernels of the characters λ_i. It is easy to see that the definition of the λ_i's implies that the coordinate ring of $Y(v_1, \dots, v_s, \Omega_1(\lambda_1), \dots, \Omega_s(\lambda_s))$ can be identified with the ring of Γ invariants in

$$\mathcal{R} = \bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{Y})} H^0(\mathcal{Y}, L_\lambda),$$

the $\mathcal{T}$ action being given by $ts = \lambda(t)s$, if $s \in H^0(\mathcal{Y}, L_\lambda),\ t \in \mathcal{T}$.

This immediately implies that $Y(v_1, \dots, v_s, \Omega_1(\lambda_1), \dots, \Omega_s(\lambda_s))$ is normal with rational singularities. We leave the rest of the proof to the reader. □

5 Some Examples

In this section, using the notations of the previous sections, we are going to make a number of examples of varieties of the form $Y(v_1, \dots, v_s; \Omega_1, \dots, \Omega_s)$.

Example 5.1. Assume $\mathcal{G}$ is arbitrary and $\mathcal{P} = \mathcal{B}$ is a Borel subgroup. Then necessarily $G = \{e\}$ the trivial group. We can then take any dominant weight $\lambda \in \tilde{P}$ and consider the irreducible $\mathcal{G}$-module M_λ. Then necessarily v is a highest weight vector and the variety $Y(v, \{\lambda\})$ is just the affine cone over the unique closed orbit in $\mathbb{P}(M_\lambda)$. Notice that in this case, $Y(v, \{\lambda\})$ is normal with rational singularities in accordance with our result (every representation of the trivial group is minuscule). However, we remark that we have used this fact to prove our result.

Example 5.2. Let us start with the case of one of the semigroups Z_V. Take $G = \mathrm{SL}(n)$ and let V be its fundamental representation $\bigwedge^h k^n$. Then it is easy to see, as we have already remarked in Section 3 in a special case, that the coordinate ring $k[Z_V]$ is nothing else than the subring of the polynomial ring $k[x_{i,j}]$, $i, j = 1, \dots, n$, generated by the determinants of the $h \times h$ minors of the matrix $(x_{i,j})$. More generally, assume $\mathcal{G} = \mathrm{SL}(n) \times \mathrm{SL}(m)$ and $G = \mathrm{SL}(h)$ with $h \leq \min(m, n)$. Take $V = \mathrm{Hom}(\bigwedge^r k^n, \bigwedge^r k^m)$ for some $r \leq h$. Consider k^h as a subspace of both $k^m$and k^n in the obvious way, so that we have an inclusion of $\mathrm{End}(\bigwedge^r k^h)$ into $\mathrm{Hom}(\bigwedge^r k^n, \bigwedge^r k^m)$. Take v to be the identity map in $\mathrm{End}(\bigwedge^r k^h)$. Then, since $\bigwedge^r k^h$ is a minuscule $\mathrm{SL}(h)$-module , we deduce that the corresponding variety $Y(v, \Omega)$ is normal with rational singularities. It is clear from our description, that the coordinate ring of $Y(v, \Omega)$ can be described as follows. Consider the ring $R = k[x_{i,j}]/I_h$ with $i = 1, \dots, n$, $j = 1, \dots, m$ and I_h equal to the ideal generated by determinants of $h + 1 \times h + 1$ minors of the matrix $(x_{i,j})$. Then $k[Y(v, \Omega)]$ is the subring of R generated as a k algebra, by the determinants of $r \times r$ minors of $(x_{i,j})$ (in particular if $r = 1$, $Y(v, \Omega)$ is the determinantal variety of $n \times m$ matrices of rank less than or equal than h). The fact that this ring is normal with rational singularities, at least when $h = \min(n, m)$, has been originally shown in [Br], see also [BrC].

Example 5.3. Suppose now $\mathcal{G} = Sl(n_0) \times Sl(n_1) \times \cdots \times Sl(n_r)$, fix a sequence of non negative integers $(h_1, h_2, \dots, h_r)$ with $h_1 \leq \min(n_0, n_1)$, $h_i \leq \min(n_{i-1} - h_{i-1}, n_i)$, for $2 \leq i \leq r$. Set $v = (v_1, v_2, \dots, v_r) \in \mathrm{Hom}(k^{n_0}, k^{n_1}) \oplus \mathrm{Hom}(k^{n_1}, k^{n_2}) \oplus \cdots \oplus \mathrm{Hom}(k^{n_{r-1}}, k^{n_r})$ where, letting I_h denotes the identity $h \times h$ matrix, we have

$$v_i = \begin{pmatrix} I_{h_i} & 0 \\ 0 & 0 \end{pmatrix}$$

if i is odd,

$$v_i = \begin{pmatrix} 0 & 0 \\ 0 & I_{h_i} \end{pmatrix}$$

if i is even.

Notice that $v_{i+1}v_i = 0$ for each $i = 1, \dots, r-1$. $G = Sl(h_1) \times \cdots \times Sl(h_r)$ and the $G \times G$-module spanned by v is just $\mathrm{End}(k^{h_1}) \oplus \cdots \oplus \mathrm{End}(k^{h_r})$ so that the conditions of Theorem 4.12 are satisfied. Furthermore for each i, $\Omega_i = \{\omega_1\}$, a minuscule weight. We deduce that the variety

$$Y(v_1, v_2, \dots, v_r; \{\omega_1\}, \dots, \{\omega_1\})$$

is normal with rational singularities. The fact $v_{i+1}v_i = 0$ for each $i = 1, \dots, r-1$ clearly implies that $Y(v_1, v_2, \dots, v_r; \{\omega_1\}, \dots, \{\omega_1\})$ is nothing else than the variety of complexes with rank conditions $(h_1, h_2, \dots, h_r)$, i.e., it is the variety of sequences $(\psi_1, \dots, \psi_r)$ in $\mathrm{Hom}(k^{n_0}, k^{n_1}) \oplus \cdots \oplus \mathrm{Hom}(k^{n_{r-1}}, k^{n_r})$ such that $\psi_{i+1}\psi_i = 0$ for $i = 1, \dots, r-1$ and $\mathrm{rk}\, \psi_i \leq h_i$ for $i = 1, \dots, r$. The fact that varieties of complexes have rational singularities is well known (see [Ke], [DS] or, for a more recent reference [MT]).

Example 5.4. Suppose now $\mathcal{G} = \mathrm{SL}(n)\times\mathrm{SL}(m)$, and fix a pair of non negative integers (h, s) with $h + s \le \min(n, m)$. Define $v = (v_1, v_2) \in \mathrm{Hom}(k^n, k^m) \oplus \mathrm{Hom}(k^m, k^n)$ by

$$v_1 = \begin{pmatrix} I_h & 0 \\ 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_s \end{pmatrix}.$$

(As before, I_h and I_s are identity matrices of size h and k respectively). $G = \mathrm{SL}(h)\times\mathrm{SL}(s)$ and the $G\times G$-module spanned by v is just $\mathrm{End}(k^h)\oplus\mathrm{End}(k^s)$, so $\Omega_i = \{\omega_1\}$ for $i = 1, 2$. Reasoning as in the previous example, we deduce that $Y(v_1, v_2; \{\omega_1\}, \{\omega_1\})$ is normal with rational singularities. Now notice that $v_1v_2 = 0$ and $v_2v_1 = 0$. From this, it is immediate that $Y(v_1, v_2; \{\omega_1\}, \{\omega_1\})$ is the variety of circular complexes with rank conditions (h, s), i.e., it is the variety of pairs (ψ_1, ψ_2) in $\mathrm{Hom}(k^n, k^m) \oplus \mathrm{Hom}(k^m, k^n)$ such that $\psi_1\psi_2 = 0$, $\psi_2\psi_1 = 0$ and $\mathrm{rk}\,\psi_1 \le h,\ \mathrm{rk}\,\psi_2 \le s$. These varieties have been studied in [S] and in [MT1].

Example 5.5. As a final example, let V be an n-dimensional vector space with a non degenerate symmetric or antisymmetric bilinear form (in this case $\dim V$ is even). We let G be the group of isometries with respect to the form: i.e., $G = \mathrm{SO}(V)$ if our form is symmetric, $G = \mathrm{Sp}(V)$ if it is antisymmetric. Given a linear tranformation $A \in \mathrm{End}\,(V)$, we denote by ${}^tA$ its adjoint with respect to the form. Then the variety Z_V is the variety of linear tranformations $A \in \mathrm{End}\,(V)$ such that ${}^tAA = A{}^tA = tI$, for some $t \in k$ (I is the identity) and, if $G = \mathrm{SO}(V)$ and $\dim V$ is even, $\det A = t^n$. Since V is a minuscule representation if and only if $\dim V$ is even, we deduce that in this case Z_V is normal with rational singularities. If on the other hand $\dim V$ is odd, the normalization of Z_V is given by Z_W, with $W = V \oplus k$.

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