Weakly hyperbolic systems with Hölder continuous coefficients

Piero D'Ancona, Tamotu Kinoshita and Sergio Spagnolo

§1. Introduction

We consider the Cauchy Problem on $[0,T] \times \mathbf{R}_x$

(1)
$$\begin{cases} \partial_t U = A(t)\partial_x U + B(t)U \\ U(0,x) = U_0(x), \end{cases}$$

where $A(t), B(t)$ are $m \times m$ matrices, and $A(t)$ has real eigenvalues

$$\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_m(t) \ .$$

If the entries of $A(t)$ are sufficiently regular in t, say, of class C^k with $k \ge k(m)$, we know ([B], [K1]) that (1) is well posed in the Gevrey classes $\gamma^s = \gamma^s(\mathbf{R})$ for

$$1 \le s < 1 + 1/(m-1)$$

[actually, using the techniques of [DS], one can reach such a conclusion assuming $A(t) \in C^2$].

When the leading coefficients are only Hölder continuous, i.e., $A(t)$ belongs to $C^{0,\alpha}([0,T])$ with $0 < \alpha \le 1$, we espect that (1) is γ^s well posed for $1 \le s < \bar{s}$, for some $\bar{s} = \bar{s}(m,\alpha) > 1$. The first result in this direction concerned the scalar equations of order two, i.e.,

$$\partial_t^2 u \ = \ a(t)\partial_x^2 u + b(t)\partial_x u, \qquad \text{where} \quad a(t) \ge 0, \quad a(t) \in C^{0,\alpha}([0,T]),$$

for which the well-posedness was proved to hold for $s < 1 + \alpha/2$ ([CJS]). This bound is sharp.

This result has been extended to the second order equations with coefficients depending also on x ([N]), and then to any scalar equation of order m ([OT]). In the last case, one has γ^s well-posedness is

$$1 \leq s < 1 + \alpha/m \; .$$

The purpose of this paper is to prove the same range of Gevrey well-posedness for any $m \times m$ system of type (1), at least when $m = 2, 3$. It should be mentioned that a (partially) weaker result was proved to hold for any system of size m ([K2], see also [Y]), namely the well-posedness for $1 \leq s < 1 + \alpha/(m+1)$.

Our main result is the following :

Theorem 1. *Let* $m = 2, 3$, *and let* $T > 0$. *Assume that* (1) *is hyperbolic, i.e., the eigenvalues* $\lambda_1(t), \cdots, \lambda_m(t)$ *are real, with maximum multiplicity* r ($1 \leq r \leq m$), *and that* $A(t) \in C^{0,\alpha}([0,T])$, $B(t) \in L^1(0,T)$. *Then, the Cauchy Problem* (1) *is well posed in* γ^s *provided*

$$1 \leq s < \begin{cases} \dfrac{1}{1-\alpha} & \text{if} \quad r = 1, \\ 1 + \dfrac{\alpha}{r} & \text{if} \quad r = 2, 3. \end{cases}$$

We also prove a result of Gevrey well-posedness for systems with arbitrary size m, under the additional assumption that the square of the matrix $A(t)$ is Hermitian. Note that if $A(t)$ is Hermitian, then (1) is a symmetric system, hence the Cauchy problem is well posed in C^∞ no matter how regular the coefficients are. However, A^2 may be Hermitian even if A is not: for instance, every 2×2 hyperbolic matrix A with trace zero has an Hermitian square A^2.

Theorem 2. *Let* $T > 0$. *Assume* (1) *is hyperbolic, and* $A(t)$ *belongs to* $C^{0,\alpha}([0,T])$, *while* $B(t) \in L^1(0,T)$; *also assume*

$$A(t)^2 \text{ is Hermitian.} \tag{2}$$

Therefore, the Cauchy Problem (1) *is well posed in* γ^s *for*

$$1 \leq s < 1 + \frac{\alpha}{2} \; .$$

If, in addition, $\lambda_1(t)^2 + \cdots + \lambda_m(t)^2 \neq 0$ *for all* t, *then* (1) *is well posed for*

$$1 \leq s < \frac{1}{1-\alpha} \; .$$

REMARK 1 : Thanks to (2), the condition $\sum \lambda_j(t)^2 \neq 0$ is equivalent to the condition that $A(t)^2$ is not the zero matrix, for any t.

REMARK 2 : For $m = 2$, Theorem 1 can be directly derived from Theorem 2: indeed, it is not restrictive to assume that the 2×2 matrix $A(t)$ has trace zero (see §2 below), which implies that $A(t)^2$ is Hermitian. Moreover, any 2×2 system can be viewed as a 3×3 system with maximum multiplicity $r \leq 2$, thus the case $m = 2$, in Theorem, is a special case of $m = 3$. However, we prefer to give here a direct proof of Theorem 1 even for $m = 2$.

REMARK 3 : The conclusions of Theorems 1 and 2 can be easily extended to spatial dimension $n > 1$. Here, for the simplicity in the proofs, we shall consider only the one dimensional case.

The proof of Theorem 1 relies on a suitable choice of the energy function, based on an approximation of the characteristic invariants and the Hamilton-Cayley equation of the matrix $A(t)$. This energy is rather simple in the case $m = 2$ (see §3 below), and will be proposed in a direct way, while for $m = 3$ (see §5) it can be better understood in the framework of the theory of the quasi-symmetrizers ([DS], [J1], [J2]).

§2. Preliminaries

In order to prove Theorem 1, we can assume that the matrix $A(t)$ satisfies

(3) $$\mathrm{tr}\,(A(t)) = 0, \qquad \forall\, t \in [0, T].$$

Indeed, if we put $U(t,x) = \widetilde{U}\big(t, x + \int_0^t \mathrm{tr}\,(A(\tau))d\tau/m\big)$, we can reduce (1) to

$$\begin{cases} \partial_t \widetilde{U} = & \widetilde{A}(t)\partial_x \widetilde{U} + B(t)\widetilde{U} \\ \widetilde{U}(0,x) = & U_0(x), \end{cases}$$

where the matrix $\widetilde{A}(t) \equiv A(t) - \{\mathrm{tr}\,(A(t))/m\}I$ has trace zero. Note that, if $\widetilde{U}$ belongs to $C^1\big([0,T]; \gamma^s(\mathbf{R}_x)\big)$, then also U belongs to $C^1\big([0,T]; \gamma^s(\mathbf{R}_x)\big)$.

We look for an a priori estimate for a solution $U(t,x)$ to (1), thus it is not restrictive to assume that $U(t,x)$ is a smooth function with compact support in $\mathbf{R}_x$ for all $t \in [0,T]$. By Fourier transform $U(t,x) \mapsto V(t,\xi) \equiv \hat{U}(t,\xi)$, (1) is changed to the Cauchy problem on $[0,T] \times \mathbf{R}_\xi$

(4) $$\begin{cases} V' = & i\xi A(t)V + B(t)V \\ V(0,\xi) = V_0(\xi) \; . \end{cases}$$

Now, $U(t,.)$ belongs to $\gamma^s(\mathbf{R}_x)$ if and only if its Fourier transform satisfies

$$|V(t,\xi)| \leq \; Ce^{-\delta|\xi|^{1/s}} \qquad \text{for } |\xi| \geq r\,,$$

for some $C, \delta, r > 0$. Thus, in order to prove that $U \in \gamma^s(\mathbf{R}_x)$ for all $s < \sigma$, it will be sufficient to prove that

(5) $$|V(t,\xi)| \;\leq\; |\xi|^{\nu}\,|V_0(\xi)|\,e^{C_1|\xi|^{1/\sigma}} \qquad \text{for } |\xi| \geq r\,.$$

Given a non-negative function $\varphi \in C_0^\infty(\mathbf{R})$ with $\int_{-\infty}^{\infty} \varphi(t)dt = 1$, and $0 < \varepsilon < 1$, we define the mollified matrix

(6) $$A_\varepsilon(t) \;=\; \int_{-\infty}^{\infty} A(t+\tau/\varepsilon)\varphi(\tau)d\tau.$$

Then, we put

$$h_A(t) = (-1)^{m-1}\det(A(t)), \qquad h_{A_\varepsilon}(t) = (-1)^{m-1}\det(A_\varepsilon(t)), \qquad h_\varepsilon(t) = \Re h_{A_\varepsilon}(t).$$

Note that $h_A \geq 0$, since A has trace zero, whereas h_{A_ε} is complex valued. Denoting by $\|\cdot\|$ the matrix norm, there exists a constant M for which

(7) $$\| A_\varepsilon(t) \| \leq \; M, \qquad \| A'_\varepsilon(t) \| \leq \; M\varepsilon^{\alpha-1}, \qquad \| A_\varepsilon(t) - A(t) \| \leq \; M\varepsilon^{\alpha},$$

for all $t \in [0,T]$. Consequently we obtain, for a possibly larger constant M,

$$|h'_{A_\varepsilon}(t)| \leq M\varepsilon^{\alpha-1}, \qquad |h_{A_\varepsilon}(t) - h_A(t)| \leq M\varepsilon^{\alpha},$$

which also gives

(8) $$|\,h'_\varepsilon(t)| \leq M\varepsilon^{\alpha-1}, \qquad |\,h_\varepsilon(t) - h_A(t)| \leq M\varepsilon^{\alpha}, \qquad |\Im h_{A_\varepsilon}(t)| \leq M\varepsilon^{\alpha}.$$

§3. Proof of Theorem 1 in the case $m = 2$

For the sake of brevity, we'll confine oourselves to the case when $B(t) \equiv 0$, the general case requiring only minor changes. By (3), the characteristic equation and the Hamilton-Cayley equality take, respectively, the following forms:

$$\lambda^2 - h_A(t) = 0, \qquad A(t)^2 - h_A(t)I = 0.$$

Since $\mathrm{tr}\,(A_\varepsilon(t)) = \mathrm{tr}\,(A(t)) = 0$, we also have

(9) $$A_\varepsilon(t)^2 - h_{A_\varepsilon}(t)I = 0.$$

Now, having fixed the constant M as above (see (7), (8)), we define, for any solution $V(t,\xi)$ of (4) and for any ϵ, the energy

$$E(t,\xi) = |A_\varepsilon(t)V|^2 + \big\{h_\varepsilon(t) + 2M\varepsilon^\alpha\big\}|V|^2. \tag{10}$$

By (8) we have

$$h_\varepsilon(t) + 2M\varepsilon^\alpha \;\geq\; h_A(t) + M\varepsilon^\alpha \;\geq\; \begin{cases} c & \text{if } \; r=1, \\ M\varepsilon^\alpha & \text{if } \; r=2, \end{cases}$$

since $h_A(t) \geq c > 0$ in the strict hyperbolic case, hence

$$M|V|^2 \;\geq\; E(t,\xi) \;\geq\; \begin{cases} |A_\varepsilon(t)V|^2 + c\,|V|^2 & \text{if } \; r=1, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha|V|^2 & \text{if } \; r=2. \end{cases} \tag{11}$$

Differentiating in time the energy, and using (4), we find the equality

$$\begin{aligned} E'(t,\xi) =\; & 2\Re\big(A_\varepsilon V', A_\varepsilon V\big) + 2\Re\big(A'_\varepsilon V, A_\varepsilon V\big) + h'_\varepsilon|V|^2 + 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\Re\big(V',V\big) \\ =\; & -2\xi\,\Im\big(A^2_\varepsilon V, A_\varepsilon V\big) - 2\xi\,\Im\big(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V\big) + 2\Re\big(A'_\varepsilon V, A_\varepsilon V\big) + h'_\varepsilon|V|^2 \\ & - 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\xi\,\Im\big(A_\varepsilon V, V\big) - 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\xi\,\Im\big(\{A - A_\varepsilon\}V, V\big) \\ \equiv\; & I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Tking into accouny that $\Re h_{A_\varepsilon} = h_\varepsilon$, by (9) we see that

$$\Im\big(A^2_\varepsilon V, A_\varepsilon V\big) \;=\; h_\varepsilon \Im\big(V, A_\varepsilon V\big) + \Im h_{A_\varepsilon}\,\Re\big(V, A_\varepsilon V\big)\,,$$

hence, by (7) and (10), we find

$$\begin{aligned} I_1 + I_5 \;&=\; -2\xi\,\Im h_{A_\varepsilon}\,\Re\big(V, A_\varepsilon V\big) - 4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) \leq\; 6M\varepsilon^\alpha|\xi||V||A_\varepsilon V| \\ I_2 \;&\leq\; 2\,|\xi|\,\|\,A_\varepsilon\,\|\|\,A - A_\varepsilon\,\|\,|V||A_\varepsilon V| \leq\; 2M^2\varepsilon^\alpha|\xi||V||A_\varepsilon V| \\ I_3 \;&\leq\; 2\,\|\,A'_\varepsilon\,\|\,|V||A_\varepsilon V| \leq\; 2M\varepsilon^{\alpha-1}|V||A_\varepsilon V| \\ I_4 \;&\leq\; |h'_\varepsilon||V|^2 \leq\; M\varepsilon^{\alpha-1}|V|^2 \\ I_6 \;&\leq\; 2\,|\xi|\,\|\,A - A_\varepsilon\,\|\,\Big[\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}|V|^2\Big] \leq\; 2M\varepsilon^\alpha|\xi|E(t,\xi). \end{aligned}$$

Thus, if we choose

$$\varepsilon \;=\; \begin{cases} |\xi|^{-1} & \text{if } \; r=1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \; r=2, \end{cases}$$

and recall (11), we get, for some constant $C = C(M)$,

$$E'(t,\xi) \;\leq\; \begin{cases} CE(t,\xi)\big\{\varepsilon^\alpha|\xi| \,+\, \varepsilon^{\alpha-1}\big\} \;\leq\; CE(t,\xi)|\xi|^{1-\alpha} & \text{if } \; r=1, \\ CE(t,\xi)\big\{\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\,\big\} \;\leq\; CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } \; r=2. \end{cases}$$

Gronwall's inequality, together with (11), yields the apriori estimate (5) with $\sigma = 1/(1-\alpha)$, or $\sigma = 1 + \alpha/2$, hence the proof of Theorem 1 for $m = 2$. $\square$

§4. Proof of Theorem 2

Theorem 2 can be proved in a similar way than Theorem 1 in the case of $m = 2$, but we need not suppose (3). We still assume $B \equiv 0$.

Let us first observe that $\| A_\varepsilon^2 - A^2 \| \leq (\| A_\varepsilon \| + \| A \|) \| A_\varepsilon - A \|$, thus we can take the constant M large enough to satisfy, besides (7) and (8),

$$\| A_\varepsilon(t)^2 - A(t)^2 \| \leq M\epsilon^\alpha \ . \tag{12}$$

Then we define, instead of (10), the following energy:

$$E(t,\xi) = |A_\varepsilon(t)V|^2 + \big(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V, V\big).$$

By (12) we have

$$\big(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V, V\big) \geq \big(A(t)^2 V, V\big) + M\varepsilon^\alpha |V|^2.$$

But the Hermitian matrix $A(t)^2$ has eigenvalues $\lambda_j(t)^2 \geq 0$, hence we see that $\big(A(t)^2 V, V\big) \geq 0$, while $\big(A(t)^2 V, V\big)|V|^{-2} \geq c > 0$ in the special case when $\sum \lambda_j(t)^2 \neq 0$; thus, we obtain the estimates

$$C(M)|V|^2 \geq E(t,\xi) \geq \begin{cases} |A_\varepsilon(t)V|^2 + c\,|V|^2 & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha |V|^2 & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geq 0. \end{cases} \tag{13}$$

We differentiate the energy: by (4), we get the equality

$$\begin{aligned} E'(t,\xi) &= 2\Re\big(A_\varepsilon V', A_\varepsilon V\big) + 2\Re\big(A_\varepsilon' V, A_\varepsilon V\big) + \big(\{A_\varepsilon^2\}' V, V\big) + 2\Re\big(\{A_\varepsilon^2 + 2M\varepsilon^\alpha\}V', V\big) \\ &= -2\xi\,\Im\big(A_\varepsilon^2 V, A_\varepsilon V\big) - 2\xi\,\Im\big(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V\big) + 2\Re\big(A_\varepsilon' V, A_\varepsilon V\big) + \big(\{A_\varepsilon^2\}' V, V\big) \\ &\quad - 2\xi\,\Im\big(\{A_\varepsilon^2 + 2M\varepsilon^\alpha\}A_\varepsilon V, V\big) - 2\xi\Im\big(\{A_\varepsilon^2 + 2M\varepsilon^\alpha\}(A - A_\varepsilon)V, V\big) \\ &\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Using (2) and (7), we find some constant $C = C(M)$ for which

$$\begin{aligned} I_1 + I_5 &= -2\xi\Im\big[\big(A_\varepsilon^2 V, A_\varepsilon V\big) + \big(A_\varepsilon^3 V, V\big)\big] - 4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) = -4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) \\ &\leq C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \\ I_2 &\leq C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \qquad I_3 \leq C\varepsilon^{\alpha-1}|V||A_\varepsilon V|, \qquad I_4 \leq C\varepsilon^{\alpha-1}|V|^2, \\ I_6 &= -2\xi\Im\big((A - A_\varepsilon)V, A_\varepsilon^2 V\big) - 4M\xi\varepsilon^\alpha\Im\big((A - A_\varepsilon)V, V\big) \leq C\varepsilon^\alpha|\xi||V||A_\varepsilon V| + C\varepsilon^{2\alpha}|\xi||V|^2. \end{aligned}$$

We have used the fact that A_ε^2 is Hermitian, by (2), and that $|A_\varepsilon^2 V| \le C|A_\varepsilon V|$. Recalling (13), and choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \ne 0, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \ge 0, \end{cases}$$

we find the estimate

$$E'(t) \le \begin{cases} CE(t,\xi)\left[\varepsilon^\alpha|\xi| + \varepsilon^{\alpha-1}\right] \le CE(t,\xi)|\xi|^{1-\alpha} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \ne 0, \\ CE(t,\xi)\left[\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\right] \le CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \ge 0. \end{cases}$$

which yields (5) with $\sigma = 1/(1-\alpha)$, or $\sigma = 1 + \alpha/2$. Hence, the conclusion of Theorem 2 follows. □

§5. Proof of Theorem 1 in the case $m = 3$

By (3), the characteristic equation and the Hamilton-Cayley equality have the forms :

$$\lambda^3 - k_A(t)\lambda - h_A(t) = 0, \qquad A(t)^3 - k_A(t)A(t) - h_A(t)I = 0,$$

where $h_A(t) = \det(A(t)) = \lambda_1(t)\lambda_2(t)\lambda_3(t)$, while

$$k_A(t) = \sum_{1\le i,j\le 3} \left\{a_{ij}(t)a_{ji}(t) - a_{ii}(t)a_{jj}(t)\right\} = \frac{1}{2}\sum_{j=1}^{3}\lambda_j(t)^2.$$

By the hyperbolicity assumption, the function $k_A(t)$ is non-negative, and in particular satisfies $k_A(t) \ge c > 0$ when $r \le 2$, moreover

$$\triangle_A(t) \equiv \prod_{1\le i<j\le 3} (\lambda_i(t) - \lambda_j(t))^2 = 4k_A(t)^3 - 27h_A(t)^2 \ge 0$$

Similarly, since $\mathrm{tr}\,(A_\varepsilon(t)) = \mathrm{tr}\,(A(t)) = 0$, we see that the regularized matrix (6) satisfies the equality

$$A_\varepsilon(t)^3 - k_{A_\varepsilon}(t)A_\varepsilon(t) - h_{A_\varepsilon}(t)I = 0. \tag{14}$$

However, the eigenvalues of $A_\varepsilon(t)$ may be non real, thus $k_{A_\varepsilon}(t)$ and $h_{A_\varepsilon}(t)$ are complex valued. To overcome this difficulty, we introduce the real functions

$$h_\varepsilon(t) = \Re h_{A_\varepsilon}(t), \qquad k_\varepsilon(t) = \left\{\left\{\Re k_{A_\varepsilon}(t) + M\varepsilon^\alpha\right\}^{3/2} + 12\,M^{3/2}\varepsilon^\alpha\right\}^{2/3}. \tag{15}$$

Here M is constant ≥ 1, which is choosen large enough to fulfil, besides (7), the following inequalities on $[0,T]$:

$$(16)\qquad \begin{cases} |\, h_\varepsilon(t) - h_A(t)| \,\leq\, M\varepsilon^\alpha, & |\Im h_{A_\varepsilon}(t)| \leq M\varepsilon^\alpha, \quad |\, h'_\varepsilon(t)| \leq M\varepsilon^{\alpha-1}, \\ |\, k_{A_\varepsilon}(t)| \,\leq\, M, & |\, k_{A_\varepsilon}(t) - k_A(t)| \,\leq\, M\varepsilon^\alpha, \quad |\, k'_{A_\varepsilon}(t)| \,\leq\, M\varepsilon^{\alpha-1}, \end{cases}$$

which imply, in particular,

$$(17)\qquad |\,\Re k'_{A_\varepsilon}(t)| \leq\, M\varepsilon^{\alpha-1}, \qquad |\,\Re k_{A_\varepsilon}(t) - k_A(t)| \leq\, M\varepsilon^\alpha, \qquad |\Im k_{A_\varepsilon}(t)| \leq\, M\varepsilon^\alpha.$$

We also define

$$(18)\qquad \triangle_\varepsilon(t) \,=\, 4k_\varepsilon(t)^3 - 27\, h_\varepsilon(t)^2.$$

Next we show that $z^3 - k_\varepsilon(t)z + h_\varepsilon(t)$ is a hyperbolic polynomial, i.e., $\triangle_\varepsilon(t) \geq 0$, and also prove some crucial estimates on $k_\varepsilon(t)$:

Lemma 1. *There exists a constant $C = C(M)$, and $c > 0$, such that*

$$(19)\qquad k_\varepsilon(t) \,\geq\, \begin{cases} c & \text{if} \;\; r = 1,2, \\ M\varepsilon^{2\alpha/3} & \text{if} \;\; r = 3, \end{cases}$$

$$(20)\qquad |k'_\varepsilon(t)| \,\leq\, C\varepsilon^{\alpha-1}, \qquad |k_\varepsilon(t) - k_{A_\varepsilon}(t)| \,\leq\, C\varepsilon^\alpha k_\varepsilon(t)^{-1/2},$$

$$(21)\qquad \triangle_\varepsilon(t) \,\geq\, \begin{cases} c & \text{if} \;\; r = 1, \\ M^{3/2}\,\varepsilon^\alpha k_\varepsilon(t)^{3/2} & \text{if} \;\; r = 2,3, \end{cases}$$

$$(22)\qquad |h_\varepsilon(t)| \,\leq\, \sqrt{\frac{4}{27}}\; k_\varepsilon(t)^{3/2}.$$

Proof : We write for brevity (15) in the form

$$k_\varepsilon(t) \,=\, \left\{\widetilde{k}_\varepsilon(t)^{3/2} + 12M^{3/2}\varepsilon^\alpha\right\}^{2/3}, \quad \text{where} \quad \widetilde{k}_\varepsilon(t) \,=\, \Re k_{A_\varepsilon}(t) + M\varepsilon^\alpha,$$

and observe that, by (17),

$$\widetilde{k}_\varepsilon(t) \,=\, \left\{\Re k_{A_\varepsilon}(t) - k_A(t)\right\} + k_A(t) \,+ M\varepsilon^\alpha \,\geq\, k_A(t) \,\geq\, \begin{cases} c & \text{if} \;\; r = 1,2, \\ 0 & \text{if} \;\; r = 3. \end{cases}$$

This yelds (19). Let us prove (20): By (15) and (17) it follows

$$|\, k'_\varepsilon| \,=\, |\,\widetilde{k}'_\varepsilon\,|\; \widetilde{k}_\varepsilon^{1/2}\; \{\widetilde{k}_\varepsilon^{3/2} + 12M^{3/2}\varepsilon^\alpha\}^{-1/3} \leq\, |\,\widetilde{k}'_\varepsilon| \,=\, |\,\Re k'_{A_\varepsilon}| \,\leq\, M\varepsilon^{\alpha-1}.$$

Moreover we get, since $k_\varepsilon(t) \geq \widetilde{k}_\varepsilon(t)$,

$$|\, k_\varepsilon - \widetilde{k}_\varepsilon | = \frac{\{k_\varepsilon^{3/2} - \widetilde{k}_\varepsilon^{3/2}\}\{k_\varepsilon^{3/2} + \widetilde{k}_\varepsilon^{3/2}\}}{k_\varepsilon^2 + k_\varepsilon \widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^2} \leq \frac{12M^{3/2}\varepsilon^\alpha \cdot 2k_\varepsilon^{3/2}}{k_\varepsilon^2} = 24\, M^{3/2}\varepsilon^\alpha k_\varepsilon^{-1/2} \;,$$

and hence, using again (17),

$$|\, k_\varepsilon - k_{A_\varepsilon}| \;\leq\; \big| k_\varepsilon(t) - \widetilde{k}_\varepsilon(t)\big| + \big|\widetilde{k}_\varepsilon(t) - \Re k_{A_\varepsilon}(t)\big| + |\, \Im k_{A_\varepsilon}(t)| \;\leq\; C\varepsilon^\alpha k_\varepsilon^{-1/2}.$$

This completes the proof of (20).

To prove (21) we first derive, using (16), (17), and recalling that $\widetilde{k}_\varepsilon(t) \geq k_A(t)$, $M > 1$, $\varepsilon < 1$, the following estimate

$$\begin{aligned} (23)\quad \big|\widetilde{k}_\varepsilon^{3/2} - k_A^{3/2}\big| &= \big|\widetilde{k}_\varepsilon - k_A\big| \cdot \frac{\widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^{1/2} k_A^{1/2} + k_A}{\widetilde{k}_\varepsilon^{1/2} + k_A^{1/2}} \;\leq \Big\{|\Re k_{A_\varepsilon} - k_A| + M\varepsilon^\alpha\Big\} \cdot \frac{3\,\widetilde{k}_\varepsilon}{\widetilde{k}_\varepsilon^{1/2}} \\ &\leq\; 2M\varepsilon^\alpha \cdot 3\,\widetilde{k}_\varepsilon^{1/2} \;\leq\; 2M\varepsilon^\alpha \cdot 3\,(|\Re k_{A_\varepsilon}| + M\varepsilon^\alpha)^{1/2} \;\leq\; 6\sqrt{2}\, M^{3/2}\varepsilon^\alpha, \end{aligned}$$

Then, we write

$$(24)\qquad \triangle_\varepsilon \;=\; 4\big\{2k_\varepsilon^{3/2} + \sqrt{27}\, h_\varepsilon\big\}\big\{2k_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon\big\}.$$

We know that

$$\big\{2k_A^{3/2} + \sqrt{27}\, h_A\big\}\big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} \;=\; \triangle_A(t) \;\geq\; 0, \qquad \text{and} \qquad k_A(t) \geq 0\,,$$

thus

$$(25)\qquad \big\{2k_A(t)^{3/2} \pm \sqrt{27}\, h_A(t)\big\} \;\geq\; 0\,.$$

For each fixed $t \in [0, T]$, we have either $h_\varepsilon(t) \geq 0$, or $h_\varepsilon(t) \leq 0$. In the first case, we have $\big\{2k_\varepsilon(t)^{3/2} + \sqrt{27}\, h_\varepsilon(t)\big\} \geq k_\varepsilon(t)^{3/2}$, while, by (16), (22), (23) and (25), we obtain

$$\begin{aligned} \big\{2k_\varepsilon(t)^{3/2} - \sqrt{27}\, h_\varepsilon(t)\big\} \;&=\; 24\, M^{3/2}\varepsilon^\alpha + \big\{2\widetilde{k}_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon\big\} \\ &=\; 24\, M^{3/2}\varepsilon^\alpha + 2\big\{\widetilde{k}_\varepsilon^{3/2} - k_A^{3/2}\big\} + \big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} + \sqrt{27}\,(h_A - h_\varepsilon) \\ &\geq\; 24\, M^{3/2}\varepsilon^\alpha - 2\,\big|\,\widetilde{k}_A^{3/2} - k_\varepsilon^{3/2}\,\big| + \big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} - \sqrt{27}\,|\, h_A - h_\varepsilon| \\ &\geq\; \big[24 - 12\sqrt{2} - \sqrt{27}\,\big] M^{3/2}\varepsilon^\alpha + \big\{2k_A^{3/2} - \sqrt{27} h_A\big\} \\ &\geq\; M^{3/2}\varepsilon^\alpha. \end{aligned}$$

In the same way, when $h_\varepsilon(t) \leq 0$ we obtain

$$\big\{2k_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon(t)\big\} \;\geq\; k_\varepsilon(t)^{3/2}, \qquad \big\{2k_\varepsilon(t)^{3/2} + \sqrt{27}\, h_\varepsilon(t)\big\} \;\geq\; M^{3/2}\varepsilon^\alpha.$$

Thus, in both cases we get (see (24))

$$\triangle_\varepsilon(t) \;\geq\; M^{3/2}\,\varepsilon^\alpha\, k_\varepsilon(t)^{3/2}.$$

In the special case when $r = 1$, the discriminant $\triangle_A(t)$ is strictly positive, hence both the inequalities in (25) are strict, and we conclude that $\triangle_\varepsilon(t) \;\geq c > 0$.

Finally, (22) follows directly from (21) and the definition (18) of $\triangle_\varepsilon(t)$. □

In the following Lemma, we consider the 3×3 Sylvester matrix $A^\sharp_\varepsilon$ which has characteristic polynomial $z^3 - k_\varepsilon(t)z + h_\varepsilon(t)$, and exhibit an exact (but possibly non-coercive) symmetrizer for this matrix. We also prove a lower estimate of the symmetrizer.

Lemma 2. *Let $A^\sharp_\varepsilon(t)$ and $Q_\varepsilon(t)$ be defined by*

$$A^\sharp_\varepsilon(t) \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_\varepsilon(t) & k_\varepsilon(t) & 0 \end{pmatrix}, \quad Q_\varepsilon(t) \;=\; \begin{pmatrix} k_\varepsilon(t)^2 & 3h_\varepsilon(t) & -k_\varepsilon(t) \\ 3h_\varepsilon(t) & 2k_\varepsilon(t) & 0 \\ -k_\varepsilon(t) & 0 & 3 \end{pmatrix}.$$

Therefore, $Q_\varepsilon(t)$ is Hermitian and satisfies the equality

$$Q_\varepsilon(t)\,A^\sharp_\varepsilon(t) \;=\; A^\sharp_\varepsilon(t)^*\,Q_\varepsilon(t). \tag{26}$$

Moreover we have, for all $W \in \mathbf{C}^3$, and for some $c > 0$,

$$\big(Q_\varepsilon(t)W, W\big) \;\geq\; c\,|L_\varepsilon(t)W|^2, \tag{27}$$

where

$$L_\varepsilon(t) \;=\; \triangle_\varepsilon(t)^{1/2} \begin{pmatrix} k_\varepsilon(t)^{-1/2} & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1} & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2} \end{pmatrix}.$$

Proof : (26) follows directly from the definitions. As to (27), we observe that

$$L_\varepsilon^{-1} \;=\; (L_\varepsilon^{-1})^* \;=\; \triangle_\varepsilon^{-1/2} \begin{pmatrix} k_\varepsilon^{1/2} & 0 & 0 \\ 0 & k_\varepsilon & 0 \\ 0 & 0 & k_\varepsilon^{3/2} \end{pmatrix},$$

hence

$$\big(L_\varepsilon^{-1}\big)^* Q_\varepsilon L_\varepsilon^{-1} \;=\; \frac{k_\varepsilon^3}{\triangle_\varepsilon}\,\widetilde{Q}_\varepsilon, \tag{28}$$

where

$$\widetilde{Q}_\varepsilon(t) \;\equiv\; \big[\widetilde{q}_{ij}(t)\big]_{1\leq i,j\leq 3} \;=\; \begin{pmatrix} 1 & 3h_\varepsilon k_\varepsilon^{-3/2} & -1 \\ 3h_\varepsilon k_\varepsilon^{-3/2} & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

By (22) it follows that $\| \widetilde{Q}_\varepsilon(t) \| \le C$ on $[0,T]$. Moreover, by (19) and (20), we see the determinant and the minor determinants of $\widetilde{Q}_\varepsilon(t)$ satisfy

$$\det(\widetilde{Q}_\varepsilon(t)) \;=\; 4 - \frac{27h_\varepsilon^2}{k_\varepsilon^3} \;=\; \frac{\triangle_\varepsilon}{k_\varepsilon^3} \;>\; 0$$

$$\Big\{\widetilde{q}_{11}(t)\widetilde{q}_{22}(t) - \widetilde{q}_{12}(t)\widetilde{q}_{21}(t)\Big\} \;=\; 2 - \frac{9h_\varepsilon^2}{k_\varepsilon^3} \;=\; \frac{2}{3} + \frac{\triangle_\varepsilon}{3k_\varepsilon^3} \;>\; 0, \qquad \widetilde{q}_{11}(t) \;=\; 1 \;>\; 0.$$

This implies that the eigenvalues $\mu_1(t), \mu_2(t), \mu_3(t)$ of $\widetilde{Q}_\varepsilon(t)$ are non-negative, and thus we have, for $i = 1, 2, 3$,

$$\mu_i(t) \;=\; \frac{\mu_i(t)\mu_j(t)\mu_k(t)}{\mu_j(t)\mu_k(t)} \;\ge\; \frac{\det(\widetilde{Q}_\varepsilon(t))}{\| \widetilde{Q}_\varepsilon(t) \|^2} \;\ge\; c\,\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3} \qquad (c > 0).$$

Hence we get, for all $\widetilde{W} \in \mathbf{C}^3$,

$$\big(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}\big) \;\ge\; c\,\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3}\,|\widetilde{W}|^2,$$

and consequently, taking $W = L_\varepsilon(t)^{-1}\widetilde{W}$ and recalling (28),

$$\big(Q_\varepsilon(t)W, W\big) \;=\; \frac{k_\varepsilon(t)^3}{\triangle_\varepsilon(t)}\,\big(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}\big) \;\ge\; c\,|\widetilde{W}|^2 \;=\; c\,|L_\varepsilon(t)W|^2. \qquad \square$$

Lemma 2 applies also to the 9×9 block matrices whose blocks are 3×3 scalar matrices :

Lemma 3. *Let I be the 3×3 identity matrix, and $\mathcal{A}_\varepsilon(t), \mathcal{Q}_\varepsilon(t), \mathcal{L}_\varepsilon(t)$ be the 9×9 matrices defined by*

$$\mathcal{A}_\varepsilon(t) \;=\; \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ h_\varepsilon(t)I & k_\varepsilon(t)I & 0 \end{pmatrix}, \quad \mathcal{Q}_\varepsilon(t) \;=\; \begin{pmatrix} k_\varepsilon(t)^2 I & 3h_\varepsilon(t)I & -k_\varepsilon(t)I \\ 3h_\varepsilon(t)I & 2k_\varepsilon(t)I & 0 \\ -k_\varepsilon(t)I & 0 & 3I \end{pmatrix},$$

and

$$\mathcal{L}_\varepsilon(t) \;=\; \triangle_\varepsilon(t)^{1/2} \begin{pmatrix} k_\varepsilon(t)^{-1/2}I & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1}I & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2}I \end{pmatrix}.$$

Then $\mathcal{Q}_\varepsilon(t)$ is Hermitian and satisfies

(29) $$\mathcal{Q}_\varepsilon(t)\mathcal{A}_\varepsilon(t) \;=\; \mathcal{A}_\varepsilon(t)^*\mathcal{Q}_\varepsilon(t),$$

(30) $$\big(\mathcal{Q}_\varepsilon(t)\mathcal{W}, \mathcal{W}\big) \;\ge\; c\,|\mathcal{L}_\varepsilon(t)\mathcal{W}|^2, \qquad \forall\, \mathcal{W} \in \mathbf{C}^9.$$

Proof : Since the 3×3 submatrices in $\mathcal{A}_\varepsilon(t)$, $\mathcal{Q}_\varepsilon(t)$ and $\mathcal{L}_\varepsilon(t)$ consist of the 3×3 identity matrix I, (29) and (30) can be easily derived from (26) and (27) respectively. $\square$

Now, we transform our system (4) in a 9×9 system having for principal part the block Sylvester matrix $\mathcal{A}_\varepsilon(t)$ of Lemma 3. ¿From (4) we deduce that

$$
\begin{aligned}
(i) \qquad\quad V' &= i\xi AV + BV = i\xi A_\varepsilon V + i\xi(A - A_\varepsilon)V + BV,\\
(ii) \quad (A_\varepsilon V)' &= i\xi A_\varepsilon^2 V + i\xi A_\varepsilon(A - A_\varepsilon)V + A_\varepsilon' V + A_\varepsilon BV,\\
(iii) \quad (A_\varepsilon^2 V)' &= i\xi A_\varepsilon^3 V + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV\\
&= \big[i\xi\, h_\varepsilon V + i\xi k_\varepsilon A_\varepsilon V\big] - \xi\, \Im h_{A_\varepsilon} V + i\xi(k_{A_\varepsilon} - k_\varepsilon)A_\varepsilon V\\
&\qquad + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV.
\end{aligned}
$$

In the last equality, we used the Hamilton-Cayley equality (14).

If we put

$$
\mathcal{V} \equiv \mathcal{V}(t,\xi) = \begin{pmatrix} V \\ A_\varepsilon V \\ A_\varepsilon^2 V \end{pmatrix} \in \mathbf{C}^9,
$$

we are able to combine (i), (ii) and (iii), to get the following 9×9 system :

$$
\mathcal{V}' = i\xi\mathcal{A}_\varepsilon(t)\mathcal{V} + i\xi\mathcal{R}_\varepsilon(t)\mathcal{V} - \xi\mathcal{P}_\varepsilon(t)\mathcal{V} + \mathcal{D}_\varepsilon(t)\mathcal{V} + \mathcal{B}_\varepsilon(t)\mathcal{V}, \tag{31}
$$

where $\mathcal{A}_\varepsilon(t)$ is the matrix of Lemma 3, while

$$
\mathcal{R}_\varepsilon(t) = \begin{pmatrix} A - A_\varepsilon & 0 & 0 \\ A_\varepsilon(A - A_\varepsilon) & 0 & 0 \\ A_\varepsilon^2(A - A_\varepsilon) & 0 & 0 \end{pmatrix}, \quad \mathcal{P}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Im h_{A_\varepsilon} I & -i(k_{A_\varepsilon} - k_\varepsilon)I & 0 \end{pmatrix}
$$

$$
\mathcal{D}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ A_\varepsilon' & 0 & 0 \\ (A_\varepsilon^2)' & 0 & 0 \end{pmatrix}, \quad \mathcal{B}_\varepsilon(t) = \begin{pmatrix} B & 0 & 0 \\ A_\varepsilon B & 0 & 0 \\ A_\varepsilon^2 B & 0 & 0 \end{pmatrix}.
$$

Then, we define the energy:

$$
E(t,\xi) = \big(\mathcal{Q}_\varepsilon(t)\mathcal{V}, \mathcal{V}\big).
$$

By the definition of $\mathcal{L}_\varepsilon(t)$, using (19) and (21), we see that

$$
\big(\mathcal{L}_\varepsilon(t)\mathcal{W}, \mathcal{W}\big) \geq c_1 \triangle_\varepsilon(t) k_\varepsilon(t)^{-1}|V|^2 \geq c_2\, \varepsilon^{\alpha/3}|V|^2,
$$

hence, remarking that $\mathcal{Q}_\varepsilon(t)$ is bounded on $[0,T]$, we derive by (30) :

$$
c\,\varepsilon^{\alpha/3}|V|^2 \leq E(t,\xi) \leq C\,|V|^2. \tag{32}
$$

By (29) and (31), considering that $\mathcal{Q}_\varepsilon$ is Hermitian, we get the equality

$$
\begin{aligned}
E'(t,\xi) &= \big(\mathcal{Q}_\varepsilon'\mathcal{V}, \mathcal{V}\big) + \big(\mathcal{Q}_\varepsilon\mathcal{V}', \mathcal{V}\big) + \big(\mathcal{Q}_\varepsilon\mathcal{V}, \mathcal{V}'\big)\\
&= \big(\mathcal{Q}_\varepsilon'\mathcal{V}, \mathcal{V}\big) + i\xi\big(\{\mathcal{Q}_\varepsilon\mathcal{A}_\varepsilon - \mathcal{A}_\varepsilon^*\mathcal{Q}_\varepsilon^*\}\mathcal{V}, \mathcal{V}\big)\\
&\quad + \big(\mathcal{Q}_\varepsilon\{i\xi\mathcal{R}_\varepsilon - \xi\mathcal{P}_\varepsilon + \mathcal{D}_\varepsilon + \mathcal{B}_\varepsilon\}\mathcal{V}, \mathcal{V}\big) + \overline{\big(\mathcal{Q}_\varepsilon\{i\xi\mathcal{R}_\varepsilon - \xi\mathcal{P}_\varepsilon + \mathcal{D}_\varepsilon + \mathcal{B}_\varepsilon\}\mathcal{V}, \mathcal{V}\big)}\\
&= \big(\mathcal{Q}_\varepsilon'\mathcal{V}, \mathcal{V}\big) - 2\xi\Im\big(\mathcal{Q}_\varepsilon\mathcal{R}_\varepsilon\mathcal{V}, \mathcal{V}\big) - 2\xi\Re\big(\mathcal{Q}_\varepsilon\mathcal{P}_\varepsilon\mathcal{V}, \mathcal{V}\big) + 2\Re\big(\mathcal{Q}_\varepsilon\mathcal{D}_\varepsilon\mathcal{V}, \mathcal{V}\big) + 2\Re\big(\mathcal{Q}_\varepsilon\mathcal{B}_\varepsilon\mathcal{V}, \mathcal{V}\big).
\end{aligned}
$$

In order to prove the energy estimate, we'll use the following

Lemma 4. *Let $\mathcal{S}$ be a 9×9 matrix. Then we have, for all $\mathcal{W}\in \mathbf{C}^9$,*

$$\big(\mathcal{S}\mathcal{W},\mathcal{W}\big) \le C \parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1} \parallel \big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big), \tag{33}$$

$$\big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{W}\big) \le C \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S})\mathcal{L}_\varepsilon^{-1} \parallel^{1/2} \big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big), \tag{34}$$

where $C=1/c$, and $c>0$ is given by (30).

Proof : (33) follows directly from (30), noting that $\mathcal{L}_\varepsilon^*=\mathcal{L}_\varepsilon$, indeed :

$$\begin{aligned}\big(\mathcal{S}\mathcal{W},\mathcal{W}\big) &= \big(\mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\mathcal{L}_\varepsilon\mathcal{W},\mathcal{L}_\varepsilon^*\mathcal{W}\big) \le \parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1} \parallel |\mathcal{L}_\varepsilon(t)\mathcal{W}|^2 \\ &\le \frac{1}{c} \parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1} \parallel \big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big).\end{aligned}$$

To prove (34), we use the Schwarz inequality for the scalar product $\langle\mathcal{Y},\mathcal{W}\rangle\equiv \big(\mathcal{Q}_\varepsilon\mathcal{Y},\mathcal{W}\big)$, and (33) with $\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S}$ in place of $\mathcal{S}$. Thus we obtain :

$$\begin{aligned}\big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{W}\big) &= \big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{S}\mathcal{W}\big)^{1/2}\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big)^{1/2} \\ &\le C \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S})\mathcal{L}_\varepsilon^{-1} \parallel^{1/2} \big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big). \qquad \square\end{aligned}$$

By (33) and (34), it follows

$$\begin{aligned}E'(t,\xi) \le C\,E(t,\xi)\Big\{& \parallel \mathcal{L}_\varepsilon^{-1}\mathcal{Q}_\varepsilon'\mathcal{L}_\varepsilon^{-1} \parallel + |\xi| \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^*\mathcal{Q}_\varepsilon\mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \parallel^{1/2} \\ &+ |\xi| \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{P}_\varepsilon^*\mathcal{Q}_\varepsilon\mathcal{P}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \parallel^{1/2} + \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^*\mathcal{Q}_\varepsilon\mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \parallel^{1/2} + \parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^*\mathcal{Q}_\varepsilon\mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \parallel^{1/2}\Big\}.\end{aligned}$$

Now we estimate the five summands in the left side. To this end we first observe that, for any 9×9 block matrix $\mathcal{S}=\big[S_{ij}\big]_{1\le i,j\le 3}$, one has

$$\mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1} = \frac{1}{\triangle_\varepsilon}\,[S_{ij}]_{1\le i,j\le 3} \tag{35}$$

1) Estimate of $\parallel \mathcal{L}_\varepsilon^{-1}\mathcal{Q}_\varepsilon'\mathcal{L}_\varepsilon^{-1} \parallel$: Using (35), we see that

$$\mathcal{L}_\varepsilon^{-1}\mathcal{Q}_\varepsilon'\mathcal{L}_\varepsilon^{-1} = \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}\begin{pmatrix} 2k_\varepsilon^{1/2}k_\varepsilon' I & 3h_\varepsilon' I & -k_\varepsilon^{1/2}k_\varepsilon' I \\ 3h_\varepsilon' I & 2k_\varepsilon^{1/2}k_\varepsilon' I & 0 \\ -k_\varepsilon^{1/2}k_\varepsilon' I & 0 & 0\end{pmatrix},$$

thus, by (16) and (20), we get

$$\parallel \mathcal{L}_\varepsilon^{-1}\mathcal{Q}_\varepsilon'\mathcal{L}_\varepsilon^{-1} \parallel \le \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}\,C\left\{k_\varepsilon^{1/2}|\,k_\varepsilon'| + |\,h_\varepsilon'|\right\} \le \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon}\,C_1\varepsilon^{\alpha-1}. \tag{36}$$

2) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: By the equality

$$\begin{pmatrix} 0 & 0 & Y_1^* \\ 0 & 0 & Y_2^* \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k^2 I & 3hI & -I \\ 3hI & 2kI & 0 \\ -kI & 0 & 3I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y_1 & Y_1 & 0 \end{pmatrix} = 3 \begin{pmatrix} Y_1^* Y_1 & Y_1^* Y_2 & 0 \\ Y_2^* Y_1 & Y_2^* Y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by (35), we find

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon)\mathcal{L}_\varepsilon^{-1} = \frac{3k_\varepsilon}{\triangle_\varepsilon} \begin{pmatrix} (\Im h_{A_\varepsilon})^2 I & -ik_\varepsilon^{1/2}(k_{A_\varepsilon} - k_\varepsilon)\,\Im h_{A_\varepsilon} I & 0 \\ ik_\varepsilon^{1/2}\overline{(k_{A_\varepsilon} - k_\varepsilon)}\,\Im h_{A_\varepsilon} I & k_\varepsilon\,|k_{A_\varepsilon} - k_\varepsilon|^2 I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by (16),

$$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} C\Big\{\varepsilon^{2\alpha} + k_\varepsilon^{1/2} |\, k_{A_\varepsilon} - k_\varepsilon|\varepsilon^\alpha + k_\varepsilon |\, k_{A_\varepsilon} - k_\varepsilon|^2\Big\} \le \frac{k_\varepsilon}{\triangle_\varepsilon} C\varepsilon^{2\alpha}. \tag{37}$$

To compute the products $\mathcal{X}^* \mathcal{Q}_\varepsilon \mathcal{X}$ with $\mathcal{X} = \mathcal{R}_\varepsilon, \mathcal{D}_\varepsilon, \mathcal{B}_\varepsilon$, we note that

$$\begin{pmatrix} X_1^* & X_2^* & X_3^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_\varepsilon^2 I & 3h_\varepsilon I & -k_\varepsilon I \\ 3h_\varepsilon I & 2k_\varepsilon I & 0 \\ -k_\varepsilon I & 0 & 3I \end{pmatrix} \begin{pmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ X_3 & 0 & 0 \end{pmatrix} = Z_\varepsilon\, \mathcal{J} \tag{38}$$

where

$$\mathcal{J} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z = k_\varepsilon^2 X_1^* X_1 + 3h_\varepsilon(X_1^* X_2 + X_2^* X_1) - k_\varepsilon(X_1^* X_3 + X_3^* X_1 - 2X_2^* X_2) + 3X_3^* X_3.$$

3) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: By (38) with $X_j = A_\varepsilon^{j-1}(A - A_\varepsilon)$, $j = 1, 2, 3$, recalling (35), we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} = \frac{k_\varepsilon}{\triangle_\varepsilon} F_\varepsilon\, \mathcal{J},$$

where

$$F_\varepsilon = (A - A_\varepsilon)^* \Big\{k_\varepsilon^2 I + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3A_\varepsilon^{*2} A_\varepsilon^2\Big\} (A - A_\varepsilon).$$

Hence, by (7), we get

$$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} C \,\| A - A_\varepsilon \|^2 \le \frac{k_\varepsilon}{\triangle_\varepsilon} C_2\, \varepsilon^{2\alpha}. \tag{39}$$

4) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: By (38) with $X_1 = 0$, $X_2 = A'_\varepsilon$ and $X_3 = (A_\varepsilon^2)'$, using (35), we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \;=\; \frac{k_\varepsilon}{\triangle_\varepsilon}\, G_\varepsilon\, \mathcal{J},$$

where $G_\varepsilon = 2k_\varepsilon {A'_\varepsilon}^* A'_\varepsilon + 3(A_\varepsilon^2)'^*(A_\varepsilon^2)'$. Hence we get, by (7),

(40) $$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon}\, C \,\| A'_\varepsilon \|^2 \le \frac{k_\varepsilon}{\triangle_\varepsilon}\, C_3\, \varepsilon^{2(\alpha-1)}.$$

5) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: By (38) with $X_1 = B$, $X_2 = A_\varepsilon B$, $X_3 = A_\varepsilon^2 B$, and by (35), we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \;=\; \frac{k_\varepsilon}{\triangle_\varepsilon}\, H_\varepsilon\, \mathcal{J},$$

$$H_\varepsilon \;=\; B^*\Big\{k_\varepsilon^2 + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3{A_\varepsilon^*}^2 A_\varepsilon^2\Big\}B.$$

Hence

(41) $$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} \,\| H_\varepsilon \| \le C_5 \,\frac{k_\varepsilon}{\triangle_\varepsilon}\, \| B(t) \|^2 .$$

By (36), (37), (39), (40), (41), and (19), (21), recalling that $B(t)$ belongs to $L^1(0,T)$, and $\varepsilon < 1$, we find the following estimate, for some $\beta(t) \in L^1(0,T)$,

$$\begin{aligned}
E'(t,\xi) \;&\le\; CE\,\beta(t)\Big[\varepsilon^{\alpha-1}\frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} + \varepsilon^\alpha \frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}}|\xi| + \varepsilon^{\alpha-1}\frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}}\Big] \\
&\le \begin{cases} CE\,\beta(t)\Big[\varepsilon^{\alpha-1}k_\varepsilon^{3/2} + \varepsilon^\alpha k_\varepsilon^{1/2}|\xi| + \varepsilon^{\alpha-1}k_\varepsilon^{1/2}\Big] & \text{if } \; r=1 \\ CE\,\beta(t)\Big[\varepsilon^{-1} + \varepsilon^{\alpha/2}k_\varepsilon^{-1/4}|\xi| + \varepsilon^{\alpha/2-1}k_\varepsilon^{-1/4}\Big] & \text{if } \; r=2,3 \end{cases} \\
&\le \begin{cases} CE\,\beta(t)\Big[\varepsilon^\alpha|\xi| + \varepsilon^{\alpha-1}\Big] \;\le\; CE\,\beta(t)|\xi|^{1-\alpha} & \text{if } \; r=1, \\ CE\,\beta(t)\Big[\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\Big] \le\; CE\,\beta(t)|\xi|^{1/(1+\alpha/2)} & \text{if } \; r=2, \\ CE\,\beta(t)\Big[\varepsilon^{\alpha/3}|\xi| + \varepsilon^{-1}\Big] \le\; CE\,\beta(t)|\xi|^{1/(1+\alpha/3)} & \text{if } \; r=3. \end{cases}
\end{aligned}$$

for $|\xi| > 1$, by choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } \; r=1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \; r=2, \\ |\xi|^{-1/(1+\alpha/3)} & \text{if } \; r=3. \end{cases}$$

Thus, by (32), we get the wished a priori estimate (5), where σ is equal, respectively, to $1/(1-\alpha)$, $1+\alpha/2$, $1+\alpha/3$. This concludes the proof of Theorem 1 in the case $m=3$. □

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