

Morse Index and Liouville Property for Superlinear Elliptic Equations on the Heisenberg Group

I. Birindelli, I. Capuzzo Dolcetta*

1 Introduction

Let us consider the semilinear partial differential equation

$$\Delta_{H^n} u + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^{2n+1} \quad (1.1)$$

where Δ_{H^n} is the second order degenerate elliptic operator often called the Kohn Laplacian on the Heisenberg group H^n and the exponent p satisfies $1 < p < 1 + \frac{2}{n}$. We restrict our attention in this paper to bounded classical solutions of equation (1.1) having finite Morse index. Let us recall that the Morse index $i(u)$ of a solution u of

$$\Delta_{H^n} u + f(x, u) = 0 \quad \text{in } A \quad (1.2)$$

$$u = 0 \quad \text{on } \partial A \quad (1.3)$$

in a general open set A is the number of negative eigenvalues of the linear operator

$$L_u := -\Delta_{H^n} - \frac{\partial f}{\partial t}(x, u(x)) \quad (1.4)$$

on a suitable energy space incorporating the homogeneous Dirichlet boundary condition. In Section 3 we prove a few results , which extend to the present

*Dipartimento di Matematica, Università di Roma "La Sapienza", Piazzale A.Moro 2 - I 00185 Roma, Italy; e-mails: capuzzo@mat.uniroma1.it, isabeau@mat.uniroma1.it

setting similar ones known in the uniformly elliptic case (see [1], [17], [7], [19]), concerning bounded classical solutions of (1.1) having finite Morse index. The main result in that section is Theorem 3.1 stating that bounded classical solutions of (1.1) have in fact finite energy and belong to $L^{p+1}(\mathbb{R}^{2n+1})$.

As a consequence of this result we obtain in Section 4 more precise estimates on the decay at infinity of such solutions as well as some Liouville type non existence results for equation (1.1) . We prove the following result (see Section 2 for the notations) which synthetizes Propositions 4.2 , 4.3 , 4.4:

Theorem 1.1 *Let u be a bounded classical solution of (1.1) with $1 < p < 1 + \frac{2}{n}$ having finite Morse index . Then $u \equiv 0$ provided that one of the following conditions hold :*

$$(i) \quad u(\xi) = 0(|\xi|_{H^n}^{-2n}) \text{ as } |\xi|_{H^n} \rightarrow +\infty \quad ,$$

$$(ii) \quad \frac{X \cdot \nabla u}{|\xi|_{H^n}} \in L^2(\mathbb{R}^{2n+1}) \quad ,$$

$$(iii) \quad u \in L^{(p-1)q}(\mathbb{R}^{2n+1}) \quad , \text{ for some } q < n+1 \quad .$$

The above can be seen as an analogue of the Liouville theorem proved by Bahri-Lions [1] (see also [17]) for equation (1.1) in the euclidean case , that is when Δ_{H^n} is replaced by the standard laplacian Δ .

It is important to point out the role played by the Liouville theorems in some compactness issues arising in connection with the application of the classical Leray-Schauder degree method to the existence of solutions of semilinear boundary value problems such as

$$\Delta_{H^n} u + f(x, u) = 0 \quad , x \in \Omega \tag{1.5}$$

$$u = 0 \text{ on } \partial\Omega \tag{1.6}$$

in an open bounded subset Ω of \mathbb{R}^{2n+1} (see [3]) . Indeed, Liouville theorems for equation (1.1) in the whole space or in a cone (this is typical of the degenerate elliptic case since the limit blow up point may well be a characteristic point of the boundary) allow, in combination with blow-up techniques, to deduce an L^∞ a priori bound for the set of all classical solutions of (1.5) , (1.6) having finite Morse index .

In order to illustrate this, suppose that u_k is a sequence of bounded solutions of (1.5) , (1.6) with $i(u_k) \leq K$, for some integer K , and assume that the nonlinearity f satisfies

$$f, \frac{\partial f}{\partial t} \in C(\overline{\Omega} \times \mathbb{R}) ,$$

$$\lim_{t \rightarrow \pm\infty} \frac{f(x, t)}{t} = +\infty , \lim_{t \rightarrow \pm\infty} \frac{f(x, t)}{t|t|^{\frac{2}{n}}} = 0 ,$$

uniformly for $x \in \overline{\Omega}$. It can be proved (see [3] where the argument below is performed also in the more general case where f may vanish and change sign) that if

$$\|u_k\|_{\infty} \rightarrow +\infty \quad \text{as } k \rightarrow +\infty,$$

then for suitable rescalings \overline{u}_k of functions u_k , two cases can occur in the limit : either \overline{u}_k converge to some function u satisfying

$$\Delta_{H^n} u + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^{2n+1} , u(0) = \|u\|_{\infty} = 1 , \quad i(u) \leq K \quad (1.7)$$

or \overline{u}_k converge to a solution u of

$$\Delta_{H^n} u + |u|^{p-1} u = 0 \quad \text{in } \Sigma , u(0) = \|u\|_{\infty} = 1 , \quad i(u) \leq K \quad (1.8)$$

in a cone Σ with the boundary condition $u = 0$ on $\partial\Sigma$.

If it is known by a Liouville type result that the only function satisfying (1.7) or (1.8) is $u \equiv 0$, then a contradiction arises with the fact that $u(0) = 1$ and therefore $\|u_k\|_{\infty}$ is uniformly bounded.

Our results in Section 4 depend heavily on the papers [10], [15] , [18] . In particular , [15] and [16] also deal with equation (1.1) in the "critical" case $p = 1 + \frac{2}{n}$, arising in connection with the Yamabe problem for CR manifolds (see [14]), as well in the "supercritical" case $p > 1 + \frac{2}{n}$.

Let us conclude by pointing out that the results of the present paper could be extended to the more general framework of sublaplacians on stratified Lie groups (see [6]).

2 Some known facts about Δ_{H^n}

For the sake of completeness, we collect in this section a few basic properties of the Heisenberg group H^n and the operator Δ_{H^n} . For proofs and more information we refer for example to [5, 6, 8, 9, 10, 13].

The Heisenberg group H^n is the space \mathbb{R}^{2n+1} endowed with the group action

$$\xi \circ \eta = (\xi_1 + \eta_1, \dots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2 \sum_{i=1}^n (\xi_{i+n} \eta_i - \xi_i \eta_{i+n})) .$$

The Kohn Laplacian Δ_{H^n} is the second order degenerate elliptic operator

$$\Delta_{H^n} = \sum_{i=1}^{2n} \left(\frac{\partial^2}{\partial \xi_i^2} + 4 \xi_i^2 \frac{\partial^2}{\partial \xi_{2n+1}^2} \right) + 4 \sum_{i=1}^n \left(\xi_{i+n} \frac{\partial^2}{\partial \xi_i \partial \xi_{2n+1}} - \xi_i \frac{\partial^2}{\partial \xi_{i+n} \partial \xi_{2n+1}} \right) \quad (2.9)$$

acting on functions $u = u(\xi)$, where $\xi = (\xi_1, \dots, \xi_{2n}, \xi_{2n+1}) \in \mathbb{R}^{2n+1}$. We shall also use the notation $x = (\xi_1, \dots, \xi_n)$, $y = (\xi_{n+1}, \dots, \xi_{2n})$, $t = \xi_{2n+1}$. Let us consider the anisotropic dilation δ_λ

$$\delta_\lambda(\xi) = (\lambda \xi_1, \dots, \lambda \xi_{2n}, \lambda^2 \xi_{2n+1}) \quad (\lambda > 0)$$

and the homogeneous norm

$$|\xi|_{H^n} = \left(\left(\sum_{i=1}^{2n} \xi_i^2 \right)^2 + \xi_{2n+1}^2 \right)^{\frac{1}{4}} . \quad (2.10)$$

Since $\xi \rightarrow |\xi|_{H^n}$ is homogeneous of degree one with respect to the dilation δ_λ and the Jacobian of δ_λ equals λ^Q with $Q = 2n + 2$ then the Lebesgue measure $|\cdot|$ scales as

$$|B_{H^n}(0, R)| = R^Q |B_{H^n}(0, 1)| \quad (2.11)$$

where $B_{H^n}(0, R)$ is the open ball of radius R centered at 0, that is

$$B_{H^n}(0, R) = \{\eta \in H^n : |\eta|_{H^n} < R\}.$$

For simplicity in the rest of the article we will use B_R to indicate the ball $B_{H^n}(0, R)$.

It is sometimes convenient to look at the expression of Δ_{H^n} as an Hormander square operator, that is

$$\Delta_{H^n} = \sum_{i=1}^{2n} X_i^2 \quad (2.12)$$

with

$$X_i = \frac{\partial}{\partial \xi_i} + 2\xi_{i+n} \frac{\partial}{\partial \xi_{2n+1}}, \quad X_{i+n} = \frac{\partial}{\partial \xi_{i+n}} - 2\xi_i \frac{\partial}{\partial \xi_{2n+1}} \quad (2.13)$$

for $i = 1, \dots, n$. It is easy to check that the vectorfields X_i satisfy

$$[X_i, X_{i+n}] = -4X_{2n+1}, \quad [X_i, X_j] = 0$$

for any $i, j \in \{1, \dots, n\}$. This implies that Δ_{H^n} , although degenerate elliptic, satisfies the Hormander rank condition (see [13]) so that, in particular, Δ_{H^n} is hypoelliptic and it satisfies the Bony's maximum principle (see [5]).

A further useful representation of the Kohn Laplacian is in divergence form as

$$\Delta_{H^n} = \operatorname{div}(\sigma^*(\xi)\sigma(\xi)\nabla) = \operatorname{div}(\sigma^*(\xi)\nabla_{H^n})$$

where $\sigma(\xi)$ is the $(2n) \times (2n+1)$ matrix

$$\sigma(\xi) = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}$$

I_n is the identity $n \times n$ matrix and $*$ denotes transposition. The notation ∇ is for the standard gradient while the Heisenberg gradient ∇_{H^n} of a function φ is defined as

$$\nabla_{H^n}\varphi = (X_1\varphi, \dots, X_{2n}\varphi) = \sigma(\xi)\nabla.$$

We will consider the following functional spaces:

$$\Gamma^1(A) = \{\varphi \in L^\infty(A) \cap C^o(A) : \sup_{\xi, \eta} \frac{|\varphi(\eta \circ \xi) + \varphi(\eta \circ \xi^{-1}) - 2\varphi(\eta)|}{|\xi|_{H^n}} < \infty\},$$

$$\Gamma^2(A) = \{\varphi \in \Gamma^1(A) : X_i\varphi \in \Gamma^1(A) \text{ for } i = 1, \dots, 2n\}.$$

The Sobolev - Stein space $S_o^1(A)$ is the completion of $C_o^\infty(A)$ in the norm

$$\|\varphi\|_{S_o^1} := \left\{ \int_A |\nabla_{H^n}\varphi|^2 \right\}^{\frac{1}{2}}.$$

Finally, the space $S^2(A)$ consists of all functions $\varphi \in L^2(A)$ such that $X_i X_j \varphi \in L^2(A)$ for all $i, j = 1, \dots, 2n$.

3 Some properties of solutions with finite Morse index

Consider the equation

$$\Delta_{H^n} u + |u|^{p-1} u = 0 \quad \text{in } H^n. \quad (3.14)$$

A classical solution of (3.14) is a function $u \in \Gamma^2(A) \cap C^o(H^n)$ satisfying (3.14) pointwise; on the other hand, u is a weak solution of (3.14) if $u \in S_o^1(H^n)$ and

$$\int_{H^n} \nabla_{H^n} u \cdot \nabla_{H^n} \varphi = \int_{H^n} |u|^{p-1} u \varphi \quad (3.15)$$

for any $\varphi \in C_o^\infty(H^n)$.

By definition, the Morse index $i(u)$ of a bounded classical solution u of (3.14) is the number of negative eigenvalues of the operator

$$L_u(v) := -\Delta_{H^n} v - p|u|^{p-1} v$$

in $S_o^1(H^n) \cap L^2(H^n)$ counted with their multiplicity.

In particular, there exist $i(u)$ negative eigenvalues λ_j and $i(u)$ independent functions $\Phi_j \in S_o^1(H^n) \cap L^2(H^n)$ such that

$$\int_{H^n} \nabla_{H^n} \Phi_j \cdot \nabla_{H^n} \varphi - p \int_{H^n} |u|^{p-1} \Phi_j \varphi = \lambda_j \int_{H^n} \Phi_j \varphi \quad (3.16)$$

for any $\varphi \in S_o^1(H^n) \cap L^2(H^n)$. Clearly the quadratic form

$$Q_u(\varphi, \varphi) := \int_{H^n} [|\nabla_{H^n} \varphi|^2 - p|u|^{p-1} |\varphi|^2], \quad (3.17)$$

restricted to the subspace generated by $\Phi_1, \dots, \Phi_{i(u)}$ is negative definite.

In the next two simple propositions we will characterize some subspace of $S_o^1(H^n) \cap L^2(H^n)$ where Q_u is positive definite and we will connect the Morse index with the number of regions where u has a constant sign.

Proposition 3.1 *Let u be a classical solution of (3.14) with $i(u) < +\infty$. Then there exists $R_o \gg 1$ such that*

$$Q_u(\varphi, \varphi) \geq 0$$

for all $\varphi \in S_o^1(H^n \setminus B_{R_o}) \cap L^2(H^n \setminus B_{R_o})$.

Proof: Let $K = i(u)$ and $\lambda_1, \lambda_2, \dots, \lambda_K, \Phi_1, \dots, \Phi_K$ be as in the definition of Morse index. By a standard spectral argument for any ψ in the orthogonal of $\text{span}\{\Phi_1, \dots, \Phi_K\}$, the following properties hold

$$Q_u(\Phi_i, \psi) = 0 \text{ and } Q_u(\psi, \psi) \geq 0.$$

Moreover any $\phi \in S_o^1(H^n) \cap L^2(H^n)$ can be decomposed as $\phi = \sum_{i=1}^K c_i \Phi_i + \psi$ with ψ as above.

The density of $C_o^\infty(H^n)$ in $S_o^1(H^n)$ implies that for any $\varepsilon > 0$ one can find linearly independent functions $\tilde{\Phi}_1, \dots, \tilde{\Phi}_K$ in $C_o^\infty(H^n)$ with compact support such that

$$\|\Phi_j - \tilde{\Phi}_j\|_{S_o^1} + \|\Phi_j - \tilde{\Phi}_j\|_{L^2} \leq \varepsilon, \quad Q_u(\tilde{\Phi}_j, \tilde{\Phi}_j) < 0.$$

Choose now $R_o > 0$ such that

$$\cup_{i=1}^K \text{supp } \tilde{\Phi}_i \subset B_{R_o}.$$

Since any $\varphi \in S_o^1(H^n \setminus B_{R_o})$ is independent of the $\tilde{\Phi}_i$ then a simple computation shows that

$$|c_i| \leq \left| \int_{H^n} \nabla_{H^n} \Phi_i \cdot \nabla_{H^n} \varphi \right| \leq C\varepsilon,$$

for some constant C independent of ε . Hence

$$Q_u(\varphi, \varphi) = \sum_{i=1}^K \lambda_i c_i^2 + Q_u(\psi, \psi) \geq C\varepsilon^2 \sum_{i=1}^K \lambda_i + Q_u(\psi, \psi)$$

for any $\varepsilon > 0$. This completes the proof.

Proposition 3.2 *Let u be a classical solution of equation (3.14) with $p > 1$ and let $n(u)$ be the number of the bounded connected components of the set $\{\xi \in H^n : u(\xi) \neq 0\}$. Then*

$$i(u) \geq n(u).$$

Proof: Let us denote by A_j ($j = 1 \dots n(u)$) a bounded connected component of $\{\xi \in H^n : u(\xi) \neq 0\}$ and observe that A_j is open and that $u \in S_o^1(A_j) \subseteq$

$S_o^1(H^n)$. Then we can choose u as a test function in the definition of weak solution of (3.14) to obtain

$$\int_{A_j} |\nabla_{H^n} u|^2 = \int_{A_j} |u|^{p-1} u^2 < p \int_{A_j} |u|^{p-1} u^2$$

since $p > 1$. Hence we have constructed $n(u)$ linearly independent functions $\varphi_j \equiv u|_{A_j}$ such that $\varphi_j \in S_o^1(H^n)$ and $Q_u(\varphi_j, \varphi_j) < 0$. The conclusion follows by the argument in Proposition 3.1.

Remark. The same result is true assuming that $u \in C^0(H^n) \cap S_o^1(H^n)$ as it follows from the next Theorem.

The main result of this section is the following :

Theorem 3.1 *Let $1 < p < 1 + \frac{2}{n}$ and let u be a bounded classical solution of*

$$\Delta_{H^n} u + |u|^{p-1} u = 0 \quad \text{in } H^n \quad (3.18)$$

with finite Morse index. Then $u \in L^{p+1}(H^n) \cap S_o^1(H^n)$.

Proof: Let R_o be as in Proposition 3.1 and let $\varphi \in C_o^\infty(H^n \setminus B_{R_o})$ be a non-negative function. Then, for $\alpha > 1$, $u\varphi^\alpha \in S_o^1(H^n \setminus B_{R_o})$ and consequently by Proposition 3.1

$$Q_u(u\varphi^\alpha, u\varphi^\alpha) = \int_{H^n} [|\nabla_{H^n}(u\varphi^\alpha)|^2 - p|u|^{p+1}\varphi^{2\alpha}] \geq 0$$

or, which is the same,

$$\int_{H^n} u^2 |\nabla_{H^n} \varphi^\alpha|^2 + \varphi^{2\alpha} |\nabla_{H^n} u|^2 + 2u\varphi^\alpha \nabla_{H^n} \varphi^\alpha \cdot \nabla_{H^n} u \geq p \int_{H^n} |u|^{p+1} \varphi^{2\alpha} \quad (3.19)$$

On the other hand, multiplying equation (3.14) by $u\varphi^{2\alpha}$ and integrating by parts we obtain

$$\int_{H^n} |u|^{p+1} \varphi^{2\alpha} = \int_{H^n} \nabla_{H^n} u \cdot \nabla_{H^n} (u\varphi^{2\alpha}) = \int_{H^n} \varphi^{2\alpha} |\nabla_{H^n} u|^2 + 2u\varphi^\alpha \nabla_{H^n} u \cdot \nabla_{H^n} \varphi^\alpha \quad (3.20)$$

From (3.19) and (3.20) it follows that

$$(p-1) \int_{H^n} |u|^{p+1} \varphi^{2\alpha} \leq \int_{H^n} u^2 |\nabla_{H^n} \varphi^\alpha|^2 \quad (3.21)$$

At this point we take any $R > 2R_o$ and we choose φ such that $0 \leq \varphi \leq 1$ and

$$\varphi(\xi) = \begin{cases} 0 & \text{for } |\xi|_{H^n} \leq R_o \\ 1 & \text{for } 2R_o \leq |\xi|_{H^n} \leq R \\ 0 & \text{for } |\xi|_{H^n} \geq 2R. \end{cases}$$

Observe that $\nabla_{H^n} \varphi$ is supported in $(B_{2R} \setminus B_R) \cup (B_{2R_o} \setminus B_{R_o})$ and, furthermore, that we can choose φ satisfying

$$|\nabla_{H^n} \varphi| \leq \frac{C}{R} \quad \text{in } B_{2R} \setminus B_R \quad |\nabla_{H^n} \varphi| \leq \frac{C}{R_o} \quad \text{in } B_{2R_o} \setminus B_{R_o}$$

for some constant C . From now on we shall denote different constants independent of R by the same letter C .

With this choice of φ inequality (3.21) becomes

$$(p-1) \int_{B_{2R}} |u|^{p+1} \varphi^{2\alpha} \leq \frac{C}{R_o^2} \int_{B_{2R_o} \setminus B_{R_o}} |u|^2 \varphi^{2(\alpha-1)} + \frac{C}{R^2} \int_{B_{2R} \setminus B_R} |u|^2 \varphi^{2(\alpha-1)} \quad (3.22)$$

Since $p > 1$ then by the properties of φ

$$\int_{B_R \setminus B_{2R_o}} |u|^{p+1} \varphi^{2\alpha} \leq C_0 + \frac{C}{R^2} \int_{B_{2R} \setminus B_R} u^2 \varphi^{2(\alpha-1)} \quad (3.23)$$

where

$$C_0 = \frac{C}{R_o^2} \int_{B_{2R_o} \setminus B_{R_o}} u^2 \varphi^{2(\alpha-1)}.$$

We claim now that (3.23) implies

$$\lim_{R \rightarrow +\infty} \int_{B_{2R}} |u|^{p+1} \varphi^{2\alpha} < +\infty \quad (3.24)$$

Indeed, were this false we would have

$$\frac{1}{2} \int_{B_{2R}} \varphi^{2\alpha} |u|^{p+1} \geq C_0$$

for R large enough and consequently,

$$\frac{1}{2} \int_{B_{2R}} \varphi^{2\alpha} |u|^{p+1} \leq \frac{C}{R^2} \int_{B_{2R}} u^2 \varphi^{2(\alpha-1)}. \quad (3.25)$$

Set now $I_R := \int_{B_{2R}} |u|^{p+1} \varphi^{2\alpha}$, choose $\alpha = \frac{p+1}{p-1}$ and majorize the right hand side of (3.25) by use of the Holder inequality to obtain, recalling (2.11), the estimate

$$I_R \leq C I_R^{\frac{2}{p+1}} R^{\frac{Q(p-1)}{p+1}-2}. \quad (3.26)$$

Since $1 < p < 1 + \frac{2}{n} \equiv \frac{Q+2}{Q-2}$ by assumption, the exponent of R is negative and hence

$$I_R^{1-\frac{2}{p+1}} \leq C R^{\frac{Q(p-1)}{p+1}-2}, \quad (3.27)$$

implies that I_R tends to zero as $R \rightarrow +\infty$. This contradiction proves the validity of (3.23). From (3.23) it follows easily, taking the boundedness of u into account, that $\int_{H^n} |u|^{p+1} < +\infty$.

It remains to be proved that $u \in S_o^1(H^n)$. We choose at this purpose a cut-off function

$$\varphi(\xi) = \begin{cases} 1 & \text{for } |\xi|_{H^n} \leq R \\ 0 & \text{for } |\xi|_{H^n} \geq 2R \end{cases}$$

and such that $|\nabla_{H^n} \varphi| \leq \frac{C}{R}$.

After multiplication of equation (3.14) by $u\varphi^2$ and integration by parts we obtain

$$\int_{B_{2R}} |\nabla_{H^n} u|^2 \varphi^2 = \int_{B_{2R}} |u|^{p+1} \varphi^2 - 2 \int_{B_{2R}} u \varphi \nabla_{H^n} u \cdot \nabla_{H^n} \varphi$$

By Young and Holder inequalities we deduce

$$\int_{B_{2R}} |\nabla_{H^n} u|^2 \leq C \left\{ \int_{B_{2R}} |u|^{p+1} \varphi^2 + R^{\frac{Q(p-1)}{p+1}-2} \left\{ \int_{B_{2R}} |u|^{p+1} \varphi^2 \right\}^{\frac{2}{p+1}} \right\} \quad (3.28)$$

Since we know from the first part of the proof that $\int_{H^n} |u|^{p+1} < +\infty$, letting $R \rightarrow +\infty$ in the above concludes the proof of Theorem 3.1.

4 Decay at infinity and Liouville type results

In this section we shall always assume

$$1 < p < 1 + \frac{2}{n} \quad (4.29)$$

and that

$$u \text{ is bounded classical solution of (1.1) with } i(u) < +\infty \quad (4.30)$$

and draw some relevant consequences of Theorem 3.1. The first one comprises a summability and a decay result (see [10] for a related result):

Proposition 4.1 *Assume (4.29) and (4.30). Then $u \in L^q(H^n)$ for every $q \in [p+1, +\infty)$ and $u \rightarrow 0$ as $|\xi|_{H^n} \rightarrow +\infty$.*

Proof. Let us recall (see [9]) that the solution of

$$\Delta_{H^n} v = f$$

is given by

$$v = \Gamma * f := \int_{H^n} \Gamma(\xi, \eta) f(\eta) d\eta$$

where the function

$$\Gamma(\xi, \eta) := C_Q |\eta^{-1} \circ \xi|_{H^n}^{2-Q}$$

is the fundamental solution for Δ_{H^n} . Moreover if $f \in L^q(H^n)$ then

$$v = \Gamma * f \in L^r(H^n) \text{ for } \frac{1}{r} = \frac{1}{q} - \frac{2}{Q}.$$

Let u satisfy (4.29), (4.30), by Theorem 3.1, $u \in L^{p+1}(H^n)$ hence $f := |u|^{p-1}u \in L^{\frac{p+1}{p}}$; . Therefore

$$u = \Gamma * (|u|^{p-1}u) \in L^r(H^n) \text{ with } \frac{1}{r} = \frac{p}{q} - \frac{2}{Q}, \quad (4.31)$$

since $\frac{p}{p+1} > \frac{2}{Q}$. Moreover, by the immersion $S_O^1(H^n) \subset L^{\frac{2Q}{Q-2}}(H^n)$, using again Theorem 3.1 we conclude that

$$u \in L^q(H^n) \text{ for any } q \in [p+1, \frac{2Q}{Q-2}]. \quad (4.32)$$

It is easy to see that $\frac{(p-1)Q}{2} \leq p+1 \leq \frac{pQ}{2}$. Now two cases are possible:

$$\text{either } \frac{pQ}{2} < \frac{2Q}{Q-2} \text{ or } \frac{pQ}{2} \geq \frac{2Q}{Q-2}.$$

Case 1: $\frac{pQ}{2} < \frac{2Q}{Q-2}$

In this case we apply (4.31) to $q \in [p+1, \frac{pQ}{2}]$ to obtain that $u \in L^r(H^n)$ for any r such that $\frac{1}{r} \in (0, \frac{p+1}{p}]$ i.e. $u \in L^q(H^n)$ for any $q \in [p+1, +\infty]$.

Case 2: $\frac{2Q}{Q-2} \leq \frac{pQ}{2}$

In this case, by (4.31), $u \in L^r(H^n)$ for any r such that $\frac{1}{r} \in [\frac{p(Q-2)}{2Q} - \frac{2}{Q}, \frac{p}{p+1} - \frac{2}{Q}]$. Since $\frac{2Q}{p(Q-2)-4} > \frac{pQ}{2}$, we are back in Case 1. This completes the proof of the first part of the Proposition.

For the second statement we use the following weak-Harnack type inequality

$$\sup_{B_{H^n}(\xi, R)} u \leq CR^{-Q/q} \|u\|_{L^q(B_{H^n}(\xi, 2R))}$$

for any ball $B_{H^n}(\xi, 2R)$, $q > 1$ and some $C > 0$. This holds for if $u \in S_o^1(\Omega)$ satisfies

$$\Delta_{H^n} u + Vu = 0 \in H^n$$

with V bounded (see [15]). By choosing $V = |u|^{p-1}$ and $q = p+1$ we obtain

$$\sup_{B_{H^n}(\xi, R)} u \leq CR^{-Q/p+1} \|u\|_{L^{p+1}(B_{H^n}(\xi, 2R))}$$

Since $u \in L^{p+1}(H^n)$ the right hand side converges to zero as $|\xi|_{H^n}$ goes to infinity and this completes the proof of the Proposition 4.1.

The next two propositions state some non - existence Liouville type results for equation (1.1) under different energy or decay conditions. Observe preliminarily that Theorem 3.1 implies that under the assumptions (4.29) and (4.30) any solution u of (1.1) satisfies the identity

$$\int_{H^n} |\nabla_{H^n} u|^2 = \int_{H^n} |u|^{p+1} \quad (4.33)$$

Proposition 4.2 *Assume (4.29) and (4.30) . Then $u \equiv 0$, provided that*

$$\frac{(x, y, 2t) \cdot \nabla u}{|\xi|_{H^n}} \in L^2(H^n)$$

Proof. Let us observe first that u satisfies for each $R > 0$ the following Rellich-Pohozaev type identity due to Garofalo - Lanconelli [10] :

$$2 \int_{\partial B_R} (A(\xi) \nabla u \cdot N) X \cdot \nabla u d\sigma - \int_{\partial B_R} |\nabla_{H^n} u|^2 X \cdot N d\sigma = \quad (4.34)$$

$$(2 - Q) \int_{B_R} |\nabla_{H^n} u|^2 + 2 \int_{B_R} X \cdot \nabla u \Delta_{H^n} u.$$

Here , $A = \sigma^* \sigma$ (see Section 2) , $X = (x, y, 2t)$ and N is the outer unit normal to ∂B_R . Equation (1.1) yields immediately

$$(X \cdot \nabla u) \Delta_{H^n} u = -(X \cdot \nabla u) |u|^{p-1} u = -X \cdot \nabla \left(\frac{|u|^{p+1}}{p+1} \right).$$

The above equality, the divergence theorem, and the fact that $\operatorname{div} X = Q$ yield

$$(2 - Q) \int_{B_R} |\nabla_{H^n} u|^2 + \frac{2Q}{p+1} \int_{B_R} |u|^{p+1} = \quad (4.35)$$

$$2 \int_{\partial B_R} (A \nabla u \cdot N) X \cdot \nabla u d\sigma - \int_{\partial B_R} |\nabla_{H^n} u|^2 X \cdot N d\sigma - \frac{2}{p+1} \int_{\partial B_R} |u|^{p+1} X \cdot N d\sigma$$

Since by Theorem 3.1 $u \in L^{p+1}(H^n) \cap S_o^1(H^n)$, the second and third term on the right hand side of (4.35) tend to zero as $R \rightarrow +\infty$.

The assumption

$$\frac{(x, y, 2t) \cdot \nabla u}{|\xi|_{H^n}} \in L^2(H^n)$$

implies (see Theorem 2.3 in [10]) that the first term tends to zero as well.

Hence , letting $R \rightarrow +\infty$ in (4.35) we obtain

$$(2 - Q) \int_{H^n} |\nabla_{H^n} u|^2 + \frac{2Q}{p+1} \int_{H^n} |u|^{p+1} = 0$$

Comparing this with (4.33) we conclude , since $p < \frac{Q+2}{Q-2}$, that $\int_{H^n} |u|^{p+1} = 0$, that is $u \equiv 0$.

Proposition 4.3 *Assume (4.29) and (4.30) . Then $u \equiv 0$, provided that*

$$|u(\xi)| = O(|\xi|_{H^n}^{2-Q}). \quad (4.36)$$

Proof: From (4.36) it follows that for any $s < Q - 2$, $|u(\xi)| = O(|\xi|_{H^n}^{-s})$ for $|\xi|_{H^n}$ large. Hence $\frac{|X \cdot \nabla u(\xi)|}{|\xi|_{H^n}} = O(|\xi|_{H^n}^{-s-1})$; . On the other hand, $|\xi|_{H^n}^{-s-1} \in L^2(H^n \setminus B_R)$ for $(s+1)2 > Q$. Since this holds true for some $s < Q - 2$, the result follows from Proposition 4.2.

Our last result is:

Proposition 4.4 *Assume (4.29) and (4.30) . Then $u \equiv 0$, provided that*

$$u \in L^q(H^n) \text{ for some } q < \frac{Q}{2(p-1)} . \quad (4.37)$$

Proof. By Theorem 3.1 we know that $u \in S_o^1(H^n)$. Proposition 4.1 implies on one hand that $u(\xi)$ tends to 0 as $|\xi|_{H^n}$ tends to infinity and on the other, using (4.37), we deduce that $|u|^{p-1} \in L^{q_1}(H^n) \cap L^{\frac{Q}{2}}(H^n) \cap L^{q_2}(H^n)$ for some $q_1 \leq \frac{Q}{2} \leq q_2$.

By Theorem 1.3 in [18], $u(\xi) = O(|\xi|_{H^n}^{-s})$ for any $s < Q - 2$. At this point the conclusion follows from Proposition 4.2.

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