NoDEA
Nonlinear differ. equ. appl. 0 (2003) 000–000
1021–9722/03/000000–00 $ 1.50+0.20/0

Nonlinear Differential Equations
and Applications NoDEA

Superharmonic functions in the Heisenberg group: estimates and Liouville theorems

Isabeau BIRINDELLI
Dipartimento di Matematica
Universita di Roma"La Sapienza"
Piazzale Aldo Moro 2
00185 Roma
e-mail: isabeau@mat.uniroma1.it

Abstract. We state some growth conditions on non negative superharmonic functions of the Heisenberg group in intrinsic cones and prove some non existence results for non negative Δ_H polyharmonic functions and L^1 Δ_H superharmonic functions.

2000 *Mathematics Subject Classification*: 35J70.
Key words: Heisenberg group, Kohn Laplacian, superharmonic functions, half-spaces, Liouville Theorems.

1 Introduction

We denote by H^n the vector space $\mathbb{R}^{2n+1}$ endowed with the group action:

$$\xi' \circ \xi = \left(x + x', y + y', t + t' + 2\sum_{i=1}^{n}(x_i y_i' - y_i x_i') \right)$$

where $\xi := (x_1, \ldots, x_n, y_1, \ldots, y_n, t) := (x, y, t)$. H^n, called the Heisenberg group, is a Lie group and the corresponding Lie Algebra of left-invariant vector fields is generated by the following vector fields

$$\begin{cases} X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \\ T = \frac{\partial}{\partial t}, \end{cases}$$

for $i = 1, \ldots, n$. The second order self-adjoint operator:

$$\Delta_H := \sum_{i=1}^{n} (X_i)^2 + (Y_i)^2 \tag{1}$$

i.e.

$$\Delta_H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2}$$

is usually called the Heisenberg Laplacian. Let us define the "norm" in H^n introduced by Folland and Stein, see [6]:

$$|\xi|_H = \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right)^2 + t^2 \right)^{\frac{1}{4}}.$$

Using the group action, the "intrinsic" distance in H^n is defined by: $d(\xi_1, \xi_2) = |\xi_2^{-1} \circ \xi_1|_H$. The homogeneous dimension of H^n is $Q = (2n+1) + 1 = 2n + 2$. It is a known fact that if u is a non negative superharmonic function in $\mathbb{R}^N$ and $u \in C^2$ then there exists a constant C such that

$$u(x) \geq C|x|^{2-N},$$

for any x such that $|x| \geq 1$ it is less known that there are equivalent results for superharmonic functions in cones. Indeed suppose u is a positive superharmonic function in a cone $\Sigma \subset \mathbb{R}^n$ with e.g. vertex the origin, let λ_1 be the principal eigenvalue of $-\Delta$ restricted to Σ intersection with the unit sphere, then for any $\alpha > 0$ such that $\alpha(\alpha + n - 2) > \lambda_1$ and for any cone Σ_o strictly contained in Σ, there exists a positive constant C such that

$$u(x) \geq C|x|^{2-N-\alpha}, \tag{2}$$

for any $x \in \Sigma_o$ such that $|x| \geq 1$, see e.g. [3]. Let us recall that if ϕ is the eigenfunction corresponding to λ_1 then $v = |x|^{2-n-\alpha}\phi$ is a positive harmonic function in the cone Σ, null on the boundary of Σ. In this paper we shall prove similar results in the Heisenberg space, and we shall deduce some Liouville theorems for L^1 superharmonic functions in some half spaces. Our main result in the whole space and in half spaces reads as follows

Theorem 1.1 *If u is a non negative Δ_H superharmonic function in H^n i.e $u \in C^2$ and*

$$-\Delta_H u \geq 0 \text{ in } H^n$$

then, if u is not identically zero, there exists a constant C such that

$$u(\xi) \geq C|\xi|_H^{2-Q}, \tag{3}$$

for any ξ such that $|\xi|_H \geq 1$. Suppose ν is a non trivial C^2 non negative Δ_H superharmonic function in $\Sigma = \{\xi : t > 0\}$ i.e.

$$\nu \geq 0 \quad \textit{and} \quad -\Delta_H \nu \geq 0 \in \Sigma$$

then there exists $\overline{\varepsilon} > 0$ and $\overline{\delta} > 0$ such that, for any $0 < \varepsilon < \overline{\varepsilon}$ and any $0 < \delta < \overline{\delta}$, there exists a constant C such that

$$\nu(\xi) \geq C|\xi|_H^{-Q-\varepsilon},$$

for any $\xi \in D_\delta := \{\xi : \ t > \delta(x^2 + y^2)\}$.

Remark 1.2 For simplicity we have stated Theorem 1.1 for the whole space and the half space $t \geq 0$. However, as was pointed out to us by F. Uguzzoni, Δ_H is invariant with respect to the linear application T defined by $T(x, y, t) = (y, x, -t)$. Since the image through T of the half space $t \geq 0$ is the half space $t \leq 0$, Theorem 1.1 holds also there. Furthermore if $\Gamma := \{\xi : a \cdot x + b \cdot y + ct > d$ with $c \neq 0\}$ then there exists $\xi_o \in H^n$ such that

$$\xi_o \circ \Gamma = \{\xi : \ \ t > 0\} \ \text{ or } \xi_o \circ \Gamma = \{\xi : \ \ t < 0\}.$$

Hence, by the invariance of the Δ_H with respect to the group action $\circ$, Theorem 1.1 holds true for any half space Γ.

Remark 1.3 However, for half spaces $\Gamma_x := \{\xi \in H^n$ such that $a \cdot x + b \cdot y \geq 0\}$, we cannot prove any similar result, in fact it would be interesting to know if the result is still true.

Let $S_1 = \{\xi : |\xi|_H = 1\}$ and $\phi = \frac{t}{\rho^2}$, we can define the radialized ν by

$$\nu_\#(\rho) := \int_{\Sigma \cap S_1} \nu(\rho, \theta)\phi(\theta)h(\theta)d\theta,$$

where θ are the coordinates on S_1 and $h(\theta) = \frac{x^2+y^2}{\rho^2}$. We obtain this other growing condition

Proposition 1.4 *Suppose ν is a non negative H^n super harmonic function in Σ then there exists $C > 0$ such that*

$$\nu_\#(\rho) \geq C\rho^{-Q}$$

for any $\rho > 1$.

Clearly, since $\phi h \leq 1$, Proposition 1.4 implies

Corollary 1.5 *Suppose ν is a non negative Δ_H superharmonic function in a half-space Γ such that $\nu \in C^2 \cap L^1(\Gamma)$ then $\nu \equiv 0$.*

F. Uguzzoni in [14] proved that in the half-spaces Γ_x (i.e. those not covered by corollary 1.5), the only non negative L^1, Δ_H superharmonic functions is the trivial one (see also [12] for more recent results). Other non-existence theorems have been given in [1] and [2] for semi-linear equations in the Heisenberg group. Those results differ completely in nature with corollary 1.5 because there the solutions are not required to decay at infinity. Other non existence results for semi-linear equations where given by E. Lanconelli and F. Uguzzoni (see e.g. [11]) they lead to the work F. Uguzzoni and G. Citti in [15] where they obtain the equivalent of the famous Bahri Coron result in the setting of the Heisenberg group. Another application of Theorem 1.1 concerns higher order operators. We will say that ν is a Δ_H super polyharmonic function of order m in Σ if

$$(-1)^s {\Delta_H}^s \nu \geq 0 \text{ for } s = 1, \ldots, m > 1 \text{ in } \Sigma, \tag{4}$$

where ${\Delta_H}^s$ denotes the Heisenberg laplacian applied s times.

Theorem 1.6 *If $2m \geq Q$ then the only non negative Δ_H super polyharmonic functions of order $2m$ in H^n are constants. If $2m \geq Q + 2$ then the only non negative Δ_H super polyharmonic functions in Σ are constants.*

In fact we will prove that results similar to those stated above (such as Theorem 1.1 and Proposition 1.4) hold for "intrinsic cones" symmetric in x and y i.e. for

$$D_\delta := \{\xi : \ t > \delta(x^2 + y^2)\},$$

for all $\delta \leq 0$. The lower bound for Δ_H superharmonic functions will be

$$\nu(\xi) \geq C|\xi|_H^{2-Q-\alpha_\delta-\varepsilon},$$

where $0 < \alpha_\delta \leq 2$ is a constant that will be defined later. α_δ plays the role of α in the euclidean case as seen in (2). Indeed α_δ is such that there exists some function ψ on the unit sphere $S^1 = \{\xi \text{ such that } |\xi|_H = 1\}$ that satisfies: $v(\xi) = |\xi|_H^{2-Q-\alpha_\delta}\psi$ is Δ_H-harmonic and positive in the cone, and zero on the boundary of the cone. However, while in the euclidean setting α is just the solution of a second order algebraic equation, here it is a solution of a non linear "eigenvalue" problem on $\Sigma \cap S^1$. It is easy to see that for $\delta = 0$ (i.e. in half spaces) $\alpha_\delta = 2$, hence the statement of 1.1.

Remark 1.7 With similar arguments and constructions, the author and J. Prajapat in [4] have proved a maximum principle in unbounded domains of H^n.

2 Proofs

The first part of this section is dedicated to prove the existence of α_δ as defined in the introduction, which is the core of this paper. Then we will state the theorem concerning the growths of superharmonic function in cones i.e. the generalization to cones of Theorem 1.1. We begin by introducing some notions and some notations. We will denote by

$$B_H(\eta, r) = \{\xi \in \mathbb{R}^{2n+1} \ : \ d(\xi, \eta) < r\} \tag{5}$$

the Heisenberg ball, also called "Boule de Korànyi", which will play the role of the euclidean ball in H^n. Clearly the vector fields X_i, Y_i and T are homogeneous with respect to the distance $d(.,.)$ of degree -1, -1, -2, respectively. Therefore, if we consider in this metric the polar coordinates $\rho = d(\xi, 0)$ and θ the coordinates on $\partial B_H(0,1)$, we can define, see e.g. Jerison in [9], the differential operators on the unit Heisenberg sphere $S^1 \equiv \partial B_H(0,1)$ R_i^α, S_i^α and Z by:

$$X_i(u(\theta)\rho^\alpha) = R_i^\alpha(u(\theta))\rho^{\alpha-1},$$

$$Y_i(u(\theta)\rho^\alpha) = S_i^\alpha(u(\theta))\rho^{\alpha-1},$$

$$T(u(\theta)\rho^\alpha) = Z(u(\theta))\rho^{\alpha-2}.$$

Of course,

$$R_i^\alpha = \hat{R}_i + \alpha a_i,$$

$$S_i^\alpha = \hat{S}_i + \alpha b_i,$$

$$Z = \hat{Z} + \alpha \rho T \rho,$$

where $a_i \equiv X_i(\rho)$, $b_i \equiv Y_i(\rho)$ and $\hat{R}_i$, $\hat{S}_i$, $\hat{Z}$ are vector fields on the Heisenberg sphere (they are null on constant functions) satisfying $[\hat{R}_i, \hat{S}_i] = -4\hat{Z}$. We consider the differential operator L^α defined in the following way:

$$\Delta_H(u(\theta)\rho^\alpha) = [L^\alpha(u(\theta))]\rho^{\alpha-2}. \tag{6}$$

Clearly,

$$\begin{aligned} L^\alpha &= \sum_{i=1}^n R_i^{\alpha-1}R_i^\alpha + S_i^{\alpha-1}S_i^\alpha \\ &= \sum_{i=1}^n \hat{R}_i^2 + \hat{S}_i^2 + (2\alpha-1)(a_i\hat{R}_i + b_i\hat{S}_i) \\ &\quad + \alpha(\alpha-1)(a_i^2+b_i^2) + \alpha(\hat{R}_i a_i + \hat{S}_i b_i). \end{aligned}$$

On the other hand, to determine $(\hat{R}_i a_i + \hat{S}_i b_i)$, we can use the fact that $\Delta_H \rho^2 = 2Q\sum_{i=1}^n (a_i^2 + b_i^2)$, and therefore:

$$2Q\sum_{i=1}^n (a_i^2+b_i^2) = L^2(1) = \sum_{i=1}^n 2(a_i^2+b_i^2) + 2(\hat{R}_i a_i + \hat{S}_i b_i).$$

Hence, we obtain

$$\sum_{i=1}^{n}(\hat{R}_i a_i + \hat{S}_i b_i) = (Q-1)\sum_{i=1}^{n}(a_i^2 + b_i^2) \tag{7}$$

i.e.

$$L^\alpha u = \sum_{i=1}^{n} \hat{R}_i^2 u + \hat{S}_i^2 u + (2\alpha - 1)(a_i \hat{R}_i + b_i \hat{S}_i)u + \alpha(Q-2+\alpha)hu, \tag{8}$$

where $h = \sum_{i=1}^{n}(a_i^2 + b_i^2) = \frac{x^2+y^2}{\rho^2}$. We can define the following $2n+1$ functions on S^1:

$$\phi_1^i(\theta) := \frac{x_i}{\rho}, \quad \phi_2^i(\theta) := \frac{y_i}{\rho},$$
$$\phi(\theta) := \frac{t}{\rho^2}.$$

Clearly, since

$$X_i(t) = 2y_i \text{ and } Y_i(t) = -2x_i,$$

using the definitions of the vector fields on S^1 we obtain

$$X_i(\rho^2\phi) = \rho(\hat{R}_i\phi + 2a_i\phi) = 2\rho\phi_2^i, \;\; Y_i(\rho^2\phi) = \rho(\hat{S}_i\phi + 2b_i\phi) = -2\rho\phi_1^i. \tag{9}$$

Hence,

$$\hat{R}_i\phi = -2a_i\phi + 2\phi_2^i(\theta), \quad \hat{S}_i\phi = -2b_i\phi - 2\phi_1^i. \tag{10}$$

Similarly, it is easy to prove that, since $X_i(y_i) = 0$ and $Y_i(x_i) = 0$

$$\hat{R}_i\phi_2^i = -2a_i\phi_2^i, \quad \text{and } \hat{S}_i\phi_1^i = -2b_i\phi_1^i. \tag{11}$$

Let us prove the following

Proposition 2.1 *There exists $\overline{\delta} > 0$ such that, for any $\delta < \overline{\delta}$, there exist $\alpha = \alpha(\delta) > 0$ and $u_\alpha(\phi) \in C^2$ such that $v = \rho^\alpha u_\alpha(\phi)$ satisfies*

$$\begin{cases} \Delta_H v = 0 \quad in \quad D_\delta := \{\xi : t > \delta(x^2+y^2)\}, \\ v > 0 \quad in\ D_\delta, \;\; v = 0 \;\; on\ \partial D_\delta. \end{cases} \tag{12}$$

Proof. Let $v(\rho,\phi) = \rho^\alpha u(\phi)$ with α and u to be determined. Using (6) and (8), we get that

$$\begin{aligned}\Delta_H v &= \rho^{\alpha-2}\sum_{i=1}^{n}(\hat{R}_i^2 u + \hat{S}_i^2 u + (2\alpha-1)(a_i\hat{R}_i + b_i\hat{S}_i)u + \alpha(Q-2+\alpha)hu) \\ &= \rho^{\alpha-2}\sum_{i=1}^{n}(u''((\hat{R}_i\phi)^2 + (\hat{S}_i\phi)^2) + u'((\hat{R}_i^2 + \hat{S}_i^2)\phi \\ &\quad + (2\alpha-1)(a_i\hat{R}_i + b_i\hat{S}_i)\phi) + \alpha(Q-2+\alpha)hu).\end{aligned}$$

(10) implies:

$$\sum_{i=1}^{n}(a_i\hat{R}_i + b_i\hat{S}_i)\phi = -2h\phi + 2\sum_{i=1}^{n} a_i\phi_2^i - b_i\phi_1^i = h\phi(-2+2) = 0. \tag{13}$$

Again from (10), (11), (7) and from the fact that $a_i\phi_2^i - b_i\phi_1^i = \phi(a_i^2 + b_i^2)$ one gets

$$\sum_{i=1}^{n}(\hat{R}_i^2\phi + \hat{S}_i^2\phi) \;=\; -2Qh\phi. \tag{14}$$

Hence, using (14) and (13), we obtain that v satisfies:

$$\Delta_H v = \rho^{\alpha-2}(4(1-\phi^2)u''(\phi) - 2Q\phi u'(\phi) + \alpha(Q-2+\alpha)u(\phi)). \tag{15}$$

In order to prove Proposition 2.1 we have to find a function u that satisfies the following equation

$$\begin{cases} \mathcal{L}_\alpha u(\phi) = 0,\ u > 0 & \text{for } \frac{\delta}{\sqrt{1+\delta^2}} < \phi \le 1 \\ u = 0 & \text{for } \phi = \frac{\delta}{\sqrt{1+\delta^2}}; \end{cases}$$

where the operator $\mathcal{L}_\alpha u(\phi) = (1-\phi^2)u''(\phi) - \frac{1}{2}Q\phi u'(\phi) + \frac{1}{4}\alpha(Q-2+\alpha)u(\phi)$ is a Jacobi operator (see Jerison [10]). From standard results, the solutions of $\mathcal{L}_\alpha u(\phi) = 0$ are hypergeometric series, precisely:

$$u_\alpha(\phi) = F\left(-\frac{\alpha}{2}, \frac{Q-2+\alpha}{2}, \frac{Q}{4}, \frac{1}{2}(1-\phi)\right). \tag{16}$$

Let us recall that

$$F(a,b,c,x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}$$

where $(a)_k = \prod_{j=0}^{k-1}(a+j)$. Observe that $u_\alpha(1) = 1$, hence u_α is regular in 1 and positive in a left neighborhood of 1. We have to check that for each $\delta \le \overline{\delta}$ there exists α such that $u_\alpha(\frac{\delta}{(1+\delta^2)^{\frac{1}{2}}}) = 0$. Before going ahead, to simplify notations let us introduce $\beta := -\frac{\alpha}{2}$ and $x := \frac{1}{2}(1-\phi)$ i.e.:

$$u_\alpha(\phi) = F\left(\beta, n-\beta, \frac{n+2}{2}, x\right) \equiv G(\beta, x).$$

In terms of G, we have that $G(\beta,0)=1$ and we want to prove **Claim:** *For each $\beta\in[-1,0)$, there exists $x_\beta\in[\frac{1}{2},1)$ such that $G(\beta,x)>0$ for $x\in[1,x_\beta)$ and* $G(\beta,x_\beta)=0$. First observe that $G(-1,x)=1-2x$, therefore

$$x_{-1}=\frac{1}{2}. \tag{17}$$

Second observation, since

$$G(\beta,x)=1+\sum_{k=1}^{\infty}\frac{(\beta)_k(n-\beta)_k}{(\frac{n+2}{2})_k}\frac{x^k}{k!},$$

it is easy to see that

$$\frac{\partial G}{\partial x}=\beta\left(\sum_{k=1}^{\infty}\frac{(\beta+1)_{k-1}(n-\beta)_k}{(\frac{n+2}{2})_k}\frac{x^{k-1}}{(k-1)!}\right).$$

Therefore if $-1<\beta<0$ then $\frac{\partial G}{\partial x}<0$ for $x>0$. Furthermore $\lim_{x\to 1}G(\beta,x)=-\infty$, hence there exits $0<x_\beta<1$ such that $G(\beta,x_\beta)=0$. The first part of the claim is proved. Using the fact that G is monotone decreasing if $G(\beta,\frac{1}{2})>0$ then we will know that $\frac{1}{2}<x_\beta<1$ which proves the claim. Let us compute $G(\beta,\frac{1}{2})$. It is easy to see that

$$G\left(\beta,\frac{1}{2}\right)=1+\beta\left(\frac{n-\beta}{n+2}\right)+\beta\left[\sum_{k=2}^{\infty}I_kJ_k\right] \tag{18}$$

where

$$I_k=\prod_{j=0}^{k-1}\left(\frac{n-\beta+j}{n+2j}\right)$$

and for $k\geq 3$

$$J_2=\frac{\beta+1}{2};\ \ J_k=(\beta+1)(\beta+2)\left[\frac{(\beta+3)\dots(\beta+k-1)}{k!}\right].$$

Clearly for $-1<\beta<0$, $I_k\leq 1$ and for $k\geq 3$

$$J_k\leq(\beta+1)(\beta+2)\cdot\frac{1}{k(k-1)}.$$

Recalling that

$$\sum_{k=3}^{+\infty}\frac{1}{k(k-1)}=\frac{1}{2},$$

and using the above inequalities, (18) becomes

$$G\left(\beta, \frac{1}{2}\right) \geq 1 + \beta + \beta\frac{(\beta+1)}{2} + \beta(\beta+1)(\beta+2)\frac{1}{2}$$

i.e.

$$G\left(\beta, \frac{1}{2}\right) = \frac{1}{2}(1+\beta)(2+3\beta+\beta^2) > 0$$

for $-1 < \beta < 0$. This proves the claim. Coming back to $u_\alpha(\phi)$, we have found that, for $\alpha = 2$, $u_2(\phi) = \phi$ and for any $0 < \alpha < 2$ there exists a value $\phi_\alpha < 0$ such that $u_\alpha(\phi_\alpha) = 0$. Furthermore $\lim_{\alpha\to 0}\phi_\alpha = -1$ Now that we have defined ϕ_α, let α_δ be such that

$$\frac{\phi_\alpha}{(1-\phi_\alpha^2)^{\frac{1}{2}}} = \delta \ \text{ i.e. } \ \phi_\alpha = \frac{\delta}{(1+\delta^2)^{\frac{1}{2}}}.$$

For any $\delta \leq 0$, $0 < \alpha(\delta) \leq 2$ is the required value. We have proved Proposition 2.1 for $\delta \leq 0$. Now, as mentioned above, we have that $G(-1, \frac{1}{2}) = 0$ and after some tedious computation it is easy to see that

$$\frac{\partial G}{\partial \beta}\left(-1, \frac{1}{2}\right) = \frac{1}{4}\left(1 + \frac{1}{n+1}\right) - \frac{1}{4}\sum_{k=2}^{\infty}\prod_{j=0}^{k-1}\left(\frac{(n+j)}{(n+2j+1)}\right)\frac{1}{k(k-1)}.$$

Therefore

$$\sum_{k=2}^{\infty}\prod_{i=0}^{k-1}\frac{(n+i)}{(n+2i+1)}\frac{1}{k(k-1)} \leq \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = 1 < 1 + \frac{1}{n+1}.$$

We have proved that $\frac{\partial G}{\partial \beta}(-1, \frac{1}{2}) > 0$ and by the implicit function theorem, there exist $\overline{\varepsilon} > 0$ and $\overline{\gamma} > 0$ such that for any $\beta \in (-1-\overline{\varepsilon}, -1+\overline{\varepsilon})$ there exists $x(\beta) \in (\frac{1}{2}-\overline{\gamma}, \frac{1}{2}+\overline{\gamma})$ such that $G(\beta, x(\beta)) = 0$. Furthermore, since $\frac{\partial G}{\partial x}(-1, x) = -2$ we get that $x'(-1) > 0$ i.e. $x(\beta) < \frac{1}{2}$ for $\beta \in (-1-\overline{\varepsilon}, -1)$. Recalling that $\phi(x) = 1 - 2x$, let $\overline{\delta} := \frac{2\overline{\gamma}}{\sqrt{1-4\overline{\gamma}^2}}$. We have just proved that for any $0 < \delta < \overline{\delta}$ there exists $\alpha \in (2, 2+2\overline{\varepsilon})$ such that $\phi_\alpha = \frac{\delta}{\sqrt{1+\delta^2}}$ satisfies $u_\alpha(\phi_\alpha) = 0$. This completes the proof of Proposition 2.1.

Theorem 2.2 *Suppose ν is a non negative Δ_H superharmonic function in $D_\delta = \{\xi : \ t \geq \delta(x^2+y^2)\}$ with $\delta \leq 0$ i.e.*

$$\nu \geq 0 \quad \textit{and} \quad -\Delta_H \nu \geq 0 \quad \in D_\delta$$

then, if ν is not identically zero, there exists $\overline{\varepsilon} > 0$ and $\overline{\delta} > 0$ such that, for any $\alpha \in (\alpha(\delta), 2 + \overline{\varepsilon})$ and any $\delta_o \in (\delta, \overline{\delta})$, there exists a constant C such that

$$\nu(\xi) \geq C|\xi|_H^{2-Q-\alpha},$$

for any $\xi \in D_{\delta_o}$ such that $|\xi|_H \geq 1$.

Remark 2.3 Since $\alpha(0) = 2$, it is clear that the second part of Theorem 1.1 is just a corollary of Theorem 2.2 with $\alpha = 2 + \varepsilon$.

Remark 2.4 We have stated the theorem only for cones D_δ. Clearly, using the linear application T defined in Remark 1.2, Theorem 2.2 holds in any cone $C_\delta = \{\xi :\ t \leq \delta(x^2 + y^2)\}$ for $\delta \geq 0$. Similarly if D is an unbounded domain such that for some $\xi_o \in H^n$ and $\delta \in \mathbb{R}$:

$$\xi_o \circ D = D_\delta \ \ (\text{resp.} \xi_o \circ D = C_\delta)$$

then if $\delta \leq 0$ (resp. ≥ 0), the equivalent lower bound holds true for D.

Proof. Let us choose $\overline{\varepsilon}$ and $\overline{\delta}$ as in Proposition 2.1, hence $\alpha(\delta_o)$ is well defined. Then there are two possible cases: Case 1: $\alpha(\delta_o) \leq \alpha$ Case 2: $\alpha(\delta_o) > \alpha$. Let us consider Case 1. Let δ' such that $\delta_o > \delta' > \delta$ i.e. $D_{\delta_o} \subset D_{\delta'} \subset D_\delta$. Let us choose $v(\rho, \phi) = \rho^{2-Q-\alpha(\delta')} u_{\alpha(\delta')}(\phi)$. Now clearly, from (15), if $\mathcal{L}_\alpha u = 0$ then $\mathcal{L}_{2-Q-\alpha} u = 0$, hence $\Delta_H v(\rho, \phi) = 0$. Let $\Omega = D_{\delta'} \cap \{|\xi|_H = 1\}$ and let us consider $w = \nu - \tau v$, where $\tau = \dfrac{\inf_\Omega \nu}{\sup_\Omega v}$. We are in the hypothesis that ν is not identically zero hence, by the maximum principle, $\nu > c > 0$ in Ω and therefore $\tau > 0$. Clearly w satisfies

$$\begin{cases} -\Delta_H w \geq 0 & \text{in } \ D_{\delta'} \cap \{|\xi|_H \geq 1\}, \\ w = u > 0 & \text{in } \ \partial D_{\delta'} \cap \{|\xi|_H \geq 1\} \end{cases}$$

and by definition of τ

$$w \geq 0 \ \text{ in } \Omega.$$

Furthermore since $\alpha \leq 2 + \overline{\varepsilon}$, $\lim_{|\xi|_H \to \infty} w \geq 0$. We can apply the maximum principle and we obtain $w \geq 0$ in $D_{\delta'} \cap \{|\xi|_H \geq 1\}$. We have proved that for $\xi \in D_{\delta_o}$,

$$\nu(\xi) \geq C\rho^{2-Q-\alpha(\delta')}$$

where $C = \tau \inf\limits_{D_{\delta_o}} u_\alpha$, this completes the Case 1. In the Case 2 i.e. $\alpha(\delta_o) > \alpha$, we choose δ' such that $\alpha = \alpha(\delta')$, hence $\delta_o > \delta' > \delta$. Now we can proceed as in the first case. This completes the proof of Theorem 2.2.

Remark 2.5 By choosing $v = \rho^{2-Q}$ and repeating the above argument, we obtain the first part of Theorem 1.1.

Again, since we have defined $\alpha(\delta)$ for "intrinsic cones", we will state and prove a more general version of Theorem 1.6.

Theorem 2.6 *If $2m \geq Q + \alpha(\delta)$ then the only non negative Δ_H polyharmonic function in D_δ are constants.*

Here again the second claim of Theorem 1.6, is just Theorem 2.6 with $\delta = 0$ and $\alpha_0 = 2$. While the first claim is proved exactly as Theorem 2.6, using the fact that superharmonic functions in H^n satisfy (3). Before proving Theorem 2.6, let us introduce the notion of "radialized" function in D_δ, precisely we will consider the weighted spherical mean. Suppose ν is a function locally measurable in D_δ, $\rho := |\xi|_H$ then we will call

$$\nu_{\#}(\rho) := \int_{D_\delta \cap S_1} \nu(\rho, \theta) u_{\alpha(\delta)}(\theta) h(\theta) d\theta,$$

where $u_{\alpha(\delta)}$ is defined in Lemma 2.1. In the euclidean setting analogous functions have been considered to prove Liouville theorems for semi-linear equations in cones, see e.g. [3], [13]. We will prove now the following proposition, which is the generalization to cones of Proposition 1.4.

Proposition 2.7 *Suppose ν is a non negative Δ_H superharmonic function in D_δ then there exists $C > 0$ such that*

$$\nu_{\#}(\rho) \geq C \rho^{2-Q-\alpha(\delta)}$$

for any $\rho > 1$.

To prove this result we will use the following

Lemma 2.8 *For $M \leq 0$, suppose ν is a regular solution of*

$$-\Delta_H \nu(\xi) = f(\xi) \text{ in } D_\delta, \tag{19}$$

then the following equality holds true

$$-(\rho^{Q-1+2\alpha_\delta}(\rho^{-\alpha_\delta} \nu_{\#}(\rho))')' = \rho^{Q-1+\alpha_\delta} \int_{D_\delta \cap S_1} f(\rho,\, \theta) u_{\alpha_\delta}(\theta) d\theta, \tag{20}$$

where u_{α_δ} is defined in (16).

Proof of Lemma 2.8. Suppose Ω is a regular subset of S_1, then for u and v in C^1 and $v = 0$ on the boundary of Ω, following Jerison in [9] (proof of Proposition 3.1), it is easy to see that:

$$\int_\Omega \hat{R}_i u v d\theta = -\int_\Omega u \hat{R}_i v d\theta + (Q-1) \int_\Omega u v a_i d\theta, \tag{21}$$

and similarly for $\hat{S}_i$. On the other hand from the definitions of Δ_H, $\hat{R}_i$ and $\hat{S}_1$ it is easy to see that

$$\begin{aligned}\Delta_H v &= h\left(\frac{\partial^2 v}{\partial\rho^2}+\frac{Q-1}{\rho}\frac{\partial v}{\partial\rho}\right)+\frac{2}{\rho}\sum_{i=1}^{n}(a_i\hat{R}_i+b_i\hat{S}_i)\frac{\partial v}{\partial\rho}\\ &\quad+\frac{1}{\rho^2}\left(\sum_{i=1}^{n}\hat{R}_i^2 v+\hat{S}_i^2 v-(a_i\hat{R}_i v+b_i\hat{S}_i v)\right).\end{aligned}$$

Using (21), after a long but easy computation, we obtain for any $\nu\in C^2$ and any $v\in C^2$ independent of ρ such that $v=0$ on the boundary of Ω

$$\begin{aligned}\int_\Omega \Delta_H\nu v d\theta &= \left[\int_\Omega \nu v h d\theta\right]_{\rho\rho}+\frac{Q-1}{\rho}\left[\int_\Omega \nu v h d\theta\right]_{\rho}\\ &\quad+-\frac{2}{\rho}\left[\int_\Omega \nu\sum_{i=1}^{n}(a_i\hat{R}_i v+b_i\hat{S}_i v)hd\theta\right]_\rho\\ &\quad+\frac{1}{\rho^2}\left(\int_\Omega \nu\sum_{i=1}^{n}\hat{R}_i^2 v+\hat{S}_i^2 v-(4n+1)(a_i\hat{R}_i v+b_i\hat{S}_i v)\right).\end{aligned}$$

Clearly, from what we have seen above, $\displaystyle\sum_{i=1}^{n}(a_i\hat{R}_i u_{\alpha_\delta}(\phi)+b_i\hat{S}_i u_{\alpha_\delta}(\phi))=0$. So, in particular, if ν is a solution of (19) and $v=u_{\alpha_\delta}$ and $\Omega=D_\delta\cap S_1$, we obtain

$$\begin{aligned}\int_\Omega \Delta_H\nu u_{\alpha_\delta}d\theta &= \frac{\partial^2\nu_\#}{\partial\rho^2}+\frac{Q-1}{\rho}\frac{\partial\nu_\#}{\partial\rho}\\ &\quad+\frac{1}{\rho^2}\left(\int_\Omega \nu((1-\phi^2)u''_{\alpha_\delta}(\phi)-\frac{1}{2}Qu'_{\alpha_\delta}(\phi))4hd\theta\right).\end{aligned}$$

Finally, since u_{α_δ} is a solution of $\mathcal{L}_{\alpha_\delta}u_{\alpha_\delta}=0$, we have

$$\int_{D_\delta\cap S_1} f(\rho,\theta)u_{\alpha_\delta}d\theta=\frac{\partial^2\nu_\#}{\partial\rho^2}+\frac{Q-1}{\rho}\frac{\partial\nu_\#}{\partial\rho}-\frac{\alpha_\delta(\alpha_\delta+Q-2)}{\rho^2}\nu_\#. \tag{22}$$

Let us recall the simple observation that for any $a,\ b\in\mathbb{R}$

$$\rho^{b-a}\left(\rho^a(\rho^{-b}\nu_\#)'\right)'=\nu''_\#+\frac{(a-2b)}{r}\nu'_\#+\frac{b(b+1-a)}{r^2}\nu_\#. \tag{23}$$

Then (20) is a consequence of (22) and (23) where we have chosen $b=\alpha_\delta$ and $a=Q-1+2\alpha_\delta$. *Proof of Proposition* 2.7. We are in the hypotheses that ν is not identically zero, hence in D_δ it is positive. From Lemma 2.8 if ν is a positive Δ_H superharmonic function in D_δ then

$$-(\rho^{Q-2+2\alpha_\delta}(\rho^{-\alpha_\delta}\nu_\#)')'\geq 0. \tag{24}$$

Let us state now the following simple fact, see [5] Lemma 2.1: if v is a non negative solution of $(r^a v')' \leq 0$ in $(R, +\infty)$ for some $R \geq 0$ and for some $a > 1$ and $v'(R) \leq 0$ then

$$\text{then } v'(r) \leq 0 \text{ and } \; rv'(r) + (a-1)v(r) \geq 0 \;\; \forall r \geq R. \tag{25}$$

Therefore in particular (25) holds true for $v(\rho) := \rho^{-\alpha_\delta}\nu_\#(\rho)$. Let us integrate (24) and use (25), we obtain for $\rho_2 \geq \rho_1$:

$$\begin{aligned}&(Q-2+2\alpha_\delta)\rho_2^{Q-2+2\alpha_\delta}(\rho_2^{-\alpha_\delta}\nu_\#(\rho_2)\\ &\quad \geq -\rho_2^{Q-1+2\alpha_\delta}(\rho_2^{-\alpha_\delta}\nu_\#(\rho_2))' \geq -\rho_1^{Q-1+2\alpha_\delta}(\rho_1^{-\alpha_\delta}\nu_\#(\rho_1))' := C\end{aligned}$$

We have proved that for $\rho \geq 1$ there exists $C > 0$ such that

$$\nu_\#(\rho) \geq C\rho^{2-Q-\alpha_\delta}.$$

End of Proposition 2.7*'s proof. Proof of Theorem 1.6.* We argue by contradiction.

Let ν be a positive function satisfying (4). Let us call $(u_0, u_1, \ldots, u_{m-1})$ the vector of functions defined by

$$u_0 = \nu, \quad -\Delta u_0 = u_1, \quad -\Delta u_1 = u_2, \ldots, \quad -\Delta u_{m-1} \geq 0 \text{in} D_\delta. \tag{26}$$

Applying Lemma 2.8 to (26) we obtain

$$-(\rho^{Q-1+2\alpha_\delta}(\rho^{-\alpha_\delta}u_{i\#})')' \geq \rho^{Q-1+\alpha_\delta}u_{i+1\#}(\rho). \tag{27}$$

for $i = 0, \ldots, m-2$ and $-(\rho^{Q-1+2\alpha_\delta}(\rho^{-\alpha_\delta}u_{m-1\#})')' \geq 0$. Let $v_i(\rho) = \rho^{-\alpha_\delta}u_{i\#}(\rho)$. Clearly from (25), the functions v_i are decreasing for $i = 0, \ldots, m-1$. Therefore the following limit exists and is finite:

$$\lim_{\rho\to\infty} v_i(\rho) = v_{\infty,i} \geq 0.$$

In particular it follows that, for $i = 0 \ldots m-2$:

$$-(\rho^{Q-1+2\alpha_\delta}(v_i(\rho) - v_{\infty,i})')' \geq \rho^{Q-1+2\alpha_\delta}v_{i+1}(\rho)$$

Therefore using (25), the fact that v_i are decreasing and reasoning as in the proof of Proposition 2.7 we obtain:

$$v_i(\rho) - v_{\infty,i} \geq C\rho^2 v_{i+1}(\rho).$$

It follows that for $\rho > 0$,

$$v_o(\rho) - v_{\infty,o} \geq C\rho^{2(m-1)}v_{m-1}(\rho).$$

Applying proposition 2.7 we know that for some positive constant C

$$v_{m-1}(\rho) \geq C\rho^{2-Q-2\alpha_\delta}.$$

Hence

$$v_o(\rho) - v_{\infty,o} \geq C\rho^{2m-Q-2\alpha_\delta}. \tag{28}$$

We have reached a contradiction since by assumption $2m \geq Q + 2\alpha_\delta$ while the left hand side of (28) goes to zero.

Acknowledgment I would like to thank Thierry Paul and Francesco Uguzzoni for very useful conversations. The main proofs of this paper were completed while the author was visitng the University of Cergy Pontoise with a *Bourse de longue durée* hence she would like to thank the Laboratoire for its kind hospitality. A preliminary version of these results appeared in a preprint of that same Laboratoire.

References

[1] I. BIRINDELLI, I. CAPUZZO DOLCETTA, A. CUTRI, Liouville theorems for semilinear equations on the Heisenberg group, *Annales de l'I.H.P.-Analyse non linéaire*, **14** (1997), 295–308.

[2] I. BIRINDELLI, I. CAPUZZO DOLCETTA, A. CUTRI, Indefinite semilinear equations on the Heisenberg group, *Comm. Partial Differential Equations* **23**(7–8) (1998), 1123–1157.

[3] I. BIRINDELLI, E. MITIDIERI, Liouville theorems for elliptic inequalities and applications, *Proc. Roy. Soc. Edinburgh* Sect. A **128**(6) (1998), 1217–1247.

[4] I. BIRINDELLI, J. PRAJAPAT, One dimensional symmetry in the Heisenberg group, preprint.

[5] P. CLEMENT, R. MANÁSEVICH, E. MITIDIERI, Positive solutions for a quasilinear system via blow up, *Comm. Partial Differential Equations*, **18** (1993), 2071–2106.

[6] G.B. FOLLAND, E.M. STEIN, Estimates for the ∂_h complex and analysis on the Heisenberg Group, *Comm. Pure Appl. Math.* **27** (1974), 492–522.

[7] P.C. GREINER, Spherical harmonics in the Heisenberg group, *Canad. Math. Bull.* **23**(4) (1980), 383–396.

[8] L. HORMANDER, Hypoelliptic second order differential equations, *Acta Math.* Uppsala, **119** (1967), 147–171.

[9] D.S. JERISON, Boundary regularity in the Dirichlet problem for $\Box_b$ on *CR* manifolds *Comm. Pure Appl. Math.* **36** (1983), 143–181.

[10] D.S. JERISON, The Dirichlet Problem for Kohn Laplacian on the Heisenberg Group II *J. of Funct. Anal.* **43** (1981), 224–257.

[11] E. LANCONELLI, F. UGUZZONI, Asymptotic behavior and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group, *Boll. Un. Mat. Ital.* **8**(1-B) (1998), 139–168.

[12] E. LANCONELLI, F. UGUZZONI, Non-existence results for semilinear Kohn-Laplace equations in unbounded domains, *Comm. Partial Differential Equations* **25** (2000), 1703–1739.

[13] M. RIGOLI, A.G. SETTI, A Liouville theorem for a class of superlinear elliptic equations on cones, Preprint.

[14] F. UGUZZONI, A Liouville-type theorem on halfspaces for the Kohn Laplacian. *Proc. Amer. Math. Soc.* **127** (1999), 117–123.

[15] F. UGUZZONI, A. CITTI, Critical semilinear equations on the Heisenberg group: the effect of the topology of the domain, to appear in *Nonlinear Anal.*