

# ON SOME PARTIAL DIFFERENTIAL EQUATION FOR NON COERCIVE FUNCTIONAL AND CRITICAL SOBOLEV EXPONENT

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## 1. INTRODUCTION AND RESULTS

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^n$ ,  $n > 2$ . We assume that  $q$  and  $f$  are some continuous functions on  $\overline{\Omega}$  which change sign in  $\Omega$ .

We are interested in the following partial differential equation

$$\begin{cases} -\Delta u + (q - \lambda)u = f(x)u^{2^*-1}, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\lambda$  is a real parameter and  $2^* = \frac{2n}{n-2}$  is the critical exponent for the Sobolev embedding.

Let  $\lambda_1$  be the principal eigenvalue of  $-\Delta + q$  in  $\Omega$ , or equivalently

$$\lambda_1 = \inf_{\{v \in H_0^1(\Omega), |v|_2=1\}} \left\{ \int_{\Omega} |\nabla v|^2 + \int_{\Omega} q|v|^2 \right\}. \quad (2)$$

It is well known that there exists  $\phi \geq 0$  which realizes the infimum in (2); it satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta \phi + (q - \lambda_1)\phi = 0, & \phi \geq 0 \text{ in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

The strict maximum principle implies that  $\phi > 0$  in  $\Omega$ . In the sequel we shall choose  $\phi$  normalized in  $L^2(\Omega)$ .

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In this paper, we will study the existence of classical solutions of (1) when  $\lambda > \lambda_1$ .

In the sub-critical case (i.e., the exponent of  $u$  on the right hand side of (1) is less than  $2^* - 1$ ), the problem has been studied by Berestycki, Capuzzo-Dolcetta, and Nirenberg [4], [5] and Alama and Tarantello [1]. The same problem (with sub-critical exponent) and the Heisenberg Laplacian has been studied in [6]. On another hand the case where  $\lambda < \lambda_1$  and the right hand side is critical has been studied by Hebey and Vaugon in [12], see also Demengel and Hebey [11] for the  $p$ -Laplacian case.

In [1], [5], [6], the authors establish the following necessary condition

**Theorem 1.** *Assume that (1) has a solution. Then the following conditions are satisfied:*

- (i)  $\int_{\Omega} f \phi^{2^*} < 0$  if  $\lambda > \lambda_1$
- (ii)  $\Omega^+ := \{x \in \Omega, f(x) > 0\} \neq \emptyset$  if  $\lambda < \lambda_1$
- (iii)  $\Omega^+ \neq \emptyset, \Omega^- := \{x \in \Omega, f(x) < 0\} \neq \emptyset$  if  $\lambda = \lambda_1$ .

The main result of the present article is the following:

**Theorem 2.** *Let  $K(n, 2)$  be the best constant for the Sobolev embedding of  $H^1(\mathbb{R}^n)$  into  $L^{2^*}(\mathbb{R}^n)$ . We assume that  $\int_{\Omega} f \phi^{2^*} < 0$  and that*

$$M_1 = \sup_{\{\int_{\Omega} |\nabla v|^2 + \int_{\Omega} (q - \lambda_1) |v|^2 = 1, \}} \left\{ \int_{\Omega} f |v|^{2^*} \right\} > K(n, 2)^{2^*} \sup f \quad (4)$$

*then there exists a real  $\lambda^* > \lambda_1$ , such that for every  $\lambda \in [\lambda_1, \lambda^*[$ , (1) possesses a solution, and there is no solution for  $\lambda > \lambda^*$ .*

**Remark 1.** 1) The same problem with  $q = 0$  was considered in [14] and [1]. In particular in [14] Ouyang, through a bifurcation method, proved the existence of a branch of solutions of

$$\begin{cases} -\Delta u - \lambda u = f u^p, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (5)$$

bifurcating on the right from  $\lambda_1$  under conditions similar to i), ii), iii) in Theorem 1,  $\lambda \neq \lambda_1$ , and for  $p$  large (i.e.  $p \leq \frac{n-6}{n-10}$  if  $n \geq 10$  and for any  $p$  if  $n \leq 10$ ). These solutions  $u(\lambda)$  don't coincide with those considered here, since they are "small" and  $\int_{\Omega} f u(\lambda)^{p+1} < 0$ . On the other hand, for  $p + 1$  sub-critical, through variational methods, he proved the existence of another solution, with  $\lambda$  sufficiently close to  $\lambda_1$ .

In [1], Alama and Tarantello completed these results by perturbing the bifurcation solution and applying the mountain pass lemma, obtaining in

that fashion the second solution, even in the critical case. This requires a condition similar to our condition (4), adapted to the case  $q = 0$ .

Our proof is direct. It uses the geometry of the functionals involved and the concentration compactness Lemma, which permits to overcome the lack of compactness.

2) The value  $K(n, 2)$  has been computed independently by Aubin in [2, 3] and Talenti [15]. It is achieved on functions of the form

$$u_\epsilon(x) = (\epsilon^2 + |x|^2)^{1-\frac{n}{2}}$$

and then

$$K(n, 2)^2 = \frac{4}{n(n-2)\omega_{n-1}^{\frac{2}{n}}}.$$

The condition (4) is comparable with the conditions in [12], [2], [11].

Furthermore it is not very restrictive since, in Section 3, we prove that one always has

$$M_1 \geq K(n, 2)^{2^*} \sup f.$$

3) Let us remark that  $I_\lambda$  defined as  $I_\lambda(u) := \int_\Omega |\nabla u|^2 + \int_\Omega (g - \lambda)u^2$  is not coercive on the space  $H_0^1$ . But it is coercive on the hyperplan  $\phi^\perp$  as soon as  $\lambda$  is close to  $\lambda_1$ . Indeed, let  $v = u - (\int_\Omega u\phi)\phi$ , where  $\phi$  is supposed to be normalized. Let us consider the set of eigenfunctions,  $\{\phi_k\}_{k \in \mathbb{R}}$  ( $\phi_1 = \phi$ ) for  $-\Delta + q$ , which is complete in  $L^2$ . Using the fact that  $\phi$  is simple, one can write  $v = \sum_{k \geq 2} t_k \phi_k$  where  $t_k = \langle v, \phi_k \rangle$ , hence

$$I_\lambda(v) = \sum_k (\lambda_k - \lambda) t_k^2 \geq (\lambda_2 - \lambda) |v|_2^2 \quad (6)$$

taking  $\lambda$  less than  $\lambda_2$ , one gets the result.

4) It would be interesting to have a better estimate for  $\lambda^*$ .

While working on this case, we also considered similar equations for the  $p$ -Laplacian. The results we obtained will be published in a forthcoming paper [7].

## 2. PROOF OF THEOREM 2

The proof of Theorem 2 is based on some variational methods similar to those used in [5].

One of the key ingredient in the proof of Theorem 2 will be the famous concentration compactness principle of P.L. Lions [13], that we enounce here:

**Lemma 1.** *Let  $\Omega$  be some bounded open set in  $\mathbb{R}^n$ , and  $(u_k)$  be some sequence in  $H_0^1(\Omega)$ , which is bounded in  $H^1(\Omega)$ . Then there exists a subsequence of  $(u_k)$ , still denoted  $(u_k)$  for simplicity, two nonnegative measures  $\mu$  and  $\nu$  on  $\overline{\Omega}$ , a sequence of points  $x_i$  in  $\overline{\Omega}$ , two sequences of nonnegative reals  $(\mu_i)$  and  $(\nu_i)$  and a function  $u$  in  $H_0^1$ , such that*

$$|\nabla u_k|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_i \mu_i \delta_{x_i}$$

(the convergence being tight i.e.,  $\int_{\Omega} |\nabla u_k|^2$  towards  $\int_{\overline{\Omega}} \mu$ ),

$$|u_k|^{2^*} \rightharpoonup \nu = |u|^{2^*} + \sum_i \nu_i \delta_{x_i}$$

(the convergence being tight on  $\overline{\Omega}$ , i.e.,  $\int_{\Omega} |u_k|^{2^*}$  towards  $\int_{\overline{\Omega}} \nu$ ), with the inequality

$$\nu_i^{\frac{2}{2^*}} \leq K(n, 2)^2 \mu_i. \quad (7)$$

**Remark 2.** In the sequel we shall denote the space of bounded measures on  $\Omega$  by  $M^1(\Omega)$ . This space will be endowed with the weak topology or the tight one. Recall that the tight convergence is equivalent to weak convergence and convergence of the total variations.

The proof of Theorem 2 is divided in several steps.

**Step 1.** For large  $\lambda > \lambda_1$  (1) has no solution.

**Step 2.** If there exists a solution for a certain  $\lambda' > \lambda_1$ , then (1) has a solution for any  $\lambda_1 < \lambda \leq \lambda'$ .

**Step 3.** For  $\lambda = \lambda_1$  and for  $\lambda > \lambda_1$  sufficiently close to  $\lambda_1$  there exists a solution to (1).

In the first two steps, we follow the proofs given in [5, 6].

**Step 1.** Let  $x_0 \in \Omega^+$  and  $R > 0$  such that  $B(x_0, R) \subset \Omega^+$ . Let  $\mu^*$  and  $\psi$  be respectively the principal eigenvalue and eigenfunction for  $-\Delta$  with Dirichlet conditions.

$$\begin{cases} -\Delta \psi - \mu^* \psi = 0 & \text{in } B(x_0, R) \\ \psi = 0 & \text{on } \partial B(x_0, R). \end{cases}$$

Suppose that there exists a solution of (1) for  $\lambda > \|q\|_{\infty} + \mu^*$ . Then since  $f > 0$  in  $B(x_0, R)$ , one has

$$\begin{cases} -\Delta u + (q - \lambda)u \geq 0 & \text{in } B(x_0, R) \\ u > 0 & \text{on } \partial B(x_0, R). \end{cases}$$

On the other hand,  $\frac{\partial \psi}{\partial n} \leq 0$  on  $\partial B(x_0, R)$  implies

$$0 \leq \int_{\partial B(x_0, R)} -u \frac{\partial \psi}{\partial n} = \int_{B(x_0, R)} (-u \Delta \psi + \psi \Delta u) \leq \int_{B(x_0, R)} (\mu^* + q - \lambda) u \psi < 0$$

from which one derives a contradiction.

**Step 2.** One uses a sub and super solution argument. Namely, let  $\lambda' > \lambda_1$ , and denote by  $w$  a solution for  $\lambda = \lambda'$ . For  $\lambda \in ]\lambda_1, \lambda'[, w$  satisfies

$$\begin{cases} -\Delta w + (q - \lambda)w \geq -\Delta w + (q - \lambda')w = fw^{2^*-1} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and  $w$  is a super solution of (1) for  $\lambda$ . On the other hand, if  $\phi$  is defined by (3),  $\epsilon\phi$  is a sub solution of (1) as soon as  $\epsilon$  is sufficiently small. Indeed

$$-\Delta(\epsilon\phi) + (q - \lambda)(\epsilon\phi) = (\lambda_1 - \lambda)\epsilon\phi \leq f(x)\epsilon^{2^*}\phi^{2^*}$$

since  $f$  is bounded from below. Since by Hopff's maximum principle,  $\epsilon\phi \leq w$  for  $\epsilon$  small enough one obtains that there exists a solution for  $\lambda$ .

**Step 3.** One defines as in [5, 6] the following

$$S_\lambda = \{u \in H^1(\Omega), I_\lambda(u) = 1\}$$

and for  $\epsilon$  small and positive,

$$M_{\epsilon, \lambda} = \sup_{u \in S_\lambda} \left\{ \int_{\Omega} f|u|^{2^*-\epsilon} \right\}, \quad M_\lambda = \sup_{u \in S_\lambda} \left\{ \int_{\Omega} f|u|^{2^*} \right\}.$$

Let us observe that according to [5, 6],  $S_\lambda \neq \emptyset$ ,  $M_{\epsilon, \lambda} > 0$  and  $M_{\epsilon, \lambda}$  is achieved by some maximizer  $u_\epsilon$ .  $u_\epsilon$  can be chosen positive since  $|u_\epsilon|$  is a maximizer as well. The proof that  $M_\lambda > 0$  is identical to the sub-critical case. Furthermore, it is not difficult to see that for  $\lambda \geq \lambda_1$ ,  $M_\lambda \geq M_{\lambda_1} = M_1$ .

**Claim 1:** *The sequences  $(M_{\epsilon, \lambda})$ ,  $(M_\lambda)$  and  $\|u_\epsilon\|_{H^1(\Omega)}$  are bounded uniformly with respect to  $\epsilon$  and  $\lambda$  for  $\epsilon$  small enough and  $\lambda$  close to  $\lambda_1$ .*

Let  $u$  be such that  $I_\lambda(u) = 1$ . Remarking that  $I_\lambda(|u|) = I_\lambda(u)$  and that  $\|u_\epsilon\|_{H^1}$  is bounded if and only if  $\|(|u_\epsilon|)\|_{H^1}$  is, one can assume in what follows that  $u \geq 0$ . Let then  $v \in H_o^1(\Omega)$ ,  $v \in \phi^\perp$ , and  $t \in \mathbb{R}$  such that  $u = v + t\phi$ , and  $\int_{\Omega} fu^{2^*} > 0$ , one gets

$$I_\lambda(v) = 1 + (\lambda - \lambda_1)t^2$$

and noticing that  $t = \int_{\Omega} u\phi > 0$ ,

$$\int_{\Omega} f(v + t\phi)^{2^*-\epsilon} \leq \tag{8}$$

$$\leq t^{2^*-\epsilon} \int_{\Omega} f \phi^{2^*-\epsilon} + |f|_{\infty} 2^{2^*} 2^{2^*-\epsilon-1} \left( \int_{\Omega} |v|^{2^*-\epsilon} + \int_{\Omega} |v|(t\phi)^{2^*-\epsilon-1} \right)$$

using the mean-value Theorem. Let  $\alpha = -\int_{\Omega} f|\phi|^{2^*}$ , and  $\epsilon_0$  be such that  $\int_{\Omega} f \phi^{2^*-\epsilon} \leq -\frac{3\alpha}{4}$  for  $\epsilon < \epsilon_0$ . By Young's inequality, there exists some constant  $C_1$  such that

$$|f|_{\infty} 2^{2^*} 2^{2^*-\epsilon-1} \int_{\Omega} v(t\phi)^{2^*-\epsilon-1} \leq C_1 |v|_{L^{2^*-\epsilon}}^{2^*-\epsilon} + \frac{\alpha}{4} t^{2^*-\epsilon}$$

hence for some  $C_2$  positive

$$\int_{\Omega} f(v+t\phi)^{2^*-\epsilon} \leq \frac{-\alpha}{2} t^{2^*-\epsilon} + C_2 |v|_{2^*-\epsilon}^{2^*-\epsilon}. \quad (9)$$

Using remark 1 on the coerciveness of  $I_{\lambda}$  on  $\phi^{\perp}$ , and choosing  $\lambda \in [\lambda_1, \frac{\lambda_1+\lambda_2}{2}]$ , one has

$$|\nabla v|_2^2 \leq I_{\lambda}(v) + |q - \lambda|_{\infty} |v|_2^2 \leq I_{\lambda}(v) \left( 1 + 2 \frac{|q|_{\infty} + |\lambda_2|}{|\lambda_1 - \lambda_2|} \right). \quad (10)$$

Using Poincaré's inequality and (10), there exists some constant  $C_3$  which does not depend on  $\lambda$ , such that

$$|v|_{2^*}^2 \leq C_3(I_{\lambda}(v)) = C_3(1 + (\lambda - \lambda_1)t^2). \quad (11)$$

Finally, through the convexity inequality:

$$\begin{aligned} (1 + (\lambda - \lambda_1)t^2)^{\frac{2^*-\epsilon}{2}} &\leq 2^{\frac{2^*-\epsilon}{2}-1} (1 + (\lambda - \lambda_1)^{\frac{2^*-\epsilon}{2}} t^{2^*-\epsilon}) \\ &\leq 2^{\frac{2^*}{2}} + 2^{\frac{2^*}{2}} (\lambda - \lambda_1)^{\frac{2^*-\epsilon}{2}} t^{2^*-\epsilon}. \end{aligned}$$

Then, we choose  $\lambda$  sufficiently close to  $\lambda_1$  in order to have

$$2^{\frac{2^*}{2}} C_2 C_3^{\frac{2^*}{2}} |\lambda - \lambda_1|^{\frac{2^*-1}{2}} \leq \frac{\alpha}{4},$$

and (9) becomes

$$\int_{\Omega} f u^{2^*-\epsilon} + t^{2^*-\epsilon} \frac{\alpha}{2} \leq C_2 |v|_{2^*-\epsilon}^{2^*-\epsilon} \leq \frac{\alpha}{4} t^{2^*-\epsilon} + C_2 C_3^{\frac{2^*}{2}} 2^{\frac{2^*}{2}}.$$

Finally there exists  $C_5$  independent of  $\epsilon$  and  $\lambda$  such that

$$\int_{\Omega} f u^{2^*-\epsilon} + t^{2^*-\epsilon} \frac{\alpha}{4} \leq C_5. \quad (12)$$

We may choose in (12)  $u = u_\epsilon$ , with  $u_\epsilon = v_\epsilon + t_\epsilon \phi$ ,  $v_\epsilon \in \phi^\perp$ , one obtains that both  $\int_\Omega f u_\epsilon^{2^*- \epsilon}$  and  $t_\epsilon$  are bounded for  $\lambda - \lambda_1$  sufficiently small. More precisely, we have proved that for  $\lambda$  sufficiently close to  $\lambda_1$ , one has

$$M_{\epsilon, \lambda} + t_\epsilon^{2^* - \epsilon} \frac{\alpha}{4} \leq C_5. \quad (13)$$

This implies that  $(M_{\epsilon, \lambda})$  and  $(t_\epsilon)$  are bounded. Using  $I_\lambda(v_\epsilon) = 1 + (\lambda - \lambda_1)t_\epsilon^2$ , one obtains that  $(v_\epsilon)$  is bounded as well. Finally  $(u_\epsilon)$  is bounded independently of  $\epsilon$ . This ends the proof of claim 1.  $\square$

Our purpose now is to let  $\epsilon$  go to zero. Since the sequences  $(u_\epsilon)$ ,  $(t_\epsilon)$ ,  $(M_{\epsilon, \lambda})$  are bounded, one can extract from them some subsequences, denoted in the same manner, such that

$$u_\epsilon \rightharpoonup u \text{ in } H^1(\Omega) \text{ weakly} \quad (14)$$

$$t_\epsilon \rightarrow t \quad (15)$$

$$M_{\epsilon, \lambda} \rightarrow \overline{M}_\lambda. \quad (16)$$

Let us note that  $\lim_{\epsilon \rightarrow 0} M_{\epsilon, \lambda} = \overline{M}_\lambda \geq M_\lambda$ . Indeed, let  $\delta > 0$  be given and  $u \in S_\lambda$ , such that  $\int_\Omega f u^{2^*} \geq M_\lambda - \delta$ . Then  $\int_\Omega f u^{2^* - \epsilon} \geq M_\lambda - 2\delta$  for  $\epsilon$  small enough. We then deduce that

$$\overline{M}_\lambda = \underline{\lim} M_{\epsilon, \lambda} \geq M_\lambda.$$

Since  $u_\epsilon$  satisfies the following partial differential equation

$$\begin{cases} -\Delta u_\epsilon + (q - \lambda)u_\epsilon = (M_{\epsilon, \lambda})^{-1} f u_\epsilon^{2^* - \epsilon - 1} & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (17)$$

by passing to the limit when  $\epsilon \rightarrow 0$ , one has

$$\begin{cases} -\Delta u + (q - \lambda)u = (\overline{M}_\lambda)^{-1} f u^{2^* - 1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (18)$$

The point here is to prove that

- i)  $u$  is not identically zero.
- ii)  $M_\lambda = \overline{M}_\lambda$  and  $u$  is a maximizer for  $M_\lambda$ .

In order to prove i) let us multiply (17) by  $u_\epsilon \varphi$ , where  $\varphi \in \mathcal{D}(\overline{\Omega})$  and integrate by parts. One obtains

$$\int_\Omega |\nabla u_\epsilon|^2 \varphi + \int_\Omega u_\epsilon \nabla u_\epsilon \cdot \nabla \varphi + \int_\Omega (q - \lambda) u_\epsilon^2 \varphi = M_{\epsilon, \lambda}^{-1} \int_\Omega f u_\epsilon^{2^* - \epsilon} \varphi. \quad (19)$$

Similarly, multiplying (18) by  $u \varphi$  one gets

$$\int_\Omega |\nabla u|^2 \varphi + \int_\Omega u \nabla u \cdot \nabla \varphi + \int_\Omega (q - \lambda) u^2 \varphi = \overline{M}_\lambda^{-1} \int_\Omega f u^{2^*} \varphi. \quad (20)$$

Using the concentration compactness principle as given in Lemma 1 we get that there exist two nonnegative measures,  $\mu$  and  $\nu$ , compactly supported in  $\overline{\Omega}$ , a numerable set of points  $x_i$  in  $\overline{\Omega}$ , some sequences of nonnegative numbers  $\nu_i$  and  $\mu_i$ , such that

$$|\nabla u_\epsilon|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_i \mu_i \delta_{x_i} \text{ in } M^1(\overline{\Omega}) \text{ tightly}$$

$$|u_\epsilon|^{2^*} \rightharpoonup \nu = |u|^{2^*} + \sum_i \nu_i \delta_{x_i} \text{ in } M^1(\overline{\Omega}) \text{ tightly}$$

with  $\nu_i^{\frac{2}{2^*}} \leq K(n, 2)^2 \mu_i$ .

Let us observe that  $\int_\Omega u_\epsilon^{2^*-\epsilon}$  is bounded and hence,  $u_\epsilon^{2^*-\epsilon}$  converges to some measure  $\tilde{\nu}$  in  $M^1(\overline{\Omega})$  tightly, up to a subsequence. By Hölder's inequality, one has  $\tilde{\nu} \leq \nu$ . Since one can extract once more a subsequence from  $u_\epsilon$ , one can also assume that  $u_\epsilon$  converges almost everywhere towards  $u$ . By Fatou's lemma, for all  $\varphi \in \mathcal{D}(\overline{\Omega})$ ,  $\varphi \geq 0$ , one has  $\int_\Omega |u|^{2^*} \varphi \leq \langle \tilde{\nu}, \varphi \rangle$ , and then  $\tilde{\nu} \geq |u|^{2^*}$ . Finally there exists some  $0 \leq \tilde{\nu}_i \leq \nu_i$  such that

$$\tilde{\nu} = |u|^{2^*} + \sum_i \tilde{\nu}_i \delta_{x_i}.$$

One then obtains, by passing to the limit in (19)

$$\langle \mu, \varphi \rangle + \int_\Omega u \nabla u \cdot \nabla \varphi + \int_\Omega (q - \lambda) |u|^2 \varphi = \frac{1}{\overline{M}_\lambda} \left( \int_\Omega f |u|^{2^*} \varphi + \sum_i \tilde{\nu}_i f(x_i) \varphi(x_i) \right). \quad (21)$$

Subtracting (20) from (21) one gets

$$\langle (\mu - |\nabla u|^2), \varphi \rangle = \overline{M}_\lambda^{-1} \sum_i \tilde{\nu}_i f(x_i) \varphi(x_i) \quad (22)$$

which implies that, if  $\mu = \mu^{ac} + \mu^S$  is the Lebesgue decomposition of  $\mu$  in its absolutely continuous part and a singular one,  $\mu^{ac} = |\nabla u|^2$  and  $\mu^S = \sum_i \tilde{\mu}_i \delta_{x_i} = \overline{M}_\lambda^{-1} \sum_i \tilde{\nu}_i f(x_i) \delta_{x_i}$ , hence

$$\mu_i \leq \tilde{\mu}_i = \overline{M}_\lambda^{-1} \tilde{\nu}_i f(x_i). \quad (23)$$

Clearly if  $f(x_i) \leq 0$  then  $\mu_i = 0$  and therefore  $\tilde{\nu}_i = 0$ . Observe that

$$\frac{1}{\overline{M}_\lambda} \int_\Omega f u^{2^*} = I_\lambda(u) = I_\lambda(v) + (\lambda_1 - \lambda) t^2 \geq (\lambda_1 - \lambda) t^2.$$



Let us define  $B = \frac{M_1 + K(n, 2)^{2^*} \sup f(x)}{2}$ . By assumption one has  $M_1 > B > K(n, 2)^{2^*} \sup f(x)$ . From claim 1, we know that  $|t| \leq K$  a constant independent of  $\lambda$ . Suppose that  $|\lambda - \lambda_1|$  is sufficiently small in order to have  $M_1^{\frac{2}{2^*}} (1 + (\lambda - \lambda_1)K^2)^{\frac{2}{2^*}-1} > B^{\frac{2}{2^*}}$ . Passing to the limit in the definition of  $M_{\epsilon, \lambda}$ , one has

$$\int_{\Omega} f u^{2^*} + \sum_i \nu_i f(x_i) = \overline{M}_{\lambda}$$

and then

$$\sum_i \nu_i f(x_i) \leq \overline{M}_{\lambda} (1 + (\lambda - \lambda_1)t^2) \quad (24)$$

then, using (4), Lemma 1, (7) and the definition of  $B$ ,

$$\begin{aligned} \tilde{\mu}_i &\leq \left( \frac{\nu_i f(x_i)}{\overline{M}_{\lambda} (1 + (\lambda - \lambda_1)t^2)} \right)^{1 - \frac{2}{2^*}} \left( \frac{\nu_i f(x_i)}{\overline{M}_{\lambda} (1 + (\lambda - \lambda_1)t^2)} \right)^{\frac{2}{2^*}} (1 + (\lambda - \lambda_1)t^2) \\ &\leq \frac{K(n, 2)^2 \sup f^{\frac{2}{2^*}}}{B^{\frac{2}{2^*}}} \tilde{\mu}_i \leq \beta \tilde{\mu}_i \end{aligned}$$

for some  $\beta < 1$ . This implies that  $\tilde{\mu}_i = 0$  and then  $\nu_i = 0$ . Finally,  $\overline{M}_{\lambda} = \int_{\Omega} f u^{2^*}$ ,  $u$  is a minimizer and  $\overline{M}_{\lambda} = M_{\lambda}$ . This ends the proof of Theorem 2.  $\square$

**Corollary.** *Under the assumptions  $\int_{\Omega} f \phi^{2^*} < 0$  and  $M_{\lambda_1} > K(n, 2)^{2^*} \sup f$ , then for  $\lambda > \lambda_1$ ,  $\lambda \rightarrow \lambda_1$ ,*

$$M_{\lambda} \rightarrow M_{\lambda_1} = M_1.$$

Moreover, from any sequence  $(u_{\lambda})_{\lambda}$  of maximizers, one can extract a sequence which converges strongly towards a maximizer for  $\lambda_1$ .

**Proof.** Clearly,  $M_{\lambda_1} \leq M_{\lambda}$  for every  $\lambda > \lambda_1$ . Let  $(u_{\lambda})$  be a sequence of maximizers obtained by Theorem 2, for  $\lambda > \lambda_1$  sufficiently close to  $\lambda_1$ . By the inequality (12) one gets that  $(u_{\lambda})$  is uniformly bounded in  $H^1$  with respect to  $\lambda$ . Extracting a subsequence from  $(M_{\lambda})$  and  $(u_{\lambda})$ , one obtains that  $u_{\lambda} \rightharpoonup w$  which satisfies

$$-\Delta w + (q - \lambda_1)w = \frac{1}{M'_1} f w^{2^*-1},$$

where  $M'_1 = \lim M_{\lambda} \geq M_{\lambda_1}$ . Arguing as in the previous proof, one gets that  $w$  cannot be zero, and by the positiveness of  $I_{\lambda_1}$ , one has  $\int_{\Omega} f w^{2^*} \geq 0$ . Since

$$\int_{\Omega} f u_{\lambda}^{2^*} = M_{\lambda} \rightarrow M'_1 = \int_{\Omega} f w^{2^*} + \sum_i \nu_i f(x_i),$$

then for every  $i$ ,  $\nu_i f(x_i) \leq M'_1$ . Proceeding as we already did in the proof of Theorem 2

$$\mu_i = \frac{\nu_i f(x_i)}{M'_1} \leq \left( \frac{\nu_i f(x_i)}{M'_1} \right)^{1-\frac{2}{2^*}} \left( \frac{\nu_i f(x_i)}{M'_1} \right)^{\frac{2}{2^*}} \leq \beta \mu_i,$$

where  $\beta < 1$ . Finally one gets that  $\mu_i = \nu_i = 0$ ,  $\int_{\Omega} f w^{2^*} = 1$ ,  $w$  is a maximizer for  $M_{\lambda_1}$ , and  $M_{\lambda_1} = M'_1$ . Moreover, there is strong convergence of  $u_{\lambda}$  towards  $w$ .

### 3. ABOUT THE CONDITION (4). TEST FUNCTIONS.

In this section, we follow the ideas developed in [11]. However, we give some details of the proof for completeness sake.

We assume from now on that  $n > 4$ . We first prove that

$$M_{\lambda_1} \geq K(n, 2)^{2^*} \sup f.$$

Let  $\delta > 0$  be given,  $\delta < \sup f$ , suppose that  $f$  achieves its supremum on a point  $x'_0 \in \bar{\Omega}$ . Let  $x_0 \in \Omega$  be such that  $f(x_0) > \sup f - \delta$  and let  $u_{\epsilon}$  be defined as

$$u_{\epsilon}(x) = (\epsilon^2 + |x - x_0|^2)^{1-\frac{n}{2}}.$$

Suppose that  $R$  is small enough to have  $f > \sup f - \delta$  on  $B(x_0, 2R)$  and choose  $\varphi \in \mathcal{D}(B(x_0, 2R))$ ,  $\varphi = 1$  on  $B(x_0, R)$ ,  $0 \leq \varphi \leq 1$ . Finally define  $v_{\epsilon} = u_{\epsilon} \varphi$ . Then

$$\begin{aligned} I_{\lambda_1}(v_{\epsilon}) &= \int_{\mathbb{R}^n} |\nabla(u_{\epsilon} \varphi)|^2 + \int_{\mathbb{R}^n} (q - \lambda_1)(u_{\epsilon} \varphi)^2 \\ &\leq \int_{\mathbb{R}^n} |\nabla u_{\epsilon}|^2 + \int_{B(x_0, 2R) \setminus B(x_0, R)} (u_{\epsilon}^2 |\nabla \varphi|^2 + 2 |\nabla u_{\epsilon}| |\nabla \varphi|) \\ &\quad + \int_{\Omega} (q - \lambda_1)(u_{\epsilon} \varphi)^2 \\ &\leq \frac{(n-2)^2 \omega_{n-1}}{2} \int_0^{\infty} \frac{r^{n+1} dr}{(\epsilon^2 + r^2)^n} + \int_{B(x_0, 2R) \setminus B(x_0, R)} u_{\epsilon}^2 |\nabla \varphi|^2 + \\ &\quad + 2 \int_{B(x_0, 2R) \setminus B(x_0, R)} |\nabla u_{\epsilon}| |\nabla \varphi| + \int_{\Omega} (|q|_{\infty} + |\lambda_1|) C |u_{\epsilon}|_2^2 \\ &\leq C_1 \epsilon^{2-n} + o(\epsilon^{2-n}) \end{aligned} \tag{25}$$

where

$$C_1 = \omega_{n-1} \frac{(n-2)^2}{2} \int_0^{+\infty} \frac{r^{n+1} dr}{(1+r^2)^n} = \omega_{n-1} \frac{(n-2)^2}{2} \frac{\Gamma(\frac{n+3}{2}) \Gamma(\frac{n-1}{2})}{n!}.$$

On the other hand

$$\int_{\Omega} f|v_{\epsilon}|^{2^*} \geq (f(x_0) - \delta) \int_{B(x_0, R)} |u_{\epsilon}\varphi|^{2^*} \geq (f(x_0) - \delta)C_2\epsilon^{-n} + O(\epsilon^{-1}) \quad (26)$$

where

$$C_2 = \frac{\omega_{n-1}}{2} \int_0^{\infty} \frac{r^{n-1}dr}{(1+r^2)^n} = \frac{\omega_{n-1}}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)}{n!}.$$

Since  $\frac{C_1}{C_2^{\frac{2}{2^*}}} = K(n, 2)^{-2}$  one gets that

$$\frac{I_{\lambda_1}(v_{\epsilon})}{(\int_{\Omega} f|v_{\epsilon}|^{2^*})^{\frac{2}{2^*}}} \leq \frac{K(n, 2)^{-2}}{(\sup f - \delta)^{\frac{2}{2^*}}}$$

which implies the desired result, since  $\delta$  is arbitrary.

Let us introduce, for  $n > 4 + 2j$ ,

$$\alpha_{n,j} = \omega_{n-1} \int_0^{+\infty} \frac{r^{2j+n-1}dr}{(1+r^2)^{n-2}} = \frac{\omega_{n-1}}{2} \frac{\Gamma(\frac{n}{2}+j)\Gamma(\frac{n}{2}-j-1)}{(n-2)!}$$

and for  $n > 2k$

$$\beta_{n,k} = \omega_{n-1} \int_0^{+\infty} \frac{r^{2k+n-1}dr}{(1+r^2)^n} = \frac{\omega_{n-1}}{2} \frac{\Gamma(k+\frac{n}{2})\Gamma(n+1-\frac{k}{2})}{n!}.$$

We have the following result:

**Theorem 3.** Suppose that  $n > 4$ , that  $q$  and  $f$  are smooth functions, and suppose that  $f$  achieves its maximum on an interior point  $x_0$ . Let also  $k_f$  and  $k_q$  be the integers defined by

$$k_q = \inf\{j \geq 0, \Delta^j(q - \lambda_1)(x_0) \neq 0\}, \quad k_f = \inf\{k \geq 1, \Delta^k(f)(x_0) \neq 0\}.$$

Then one has  $M_{\lambda_1} > K(n, 2)^{2^*} \sup f$  in each one of the following situations

(1) If  $n > 2k_q + 4$ ,  $k_f = k_q + 1$  and

$$\frac{\alpha_{n,k_q}}{(2k_q)!} \Delta^{k_q}(q - \lambda_1)(x_0) - \left(\frac{n-2}{n}\right) \frac{\beta_{n,k_f}}{f(x_0)(2k_f)!} \Delta^{k_f}f(x_0) < 0$$

(2) If  $k_f > k_q + 1$  and  $\Delta^{k_q}(q - \lambda_1)(x_0) < 0$ .

(3) If  $n > 2k_f$  and  $k_q > k_f$ , and  $\Delta^{k_f}f(x_0) > 0$ .

**Remark 3.** When  $q = 0$  the condition in (1) becomes: if  $\lambda_1 > 0$ , and if  $k_f = 1$

$$-\alpha_{n,0}\lambda_1 - \left(\frac{n-2}{2n}\right) \frac{\beta_{n,1}\Delta f(x_0)}{f(x_0)} < 0$$

and no condition when  $k_f \geq 2$ , which is exactly the condition given in [1].

**Proof of Theorem 3.** Suppose that  $\delta > 0$  is such that  $B(x_0, 2\delta) \subset\subset \Omega$  and let  $\varphi$  be some smooth function, compactly supported in  $B(x_0, 2\delta)$  whose value is 1 on a neighborhood of  $x_0$ . Let  $u_\epsilon(r) = (\epsilon^2 + r^2)^{1-\frac{n}{2}}$  and let  $v_\epsilon$  be defined as

$$v_\epsilon = u_\epsilon \varphi.$$

Suppose that  $J$  is the biggest integer such that  $n > 2J+4$  and let  $\eta = 2J+2$ , one can write

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |\nabla(u_\epsilon \varphi)|^2 - \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \right| &\leq c\epsilon^{-n+2} \int_{\frac{\delta}{\epsilon}}^{\infty} \frac{r^{n+1} dr}{(1+r^2)^n} \\ &\leq C\epsilon^{-n+2+\eta} \int_0^{+\infty} \frac{r^{\eta+n+1} dr}{(1+r^2)^n} \end{aligned}$$

this last integral being convergent. Using the definition of  $\eta$ , one gets that

$$\left| \int_{\mathbb{R}^n} |\nabla(u_\epsilon \varphi)|^2 - \int_{\mathbb{R}^n} |\nabla u_\epsilon|^2 \right| = O(\epsilon^{-n+4+2J}).$$

Let us now treat the term

$$\int_{\mathbb{R}^n} (q - \lambda_1) v_\epsilon^2 = \int_{\mathbb{R}^n} (q - \lambda_1) (u_\epsilon \varphi)^2.$$

We remark that for all  $j$  such that  $j \leq J$  one has

$$\left| \int_{\mathbb{R}^n} r^{2j} (u_\epsilon \varphi)^2 - \int_{\mathbb{R}^n} r^{2j} u_\epsilon^2 \right| = o(\epsilon^{n+2J-4}).$$

Indeed, this difference can be majorized (up to a constant) by

$$\epsilon^{2j-4+n} \int_{\frac{\delta}{\epsilon}}^{\infty} \frac{u^{2j+n-1}}{(1+u^2)^{n-2}} du.$$

Let  $\eta$  be such that  $n-4-2j > \eta > 2(J-j)$ , then one may write

$$\epsilon^{2j-4+n} \int_{\frac{\delta}{\epsilon}}^{\infty} \frac{u^{2j+n-1}}{(1+u^2)^{n-2}} = \epsilon^{2j-4+n} \int_{\frac{\delta}{\epsilon}}^{\infty} \frac{u^{2j+n+\eta-1}}{u^\eta (1+u^2)^{n-2}} du \leq \epsilon^{2j-4+n+\eta} \delta^{-\eta} C$$

where the constant  $C$  is given by the rest of some convergent integral.

Then, using a Taylor expansion around  $x_0$ , one gets that

$$\int_{\mathbb{R}^n} (q - \lambda_1) (u_\epsilon \varphi)^2 = (q(x_0) - \lambda_1) \int_{\mathbb{R}^n} u_\epsilon^2 + \sum_{1 \leq j \leq [\frac{n-4}{2}] = J} \frac{\Delta^j q(x_0) \alpha_{n,j}}{(2j)!} \epsilon^{2j} + O(\epsilon^{4-n+2J})$$

and then

$$\int_{\mathbb{R}^n} (q - \lambda_1) v_\epsilon^2 = (q(x_0) - \lambda_1) \int_{\mathbb{R}^n} u_\epsilon^2 + \sum_{1 \leq j \leq [\frac{n-4}{2}] = J} \frac{\Delta^j q(x_0) \alpha_{n,j}}{(2j)!} \epsilon^{2j} + O(\epsilon^{4-n+2J}),$$

where

$$\alpha_{n,j} = \omega_{n-1} \int_0^\infty \frac{u^{2j+n-1} du}{(1+u^2)^{n-2}}.$$

Here we have used the technical Lemma 6 in [11], which basically says that the integrals over odd order terms in the Taylor expansion are zero.

Let us now treat the denominator. Let  $K$  be the greatest integer such that  $n > 2K$ , let us see that for all  $k \leq K$

$$\int_{\mathbb{R}^n} r^{2k} u_\epsilon^{2^*} - \int_{\mathbb{R}^n} r^{2k} (u_\epsilon \varphi)^{2^*} = o(\epsilon^{-n+2K}).$$

Indeed the difference above can be majorized by

$$C \epsilon^{-n+2k} \int_{\frac{\delta}{\epsilon}}^\infty \frac{u^{2k+n-1} du}{(1+u^2)^n} \leq C \epsilon^{-n+2k} \int_{\frac{\delta}{\epsilon}}^\infty \frac{u^{2k+n+\eta-1} du}{(1+u^2)^n u^\eta},$$

hence by choosing  $\eta$  such that  $n - 2k > \eta > 2(K - k)$ , one gets

$$C \epsilon^{-n+2k} \int_{\frac{\delta}{\epsilon}}^\infty \frac{u^{2k+n+\eta-1} du}{u^\eta (1+u^2)^n} \leq C \epsilon^{-n+2k+\eta} \delta^{-\eta} = o(\epsilon^{2K-n}).$$

Using a Taylor expansion around  $x_0$  for  $f$  and Lemma 6 in [11], one gets

$$\begin{aligned} & \int_{\mathbb{R}^n} f(u_\epsilon \varphi)^{2^*} \\ &= f(x_0) \int_{\mathbb{R}^n} (u_\epsilon \varphi)^{2^*} + \sum_{k \leq [\frac{n}{2}]} \beta_{k,n} \Delta^k f(x_0) \int_{\mathbb{R}^n} r^{2k} (u_\epsilon \varphi)^{2^*} + O(\epsilon^{-n+2[\frac{n}{2}]}) \\ &= f(x_0) C_2 \epsilon^{-n} + \sum_{k \leq [\frac{n}{2}]} \Delta^{(k)} f(x_0) \beta_{k,n} \epsilon^{2k-n} + O(\epsilon^{-n+2[\frac{n}{2}]}) \\ & \left( \int_{\mathbb{R}^n} f(u_\epsilon \varphi)^{2^*} \right)^{\frac{2}{2^*}} = (f(x_0) C_2 \epsilon^{-n})^{\frac{2}{2^*}} \left( 1 + \sum_{k \leq [\frac{n}{2}]} \beta_{k,n} \frac{\Delta^k f(x_0)}{f(x_0) C_2} \epsilon^{2k} + o(\epsilon^{2[\frac{n}{2}]}) \right). \end{aligned}$$

Suppose that  $k_f \leq [\frac{n}{2}]$ , then the sum above may be reduced, say

$$\left( \int_{\mathbb{R}^n} f(u_\epsilon \varphi)^{2^*} \right)^{\frac{2}{2^*}} = (f(x_0) c_2 \epsilon^{-n})^{\frac{2}{2^*}} (1 + \beta_{k_f,n} \frac{\Delta^{k_f} f(x_0)}{f(x_0) c_2} \epsilon^{2k_f} + o(\epsilon^{2k_f}))$$

where

$$\beta_{k,n} = \omega_{n-1} \int_0^\infty \frac{u^{2k+n-1}}{(1+u^2)^n} du.$$

Suppose that  $n > 2k_q + 4$ , and  $2k_f > 2k_q + 2$ , then one may write

$$\frac{I_{\lambda_1}(v_\epsilon)}{(\int f v_\epsilon^{2^*})^{\frac{2}{2^*}}} = \frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^*}}} (1 + \epsilon^{2k_q+4} \alpha_{k_q,n} \Delta^{k_q} q(x_0) + o(\epsilon^{2k_q+4})) (1 + o(\epsilon^{2k_q+4})).$$

As a consequence, the first nonzero term which appears in this development is the term with  $\Delta^{k_q} q(x_0)$ . Since in this calculation  $\frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^*}}}$  is nothing

else that  $K(n, 2)^{-2}(\sup f)^{\frac{2}{2^*}}$ , the result follows as soon as  $\Delta^{k_q} q(x_0) < 0$ .

Suppose that  $n > 2k_f$  and  $2k_q + 4 > 2k_f$  then

$$\begin{aligned} \frac{I_{\lambda_1}(v_\epsilon)}{(\int f v_\epsilon^{2^*})^{\frac{2}{2^*}}} &= \frac{c_1}{(f(x_0)c_2)^{\frac{2}{2^*}}} \left( 1 + o(\epsilon^{2k_f}) \left( 1 - \frac{2}{2^*} \frac{\beta_{k,n} \Delta^{k_f} f(x_0) \epsilon^{2k_f}}{f(x_0) C_2} + o(\epsilon^{2k_f}) \right) \right) \\ &< \frac{K(n, 2)^{-2}}{f(x_0)^{\frac{2}{2^*}}} \end{aligned}$$

as soon as  $\Delta^{k_f} f(x_0) > 0$ .

Finally, suppose that  $n > 2k_q + 4 = 2k_f + 2$ , then the first nonzero term in the expansion above are of the same order at the numerator and the denominator. Then the result follows as soon as

$$\Delta^{k_q}(q - \lambda_1)(x_0) \alpha_{k_q,n} - \frac{2}{2^*} \frac{\Delta^{k_f} f(x_0) \beta_{k_f,n}}{f(x_0) C_2} < 0.$$

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