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HOMOGENIZATION OF HAMILTON-JACOBI EQUATIONS IN THE HEISENBERG GROUP

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Abstract. Setting the Homogenization of Hamilton Jacobi equations in the geometry of the Heisenberg group, we study the convergence toward a solution of the limit equation i.e. the solution of the effective Hamiltonian, in particular we estimate the rate of convergence. The periodicity of the fast variable and the dilation are both taken compatibly with the group.

1. **Introduction.** In this work we shall consider Hamilton-Jacobi equations involving a Hamiltonian which is not coercive. Precisely we shall study equations of the following type:

$$u + F(\xi, \sigma(\xi)\nabla u) = 0 \text{ in } \mathbb{R}^{2n+1}, \tag{1}$$

where $\sigma(\xi) = \begin{pmatrix} I & 0 & 2y^T \\ 0 & I & -2x^T \end{pmatrix}$ with I the identity $n \times n$ matrix and where $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Clearly $F(\xi, \sigma(\xi)q)$ is not coercive in q but we shall require that it is coercive in $p = \sigma(\xi)q$.

We denote by $\nabla_{\mathbb{H}^n} u := \sigma \nabla u = (X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u)$ the so called horizontal gradient in the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$. Before proceeding in this introduction we want to recall a few notions concerning the Lie group $\mathbb{H}^n$. The group action is given by

$$\xi_o \circ \xi = (x + x_o, y + y_o, t + t_o + 2(x.y_o - y.x_o))$$

and $\nabla_{\mathbb{H}^n}$ is defined through the vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t},\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

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which are left invariant with respect to $\circ$. Furthermore they satisfy the so called Hörmander condition i.e. they generate the whole Lie algebra of left invariant vector fields together with their Lie bracket.

Hamilton-Jacobi equations in the Heisenberg group have already been studied by Manfredi and Stroffolini in [11] where they give a Hopf-Lax formula for the following equation

$$u_t + F(\sigma(\xi)\nabla u) = 0.$$

For further properties concerning the Heisenberg group see the next section and [15]. We just recall that the Heisenberg group possesses a family of dilations given by $\delta_r(\xi) = (rx, ry, r^2 t)$ for $r > 0$. We shall endow $\mathbb{H}^n$ with the following "norm" with respect to $\delta_.$:

$$|\xi|_{\mathbb{H}^n} := \left(\left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 + t^2 \right)^{1/4}$$

and the associated distance:

$$d_{\mathbb{H}^n}(\xi, \eta) = |\eta^{-1} \circ \xi|_{\mathbb{H}^n}.$$

In the first part of this paper we shall establish a comparison principle for viscosity sub and super-solutions for (1) under the following assumptions on F:

There exists $C, C_3 > 0$ such that for all $\xi, \xi' \in \mathbb{H}^n$, $p, p' \in \mathbb{R}^{2n}$:

$(F1)$ $$|F(\xi', p') - F(\xi, p)| \leq C(|\xi^{-1} \circ \xi'|_{\mathbb{H}^n} + |p' - p|)$$

$(F2)$ $$|F(\xi, 0)| \leq C_3$$

$(F3)$ $$\lim_{|p| \to \infty} F(\xi, p) = +\infty \ , \quad \text{uniformly for } \xi \in \mathbb{H}^n.$$

Let us emphasize that p stands for the Heisenberg gradient of the solution, hence the norm is the Euclidean norm in $\mathbb{R}^{2n}$; furthermore even if $(F3)$ holds, for appropriate ξ one may get $F(\xi, \sigma(\xi)q) = 0$ with $q \in \mathbb{R}^{2n+1}$, $|q| >> 1$.

Also we shall need to consider the space of Lipschitz functions with respect to the group metric, namely:

$$\Lambda(\mathbb{H}^n) = \{u \in C(\mathbb{H}^n); \exists K > 0, \forall \xi, \eta \in \mathbb{H}^n,\ |u(\xi) - u(\eta)| \leq K|\eta^{-1} \circ \xi|_{\mathbb{H}^n}\}.$$

Observe that $\Lambda(\mathbb{H}^n)$ does not coincide with the set of Lipschitz functions while the topological equivalence of the Euclidean metric and the $d_{\mathbb{H}^n}$ metric implies that $C(\mathbb{H}^n) = C(\mathbb{R}^{2n+1})$.

THEOREM 1.1. *Under assumptions (F1), (F2), (F3), there exists a bounded viscosity solution in $\Lambda(\mathbb{H}^n)$ for equation (1).*
Moreover, we have the following comparison result: let u be a bounded sub solution and v be a bounded super solution of (1), then $u \leq v$ on $\mathbb{H}^n$.
In particular, the solution in $L^\infty(\mathbb{H}^n) \cap \Lambda(\mathbb{H}^n)$ of (1) is unique.

Of course the conditions on the Hamiltonian F are not the most general but our main aim is to show the particularity of the Heisenberg group in relation with viscosity solutions.

Since we want to consider a homogenization problem, we need to introduce a Hamiltonian depending both on slow and fast variables. Hence, we consider $H : \mathbb{H}^n \times \mathbb{H}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ that will be periodic in the second variable, in a sense to be specified below.

In this context, we are concerned with

$$(E_\varepsilon) \qquad u^\varepsilon(\xi) + H\left(\xi, \delta_{\frac{1}{\varepsilon}}\xi, \nabla_{\mathbb{H}^n} u^\varepsilon(\xi)\right) = 0 \ \text{ on } \mathbb{H}^n$$

where ε is a small positive parameter that will converge to zero.

The main result of this paper is to extend to the Heisenberg group the result obtained by I. Capuzzo Dolcetta, H. Ishii [8] in the coercive case where they estimate the rate of convergence of u_ε to the solution of the effective Hamiltonian. For other results in homogenization of Hamilton Jacobi equations see e.g. Evans [9] and Lions, Papanicolau, Varadhan [13]. More recently let us mention the works of O. Alvarez, M. Bardi [1, 2] and M. Arisawa [3].

Here we assume that the Hamiltonian H is $\mathbb{H}^n$-periodic in the second variable with respect to the group action i.e. we suppose that

$$H(\xi, ke_i \circ \eta, p) = H(\xi, \eta, p), \ \text{for } k \in \mathbb{N}, \forall 1 \leq i \leq 2n \tag{2}$$

with $e_i = (0, \dots, 1, \dots, 0)$ where 1 is in the i-th position.

The other conditions that we shall impose for H are morally similar to those required previously for F. Namely, we assume that there exists $C, C_3, \nu > 0$ such that for all $\xi, \xi', \eta, \eta' \in \mathbb{H}^n$, $p, p' \in \mathbb{R}^{2n}$.

$$(H1) \qquad |H(\xi', \eta', p') - H(\xi, \eta, p)| \leq C(|\xi^{-1} \circ \xi'|_{\mathbb{H}^n} + |\eta^{-1} \circ \eta'|_{\mathbb{H}^n} + |p' - p|)$$

$$(H2) \qquad \nu|p| - C_3 \leq H(\xi, \eta, p) \leq \nu|p| + C_3.$$

Following Lions, Papanicolaou, Varadhan, [13], we can formally write

$$u_\varepsilon(\xi) = u_o(\xi) + \varepsilon u_1(\xi, \delta_{\frac{1}{\varepsilon}}\xi) + o(\varepsilon).$$

Since the Heisenberg vector fields are homogeneous of degree 1 with respect to the dilation, one gets that the limit equation when ε goes to 0, satisfied by u_o and u_1 is

$$u_o(\xi) + H(\xi, \eta, \nabla_{\mathbb{H}^n, \xi} u_o(\xi) + \nabla_{\mathbb{H}^n, \eta} u_1(\xi, \eta)) = 0.$$

It is then natural to prove that the limit problem of (E_ε) is given by

$$(\overline{E}) \qquad u(\xi) + \overline{H}(\xi, \nabla_{\mathbb{H}^n} u(\xi)) = 0 \ \text{ on } \mathbb{H}^n$$

where $\overline{H}$, the so-called effective Hamiltonian, is obtained by solving the "cell-problem"

$$(CP) \qquad H(\xi, \eta, p + \nabla_{\mathbb{H}^n} v(\eta)) = \lambda \ \text{ on } \mathbb{H}^n$$

where $\xi \in \mathbb{H}^n$ and $p \in \mathbb{R}^{2n}$ are some fixed parameters.
Indeed, there exists a unique $\lambda = \lambda(\xi, p) \in \mathbb{R}$ such that (CP) admits a bounded, continuous solution v. One then defines:

$$\overline{H}(\xi, p) = \lambda \quad \forall (\xi, p) \in \mathbb{H}^n \times \mathbb{R}^{2n}.$$

We will prove, using the method of Capuzzo-Dolcetta and Ishii [8], that the sequence of solutions u^ε of (E_ε) converges uniformly on $\mathbb{H}^n$ to the solution u of $(\overline{E})$; this method will also give the rate of convergence in term of ε. Our first result is the following:

THEOREM 1.2. *Assume that* $H : \mathbb{H}^n \times \mathbb{H}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ *is* $\mathbb{H}^n$*-periodic in its second variable and satisfies* $(H1), (H2)$.
Then there exists $C > 0$, *independent of* $\varepsilon \in (0,1)$, *such that*

$$\sup_{\xi\in\mathbb{H}^n} |u^\varepsilon(\xi) - u(\xi)| \le C\varepsilon^{1/5}.$$

There are two difficulties in this new setting. One is that, even though $|.|_{\mathbb{H}^n}$ plays a role in $\mathbb{H}^n$ similar to the one played by the Euclidean norm in $\mathbb{R}^n$ it lacks a fundamental fact : it is *not* the solution of the eikonal equation $|\nabla_{\mathbb{H}^n} u| = 1$ but only a sub-solution. Hence it is necessary to prove that viscosity solutions of

$$|\nabla_{\mathbb{H}^n} u| \le C$$

are Lipschitz with respect to the norm $|.|_{\mathbb{H}^n}$. This is done in Lemma 3.1. Since the Carnot-Caratheodory distance as defined e.g. by Gromov is the solution of the eikonal equation, this result is clearly related to the fact that the Carnot-Caratheodory metric is equivalent to the metric related to the norm $|.|_{\mathbb{H}^n}$. See the works of Monti and Serra-Cassano in [12].

Here on the other hand we don't use the Carnot-Caratheodory metric because it doesn't allow explicit computations as required in the homogenization process.

The proof of Lemma 3.1 is different from the Euclidean case and it morally uses the bracket generating property of the vector fields i.e. $[X_i, Y_i] = -4\frac{\partial}{\partial t}$. Furthermore we should emphasize that this intrinsic Lipschitzianity implies only Hölder continuity.

The other difficulty rises from the fact that it is quite standard to prove comparison results for viscosity solutions by doubling the variables. In the Euclidean setting the test function is then constructed via the smoothing factor $|x-y|^2$ which has the important properties that $\nabla_x|x-y|^2 = -\nabla_y|x-y|^2$ see [6]. Hence it seems natural to replace $|x-y|^2$ with $|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n}$ but it is easy to see that

$$\nabla_{\mathbb{H}^n,\xi}|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n} \neq -\nabla_{\mathbb{H}^n,\eta}|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n}.$$

In fact $\nabla_{\mathbb{H}^n,\xi} f(\eta^{-1}\circ\xi) \neq -\nabla_{\mathbb{H}^n,\eta} f(\eta^{-1}\circ\xi)$ for any f non constant.

Hence the choice of the test functions is new and it is constructed ad hoc to this setting.

Let us mention that the converging rate is different from the one obtained in the coercive case by Capuzzo Dolcetta and Ishii since they obtain $\varepsilon^{1/3}$; this fact seems to be related to the technique adopted and the interesting question of knowing whether or not it is sharp remains open.

Let us finally consider the simpler case when the dependence on ξ is only via the matrix σ i.e. for $H(\xi,\eta,p) = H(\eta,p)$. The rate of convergence is very much improved with respect to Theorem 1.2 because the solution of the effective equation is constant:

THEOREM 1.3. *Assume that* $H(\xi,\eta,p)$ *is independent of* ξ, $\mathbb{H}^n$*-periodic in* η *and satisfies* $(H1), (H2)$.
Then there exists $C > 0$ *independent of* $\varepsilon \in (0,1)$ *such that*

$$\sup_{\xi\in\mathbb{H}^n} |u^\varepsilon(\xi) - u(\xi)| \le C\varepsilon.$$

2. **Some results on periodicity in $\mathbb{H}^n$.** In the introduction we have already given most of the notations and definitions concerning $\mathbb{H}^n$.

We define the open ball of radius R and center ξ_o by

$$B_{\mathbb{H}^n}(\xi_o, R) = \{\xi \in \mathbb{H}^n;\ d_{\mathbb{H}^n}(\xi, \xi_o) < R\}.$$

Let $Q = [-\frac{1}{2}, \frac{1}{2})^{2n} \times [-2, 2) \in \mathbb{R}^{2n+1}$. For all $k \in \mathbb{Z}^{2n}$, we define $Q_k = (k, 0) \circ Q$, the left translated cube by k with respect to translations of $\mathbb{H}^n$.

One can prove that, even though $\bigcup_{k\in\mathbb{Z}^{2n}} Q_k \neq \mathbb{H}^n$, $\{Q_k\}_{k\in\mathbb{Z}^{2n}}$ generates a tiling of $\mathbb{H}^n$ in the following sense

Lemma 2.1. *For any $\xi \in \mathbb{H}^n$ there exists $\xi_o \in Q$ and there exists a finite number of left group actions generated by elements of the form $(k, 0)$ with $k \in \mathbb{Z}^{2n}$ that applied to ξ_o give ξ.*

Proof. Indeed take any $\xi = (x, y, t)$ in $\mathbb{H}^n$. First fix $k_i = \left[x_i + \frac{1}{2}\right]$ (where $[s]$ stands for the integer part of s.) hence $\tilde{x}_i = x_i - k_i \in [-\frac{1}{2}, \frac{1}{2})$ and similarly for $h_i = \left[y_i + \frac{1}{2}\right]$ we choose $\tilde{y}_i = y_i - h_i$.

Now choose $k = (k_1, \ldots, k_n, h_1, \ldots, h_n)$ clearly :

$$(k, 0) \circ (\tilde{x}, \tilde{y}, \tilde{t}) = (x, y, \tilde{t} + 2\sum_{i=1}^{n}(h_i\tilde{x}_i - k_i\tilde{y}_i)).$$

This is true for any $\tilde{t} \in \mathbb{R}$.

Now let us note that for $i = 1, \ldots, n$:

$$s_1e_i \circ s_2e_{i+n} \circ (-s_1e_i) \circ (-s_2e_{i+n}) \circ (x, y, \tau) = (x, y, \tau - 4s_1s_2) \tag{3}$$

where e_i is the $i-th$ vector in the Euclidean base . Hence we need to find $\tilde{t} \in [-2, 2)$ and $n_1 \in \mathbb{Z}$ and $n_2 \in \mathbb{Z}$ such that

$$\tilde{t} + 2\sum_{i=1}^{n}(h_i\tilde{x}_i - k_i\tilde{y}_i) - 4n_1n_2 = t.$$

To achieved this end just choose

$$n_1n_2 = \left[\frac{t - 2\sum_{i=1}^{n}(h_i\tilde{x}_i - k_i\tilde{y}_i)}{4}\right].$$

This ends the proof of the Lemma 2.1.

Remark. In [7] a similar result was proved but $\mathbb{H}^n$ was tiled by taking the group action of any point in $\mathbb{Z}^{2n+1}$ and not only the points of the type $(k, 0)$.

In the introduction we said that a function defined on $\mathbb{H}^n$ is $\mathbb{H}^n$-periodic if

$$f((k, 0) \circ \xi) = f(\xi)\ , \forall k \in \mathbb{Z}^{2N}, \forall \xi \in Q.$$

Clearly from the considerations above it is clear that this implies that f is also periodic in the last variable.

Consider now the set Q_k dilated by a small parameter $\varepsilon > 0$:

$$Q_k^\varepsilon = \delta_\varepsilon Q_k = \delta_\varepsilon((k, 0) \circ Q) = (\varepsilon k, 0) \circ (\delta_\varepsilon Q).$$

The last equality comes from the fact that the dilation is well defined with respect to the group action $\circ$. One easily checks that $\{Q_k^\varepsilon\}_{k\in\mathbb{Z}^{2n}}$ still generates a tiling of $\mathbb{H}^n$, in the sense of Lemma 2.1 and we shall say that a function is $\varepsilon\mathbb{H}^n$-periodic if

$$f((\varepsilon k, 0)\circ\xi) = f(\xi)\ ,\forall k\in\mathbb{Z}^{2n}, \forall \xi\in Q^\varepsilon.$$

Note finally that if $\xi\mapsto f(\xi)$ is $\mathbb{H}^n$-periodic, then $\xi\mapsto f\left(\delta_{\frac{1}{\varepsilon}}\xi\right)$ is $\varepsilon\mathbb{H}^n$-periodic.

3. **Existence and comparison result for Hamilton-Jacobi equations on $\mathbb{H}^n$.**

In this section, we consider the equation:

$$\gamma u(\xi) + F(\xi, \nabla_{\mathbb{H}^n} u(\xi)) = 0 \quad \text{on } \mathbb{H}^n \tag{4}$$

where $\gamma > 0$ and F is a continuous function on $\mathbb{H}^n\times\mathbb{R}^{2n}$.
Let us first precise that $f\in C^1(\mathbb{H}^n)$ means that the horizontal gradient $\nabla_{\mathbb{H}^n} f$ is continuous on $\mathbb{H}^n$. Clearly a function may be C^1 in this sense but not in the usual sense.

Following [11] we give the following

Definition 3.1. *We say that $u\in C(\mathbb{H}^n)$ is a viscosity sub-solution of (4) if*

$\forall\phi\in C^1(\mathbb{H}^n)$, if $\xi_o\in\mathbb{H}^n$ is a point of local maximum of $u-\phi$ then:

$$\gamma u(\xi_o) + F(\xi_o, \nabla_{\mathbb{H}^n}\phi(\xi_o)) \le 0.$$

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$$\gamma u(\xi_o) + F(\xi_o, \nabla_{\mathbb{H}^n}\phi(\xi_o)) \ge 0.$$

u is a viscosity solution if it is both a super and sub solution.

The formal Taylor expansion of a function $u:\mathbb{H}^n\to\mathbb{R}$ at the point ξ_o reads (see [14]):

$$u(\xi) = u(\xi_o) + \langle\nabla_{\mathbb{H}^n} u(\xi_o), \overline{\xi_o^{-1}\circ\xi}\rangle + o(|\xi_o^{-1}\circ\xi|_{\mathbb{H}^n})$$

where we have used the following notation: if $\xi = (x,y,t)$ then $\overline{\xi} = (x,y)$.

This formula suggests the following definition of sub differential and super differential adapted to our situation.

Definition 3.2. *Let $u\in C(\mathbb{H}^n)$, we set*

$$D^+_{\mathbb{H}^n} u(\xi_o) = \{p\in\mathbb{R}^{2n}; \limsup_{\xi\to\xi_o}\frac{u(\xi)-u(\xi_o)-\langle p, \overline{\xi_o^{-1}\circ\xi}\rangle}{|\xi_o^{-1}\circ\xi|_{\mathbb{H}^n}}\le 0\},$$

$$D^-_{\mathbb{H}^n} u(\xi_o) = \{p\in\mathbb{R}^{2n}; \liminf_{\xi\to\xi_o}\frac{u(\xi)-u(\xi_o)-\langle p, \overline{\xi_o^{-1}\circ\xi}\rangle}{|\xi_o^{-1}\circ\xi|_{\mathbb{H}^n}}\ge 0\}.$$

Remark 1. *Taking* $\xi = \xi_o \circ (hp, 0)$ *with* $h \in \mathbb{R}$, *one easily checks that*

$$\limsup_{\xi \to \xi_o} \frac{\langle p, \overline{\xi_o^{-1} \circ \xi} \rangle}{|\xi_o^{-1} \circ \xi|_{\mathbb{H}^n}} = |p|,$$

$$\liminf_{\xi \to \xi_o} \frac{\langle p, \overline{\xi_o^{-1} \circ \xi} \rangle}{|\xi_o^{-1} \circ \xi|_{\mathbb{H}^n}} = -|p|.$$

As a consequence, if there exists $C > 0$ *such that for all* $\xi \in \mathbb{H}^n$

$$|u(\xi) - u(\xi_o)| \leq C|\xi_o^{-1} \circ \xi|_{\mathbb{H}^n}$$

then any $p \in D^+_{\mathbb{H}^n} u(\xi_o)$ *or any* $p \in D^-_{\mathbb{H}^n} u(\xi_o)$ *satisfies* $|p| \leq C$.
The converse is a much more delicate matter and will be considered in Lemma 3.1.

We now give an equivalent definition for a viscosity solution of (4) in term of sub and super differential (see [11] for a proof) :

Proposition 3.1. $u \in C(\mathbb{H}^n)$ *is a viscosity solution of (4) if and only if for all* $\xi_o \in \mathbb{H}^n$

$$\forall p \in D^+_{\mathbb{H}^n} u(\xi_o), \ \gamma u(\xi_o) + F(\xi_o, p) \leq 0$$

and

$$\forall p \in D^-_{\mathbb{H}^n} u(\xi_o), \ \gamma u(\xi_o) + F(\xi_o, p) \geq 0.$$

On the other hand it is possible to prove that a solution of (4) (in the sense of definition 3.1) is a solution of (4) in the classical viscosity sense since

Proposition 3.2. *Let* $\xi_o = (x_o, y_o, t_o) \in \mathbb{R}^{2n+1}$ *and* $p \in \mathbb{R}^{2n+1}$ *be an element of* $D^+ f(\xi_o)$ *(resp.* $D^- f(\xi_o)$*). Then, writing* $p = (p_1, p_2, p_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, *one has:*

$$(p_1 + 2p_3 y_o, p_2 - 2p_3 x_o) \in D^+_{\mathbb{H}^n} f(\xi_o) \ \textit{(resp. } D^-_{\mathbb{H}^n} f(\xi_o)\textit{)}.$$

Proof: Let $(p_1, p_2, p_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be an element of $D^+ f(\xi_o)$. One has:

$$\begin{aligned}
f(\xi_o \circ \eta) &= f(x_o + \eta_1, y_o + \eta_2, t_o + \eta_3 + 2(\eta_1.y_o - \eta_2.x_o)) \\
&\leq f(\xi_o) + p_1.\eta_1 + p_2.\eta_2 + p_3(\eta_3 + 2(\eta_1.y_o - \eta_2.x_o)) \\
&+ o(|\eta_1| + |\eta_2| + |\eta_3 + 2(\eta_1.y_o - \eta_2.x_o)|) \\
&= f(\xi_o) + (p_1 + 2p_3 y_o).\eta_1 + (p_2 - 2p_3 x_o).\eta_2 + p_3\eta_3 + o(|\eta_1| + |\eta_2| + |\eta_3|) \\
&= f(\xi_o) + (p_1 + 2p_3 y_o).\eta_1 + (p_2 - 2p_3 x_o).\eta_2 + o(|\eta_1| + |\eta_2| + |\eta_3|^{1/2}).
\end{aligned}$$

One easily checks that $o(|\eta|_{\mathbb{H}^n}) = o(|\eta_1| + |\eta_2| + |\eta_3|^{1/2})$ and then, by definition, one deduces that $(p_1 + 2p_3 y_o, p_2 - 2p_3 x_o) \in D^+_{\mathbb{H}^n} f(\xi_o)$.

We now want to consider the proof of Theorem 1.1. We shall give a direct proof of the comparison result because it shows the peculiarity of this Hamiltonian and it uses tools that will be used in the proof of Theorem 1.2 in a more readable context. Let us mention that the comparison result of Theorem 1.1 can be proved using Proposition 3.2 and comparison results of standard viscosity solutions (see Barles [6]). On the other hand, we skip the proof of the existence. Indeed, by the comparison principle, it is possible to use the standard Perron's method to get the existence of solutions (see Ishii [10]).

The following result will be crucial in the sequel:

LEMMA 3.1. *Any bounded viscosity sub solution of* $|\nabla_{\mathbb{H}^n} u| \le C$ *satisfies*

$$|u(\xi \circ \eta) - u(\xi)| \le C|\eta|_{\mathbb{H}^n}$$

for any $\eta \in \mathbb{H}^n$.

Proof of Lemma 3.1: For simplicity we shall write the proof when $n = 1$, but of course the proof is identical for general n.

First step: We shall prove that for any $\eta = (\eta_1, \eta_2, 0)$ we get

$$|u(\xi \circ \eta) - u(\xi)| \le C|\eta|_{\mathbb{H}^n} := C|\eta|.$$

Let $\xi = (x_1, x_2, x_3)$ and

$$\sigma(\xi) = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \end{pmatrix}$$

then $|\nabla_{\mathbb{H}^n} u(\xi)| = |\sigma(\xi)\nabla u| = \sup_{q\in\mathbb{R}^2, |q|\le 1} \sigma^T(\xi) q \cdot \nabla u$.

We now fix $q = (q_1, q_2)$ and we define $y_\xi(s; q) \in \mathbb{R}^{2n+1}$ the solution of

$$\begin{cases} y' = \sigma^T q, \\ y(0) = \xi. \end{cases}$$

If we identify $(q_1, q_2) = (q_1, q_2, 0)$ then $y_\xi(s; q) = \xi \circ sq$.

u satisfies the hypothesis of Theorem 5.21 of Bardi-Capuzzo Dolcetta [4] with $\lambda = 0$. Indeed u is a viscosity sub solution of

$$\sup_{q\in\mathbb{R}^2, |q|\le 1} \{\sigma^T(\xi) q \cdot \nabla u - C\} \le 0$$

hence u satisfies

$$u(y_\xi(s; q)) - u(y_\xi(t; q)) \le C(t - s)$$

for $s \le t$. By choosing $-q$ instead of q and observing that $-t \le -s$ we obtain that

$$|u(y_\xi(s; q)) - u(y_\xi(t; q))| \le C|(t - s)|.$$

This of course holds for any unitary $q \in \mathbb{R}^2$ and hence the first step is concluded.

Second step: Now take any $\xi = (x, y, t)$ and $\xi_1 = (x_1, y_1, t_1)$ and let $\eta = (x_1 - x, y_1 - y, 0)$, hence $\xi \circ \eta = (x_1, y_1, t + 2(yx_1 - xy_1)$. So to estimate $u(\xi_1) - u(\xi)$ we need to estimate $u(\xi_1) - u(\xi \circ \eta)$ (since we know how to estimate $u(\xi \circ \eta) - u(\xi)$) i.e. we need to estimate $u(x_1, y_1, t_1) - u(x_1, y_1, \tau)$ with $\tau = t + 2(yx_1 - xy_1)$.

Without loss of generality we can suppose that $\tau > t_1$. Recalling (3) i.e.

$$(x_1, y_1, \tau) \circ s_1 e_1 \circ s_2 e_2 \circ (-s_1 e_1) \circ (-s_2 e_2) = (x_1, y_1, \tau - 4 s_1 s_2).$$

We can choose $s_1 = s_2 = s$ and $s^2 = \frac{1}{4}(\tau - t_1)$.

Using the above result we obtain

$$u(x_1, y_1, t_1) - u(x_1, y_1, \tau) \le K(4s) = 2K(\tau - t_1)^{1/2}.$$

Putting everything together we obtain

$$|u(\xi_1) - u(\xi)| \le K(|\eta| + 2(|t - t_1 + 2(yx_1 - xy_1)|)^{1/2}) =$$
$$= K(|(x_1 - x, y_1 - y)| + 2(|t - t_1 + 2(yx_1 - xy_1)|)^{1/2}).$$

This ends the proof.

Direct proof of Theorem 1.1: Let u and v be respectively a bounded super solution and a bounded sub solution of (4). For $\varepsilon, \alpha \in (0,1)$, $\xi = (x,y,t)$ and $\eta = (x',y',t')$, we define the functions:

$$A_\varepsilon(\xi,\eta) = \left(\left(\frac{(x-x')^2+(y-y')^2}{\varepsilon}\right)^2 + (t-t'+2(x'y-xy'))^2\right)^{1/2}$$

$$\rho(\xi) = |\xi|^2_{\mathbb{H}^n}$$

and then

$$\psi(\xi,\eta) = v(\eta) - u(\xi) - \frac{1}{\varepsilon}A_\varepsilon(\xi,\eta) - \alpha\rho(\xi).$$

Since $\varepsilon \in]0,1[$, one clearly has $|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n} \leq A_\varepsilon(\xi,\eta)$; moreover, an easy calculation gives:

$$\begin{aligned} |\nabla_{\mathbb{H}^n,\xi}A_\varepsilon(\xi,\eta)| &= |\nabla_{\mathbb{H}^n,\eta}A_\varepsilon(\xi,\eta)| \leq 2\varepsilon^{-1/2}\left(\frac{(x-x')^2+(y-y')^2}{\varepsilon}\right)^{\frac{1}{2}} \\ &\leq 2\varepsilon^{-1/2}A_\varepsilon^{1/2}(\xi,\eta), \end{aligned} \tag{5}$$

$$|\nabla_{\mathbb{H}^n,\xi}A_\varepsilon(\xi,\eta) + \nabla_{\mathbb{H}^n,\eta}A_\varepsilon(\xi,\eta)| \leq 4\varepsilon^{1/2}A_\varepsilon^{1/2}(\xi,\eta), \tag{6}$$

$$|\nabla_{\mathbb{H}^n,\xi}\rho(\xi)| \leq 2|\xi|_{\mathbb{H}^n}. \tag{7}$$

Since u and v are assumed to be bounded on $\mathbb{H}^n$, one easily checks that ψ tends to $-\infty$ when $|\xi|_{\mathbb{H}^n}, |\eta|_{\mathbb{H}^n} \to \infty$. Thus, ψ attains a global maximum at some point $(\hat{\xi},\hat{\eta})$ depending of course on ε and α.
Thus, $\hat{\xi}$ is a minimum point for $\xi \mapsto u(\xi) + \frac{1}{\varepsilon}A_\varepsilon(\xi,\hat{\eta}) + \alpha\rho(\xi)$ and since u is a super solution of (4), we have:

$$\gamma u(\hat{\xi}) + F\left(\hat{\xi}, -\frac{1}{\varepsilon}\nabla_{\mathbb{H}^n,\xi}A_\varepsilon(\hat{\xi},\hat{\eta}) - \alpha\nabla_{\mathbb{H}^n,\xi}\rho(\hat{\xi})\right) \geq 0. \tag{8}$$

On the other hand, $\hat{\eta}$ is a maximum point for $\eta \mapsto v(\eta) - \frac{1}{\varepsilon}A_\varepsilon(\hat{\xi},\eta)$ and since v is a sub solution of (4), we have:

$$\gamma v(\hat{\eta}) + F\left(\hat{\eta}, \frac{1}{\varepsilon}\nabla_{\mathbb{H}^n,\eta}A_\varepsilon(\hat{\xi},\hat{\eta})\right) \leq 0. \tag{9}$$

Let $C_0 > 0$ be such that $||u||_\infty, ||v||_\infty \leq C_0$; writing $\psi(\hat{\xi},\hat{\eta}) \geq \psi(0,0)$, one gets:

$$\frac{1}{\varepsilon}A_\varepsilon(\hat{\xi},\hat{\eta}) + \alpha\rho(\hat{\xi}) \leq 4C_0.$$

In particular, we derive:

$$\rho(\hat{\xi}) = |\hat{\xi}|^2_{\mathbb{H}^n} \leq 4C_0\alpha^{-1}.$$

Then, writing $\psi(\hat{\xi},\hat{\eta}) \geq \psi(\hat{\xi},\hat{\xi})$ and using Lemma 3.1, one gets:

$$\frac{1}{\varepsilon}A_\varepsilon(\hat{\xi},\hat{\eta}) \leq v(\hat{\eta}) - v(\hat{\xi}) \leq M|\hat{\eta}^{-1}\circ\hat{\xi}|_{\mathbb{H}^n} \leq MA_\varepsilon^{1/2}(\hat{\xi},\hat{\eta})$$

which yields:

$$A_\varepsilon(\hat{\xi},\hat{\eta}) \leq M^2\varepsilon^2.$$

One then subtracts (8) from (9) to obtain:

$$\begin{aligned}
\gamma(v(\hat{\eta}) - u(\hat{\xi})) &\leq F\left(\hat{\xi}, -\frac{1}{\varepsilon}\nabla_{\mathbb{H}^n,\xi}A_\varepsilon(\hat{\xi},\hat{\eta}) - \alpha\nabla_{\mathbb{H}^n,\xi}\rho(\hat{\xi})\right) - F\left(\hat{\eta}, \frac{1}{\varepsilon}\nabla_{\mathbb{H}^n,\eta}A_\varepsilon(\hat{\xi},\hat{\eta})\right)\\
&\leq C\left(|\hat{\eta}^{-1}\circ\hat{\xi}|_{\mathbb{H}^n} + \alpha\left|\nabla_{\mathbb{H}^n,\xi}\rho(\hat{\xi})\right| + \frac{1}{\varepsilon}|\nabla_{\mathbb{H}^n,\xi}A_\varepsilon(\xi,\eta) + \nabla_{\mathbb{H}^n,\eta}A_\varepsilon(\xi,\eta)|\right)\\
&\leq C\left(A_\varepsilon^{1/2}(\hat{\xi},\hat{\eta}) + 2\alpha|\hat{\xi}|_{\mathbb{H}^n} + 4\varepsilon^{-1/2}A_\varepsilon^{1/2}(\xi,\eta)\right)\\
&\leq C(M\varepsilon + 4\sqrt{C_0\alpha} + 4M\varepsilon^{1/2}).
\end{aligned}$$

Finally, one derives, for all $\xi,\eta\in\mathbb{H}^n$:

$$\begin{aligned}
v(\eta) - u(\xi) - \frac{1}{\varepsilon}A_\varepsilon(\xi,\eta) &= \lim_{\alpha\to 0}\psi(\xi,\eta)\\
&\leq \lim_{\alpha\to 0}\psi(\hat{\xi},\hat{\eta})\\
&\leq \lim_{\alpha\to 0} v(\hat{\eta}) - u(\hat{\xi})\\
&\leq \gamma^{-1}C(M\varepsilon + 4M\varepsilon^{1/2}).
\end{aligned}$$

Taking $\xi = \eta$ in the left-hand side and then passing to the limit when $\varepsilon\to 0$, one gets:

$$v(\xi) - u(\xi) \leq 0\ ,\ \forall\xi\in\mathbb{H}^n$$

and the result is proved.

4. **Around the cell-problem.** Theorem 1.1 in the previous section immediately yields that, under assumptions $(H1)$, $(H2)$ for all $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in L^\infty(\mathbb{H}^n)\cap\Lambda(\mathbb{H}^n)$ of the equation:

$$(E_\varepsilon) \qquad u^\varepsilon(\xi) + H(\xi, \delta_{\frac{1}{\varepsilon}}\xi, \nabla_{\mathbb{H}^n}u^\varepsilon(\xi)) = 0 \ \text{ on } \mathbb{H}^n$$

with H satisfying

$$\eta\mapsto H(\xi,\eta,p) \text{ is } \mathbb{H}^n\text{-periodic for each } (\xi,p)\in\mathbb{H}^n\times\mathbb{R}^{2n}.$$

From the work of Lions, Papanicolaou, Varadhan, one guesses that the limit problem of (E_ε) is given by

$$(\overline{E}) \qquad u(\xi) + \overline{H}(\xi, \nabla_{\mathbb{H}^n}u(\xi)) = 0 \ \text{ on } \mathbb{H}^n$$

where $\overline{H}$, the so-called effective Hamiltonian, is obtained by solving the "cell-problem"

$$(CP) \qquad H(\xi,\eta,p + \nabla_{\mathbb{H}^n}v(\eta)) = \lambda, \ \text{ on } \mathbb{H}^n$$

where $\xi\in\mathbb{H}^n$ and $p\in\mathbb{R}^{2n}$ are some fixed parameters.

As announced in the introduction, our first result is the following:

THEOREM 4.1. *For every fixed $(\xi, p) \in \mathbb{H}^n \times \mathbb{R}^{2n}$, there exists a unique real number $\lambda = \lambda(\xi, p)$ such that (CP) has a bounded, continuous solution v.*

We then define the "effective Hamiltonian" $\overline{H}$ by

$$\overline{H}(\xi, p) = \lambda \ , \ \forall (\xi, p) \in \mathbb{H}^n \times \mathbb{R}^{2n}$$

and one notices that $\overline{H}$ enjoys the same structural assumptions than H.
A classical way of constructing a solution of (CP) is to introduce the approximated equation:

$$(AP) \qquad \alpha v_\alpha(\eta) + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} v_\alpha(\eta)) = 0 \ \text{ in } \mathbb{H}^n$$

where $\xi \in \mathbb{H}^n$ and $p \in \mathbb{R}^{2n}$ are fixed.
We know by the results of the previous section that there exists a unique $v_\alpha \in L^\infty(\mathbb{H}^n) \cap \Lambda(\mathbb{H}^n)$ solution of (AP). The uniqueness and the invariance of $\nabla_{\mathbb{H}^n}$ by left group action implies that v_α is $\mathbb{H}^n$-periodic.
The aim of the following proposition is to gather some estimates about the function v_α which will be useful in the next section.

PROPOSITION 4.1. *There exists $C_4 > 0$ such that, for all $\xi, \xi', \eta, \eta' \in \mathbb{H}^n$ and $p, p' \in \mathbb{R}^{2n}$:*

1. $-\sup_{\eta \in \mathbb{H}^n} H(\xi, \eta, p) \leq \alpha v_\alpha(\eta) \leq -\inf_{\eta \in \mathbb{H}^n} H(\xi, \eta, p)$.
2. $|\nabla_{\mathbb{H}^n} v_\alpha(\eta; \xi, p)| \leq C_4(1 + |p|)$ *in the viscosity sense.*
3. $|\alpha v_\alpha(\eta'; \xi', p') - \alpha v_\alpha(\eta; \xi, p)| \leq C_4(|\eta^{-1} \circ \eta'|_{\mathbb{H}^n} + |\xi^{-1} \circ \xi'|_{\mathbb{H}^n} + |p' - p|)$.
4. $|\alpha v_\alpha(\eta; \xi, p) + \overline{H}(\xi, p)| \leq \alpha C_4(1 + |p|)$.
5. $|\overline{H}(\xi', p') - \overline{H}(\xi, p)| \leq C_4(|\xi^{-1} \circ \xi'|_{\mathbb{H}^n} + |p' - p|)$.

Proof of Theorem 4.1 and Proposition 4.1: Since the constants:

$$-\alpha^{-1} \inf_{\eta \in \mathbb{H}^n} H(\xi, \eta, p) \ \text{ and } \ -\alpha^{-1} \sup_{\eta \in \mathbb{H}^n} H(\xi, \eta, p)$$

are respectively super solution and sub solution of (AP), one has:

$$-\alpha^{-1} \sup_{\eta \in \mathbb{H}^n} H(\xi, \eta, p) \leq v_\alpha(\eta) \leq -\alpha^{-1} \inf_{\eta \in \mathbb{H}^n} H(\xi, \eta, p). \tag{10}$$

This proves 1. of Proposition 4.1 and from $(H2)$, this implies that

$$\alpha ||v_\alpha||_\infty \leq \nu |p| + C_3$$

so that

$$H(\xi, \eta, p + \nabla_{\mathbb{H}^n} v_\alpha(\xi)) \leq \nu |p| + C_3. \tag{11}$$

On the other hand,

$$H(\xi, \eta, p + \nabla_{\mathbb{H}^n} v_\alpha(\xi)) \geq \nu |p + \nabla_{\mathbb{H}^n} v_\alpha| - C_3 \geq \nu |\nabla_{\mathbb{H}^n} v_\alpha| - \nu |p| - C_3. \tag{12}$$

One deduces from (11) and (12) that:

$$|\nabla_{\mathbb{H}^n} v_\alpha| \leq 2\left(|p| + \frac{C_3}{\nu}\right)$$

which proves 2. of Proposition 4.1.
Setting $\tilde{v}_\alpha = v_\alpha - \min_Q v_\alpha$, one has that $\tilde{v}_\alpha$ is periodic, bounded in $L^\infty(\mathbb{H}^n) \cap \Lambda(\mathbb{H}^n)$. Extracting a subsequence, one may assume that $(\tilde{v}_\alpha, -\alpha v_\alpha)$ converges uniformly on $\mathbb{H}^n$ to some $(v, \lambda) \in L^\infty(\mathbb{H}^n) \cap \Lambda(\mathbb{H}^n) \times \mathbb{R}$
Using the fact that $\tilde{v}_\alpha$ solves:

$$\alpha \tilde{v}_\alpha + \alpha \min_Q v_\alpha + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} \tilde{v}_\alpha) = 0 \ \text{ on } \mathbb{H}^n$$

and from the stability result, we get that v satisfies:

$$-\lambda + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} v) = 0 \ \text{ on } \mathbb{H}^n$$

which proves the existence part of Theorem 4.1.
Suppose now that there exists another couple $(\mu, w) \in \mathbb{R} \times L^\infty(\mathbb{H}^n) \cap \Lambda(\mathbb{H}^n)$ solution of the cell-problem, such that $\lambda \neq \mu$. We may assume that $\lambda < \mu$ and since $v + C$ is still solution of the cell-problem for any constant C, we may suppose that $v > w$ on $\mathbb{H}^n$.
We then choose a small α such that $\lambda + \alpha v \leq \mu + \alpha w$.
v is the unique solution of

$$\alpha u + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} u) = \lambda + \alpha v \tag{13}$$

while w is the unique solution of

$$\alpha u + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} u) = \mu + \alpha w. \tag{14}$$

In particular, w is a super solution of (13) and using the comparison result, we get $v \leq w$ on $\mathbb{H}^n$, a contradiction.
In order to prove 3., fix $h, k, \xi \in \mathbb{H}^n$, $p, l \in \mathbb{R}^{2n}$ and set $w(\eta) = v_\alpha(k \circ \eta; h \circ \xi, p + l)$ which, from the left-invariance of the Heisenberg gradient, is solution of:

$$\alpha w(\eta) + H(h \circ \xi, k \circ \eta, p + l + \nabla_{\mathbb{H}^n} w(\eta)) = 0 \ , \ \eta \in \mathbb{H}^n.$$

From (H1), one deduces that:

$$-C(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|) \leq \alpha w(\eta) + H(\xi, \eta, p + \nabla_{\mathbb{H}^n} w(\eta)) \leq C(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|).$$

In particular, this means that:

$$\eta \mapsto w(\eta) - \frac{C}{\alpha}(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|) \ \text{ and } \ \eta \mapsto w(\eta) + \frac{C}{\alpha}(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|)$$

are respectively sub solution and super solution of (AP).
Using the comparison result, one gets:

$$w(\eta) - \frac{C}{\alpha}(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|) \leq v_\alpha(\eta) \leq w(\eta) + \frac{C}{\alpha}(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|) \ , \eta \in \mathbb{H}^n$$

which may be rewritten:

$$|\alpha v_\alpha(k \circ \eta; h \circ \xi, p + l) - \alpha v_\alpha(\eta; \xi, p)| \leq C(|h|_{\mathbb{H}^n} + |k|_{\mathbb{H}^n} + |l|).$$

This proves 3. of Proposition 4.1.
We then claim that:

$$\alpha \sup_{\mathbb{H}^n} v_\alpha \geq -\overline{H}(\xi,p).$$

Indeed, if it were not the case, one would have, from the viscosity point of view:

$$H(\xi,\eta,p+\nabla_{\mathbb{H}^n}v_\alpha(\eta)) = -\alpha v_\alpha(\eta) > \overline{H}(\xi,p) = \lambda$$

Hence, v_α would be a super solution of the cell-problem, which would imply that $v_\alpha \geq w$ on $\mathbb{H}^n$, for any solution w of (CP), from the comparison result. This leads to a contradiction since, if w is a solution of the cell-problem, then so is $w+C$ for any $C \in \mathbb{R}$.
A similar consideration shows that

$$\alpha \inf_{\mathbb{H}^n} v_\alpha \leq -\overline{H}(\xi,p).$$

Then, as a consequence of 2., one has for all $\eta, \eta' \in Q$

$$|v_\alpha(\eta';\xi,p) - v_\alpha(\eta;\xi,p)| \leq C_4(1+|p|)|\eta^{-1}\circ\eta'|_{\mathbb{H}^n} \leq C_4(1+|p|)C_N$$

which yields, using the $\mathbb{H}^n$-periodicity of v_α,

$$\begin{aligned}\alpha v_\alpha(\eta;\xi,p) &\leq \alpha \inf_{\eta'\in\mathbb{H}^n} v_\alpha(\eta';\xi,p) + \alpha C_4(1+|p|)C_N \\ &\leq -\overline{H}(\xi,p) + \alpha C_4(1+|p|)C_N\end{aligned}$$

and

$$\begin{aligned}\alpha v_\alpha(\eta;\xi,p) &\geq \alpha \sup_{\eta'\in\mathbb{H}^n} v_\alpha(\eta';\xi,p) - \alpha C_4(1+|p|)C_N \\ &\geq -\overline{H}(\xi,p) - \alpha C_4(1+|p|)C_N.\end{aligned}$$

Finally, one gets

$$|\alpha v_\alpha(\eta;\xi,p) + \overline{H}(\xi,p)| \leq \alpha C_4(1+|p|)C_N$$

and 4. is proved.
From 3. and 4., one now gets, for all $\alpha > 0$:

$$\overline{H}(\xi',p') - \overline{H}(\xi,p) \leq C_4(|\xi^{-1}\circ\xi'|_{\mathbb{H}^n} + |p-p'|) + \alpha C_4(|p|+|p'|)C_N.$$

Sending α to 0, this proves 5.

5. **The rate of convergence.** In this section we shall prove Theorem 1.2 and Theorem 1.3.

Let us first recall that u^ε, u and $v_\alpha = v_\alpha(.;\xi,p)$ are respectively the solutions of:

$$(E_\varepsilon) \qquad u^\varepsilon(\xi) + H\left(\xi, \delta_{\frac{1}{\varepsilon}}\xi, \nabla_{\mathbb{H}^n}u^\varepsilon(\xi)\right) = 0\ ,\ \xi \in \mathbb{H}^n$$

$$(\overline{E}) \qquad u(\xi) + \overline{H}\left(\xi, \nabla_{\mathbb{H}^n}u(\xi)\right) = 0\ ,\ \xi \in \mathbb{H}^n$$

$$(AP) \qquad \alpha v_\alpha(\eta) + H\left(\xi,\eta,p+\nabla_{\mathbb{H}^n}v_\alpha(\eta)\right) = 0\ ,\ \eta \in \mathbb{H}^n.$$

For $\varepsilon,\delta,\beta,\theta,\lambda \in (0,1)$, $\xi = (x,y,t)$ and $\eta = (x',y',t')$, we define the functions:

$$A_{\varepsilon,\lambda}(\xi,\eta) = \left(\left(\frac{(x-x')^2+(y-y')^2}{\varepsilon^\lambda}\right)^2 + (t-t'+2(x'y-xy'))^2\right)^{1/2}$$

$$\rho(\xi) = |\xi|^2_{\mathbb{H}^n}$$

and then

$$\phi(\xi,\eta) = u^\varepsilon(\xi) - u(\eta) - \varepsilon v_{\varepsilon^\theta}\left(\delta_{\frac{1}{\varepsilon}}\xi;\xi,q_\varepsilon(\xi,\eta)\right) - \frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\eta) - \frac{\delta}{2}\rho(\xi)$$

where $q_\varepsilon(\xi,\eta) = \frac{1}{\varepsilon^\beta}\nabla_{\mathbb{H}^n,\xi}A_{\varepsilon,\lambda}(\xi,\eta)$.
Since $\varepsilon \in]0,1[$, one clearly has

$$|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n} \le A_{\varepsilon,\lambda}(\xi,\eta) \le \varepsilon^{-\lambda}|\eta^{-1}\circ\xi|^2_{\mathbb{H}^n}; \tag{15}$$

moreover, an easy calculation gives:

$$\begin{aligned}|\nabla_{\mathbb{H}^n,\xi}A_{\varepsilon,\lambda}(\xi,\eta)| &= |\nabla_{\mathbb{H}^n,\eta}A_{\varepsilon,\lambda}(\xi,\eta)| \le 2\varepsilon^{-\lambda/2}\left(\frac{(x-x')^2+(y-y')^2}{\varepsilon^\lambda}\right)^{\frac{1}{2}} \\ &\le 2\varepsilon^{-\lambda/2}A^{1/2}_{\varepsilon,\lambda}(\xi,\eta), \end{aligned}\tag{16}$$

$$|\nabla_{\mathbb{H}^n,\xi}A_{\varepsilon,\lambda}(\xi,\eta) + \nabla_{\mathbb{H}^n,\eta}A_{\varepsilon,\lambda}(\xi,\eta)| \le 4\varepsilon^{\lambda/2}A^{1/2}_{\varepsilon,\lambda}(\xi,\eta), \tag{17}$$

$$|\nabla_{\mathbb{H}^n,\xi}\rho(\xi)| \le 2|\xi|_{\mathbb{H}^n}. \tag{18}$$

In conclusion putting together (15) and (16),one has

$$|q_\varepsilon(\xi,\eta)| \le 2\varepsilon^{-\beta-\lambda/2}A^{1/2}_{\varepsilon,\lambda}(\xi,\eta) \le 2\varepsilon^{-\beta-\lambda}|\eta^{-1}\circ\xi|_{\mathbb{H}^n}. \tag{19}$$

We shall need the following estimate

$$|q_\varepsilon(\xi,\eta) - q_\varepsilon(\xi,\eta')| \le C\varepsilon^{-\beta}|(\eta')^{-1}\circ\eta|_{\mathbb{H}^n}. \tag{20}$$

Hence we need to prove that there exists some constant C such that

$$|\nabla_{\mathbb{H}^n,\eta}\nabla_{\mathbb{H}^n,\xi}A_{\varepsilon,\lambda}(\xi,\eta)| \le C. \tag{21}$$

This will be proved in the appendix since it is a simple but sort of long computation.

Let $C > 0$ be some constant such that $||u||_\infty,\ ||v||_\infty \le C$. In view of 1. of Proposition 4.1, $(H2)$ and (16), we have:

$$\begin{aligned}\phi(\xi,\eta) &\le 2C + \varepsilon^{1-\theta}\left(\nu|q_\varepsilon(\xi,\eta)| + C_3\right) - \frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\eta) - \frac{\delta}{2}\rho(\xi) \\ &\le 2C + \varepsilon^{1-\theta}\left(2\nu\frac{A^{1/2}_{\varepsilon,\lambda}(\xi,\eta)}{\varepsilon^{\beta+\lambda/2}} + C_3\right) - \frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\eta) - \frac{\delta}{2}\rho(\xi).\end{aligned}$$

Hence, ϕ attains a global maximum at some point $(\hat\xi,\hat\eta)$ depending on the various parameters that appear in the definition of ϕ.
Then, writing $\phi(\xi,\xi) \le \phi(\hat\xi,\hat\eta)$, one gets:

$$\begin{aligned}
u^\varepsilon(\xi)-u(\xi) &\leq u^\varepsilon(\hat\xi)-u(\hat\eta)\\
&+ \varepsilon\left(v_{\varepsilon^\theta}\left(\delta_{\frac{1}{\varepsilon}}\xi;\xi,0\right)-v_{\varepsilon^\theta}\left(\delta_{\frac{1}{\varepsilon}}\hat\xi;\hat\xi,q_\varepsilon(\hat\xi,\hat\eta)\right)\right)+\\
&+ \frac{\delta}{2}|\xi|^2_{\mathbb{H}^n}-\frac{\delta}{2}|\hat\xi|^2_{\mathbb{H}^n}. \qquad (22)
\end{aligned}$$

The scope now is to estimate the terms on the right hand side of (22). For this let us state a Proposition that will be proved later.

PROPOSITION 5.1. *Let* $\varepsilon,\beta\in(0,1)$, $\delta\in(0,1/2)$ *and* $\lambda\in(0,1-\beta)$, $\frac{\theta}{2}\in(0,1-\beta-\lambda)$ *then there exist some constants* L, M *and* $C_5,C_6>0$, *such that:*

$$|\hat\xi|_{\mathbb{H}^n}\leq\frac{L}{\delta^{1/2}}\quad\textit{and}\quad\frac{A^{1/2}_{\varepsilon,\lambda}(\hat\xi,\hat\eta)}{\varepsilon^\beta}\leq M, \qquad (23)$$

$$u(\hat\eta)+\overline{H}(\hat\xi,\bar q_\varepsilon(\hat\xi,\hat\eta))\geq -C_6(\varepsilon^\beta+\varepsilon^{1-\theta-\beta-\lambda}) \qquad (24)$$

where $\bar q_\varepsilon(\hat\xi,\eta)=-\frac{1}{\varepsilon^\beta}\nabla_{\mathbb{H}^n,\eta}A_{\varepsilon,\lambda}(\hat\xi,\eta)$.

$$u^\varepsilon(\hat\xi)+\overline{H}(\hat\xi,q_\varepsilon(\hat\xi,\hat\eta))\leq C_5(\varepsilon^\theta+\delta^{1/2}+\varepsilon^{1-\theta-\beta-\lambda}). \qquad (25)$$

Here and in the following $\lambda.\xi:=\delta_\lambda\xi$ with $\lambda\in\mathbb{R}$ and $\xi\in\mathbb{H}^n$ when no ambiguities arise. From 1. of Proposition 4.1, $(H2)$, (16) and (23), one has:

$$\begin{aligned}
&\varepsilon\left(v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\xi;\xi,0)-v_{\varepsilon^\theta}(\frac{1}{\varepsilon}\hat\xi;\hat\xi,q_\varepsilon(\hat\xi,\hat\eta))\right)\\
\leq\ &\varepsilon^{1-\theta}\left(\sup_{\eta\in\mathbb{H}^n}H(\frac{1}{\varepsilon}.\hat\xi,\eta,q_\varepsilon(\hat\xi,\hat\eta))-\inf_{\eta\in\mathbb{H}^n}H(\frac{1}{\varepsilon}.\xi,\eta,0)\right)\\
\leq\ &\varepsilon^{1-\theta}(\nu|q_\varepsilon(\hat\xi,\hat\eta)|+2C_3)\\
\leq\ &\varepsilon^{1-\theta}(2\nu M\varepsilon^{-\lambda/2}+2C_3). \qquad (26)
\end{aligned}$$

We now want to subtract (24) from (25), but first let us remark that from 5. of Proposition 4.1, (17) and (23),

$$|\overline{H}(\hat\xi,q_\varepsilon(\hat\xi,\hat\eta))-\overline{H}(\hat\xi,\bar q_\varepsilon(\hat\xi,\hat\eta))|\leq 4C_4\varepsilon^{-\beta+\lambda/2}A^{1/2}_{\varepsilon,\lambda}\leq\varepsilon^{\lambda/2}M'.$$

Now taking $\delta^{1/2}\leq\varepsilon^\theta$, one gets the existence of $C_7>0$ such that

$$u^\varepsilon(\hat\xi)-u(\hat\eta)\leq C_7(\varepsilon^\theta+\varepsilon^\beta+\varepsilon^{1-\theta-\beta-\lambda}+\varepsilon^{\lambda/2}) \qquad (27)$$

Using (26) and (27), (22) yields, $\forall\xi\in\mathbb{H}^n$,:

$$u^\varepsilon(\xi)-u(\xi)\leq C_7(\varepsilon^\theta+\varepsilon^\beta+\varepsilon^{1-\theta-\beta-\lambda}+\varepsilon^{\lambda/2})+\varepsilon^{1-\theta}(2\nu M\varepsilon^{-\lambda/2}+2C_3)+\frac{\delta}{2}|\xi|^2_{\mathbb{H}^n},\ .$$

Hence, sending $\delta\to 0$, one obtains:

$$u^\varepsilon(\xi)-u(\xi)\leq(C_7+2\nu M+2C_3)(\varepsilon^\theta+\varepsilon^\beta+\varepsilon^{\lambda/2}+\varepsilon^{1-\theta-\beta-\lambda})\ \forall\xi\in\mathbb{H}^n.$$

The optimal choice being $\theta=\beta=\frac{\lambda}{2}=1/5$. We have thus proved that there exists $C>0$ such that:

$$\sup_{\mathbb{H}^n}(u^\varepsilon(\xi) - u(\xi)) \le C\varepsilon^{1/5}.$$

Reversing the roles of u and u^ε, one gets the opposite inequality and Theorem 1.2 is proved.

Proof of Proposition 5.1: Let $C_o > 0$ be such that $||u^\varepsilon||_\infty$, $||u||_\infty \le C_o$. Writing $\phi(\hat{\xi}, \hat{\eta}) \ge \phi(0,0)$ and proceeding as above using 1. of Proposition 4.1, and (19) one gets:

$$\begin{aligned}\frac{\delta}{2}\rho(\hat{\xi}) + \frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\hat{\xi},\hat{\eta}) &\le 4C_o + 2\varepsilon^{1-\theta}(\nu q_\varepsilon(\hat{\xi},\hat{\eta}) + C_3)\\ &\le 4C_o + 2\varepsilon^{1-\theta}(\nu\varepsilon^{-\beta-\lambda}|\hat{\eta}^{-1}\circ\hat{\xi}|_{\mathbb{H}^n} + C_3).\end{aligned}$$

Thus, using Young's inequality, and (15)

$$\frac{\delta}{2}|\hat{\xi}|^2_{\mathbb{H}^n} \le 4C_o + \nu^2\varepsilon^{2-2\theta-\beta-2\lambda} + 2C_3\varepsilon^{1-\theta}.$$

For $0 < \theta < 1 - \frac{\beta}{2} - \lambda$, the above inequality yields:

$$\frac{\delta}{2}|\hat{\xi}|^2_{\mathbb{H}^n} \le 4C + 2\nu^2 + 2C_3$$

and the first estimate of (23) is proved.
Writing $\phi(\hat{\xi},\hat{\eta}) \ge \phi(\hat{\xi},\hat{\xi})$, one gets:

$$\frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\hat{\xi},\hat{\eta}) \le u(\hat{\xi}) - u(\hat{\eta}) + \varepsilon\left(v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\hat{\xi};\hat{\xi},0) - v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\hat{\xi};\hat{\xi},q_\varepsilon(\hat{\xi},\hat{\eta})\right).$$

Thus, from 3. of Proposition 4.1, (15) and (16), one deduces:

$$\begin{aligned}\frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\hat{\xi},\hat{\eta}) &\le C|\hat{\eta}^{-1}\circ\hat{\xi}|_{\mathbb{H}^n} + C_4\varepsilon^{1-\theta}|q_\varepsilon(\hat{\xi},\hat{\eta})|\\ &\le (C + C_4\varepsilon^{1-\theta-\beta-\frac{\lambda}{2}})A_{\varepsilon,\lambda}(\hat{\xi},\hat{\eta})^{\frac{1}{2}}.\end{aligned}$$

This conclude the proof of (23).
Proof of (24): Since ϕ has a maximum point at $(\hat{\xi},\hat{\eta})$, one deduces that the function:

$$\varphi : \eta \mapsto u(\eta) + \frac{1}{\varepsilon^\beta}A_{\varepsilon,\lambda}(\hat{\xi},\eta) + \varepsilon v_{\varepsilon^\theta}\left(\frac{1}{\varepsilon}.\hat{\xi};\hat{\xi},q_\varepsilon(\hat{\xi},\eta)\right)$$

has a minimum point at $\hat{\eta}$.

Subtracting to u a smooth positive function vanishing at $\hat{\eta}$ as well as its horizontal gradient, one may assume that φ has a strict minimum at $\hat{\eta}$.
Let us set $v_1(\eta) = u(\eta) + \frac{1}{\varepsilon^\beta}A_{\varepsilon,\lambda}(\hat{\xi},\eta)$ and $v_2(\eta) = \varepsilon v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\hat{\xi};\hat{\xi},q_\varepsilon(\hat{\xi},\eta))$ so that $\varphi = v_1 + v_2$.
Let $r > 0$ be such that $v_1(\eta) + v_2(\eta) \ge v_1(\hat{\eta}) + v_2(\hat{\eta})$ on $B_{\mathbb{H}^n}(\hat{\eta}, r)$ and define, for $\alpha > 0$, the function:

$$\psi(\xi,\eta) = v_1(\xi) + v_2(\eta) + \frac{\alpha}{2}A_{\frac{1}{\alpha},\lambda}(\xi,\eta).$$

Let $(\xi_\alpha, \eta_\alpha)$ be a minimum point of ψ on $\overline{B}_{\mathbb{H}^n}(\hat{\eta}, r)$. One has, for all $\xi \in B_{\mathbb{H}^n}(\hat{\eta}, r)$:

$$\psi(\xi,\xi) = v_1(\xi) + v_2(\xi) \geq \psi(\xi_\alpha,\eta_\alpha) \geq v_1(\xi_\alpha) + v_2(\eta_\alpha). \tag{28}$$

The first inequality with $\xi = \eta_\alpha$ yields:

$$\begin{aligned} \frac{\alpha}{2} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha) &\leq v_1(\eta_\alpha) - v_1(\xi_\alpha) \\ &\leq C|\eta_\alpha^{-1} \circ \xi_\alpha|_{\mathbb{H}^n} \end{aligned}$$

and then, from (15):

$$\alpha|\eta_\alpha^{-1} \circ \xi_\alpha|_{\mathbb{H}^n} \leq 2C. \tag{29}$$

Moreover, one may extract from (ξ_α), (η_α) a subsequence such that $\xi_\alpha \to \xi_\infty \in \overline{B}_{\mathbb{H}^n}(\hat{\eta}, r)$ and $\eta_\alpha \to \eta_\infty \in \overline{B}_{\mathbb{H}^n}(\hat{\eta}, r)$ when $\alpha \to \infty$. From (29), one clearly sees that $\xi_\infty = \eta_\infty$.
Thus, passing to the limit in (28), one gets:

$$v_1(\xi) + v_2(\xi) \geq v_1(\xi_\infty) + v_2(\xi_\infty), \ \forall \xi \in B_{\mathbb{H}^n}(\hat{\eta}, r)$$

which implies that $\xi_\infty = \hat{\eta}$ and that $\xi_\alpha \to \hat{\eta}$, $\eta_\alpha \to \hat{\eta}$ (without extracting any subsequence).
Now, the functions:

$$\xi \mapsto v_1(\xi) + \frac{\alpha}{2} A_{\frac{1}{\alpha},\lambda}(\xi,\eta_\alpha) \ \text{ and } \ \eta \mapsto v_2(\eta) + \frac{\alpha}{2} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta)$$

have respectively a minimum point at ξ_α and η_α and this implies that:

$$\frac{\alpha}{2} \nabla_{\mathbb{H}^n,\xi} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha) \in D^-_{\mathbb{H}^n} v_1(\xi_\alpha) \tag{30}$$

and

$$\frac{\alpha}{2} \nabla_{\mathbb{H}^n,\eta} \rho(\xi_\alpha,\eta_\alpha) \in D^-_{\mathbb{H}^n} v_2(\eta_\alpha). \tag{31}$$

Let us now note that from 3. of Proposition 4.1, one has:

$$\begin{aligned} |v_2(\eta') - v_2(\eta)| &= \left| \varepsilon v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\hat{\xi}; \hat{\xi}, q_\varepsilon(\hat{\xi},\eta')) - \varepsilon v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\hat{\xi}; \hat{\xi}, q_\varepsilon(\hat{\xi},\eta)) \right| \\ &\leq \varepsilon^{1-\theta} C_4 |q_\varepsilon(\hat{\xi},\eta') - q_\varepsilon(\hat{\xi},\eta)| \end{aligned}$$

and then, from (20),

$$|v_2(\eta') - v_2(\eta)| \leq \varepsilon^{1-\theta-\beta-\lambda} C' |\eta^{-1} \circ \eta'|_{\mathbb{H}^n}.$$

Finally this together with remark 1, gives:

$$|\frac{\alpha}{2} \nabla_{\mathbb{H}^n,\xi} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha)| \leq C' \varepsilon^{1-\theta-\beta-\lambda}. \tag{32}$$

On the other hand, from (30) and the definition of v_1, one has:

$$\frac{\alpha}{2} \nabla_{\mathbb{H}^n,\xi} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha) + \overline{q}_\varepsilon(\hat{\xi},\xi_\alpha) \in D^-_{\mathbb{H}^n} u(\xi_\alpha).$$

Since u is a solution of $(\overline{E})$, we obtain that:

$$u(\xi_\alpha)+\overline{H}\left(\xi_\alpha,\frac{\alpha}{2}\nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha)+\bar{q}_\varepsilon(\hat{\xi},\xi_\alpha)\right)\geq 0.$$

Then, by 5. of Proposition 4.1 and Proposition 5.1,

$$u(\xi_\alpha)+\overline{H}(\hat{\xi},\bar{q}_\varepsilon(\hat{\xi},\xi_\alpha))\geq -C_4\left(\left|\xi_\alpha^{-1}\circ\hat{\xi}\right|_{\mathbb{H}^n}+\left|\frac{\alpha}{2}\nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\eta_\alpha)\right|\right). \tag{33}$$

From (32), this yields:

$$u(\xi_\alpha)+\overline{H}(\hat{\xi},\bar{q}_\varepsilon(\hat{\xi},\xi_\alpha))\geq -C_4\left(\left|\xi_\alpha^{-1}\circ\hat{\xi}\right|_{\mathbb{H}^n}+C'\varepsilon^{1-\theta-\beta-\lambda}\right).$$

Thus, passing to the limit when α tends to $+\infty$, one obtains:

$$u(\hat{\eta})+\overline{H}(\hat{\xi},\bar{q}_\varepsilon(\hat{\xi},\hat{\eta}))\geq -C_4\left(\left|\hat{\eta}^{-1}\circ\hat{\xi}\right|_{\mathbb{H}^n}+C'\varepsilon^{1-\theta-\beta-\lambda}\right)$$

(24) is then a direct consequence of (23).

Proof of (25): First note that, adding to u^ε a smooth function vanishing at $\hat{\xi}$ as well as its horizontal gradient, one may assume that the function:

$$\xi\mapsto u^\varepsilon(\xi)-\varepsilon v_{\varepsilon^\theta}(\frac{1}{\varepsilon}.\xi;\xi,q_\varepsilon(\xi,\hat{\eta}))-\frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\hat{\eta})-\frac{\delta}{2}\rho(\xi)$$

has a strict maximum at $\hat{\xi}$. Consider next, for $\alpha>0$, the function:

$$\begin{aligned}\psi(\xi,\eta,\zeta) &= u^\varepsilon(\xi)-\varepsilon v_{\varepsilon^\theta}(\eta;\zeta,q_\varepsilon(\zeta,\hat{\eta}))-\frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\hat{\eta})-\frac{\delta}{2}\rho(\xi)+\\ &- \frac{\alpha}{2}(A_{\frac{1}{\alpha},\lambda}(\xi,\varepsilon.\eta)+A_{\frac{1}{\alpha},\lambda}(\xi,\zeta)).\end{aligned}$$

Arguing as in the proof of (24), there exists $r>0$ and a maximum point $(\xi_\alpha,\eta_\alpha,\zeta_\alpha)$ of ψ on $\overline{B}_{\mathbb{H}^n}(\hat{\xi},r)\times\overline{B}_{\mathbb{H}^n}(\frac{1}{\varepsilon}.\hat{\xi},r)\times\overline{B}_{\mathbb{H}^n}(\hat{\xi},r)$ such that $\xi_\alpha\to\hat{\xi}$, $\eta_\alpha\to\frac{1}{\varepsilon}.\hat{\xi}$, $\zeta_\alpha\to\hat{\xi}$ when $\alpha\to\infty$.
Furthermore $\alpha|(\varepsilon.\eta_\alpha)^{-1}\circ\xi_\alpha|\leq C$ for some constant independent of α and ε.

The functions:

$$\xi\mapsto u^\varepsilon(\xi)-\frac{1}{2\varepsilon^\beta}A_{\varepsilon,\lambda}(\xi,\hat{\eta})-\frac{\delta}{2}\rho(\xi)-\frac{\alpha}{2}(A_{\frac{1}{\alpha},\lambda}(\xi,\varepsilon.\eta_\alpha)+A_{\frac{1}{\alpha},\lambda}(\xi,\zeta_\alpha))$$

and

$$\eta\mapsto v_{\varepsilon^\theta}(\eta;\zeta_\alpha,q_\varepsilon(\zeta_\alpha,\hat{\eta}))-\frac{\alpha}{2\varepsilon}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta)$$

have respectively a maximum point at ξ_α and a minimum point at η_α. Since u^ε and v_{ε^θ} are respectively solutions of (E_ε) and (ACP), one may write:

$$u(\xi_\alpha)+H\left(\xi_\alpha,\Gamma\right)\leq 0 \tag{34}$$

with

$$\Gamma=\frac{1}{\varepsilon}.\xi_\alpha,q_\varepsilon(\xi_\alpha,\hat{\eta})+\frac{\delta}{2}\nabla_{\mathbb{H}^n,\xi}\rho(\xi_\alpha)+\frac{\alpha}{2}(\nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)+\nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\zeta_\alpha))$$

and

$$\varepsilon^\theta v_{\varepsilon^\theta}(\eta_\alpha;\zeta_\alpha,q_\varepsilon(\zeta_\alpha,\hat{\eta}))+H\left(\zeta_\alpha,\eta_\alpha,q_\varepsilon(\zeta_\alpha,\hat{\eta})-\frac{\alpha}{2}\nabla_{\mathbb{H}^n,\eta}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)\right)\geq 0. \tag{35}$$

Subtracting the inequalities (34) and (35) we obtain

$$\begin{aligned}
&u(\xi_\alpha) - \varepsilon^\theta v_{\varepsilon^\theta}(\eta_\alpha;\zeta_\alpha, q_\varepsilon(\zeta_\alpha,\hat\eta))\\
\leq\ & H\left(\zeta_\alpha,\eta_\alpha, q_\varepsilon(\zeta_\alpha,\hat\eta) - \frac{\alpha}{2}\nabla_{\mathbb{H}^n,\eta} A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)\right)\\
-\ & H\left(\xi_\alpha,\Gamma\right).
\end{aligned}$$

Using 4. of Proposition 4.1 and (H5), this implies:

$$\begin{aligned}
u(\xi_\alpha) &+ \overline{H}(\zeta_\alpha, q_\varepsilon(\zeta_\alpha,\hat\eta)) - \varepsilon^\theta C_4(1+|q_\varepsilon(\zeta_\alpha,\hat\eta)|)\\
&\leq C_4\left(|\xi_\alpha^{-1}\circ\zeta_\alpha|_{\mathbb{H}^n} + |\left(\frac{1}{\varepsilon}.\xi_\alpha\right)^{-1}\circ\eta_\alpha|_{\mathbb{H}^n} + |q_\varepsilon(\xi_\alpha,\hat\eta) - q_\varepsilon(\zeta_\alpha,\hat\eta)| + \right.\\
&+ \frac{\delta}{2}|\nabla_{\mathbb{H}^n,\xi}\rho(\xi_\alpha)| + \frac{\alpha}{2}|\nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\zeta_\alpha)| +\\
&+ \left.\frac{\alpha}{2}|\nabla_{\mathbb{H}^n,\eta}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha) + \nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)|\right).
\end{aligned}$$

Let us recall that from (17)

$$\begin{aligned}
|\nabla_{\mathbb{H}^n,\eta}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha) + \nabla_{\mathbb{H}^n,\xi}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)| &\leq \alpha^{-\lambda/2}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\varepsilon.\eta_\alpha)^{1/2}\\
&\leq \alpha^{-\lambda}|(\varepsilon.\eta_\alpha)^{-1}\circ\xi_\alpha|
\end{aligned}$$

and thus, from (18) and (16):

$$\begin{aligned}
u(\xi_\alpha) &+ \overline{H}(\zeta_\alpha, q_\varepsilon(\zeta_\alpha,\hat\eta)) - \varepsilon^\theta C_4(1+|q_\varepsilon(\zeta_\alpha,\hat\eta)|) \qquad (36)\\
&\leq C_4\left((1+C\varepsilon^{-\beta-\lambda}+\alpha)|\xi_\alpha^{-1}\circ\zeta_\alpha|_{\mathbb{H}^n} + \left|\left(\frac{1}{\varepsilon}.\xi_\alpha\right)^{-1}\circ\eta_\alpha\right|_{\mathbb{H}^n} + \delta|\xi_\alpha|_{\mathbb{H}^n}\right.\\
&\left. +\alpha^{1-\lambda}|(\varepsilon.\eta_\alpha)^{-1}\circ\xi_\alpha|_{\mathbb{H}^n}\right).
\end{aligned}$$

Writing $\psi(\xi_\alpha,\eta_\alpha,\zeta_\alpha) \geq \psi(\xi_\alpha,\eta_\alpha,\xi_\alpha)$ and using 3. of Proposition 4.1, one gets:

$$\begin{aligned}
\frac{\alpha}{2}A_{\frac{1}{\alpha},\lambda}(\xi_\alpha,\zeta_\alpha) &\leq \varepsilon^{1-\theta}\left(\varepsilon^\theta v_{\varepsilon^\theta}(\eta_\alpha;\xi_\alpha,q_\varepsilon(\xi_\alpha,\hat\xi)) - \varepsilon^\theta v_{\varepsilon^\theta}(\eta_\alpha;\zeta_\alpha,q_\varepsilon(\zeta_\alpha,\hat\xi))\right)\\
&\leq \varepsilon^{1-\theta}C_4(|\xi_\alpha^{-1}\circ\zeta_\alpha|_{\mathbb{H}^n} + |q_\varepsilon(\xi_\alpha,\hat\eta) - q_\varepsilon(\zeta_\alpha,\hat\eta)|)\\
&\leq \varepsilon^{1-\theta}C_4(1+C\varepsilon^{-\beta-\lambda})|\xi_\alpha^{-1}\circ\zeta_\alpha|_{\mathbb{H}^n}
\end{aligned}$$

which, from the definition (15), yields:

$$\alpha|\xi_\alpha^{-1}\circ\zeta_\alpha|_{\mathbb{H}^n} \leq 2\varepsilon^{1-\theta}C_4(1+C\varepsilon^{-\beta-\lambda}) \leq C'\varepsilon^{1-\theta-\beta-\lambda}.$$

Observing that the last term in (36) tends to 0 for α going to ∞ we can pass to the limit and then we obtain:

$$u(\hat\xi) + \overline{H}(\hat\xi, q_\varepsilon(\hat\xi,\hat\eta)) - \varepsilon^\theta C_4(1+|q_\varepsilon(\hat\xi,\hat\eta)|) \leq C_4(C'\varepsilon^{1-\theta-\beta-\lambda} + \delta|\hat\xi|_{\mathbb{H}^n})$$

and then, from Proposition 5.1, one derives:

$$u(\hat\xi) + \overline{H}(\hat\xi, q_\varepsilon(\hat\xi,\hat\eta)) \leq \varepsilon^\theta C_4(1+M) + C_4.C'\varepsilon^{1-\theta-\beta-\lambda} + C_4 L\delta^{1/2}$$

which proves (25).

Proof of Theorem 1.3

The proof follows the argument used in Theorem 1.2 in [8] ; since it is quite immediate we give it here for the sake of completeness. Since H is independent of ξ, $(\overline{HJ})$ becomes

$$u(\xi) + \overline{H}(\nabla_{\mathbb{H}^n} u(\xi)) = 0 \text{ on } \mathbb{H}^n$$

and its solution is clearly $u \equiv -\overline{H}(0)$. Let v be the unique bounded continuous solution of the cell problem

$$H(\xi, \nabla_{\mathbb{H}^n} v(\xi)) = \overline{H}(0) \text{ on } \mathbb{H}^n.$$

We then define $w^\varepsilon(\xi) = u(\xi) + \varepsilon v(\delta_{\frac{1}{\varepsilon}} \xi)$ which is a viscosity solution of

$$w^\varepsilon(\xi) + H(\delta_{\frac{1}{\varepsilon}} \xi, \nabla_{\mathbb{H}^n} w^\varepsilon(\xi)) = \varepsilon v(\delta_{\frac{1}{\varepsilon}} \xi).$$

From the comparison result, one easily obtains that

$$w^\varepsilon - \varepsilon M \leq u^\varepsilon \leq w^\varepsilon + \varepsilon M \text{ on } \mathbb{H}^n$$

where $M = ||v||_\infty$ and u^ε is the solution of (HJ^ε). This finally gives

$$||u^\varepsilon - u||_\infty \leq 2\varepsilon M.$$

□

6. **Proof of (21).** Let $\eta = (x_o, y_o, t_o) \in \mathbb{H}^n$.
We want to prove that $|\nabla_{\mathbb{H}^n,\eta} \nabla_{\mathbb{H}^n,\xi} A_{\lambda,\varepsilon}(\xi, \eta)| \leq C$, for all $\xi,\ \eta \in \mathbb{H}^n$.
Let us write $A = A_{\lambda,\varepsilon}(\xi, \eta) := (r_\varepsilon^4 + t^2)^{1/2}$ where $r_\varepsilon^2 = \frac{(x-x_o)^2+(y-y_o)^2}{\varepsilon^\lambda}$ and $t = (t - t_o + 2(x_o y - x y_o))$.
We have:

$$\begin{aligned}
X_{i,\xi} A_{\lambda,\varepsilon}(\xi, \eta) &= \frac{2(x_i - x_o^i) r_\varepsilon^2 + 2(y_i - y_o^i)t}{(r_\varepsilon^4 + t^2)^{1/2}} := \frac{N_{i,\xi}}{A} \\
Y_{j,\xi} A_{\lambda,\varepsilon}(\xi, \eta) &= \frac{2(y_j - y_o^j) r_\varepsilon^2 - 2(x_j - x_o^j)t}{(r_\varepsilon^4 + t^2)^{1/2}} := \frac{M_{j,\xi}}{A}. \\
X_{i,\eta} A_{\lambda,\varepsilon}(\xi, \eta) &= \frac{2(x_o^i - x_i) r_\varepsilon^2 + 2(y_i - y_o^i)t}{(r_\varepsilon^4 + t^2)^{1/2}} := \frac{N_{i,\eta}}{A} \\
Y_j A_{\lambda,\varepsilon}(\xi, \eta) &= \frac{2(y_o^j - y_j) r_\varepsilon^2 - 2(x_j - x_o^j)t}{(r_\varepsilon^4 + t^2)^{1/2}} := \frac{M_{j,\eta}}{A}.
\end{aligned}$$

Then,

$$\begin{aligned}
X_{j,\eta} X_{i,\xi} A_{\lambda,\varepsilon}(\xi, \eta) = \frac{A.X_{j,\eta} N_{i,\xi} - N_{i,\xi}.X_{j,\eta} A}{A^2} &= \frac{A.X_{j,\eta} N_{i,\xi} - N_{i,\xi}.\frac{N_{j,\eta}}{A}}{A^2} \\
&= \frac{A^2.X_{j,\eta} N_{i,\xi} - N_{i,\xi}.N_{j,\eta}}{A^3}.
\end{aligned}$$

One has:

$$\begin{aligned}
|X_{j,\eta} N_{i,\xi}| &= |4\frac{(x_i - x_o^i)(x_o^j - x^j)}{\varepsilon^\lambda} - 2\delta_{ij} r_\varepsilon^2 - 4(y_i - y_o^i)(y_j - y_o^j)| \\
&\leq 2\frac{((x_i - x_o^i)^2 + (x_j - x_o^j)^2)}{\varepsilon^\lambda} + 2r_\varepsilon^2 + 2((y_i - y_o^i)^2 + (y_j - y_o^j)^2) \\
&\leq 10 r_\varepsilon^2 \\
&\leq 10A
\end{aligned}$$

and similarly it is easy to see that

$$|N_{i,\xi}N_{j,\eta}| \le \frac{N_{i,\xi}^2 + N_{j,\eta}^2}{2} \le 8A^3.$$

We deduce that:

$$|X_{j,\eta}X_{i,\xi}A_{\lambda,\varepsilon}(\xi,\eta)| \le \frac{10A^3 + 8A^3}{A^3} = 18.$$

In the same manner, one gets the other estimates.

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