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A gradient flow approach to linear Boltzmann equations

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(joint work with Giada Basile, Dario Benedetto)

We consider linear Boltzmann equations of the form

$$(\partial_t + b(v)\cdot\nabla_x) f(t,x,v) = \int \pi(dv')\sigma(v,v')\big[f(t,x,v') - f(t,x,v)\big] \tag{1}$$

where $x \in \mathbb{T}^d$, the d-dimensional torus, $\pi(dv)$ is a reference probability measure on the velocity space $\mathcal{V}$, $b\colon \mathcal{V} \to \mathbb{R}^d$ is the drift, $\sigma(v,v')\pi(dv')$ is the scattering kernel and f is the density of the one-particle distribution with respect to $dx\,\pi(dv)$. We assume the detailed balance condition, i.e., $\sigma(v,v') = \sigma(v',v)$. Examples of linear Boltzmann equations of this form are the Lorentz gas [5], the evolution of a tagged particle in a Newtonian system in thermal equilibrium [6], and the propagation of lattice vibrations in insulating crystals [2].

Using the shorthand notation $f = f(t,x,v)$, $f' = f(t,x,v')$, we set

$$\eta^f = \eta^f(t,x,v,v') := \sigma(f - f') = \sigma(v,v')\big[f(t,x,v) - f(t,x,v')\big]$$

and rewrite the linear Boltzmann equation (1) in the form

$$\begin{cases} \big(\partial_t + b(v)\cdot\nabla_x\big) f(t,x,v) + \int \pi(dv')\,\eta(t,x,v,v') = 0 \\ \eta = \eta^f \end{cases}$$

where $\eta\colon [0,T]\times\mathbb{T}^d\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}$ is antisymmetric with respect to the exchange of velocities.

Referring to [1] for the details, given a time interval $[0,T]$, we rewrite the identity $\eta = \eta^f$ as the following inequality that expresses the decay of the entropy along the solutions to (1),

$$\mathcal{H}(f(T)) + \int_0^T dt\, \mathcal{E}(f(t)) + \mathcal{R}(f,\eta) \leq \mathcal{H}(f(0)). \tag{2}$$

Here $\mathcal{H}$ is the relative entropy with respect to $dx\,\pi(dv)$, i,e.,

$$\mathcal{H}(f) := \iint dx\,\pi(dv)\, f\log f,$$

$\mathcal{E}$ is the Dirichlet form of the square root of f, i.e.,

$$\mathcal{E}(f) := \int dx \iint \pi(dv)\pi(dv')\,\sigma(v,v')\big[\sqrt{f'}-\sqrt{f}\big]^2,$$

and the *kinematic term* $\mathcal{R}$ is defined by

$$\mathcal{R}(f,\eta) := \int_0^T dt \int dx \iint \pi(dv)\pi(dv')\,\Psi_\sigma(f,f';\eta)$$

in which $\sigma = \sigma(v,v')$ and

$$\Psi_\kappa(p,q;\xi) = \xi \mathrm{arcsinh}\frac{\xi}{2\kappa\sqrt{pq}} - \Big[\sqrt{\xi^2+4\kappa^2 pq} - 2\kappa\sqrt{pq}\Big].$$

Both $\mathcal{E}$ and $\mathcal{R}$ can be expressed by variational formulae that imply their lower semi-continuity and convexity on the set of density f satisfying the entropy bound $\sup_{t\in[0,T]}\mathcal{H}(f(t)) \leq \ell$, $\ell > 0$. It is then straightforward to prove existence and stability of the formulation (2). Uniqueness follows from the argument in [4].

The entropy dissipation formulation (2) of (1) allows to discuss the diffusive limit of linear Boltzmann equation, see e.g., [3], in the framework of the gradient flow formulation of the heat equation; in particular by assuming only equi-boundedness of the entropy at the initial time.

Let $\epsilon > 0$ be the scaling parameter and denote by $(f^\epsilon, \eta^\epsilon)$ the diffusively rescaled solution of the linear Boltzmann equation. According to the gradient flow formulation, the pair $(f^\epsilon,\eta^\epsilon)$ satisfies

$$\partial_t f^\epsilon(t,x,v) + \frac{1}{\epsilon} b(v)\cdot\nabla_x f^\epsilon(t,x,v) + \frac{1}{\epsilon^2}\int \pi(dv')\eta^\epsilon(t,x,v,v') = 0 \tag{3}$$

$$\mathcal{H}(f^\epsilon(T)) + \frac{1}{\epsilon^2}\int_0^T dt\,\mathcal{E}(f^\epsilon(t)) + \frac{1}{\epsilon^2}\mathcal{R}(f^\epsilon,\eta^\epsilon) \leq \mathcal{H}(f^\epsilon(0)). \tag{4}$$

We set

$$\rho^\epsilon(t,x) := \int \pi(dv) f^\epsilon(t,x,v)$$
$$j^\epsilon(t,x) := \frac{1}{\epsilon}\int \pi(dv) f^\epsilon(t,x,v) b(v).$$

Since $\eta^\epsilon(t,x,v,v')$ is antisymmetric with respect to the exchange of v and v', by integrating (3) with respect to $\pi(dv)$ we deduce the continuity equation

$$\partial_t \rho^\epsilon + \nabla \cdot j^\epsilon = 0. \tag{5}$$

Let $H(\rho) := \int dx\, \rho \log \rho$ the entropy of the probability density ρ. Assuming $\rho^\epsilon(0) \to \rho(0)$ and $\mathcal{H}(f^\epsilon(0)) \to H(\rho(0))$ we would like to take the inferior limit in the inequality (4) deducing

$$H(\rho(T)) + \int_0^T dt\, E(\rho(t)) + R(\rho, j) \leq H(\rho(0)), \tag{6}$$

that corresponds to the gradient flow formulation of the heat equation for the pair (ρ, j) satisfying the continuity equation. Here E is the Fisher information, i.e.,

$$E(\rho) = 2 \int dx\, \nabla\sqrt{\rho} \cdot D \nabla \sqrt{\rho}$$

and

$$R(\rho, j) = \frac{1}{2} \int_0^T dt\, \int dx\, \frac{1}{\rho(t)} j(t) \cdot D^{-1} j(t),$$

where the positive definite $d \times d$ matrix D is diffusion coefficient.

This step is accomplished in [1] under suitable conditions on the scattering kernel σ and the drift b implying homogenization of the velocity on the diffusive scale.

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Random walk in a non-integrable random scenery time

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(joint work with Marco Lenci, Françoise Pène)

Anomalous diffusions are stochastic processes $X(t)$, $t \in \mathbb{R}^+$, having an asymptotic variance which does not grow linearly in time, that is $\mathbb{E}(X^2(t)) \sim t^\delta$ with $\delta \neq 1$. This phenomenon is quite common in applications and it is especially related to the transport in inhomogeneous material, e.g., the motion of a light particle in an optical lattice [6, 7]. The basic mathematical models for anomalous diffusions are *Lévy flights*, which are random walks with step length provided by an i.i.d. sequence of Lévy α-stable random variables with $\alpha \in (0, 2)$ (see [10, 5]). In this simple case, the motion is indeed provided by an asymptotic super-diffusive behavior with $\delta = 2$, for $\alpha \in (0, 1]$, and $\delta = 3 - \alpha$, for $\alpha \in (1, 2)$ (Lévy scheme). To model the motion in inhomogeneous material, one would like to take also into account that steps are mutually correlated by their positions, which we may identify with the presence of scatterers in the media. To this aim, in [4] the so-called *Lévy-Lorentz gas* were introduced. This is linear interpolation of a one-dimensional random walk in a Lévy-type random environment, where scatterers are placed as a renewal point process with inter-distances having a Lévy-type distribution, and jump probabilities depend on whether the position of the walker is on a scatterer or not.

We are then interested in providing a characterization of this process under the quenched and annealed laws (LLN, scaling limits, large deviation of the empirical speed), and in determining whether (and under which law) the asymptotic behavior is super-diffusive. The theory of random walks in random environments have been intensively studied in the last forty years and many important results have been achieved, especially for one-dimensional systems that are generally quite well understood. Even so, classical results do not apply to this setting, mainly because of the non-ellipticity of the environment, and a different analysis is required.

The range of $\alpha \in (1, 2)$, when inter-distances between scatterers having finite mean but infinite variance, was first studied in [1, 8] in the annealed setting, and then extended in the quenched setting in a recent work in collaboration with Cristadoro, Lenci and Ligabò (see [3]), where we proved that the quenched law of the process satisfies a classical CLT and has moments converging to the moments of a diffusion. While the annealed CLT follows trivially from these results, there are not sharp results on the asymptotical behavior of the annealed second moment which is then still under debate, as the results in [1, 8] are not completely in agreement and may lead to different conclusions.

In the present work we investigate the annealed behavior of the process for $\alpha \in (0, 1)$, when inter-distances between scatterers having infinite mean. Under this hypothesis, some previous works where suggesting the super-diffusivity of the process, and in particular the results in [4] and in [1, 8] where some annealed quantities related to the second moment were estimated and numerically simulated. Here we confirm and extend these predictions, proving, for the first time to our knowledge, that *Lévy-Lorentz gas* is super-diffusive for $\alpha \in (0, 1)$. In particular we establish the convergence of the finite-dimensional distributions of the process under a super-diffusive scaling with exponent $1/1 + \alpha > 1/2$, and we characterize the scaling limit. This is explicitely given by the composition of three processes: The α stable process obtained as the scaling limit of the Lévy environment, the Brownian motion obtained as the scaling limit of an underlying random walk, and the inverse of the Kesten-Spitzer process. This last process, that was introduced in [9] as the scaling limit of *random walks in random scenery*, appears in this context as the scaling limit of the sequence of time-lengths between to consecutive