

Gibbsian Properties and Convergence of the Iterates for the Block Averaging Transformation*

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Abstract. We analyze the Block Averaging Transformation applied to the two-dimensional Ising model in the uniqueness region. We discuss the Gibbs property of the renormalized measure and the convergence of renormalized potential under iteration of the map. It turns out that for any temperature T higher than the critical one T_c the renormalized measure is strongly Gibbsian, whereas for $T < T_c$ we have only weak Gibbsianity. Accordingly, we have convergence of the renormalized potential in a strong sense for $T > T_c$ and in a weak sense for $T < T_c$. Since we are arbitrarily close to the coexistence region we have a diverging characteristic length of the system: the correlation length or the critical length for metastability, or both. Thus, to perturbatively treat the problem we use a scale-adapted expansion. The more delicate case is $T < T_c$ where we have a situation similar to that of a disordered system in the presence of a Griffiths' singularity. In this case we use a graded cluster expansion whose minimal scale length is diverging when approaching the coexistence line.

KEYWORDS: lattice systems, cluster expansion, disordered systems, renormalization group

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We want to analyze, from a rigorous point of view, the renormalization-group transformations in statistical mechanics. We will focus on the well-known Block

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Averaging Transformation, BAT in the sequel, for the standard Ising model. The main questions that we want to address concern the Gibbs property of the renormalized measure and the convergence of the renormalized potential under iteration of the map. We shall use perturbative techniques based on cluster expansions. We mention that, in order to study the Gibbs properties of stochastic fields, different perturbative approaches have been developed (see e.g. [8, 31]) and also percolation arguments have been successfully applied [19].

Let us start with the definition of the system, called *object system*, to which the renormalization group map is applied. We let $\mathcal{S} := \otimes_{x \in \mathcal{L}} \mathcal{S}_x$ be the state space, with $\mathcal{S}_x := \{-1, +1\}$ the single site state space, $\mathcal{L} := \mathbf{Z}^d$ the lattice, and $d \in \mathbf{N}$ its dimension. For $\Lambda \subset \mathcal{L}$ we write $\mathcal{S}_\Lambda := \otimes_{x \in \Lambda} \mathcal{S}_x$. The Hamiltonian in a finite volume $\Lambda \subset \mathcal{L}$ with boundary condition $\tau \in \mathcal{S}_{\mathcal{L} \setminus \Lambda}$ is given by

$$H_\Lambda^\tau(\sigma) := -\beta \sum_{\substack{\{x,y\} \subset \Lambda: \\ |x-y|=1}} \sigma_x \sigma_y - \beta \sum_{\substack{x \in \Lambda, y \notin \Lambda: \\ |x-y|=1}} \sigma_x \tau_y - \beta h \sum_{x \in \Lambda} \sigma_x \tag{1}$$

for any $\sigma \in \mathcal{S}_\Lambda$, where $\beta = 1/T > 0$ is the inverse of the temperature and $h \in \mathbf{R}$ the magnetic field. The corresponding finite volume Gibbs measure is

$$\mu_{\beta,h,\Lambda}^\tau(\sigma) := \frac{\exp\{-H_\Lambda^\tau(\sigma)\}}{\sum_{\eta \in \mathcal{S}_\Lambda} \exp\{-H_\Lambda^\tau(\eta)\}} \tag{2}$$

for any $\sigma \in \mathcal{S}_\Lambda$. We denote by $\mu = \mu_{\beta,h}$ the unique infinite-volume Gibbs measure in the region

$$\mathcal{U} = \{\beta < \beta_c\} \cup \{\beta > \beta_c, h \neq 0\}, \tag{3}$$

where $\beta_c = 1/T_c$ is the inverse critical temperature. The region \mathcal{U} is obtained by excluding the critical point from the uniqueness region.

We define, now, the Block Averaging Transformation. Let $\mathcal{L}^{(\ell)} := (\ell \mathbf{Z})^d$, for $\ell \in \mathbf{N}$, and partition \mathcal{L} as the disjoint union of ℓ -blocks $Q_\ell(i) := Q_\ell(0) + i$, where $i \in \mathcal{L}^{(\ell)}$ and $Q_\ell = Q_\ell(0)$ is the cube of side ℓ with the origin the site with the smallest coordinates. For $I \subset \subset \mathcal{L}^{(\ell)}$, with $\subset \subset$ meaning finite subset of, we set $Q_\ell(I) := \bigcup_{i \in I} Q_\ell(i)$. We associate with each $i \in \mathcal{L}^{(\ell)}$ a *renormalized* or *image spin* m_i taking values in

$$\mathcal{S}_i^{(\ell)} := \left\{ \frac{-\ell^d - \ell^d \bar{m}}{\sqrt{\ell^d \chi}}, \frac{-\ell^d + 2 - \ell^d \bar{m}}{\sqrt{\ell^d \chi}}, \dots, \frac{\ell^d - \ell^d \bar{m}}{\sqrt{\ell^d \chi}} \right\} \tag{4}$$

where we have denoted by $\bar{m} := \bar{m}_{\beta,h} = \mu_{\beta,h}(\sigma_0)$ the equilibrium magnetization and by $\chi := \chi(\beta, h) = \sum_{x \in \mathcal{L}} [\mu_{\beta,h}(\sigma_0 \sigma_x) - \mu_{\beta,h}(\sigma_0) \mu_{\beta,h}(\sigma_x)]$ the susceptibility. We next define the renormalized or *image* measure $\mu^{(\ell)} = \mu_{\beta,h}^{(\ell)}$ on the renormalized or image space $\mathcal{S}^{(\ell)} := \otimes_{i \in \mathcal{L}^{(\ell)}} \mathcal{S}_i^{(\ell)}$ via its finite-dimensional distributions.

Let $I \subset \subset \mathcal{L}^{(\ell)}$, set $\mathcal{S}_I^{(\ell)} := \otimes_{i \in I} \mathcal{S}_i^{(\ell)}$, and pick $\tilde{m} \in \mathcal{S}_I^{(\ell)}$, then set

$$\mu_{\beta,h}^{(\ell)}(\{m \in \mathcal{S}^{(\ell)} : m_I = \tilde{m}\}) := \int_{\mathcal{S}} d\mu_{\beta,h}(\sigma) \prod_{i \in I} \delta(M_i(\sigma_{Q_\ell(i)}) - \tilde{m}_i) \quad (5)$$

where for all $i \in \mathcal{L}^{(\ell)}$ and $\eta \in \mathcal{S}_{Q_\ell(i)}$ we have set

$$M_i(\eta) := \frac{1}{\sqrt{\ell^d \chi}} \sum_{x \in Q_\ell(i)} [\eta_x - \tilde{m}]. \quad (6)$$

We write $\mu_{\beta,h}^{(\ell)} = T^{(\ell)} \mu_{\beta,h}$ and note that $T^{(\ell)} T^{(\ell')} = T^{(\ell\ell')}$. The image measure $\mu_{\beta,h}^{(\ell)}$ represents the distribution of the empirical block magnetization $M(\sigma_{Q_\ell(i)})$, centered and normalized, under the object measure $\mu_{\beta,h}$. Various other renormalization group maps, for instance the decimation and the majority rule, have been considered in the literature and successfully applied also to other object systems different from the standard Ising model, see [18, 32].

To be concrete in this paper we focus on the standard Ising model. However, we stress that the results discussed here can be stated in the general setup of lattice spin systems with finite state space and finite-range interaction, see [3, 5, 6]. In particular the definitions concerning strong mixing and finite-size conditions will be given only for the Ising model.

Our main goal is the study of the map on the potential induced by the $T^{(\ell)}$ that has been defined on the infinite-volume measure. A preliminary condition for this program is that the renormalized measure is *strongly* or *weakly* Gibbsian with respect to a *renormalized potential*; where we say that a stochastic field is strongly resp. weakly Gibbsian if its family of conditional probabilities has the Gibbsian form with a potential absolutely *uniformly* resp. *pointwise almost surely* converging. Thus in both cases the DLR equations, see (16) below, are satisfied but with different strengths in the convergence properties of the potential. We refer to [18] for a general description of the Gibbs formalism, especially in connection with renormalization-group maps, and to [15, 31] for a discussion of the weak Gibbs property.

We introduce now the finite-volume setup. Let $I \subset \subset \mathcal{L}^{(\ell)}$ be a finite box in $\mathcal{L}^{(\ell)}$ and consider the corresponding box $\Lambda = Q_\ell(I) \subset \mathcal{L}$. We introduce the *renormalized Hamiltonian* $H_I^{(\ell),\tau}$ with boundary condition $\tau \in \mathcal{S}_{\mathcal{L} \setminus \Lambda}$ by setting

$$\exp \{ - H_I^{(\ell),\tau}(m) \} = \sum_{\sigma \in \mathcal{S}_\Lambda} \exp \{ - H_\Lambda^\tau(\sigma) \} \prod_{i \in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i) \quad (7)$$

for each $m \in \mathcal{S}_I^{(\ell)}$. In the computation of the *renormalized potential* associated with the renormalized Hamiltonian $H_I^{(\ell),\tau}$, a crucial role is played by the *constrained systems* obtained by conditioning the object system to a fixed renormalized spin configuration. More precisely, the equilibrium probability measure of

the constrained model associated with the renormalized configuration $m \in \mathcal{S}_I^{(\ell)}$ on the finite volume $\Lambda = Q_\ell(I) \subset \subset \mathcal{L}$ is given by

$$\mu_{m,\Lambda}^{(\ell),\tau}(\sigma) := \frac{\exp\{-H_\Lambda^\tau(\sigma)\} \prod_{i \in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i)}{\sum_{\eta \in \mathcal{S}_\Lambda} \exp\{-H_\Lambda^\tau(\eta)\} \prod_{i \in I} \delta(M_i(\eta_{Q_\ell(i)}) - m_i)} \quad (8)$$

for all $\sigma \in \mathcal{S}_\Lambda$. Notice that from (7) it follows that the renormalized Hamiltonian $H_I^{(\ell),\tau}(m)$ is equal to minus the logarithm of the partition function of the corresponding constrained system which is defined as

$$Z_{m,\Lambda}^{(\ell),\tau} := \sum_{\sigma \in \mathcal{S}_\Lambda} \exp\{-H_\Lambda^\tau(\sigma)\} \prod_{i \in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i). \quad (9)$$

In the case of BAT the measure $\mu_{m,I}^{(\ell),\tau}$ can be called *multicanonical*, because it is nothing but the original measure *constrained* to the assigned magnetizations in the ℓ -blocks contained in Λ . Of course $\mu_{m,I}^{(\ell),\tau}$ does not depend at all on the magnetic field h .

Sometimes it happens that the image measure, obtained by applying some renormalization group map to a Gibbs object measure corresponding, for instance, to a short range object potential, is not Gibbsian. This pathology often consists of a non-vanishing dependence of conditional probabilities of the image measure on arbitrarily far conditioning image spins.

It has been shown in [18] that for any even value of $\ell \in \mathbf{N}$ there exists $\beta_0 = \beta_0(\ell)$ (large) such that the renormalized measure $\mu_{\beta,h}^{(\ell)}$, defined in (5), at any h and any inverse temperature $\beta \geq \beta_0$ is non-Gibbsian as a consequence of violation of *quasi-locality*, a continuity property of its conditional probabilities which constitutes a necessary condition for Gibbsianity. This kind of influence from infinity is, in turn, a consequence of a first order phase transition with long range order of a particular constrained model namely, the one corresponding to $m_i = 0$ for all $i \in \mathcal{L}^{(\ell)}$. Thus the existence of just one bad image configuration, giving rise to a bad constrained system, is sufficient to induce the violation of the Gibbs property of the image measure. On the other hand it is reasonable to expect, and proven in [23] in a specific context, that when *all* the constrained systems are well-behaved the image measure is Gibbsian. As we shall see in the sequel, this will be the case for the two-dimensional Ising Model above T_c . It is clear that the above described pathology for BAT applied to the low-temperature Ising model is completely independent of the value of the magnetic field h acting on the object system. On the other hand it is also clear that this “bad” configuration inducing non-Gibbsianity, is very atypical with respect to $\mu_{\beta,h}^{(\ell)}$ for $h \neq 0$. It is thus reasonable to expect at least the validity of a weaker property of Gibbsianity. For $h = 0$ we can consider the BAT for the extremal measures μ_\pm ; we expect weak Gibbsianity also in this case.

We want to present now results on strong Gibbsianity above T_c and on weak Gibbsianity below T_c together with results on convergence of the iterated renormalized potential. All these results are based on a suitable strong mixing condition for the object system namely, exponential decay of finite-volume truncated expectations with a rate independent of the boundary conditions and of the volume chosen from a suitable class. More explicitly, given an integer ℓ_0 , we say that the measure $\mu_{\beta,h,\Lambda}^\tau$ satisfies $SM(\ell_0)$ if there exist two constants $C, \gamma > 0$ such that for every pair of local functions f, g with supports S_f, S_g , and every volume of the form $\Lambda = Q_{\ell_0}(I)$, with $I \subset\subset \mathcal{L}^{(\ell_0)}$, containing S_f, S_g , we have that

$$\sup_{\tau \in \mathcal{S}_{\mathcal{L} \setminus \Lambda}} |\mu_{\beta,h,\Lambda}^\tau(f; g)| \leq C(|S_f| \wedge |S_g|) \|f\|_\infty \|g\|_\infty \exp \{ -\gamma d(S_f, S_g) \} \quad (10)$$

where for $\Delta, \Delta' \subset \mathcal{L}$ we have set $d(\Delta, \Delta') := \inf\{|x - x'|, x \in \Delta, x' \in \Delta'\}$.

To simplify the exposition we will mainly consider the two-dimensional case; the higher-dimensional case will be briefly discussed. In two dimensions it has been proved, see [30, 36], that condition $SM(\ell_0)$ is satisfied in the whole uniqueness region \mathcal{U} for some ℓ_0 depending on β, h . We observe that there exist two characteristic lengths of the Ising model namely, the correlation length and the *critical length for metastability*, which diverge when the closure of the coexistence line $\mathcal{U}^c := \{\beta \geq \beta_c, h = 0\}$ is approached. The first diverges in the limit $h = 0$ and $\beta \rightarrow \beta_c^-$, the latter diverges as $1/h$ in the limit $\beta > \beta_c$ fixed and $h \rightarrow 0$. The critical length for metastability represents the minimal size of a droplet whose growth is energetically favorable and, at the same time, the minimal length required to *screen* the effect of a boundary condition opposite to the field. It is clear that a perturbative description outside \mathcal{U}^c , uniform in the boundary conditions, has to involve a characteristic length depending on β, h and diverging when the distance from \mathcal{U}^c tends to zero. Notice that the critical length for metastability and the correlation length can even diverge simultaneously.

To extract the renormalized potentials from the renormalized Hamiltonian $H_I^{(\ell),\tau}(m)$, where $m \in \mathcal{S}_I^{(\ell)}$, a possible strategy is to use a perturbative expansion for the constrained system corresponding to m with a procedure making sense in the thermodynamic limit. This should work for every $m \in \mathcal{S}_I^{(\ell)}$. For instance when the object system is far away from the coexistence line, then the usual high temperature or high magnetic field expansions for the constrained models are sufficient to compute the renormalized potentials, see [9, 22, 25, 26]. However, in order to get close to the coexistence line, we certainly have to use other, more powerful, perturbative theories.

We discuss now these different perturbative theories in the concrete case of systems above their critical temperature T_c . Usual high temperature expansions work only for temperatures T sufficiently larger than T_c ; they basically

involve perturbations around a universal reference system composed of independent spins; in other words the small parameter is the inverse temperature and all interactions are expanded treating, in this way, every lattice system in the same manner. In [33, 34] another perturbative expansion has been introduced, around a non-trivial model-dependent reference system, that we call *scale-adapted expansion*. The small parameter is no more $\beta = 1/T$ but, rather, the ratio between the correlation length (at the given temperature $T > T_c$) and the length scale L at which we analyze our system. The geometrical objects (polymers) involved in the scale-adapted expansion live on the scale L whereas in the usual high and low temperature or high magnetic field expansions they live on scale one. Of course the smaller is $T - T_c$, the larger has to be taken the length L . A similar situation occurs for low temperature Ising ferromagnets with generic boundary conditions at arbitrarily small but non zero magnetic field h with a diverging critical length of order $1/h$. Also in this case we have to look at our system on a scale sufficiently larger than the critical length; note that at low temperature, far from the critical point, the correlation length is of order one.

The scale-adapted expansions are based on a suitable *finite size condition* saying, roughly speaking, that if we look at the Gibbs measure in a box of sufficiently large side length L , then, uniformly in the boundary conditions, the corresponding truncated correlations decay sufficiently fast in terms of L . A possible formulation is the following, see [29, 33, 34], we say that condition $C(\varepsilon, L)$ is satisfied if there exist $\varepsilon > 0$ and $L \in \mathbf{N}$ such that

$$\sup_{\tau \in \mathcal{S}_{\mathcal{L} \setminus Q_L}} \sup_{\substack{x, y \in Q_L: \\ |x-y| \geq L-1}} |\mu_{\beta, h, Q_L}^{\tau}(\sigma_x; \sigma_y)| < \frac{\varepsilon}{L^{2(d-1)}} \quad (11)$$

where we recall $Q_L = Q_L(0)$. It has been proven (see [29, 33, 34]) that there exists a real $\varepsilon_0 = \varepsilon_0(d)$ — recall d is the dimension of the lattice — such that if there exists $L \in \mathbf{N}$ with the property that condition $C(\varepsilon, L)$ is satisfied with $\varepsilon < \varepsilon_0$, then strong mixing $\text{SM}(\ell_0)$ holds for every ℓ_0 multiple of L . Conversely, it is immediately seen that if we assume the validity of $\text{SM}(\ell_0)$ for some ℓ_0 , then condition $C(\varepsilon, L)$ holds for every $\varepsilon > 0$ and a sufficiently large L multiple of ℓ_0 . We remark that to get the above result it is not necessary to use the cluster expansion (see [28]), however this theory is needed if we want to prove complete results like analyticity properties of thermodynamic and correlation functions.

The basic idea to develop a perturbative theory on the basis of condition $C(\varepsilon, L)$ is inspired by the renormalization group theory itself and consists in applying a block decimation procedure. Consider a box in \mathcal{L} of the form $\Lambda = Q_L(I)$, with I a cube in $\mathcal{L}^{(L)}$; we identify the blocks $Q_L(i)$ in Λ with the points i in I . We look at the rescaled system as a lattice spin system on $\mathcal{L}^{(L)}$ whose spin variable associated to the site $i \in I$ is identified with the block configuration in $\mathcal{S}_{Q_L(i)}$. Then we partition $\mathcal{L}^{(L)}$ as the disjoint union of 2^d sublattices of

spacing $2L$ and enumerate these sublattices with a given order. To compute the partition function Z_Λ of the model, assuming for simplicity periodic boundary condition, we start summing over the variables in the first sublattice, keeping fixed the variables on the other sublattices. Choosing L larger than the range of the interaction, this first sum factorizes. Condition $C(\varepsilon, L)$, for small ε , implies that opposite faces of the cube $Q_L(i)$ are weakly correlated so that after the summation over the variables on the first sublattice, we get that the effective interaction between the surviving variables, sitting on the other sublattices, is weak. Then we iterate this procedure by successively summing over the L -blocks variables sitting on subsequent sublattices. In this way we end up with an expression of the partition function of the form

$$Z_\Lambda = \bar{Z}_\Lambda \Xi_\Lambda \tag{12}$$

where \bar{Z}_Λ is given as the product of partition functions on suitable domains with sizes of order L and Ξ_Λ is the partition function of a gas of polymers given as suitable unions of L -blocks whose only interaction is a hard core exclusion:

$$\Xi_\Lambda = 1 + \sum_{n \geq 1} \sum_{\substack{R_1, \dots, R_n \in \mathcal{R}_\Lambda: \\ R_i \cap R_j = \emptyset}} \zeta(R_1) \cdots \zeta(R_n) \tag{13}$$

where \mathcal{R}_Λ is the set of polymers contained in Λ and $\zeta : \mathcal{R}_\Lambda \rightarrow \mathbf{R}$ is the activity. Two overlapping polymers are called *incompatible*. The validity of condition $C(\varepsilon, L)$ with small ε implies that the polymer system is in the small activity regime. More precisely, the following estimate holds

$$\sum_{R \ni O} |\zeta(R)| \leq \delta(\varepsilon) \quad \text{with } \delta(\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \tag{14}$$

where $O = (0, \dots, 0)$ is the origin of \mathcal{L} .

Suppose that we want to treat perturbatively the free energy given as $-(1/|\Lambda|) \log Z_\Lambda$. Taking the logarithm of Z_Λ by 12 we get two terms. The logarithm of \bar{Z}_Λ gives rise to a sum of local terms whereas the logarithm of Ξ_Λ can be treated by means of the general theory of cluster expansions (see for instance [21, 27]) which is based on a condition like (14) with a sufficiently small δ . We get an expansion of the form

$$\log \Xi_\Lambda = \sum_{R_1, \dots, R_n \in \mathcal{R}_\Lambda} \varphi_T(R_1, \dots, R_n) \zeta(R_1) \cdots \zeta(R_n) \tag{15}$$

where the combinatorial factor $\varphi_T(R_1, \dots, R_n)$ vanishes when R_1, \dots, R_n do not form a globally incompatible set. We observe that the building bricks of the above described scale-adapted cluster expansion lives on scale L ; we look at our system of L -block variables and we never go below this minimal scale.

Before going back to the discussion of BAT we want to mention here that there is another, stronger notion of strong mixing, originally introduced for general short-range lattice systems by Dobrushin and Shlosman in [13, 14], before the papers [33, 34] appeared. It has been called by the authors *Complete Analyticity* (CA): it requires exponential decay of truncated correlations for *all* finite or infinite domains Λ (of arbitrary shape). This point of view contrasts the one of scale-adapted expansions. Indeed in Dobrushin–Shlosman’s approach there is no minimal scale length and there are examples where $\text{SM}(\ell_0)$ holds for ℓ_0 sufficiently large but Dobrushin–Shlosman’s Complete Analyticity fails for domains with anomalous ratio surface/volume.

Scale-adapted perturbative theory gives rise to a notion that can be called *Restricted Complete Analyticity* (RCA) or *Complete Analyticity for Regular Domains*; here “regular” means “multiple” of a sufficiently large box. Dobrushin and Shlosman also developed a finite size condition involving *all* the possible subsets of a given, sufficiently large box and not just the box itself like [29]: in some respects Dobrushin–Shlosman’s approach can be still seen as a perturbative theory on scale one. For the two-dimensional standard Ising model, as we said before, RCA has been proven in the whole region \mathcal{U} [36]. On the other hand, CA has been conjectured to hold for any $T > T_c$; however a proof is still missing. For $d \geq 3$ even RCA is not expected to hold at low-temperature for suitable nonvanishing magnetic field as a consequence of a layering phase transition [1, 10].

The Gibbs properties of the renormalized measure are particularly relevant, from the physical point of view, in a neighborhood of the critical point. Actually it is possible that even if the object system is critical it may happen that the constrained systems are in the one phase, weakly coupled regime, so that the renormalized potential is still well defined, see [2, 11, 23].

Examples of non Gibbsianity are given in [16, 17].

Let us now state the main results of [3] on strong Gibbsianity and convergence above T_c in two dimensions.

Theorem 1 ([3]). *Consider a two-dimensional Ising system with $\beta < \beta_c$ and $h \in \mathbf{R}$ given. Then there exists $\ell_0 \in \mathbf{N}$ such that for any ℓ large enough multiple of ℓ_0 , $\mu_{\beta, h}^{(\ell)}$ is Gibbsian in the sense that for each $Y \subset \subset \mathcal{L}^{(\ell)}$ and for each local function $f : \mathcal{S}_Y^{(\ell)} \rightarrow \mathbf{R}$ we have*

$$\begin{aligned} \mu^{(\ell)}(f) &= \int_{\mathcal{S}^{(\ell)}} \mu^{(\ell)}(dm') \frac{1}{Z_Y(m')} \sum_{m \in \mathcal{S}_Y^{(\ell)}} f(m) \\ &\times \exp \left\{ \sum_{X \cap Y \neq \emptyset} [\psi_X^{(\ell)}(m_{Y \cap X} m'_{Y \cap X}) + \phi_X^{(\ell)}(m_{Y \cap X} m'_{Y \cap X})] \right\} \end{aligned} \quad (16)$$

where

$$Z_Y(m') = \sum_{m \in \mathcal{S}_Y^{(\ell)}} \exp \left\{ \sum_{X \cap Y \neq \emptyset} [\psi_X(m_{Y \cap X} m'_{Y^c \cap X}) + \phi_X(m_{Y \cap X} m'_{Y^c \cap X})] \right\} \quad (17)$$

and the family $\{\phi_X^{(\ell)} + \psi_X^{(\ell)}, X \subset \subset \mathcal{L}^{(\ell)}\}$, $\phi_X^{(\ell)}, \psi_X^{(\ell)} : \mathcal{S}_X^{(\ell)} \rightarrow \mathbf{R}$ is translationally invariant and satisfies the uniform bound

$$\sum_{X \ni 0} e^{\alpha|X|} \sup_{m_X \in \mathcal{S}_X^{(\ell)}} (|\psi_X^{(\ell)}(m_X)| + |\phi_X^{(\ell)}(m_X)|) < \infty \quad (18)$$

for a suitable $\alpha > 0$. Moreover there exists $\kappa \in \mathbf{N}$: $\psi_X^{(\ell)} = 0$ if $\text{diam}(X) \geq \kappa$. Finally we have that for the same α as in (18)

$$\lim_{\ell \rightarrow \infty} \sum_{X \ni 0} e^{\alpha|X|} \sup_{m_X \in \mathcal{S}_X^{(\ell)}} |\phi_X^{(\ell)}(m_X)| = 0, \quad (19)$$

$\psi_{\{i\}}^{(\ell)}(m_i) = m_i^2/2$ for $i \in \mathcal{L}^{(\ell)}$ and there exists $a > 0$ such that

$$\lim_{\ell \rightarrow \infty} \sup_{\substack{m_X \in \mathcal{S}_X^{(\ell)} \\ |m_i| \leq \ell^a, i \in X}} |\psi_X^{(\ell)}(m_X)| = 0 \quad \text{for } |X| \geq 2.$$

Notice that, by the composition rule $T^{(\ell)}T^{(\ell')} = T^{(\ell\ell')}$, taking the limit $\ell \rightarrow \infty$ is equivalent to indefinitely iterating the map $T^{(\ell_0)}$.

To prove the above theorem we use, for the partition function of the constrained system (9), the validity of an expression like (12) with Ξ_Λ in the form (13) and $\zeta(R)$ satisfying (14). This is obtained on the basis of the validity of the finite-size condition (11) with $\varepsilon = \varepsilon(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$ *uniformly* in the renormalized configuration $m \in \mathcal{S}^{(\ell)}$. By using a delicate comparison between multi-canonical and multi-grandcanonical ensembles [3] we see that the crucial point to obtain the above uniform finite size condition is the validity of the strong mixing condition $\text{SM}(\ell_0)$ for the object system *uniformly* in the magnetic field h . This is sufficient only in the two-dimensional case. In higher dimension instead of this one needs uniformity w.r.t. variable magnetic field constant in each cube of side ℓ_0 . Uniformity of $\text{SM}(\ell_0)$ w.r.t. h fails for any $d \geq 2$ below T_c , because of the phase transition at $h = 0$. By only assuming strong mixing of the object system, without uniformity in h , we can expect only weak Gibbsianity since, as we said before, for $T < T_c$ violation of strong Gibbsianity is proven in [18]. Let us now state our main results on weak Gibbsianity and convergence of the iterates of BAT.

Theorem 2. *Consider a two-dimensional Ising model and recall \mathcal{U} is defined in (3). Given $(\beta, h) \in \mathcal{U}$, there exists ℓ_0 such that for any large enough ℓ multiple*

of $\ell_0, \mu^{(\ell)}$ is weakly Gibbsian in the sense that it satisfies the DLR equations (16) with respect to a potential $\{\psi_X^{(\ell)} + \phi_X^{(\ell)}, X \subset\subset \mathcal{L}^{(\ell)}\}$, $\psi_X^{(\ell)}, \phi_X^{(\ell)} : \mathcal{S}_X^{(\ell)} \mapsto \mathbf{R}$, satisfying the following.

There exists a measurable set $\bar{\mathcal{S}}^{(\ell)} \subset \mathcal{S}^{(\ell)}$, such that $\mu^{(\ell)}(\bar{\mathcal{S}}^{(\ell)}) = 1$, and functions $r_i^{(\ell)} : \bar{\mathcal{S}}^{(\ell)} \mapsto \mathbf{N} \setminus \{0\}$, for all $i \in \mathcal{L}^{(\ell)}$, such that for each $m \in \bar{\mathcal{S}}^{(\ell)}$ if $X \ni i$ and $\text{diam}(X) \geq r_i^{(\ell)}(m)$ then $\psi_X^{(\ell)}(m) = 0$. Furthermore, for each $i \in \mathcal{L}^{(\ell)}$ and $m \in \bar{\mathcal{S}}^{(\ell)}$ there exists a real $c_i^{(\ell)}(m) \in [0, \infty)$ such that

$$\sum_{X \ni i} |\psi_X^{(\ell)}(m_X)| \leq c_i^{(\ell)}(m). \tag{20}$$

There exists C independent of ℓ such that

$$\sup_{i \in \mathcal{L}^{(\ell)}} \sum_{X \ni i} \sup_{m_X \in \mathcal{S}_X^{(\ell)}} |\phi_X^{(\ell)}(m_X)| < C. \tag{21}$$

For each $i \in \mathcal{L}^{(\ell)}$ we have $\psi_{\{i\}}^{(\ell)}(m) = m_i^2/2$ and for each $q \in [1, +\infty)$

$$\lim_{\ell \rightarrow \infty} \sup_{i \in \mathcal{L}^{(\ell)}} \mu^{(\ell)} \left(\left| \sum_{X \ni i: |X| \geq 2} \psi_X^{(\ell)} \right|^q \right) = 0 \tag{22}$$

and

$$\lim_{\ell \rightarrow \infty} \sup_{i \in \mathcal{L}^{(\ell)}} \sum_{X \ni i} \sup_{m_X \in \mathcal{S}_X^{(\ell)}} |\phi_X^{(\ell)}(m_X)| = 0. \tag{23}$$

The above theorem is a consequence of a general result saying that in order to get weak Gibbsianity for a renormalized potential and convergence of the iterates in the above sense, we have only to check the validity, for the object system with a given value of h , of $\text{SM}(\ell_0)$ for some $\ell_0 \in \mathbf{N}$, see [6]; this, as we said before, is a general result in the region \mathcal{U} for $d=2$. Thus the result of Theorem 2 immediately extends to the case $d > 2$, $h \neq 0$, and $\beta > \beta_0(d, |h|)$ for a suitable function $\beta_0 : \mathbf{N} \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$. Indeed in this case strong mixing holds true, see [29].

Let us now discuss the result of Theorem 2; in particular we want to analyze the difficulties in computing the renormalized potential and the way to solve them in the concrete case of a low-temperature Ising system at $h \neq 0$. We know that in the bad regions with $m_i = 0$ the corresponding constrained system almost undergoes a phase transition with long range order so that the conditional probabilities of the renormalized measure in these regions are very sensitive to the values of arbitrarily far conditioning spins; in other words the corresponding renormalized potential does not decay fast enough at large distances.

For $h \neq 0$ this is a very atypical configuration; however with small but positive probability we have arbitrarily large bad regions with $m_i = 0$. In order

to prove weak Gibbsianity the key property is that bad regions are far apart: larger and larger bad regions are sparser and sparser. This situation is similar to that of disordered systems in the presence of a Griffiths' singularity [5]. A multi-scale analysis is needed.

The natural approach, quite complicated from the technical point of view, is to use a graded cluster expansion. For disordered systems there are clever methods, see [7, 12], avoiding cluster expansion, that enable to prove partial results like exponential clustering with deterministic rate and random prefactor. In the case of BAT, in order to compute renormalized potentials in the weakly Gibbsian case, the use of the full theory of graded cluster expansion (like the one in [20]) appears to be unavoidable. Since we want to study a region of parameters arbitrarily close to $\mathcal{U}^c = \{\beta \geq \beta_c, h \neq 0\}$, the closure of the coexistence line, the distinctive character of our graded cluster expansion is that the minimal scale may be chosen arbitrarily large and diverging as $T \rightarrow T_c$ and/or $h \rightarrow 0$. The minimal scale involved in our discussion being divergent, we need to use a scale-adapted cluster expansion, see [5, 33, 34], based on a finite-size mixing condition.

As we observed before in this case, contrary to low and high temperature expansion or high magnetic field expansion, the small parameter is the ratio between the diverging length and the suitably large finite size where the mixing condition holds. We want to stress again that in our approach, according to the general renormalization group ideology, we first fix the values of the thermodynamic parameters of the object system and, subsequently, the value of the scale of BAT. In other words we *take advantage* from choosing the scale ℓ of the transformation large enough. On the other side we cannot exclude that, for given values of β and h , if ℓ is not sufficiently large weak Gibbsianity ceases to be valid. In [8, 31] the authors study decimation transformation, see [18], at large β and *arbitrary* h . They first fix the scale of the transformation and, subsequently, they choose the temperature below which they get weak Gibbsianity.

It seems clear that to get results of convergence of renormalized potentials when iterating BAT, one has to use a perturbative theory based on scale-adapted cluster expansion. Even far away from the critical point, in order to get convergence, one needs to take advantage from choosing larger and larger ℓ .

In [9] Cammarota proves convergence results at high temperature by making use of a general result [24], according to which, to get convergence in a suitable sense one needs only to prove uniform boundedness in a suitable norm. In [9] the author uses a high temperature expansion giving rise to a polymer system whose activity is small uniformly in ℓ for ℓ large enough. This appears to contradict the necessity to use a scale-adapted cluster expansion, but we want to stress that he does not directly prove the convergence of the renormalized potential but only uniform boundedness. This situation is similar to the one of [35] where Shlosman uses a low-temperature expansion that converges uniformly in ℓ but does not deal with the problem of convergence.

We remark, finally, that also for a disordered lattice spin systems in the Griffiths' phase and close to criticality, it seems necessary to use a graded cluster expansion whose minimal length scale is not one as in [20] but diverges as $T \rightarrow T_c$, see [4]. We conclude by briefly discussing an example of small random perturbation of a strong mixing system possibly close to criticality: a ferromagnetic two dimensional Ising system with zero magnetic field and coupling constants given by i.i.d. random variables for different bonds with distribution

$$J = \begin{cases} 1 & \text{with probability } 1 - p, \\ J_0 & \text{with probability } p. \end{cases} \quad (24)$$

Let $T_c(1)$ be the critical temperature corresponding to a coupling constant equal to one. We expect the validity of the following result: for all $T > T_c(1)$ there exists $p_0 > 0$ such that for all $p < p_0$ and $J_0 \leq +\infty$ we have, for almost all the realizations of the disorder, a convergent multi-scale cluster expansion whose minimal scale diverges as $T \rightarrow T_c(1)$.

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