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Research article

Large deviations for a binary collision model: energy evaporation†

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Abstract: We analyze the large deviations for a discrete energy Kac-like walk. In particular, we exhibit a path, with probability exponentially small in the number of particles, that looses energy.

Keywords: Kac model; Boltzmann equation; discrete energy model; large deviations; violation of energy conservation

1. Introduction

Large deviations associated to Boltzmann-type equations have been the object of recent investigations. The most challenging case of Newtonian dynamics of hard spheres in the Boltzmann-Grad limit has been heuristically discussed in [5] and rigorously, by means of cluster expansion, in [4]. The case of microscopic stochastic dynamics has been originally analyzed in [8], where a large deviation upper bound is derived. In [15] a large deviation principle is obtained for a space inhomogeneous model with a finite set of velocities. More recently, in [2] it is considered a homogeneous model which conserves momentum but not energy. The large deviation upper bound is achieved, while the lower bound is obtained for a restricted class of paths. In [6] the analogous results are provided for the Kac walk, which conserves also the energy.

We emphasize that, except in the case of bounded velocities, a proof of a large deviation principle with matching upper and lower bound is still missing, even in the homogeneous case. A key issue is the possible occurrence of macroscopic paths with finite rate function that violate the conservation of the energy. A class of examples has been constructed in [6] by exploiting the solutions of homogeneous Boltzmann equations provided by Lu and Wennberg in [10], for which the energy is increasing.

Here we consider a Kac-like microscopic dynamics with discrete energy, that is inspired by the so-called KMP model [3, 7], and described as follows: N particles, with equally spaced energy levels, evolve via random binary collisions such that in each collision the total energy is preserved. At the kinetic level the one-particle energy distribution evolves according to a discrete homogeneous Boltzmann equation, that is an infinite system of coupled ordinary differential equations. We focus on the large deviation properties of the pair empirical measure and flux and propose a candidate rate function I when the initial distribution of the energies satisfies the micro-canonical constraint, i.e., the total energy is fixed. By the arguments in [1, 2, 6], it can be shown that the large deviation upper bound holds with rate function I, and a matching lower bound can be proven for a restricted class of paths which conserve the energy. Our main novel point is the construction of a path $(\bar{f}, \bar{Q})$ which looses the energy and whose probability is exponentially small with rate $I(\bar{f}, \bar{Q})$. This result quantifies the probability to violate the conservation of the energy, which is exponentially small in the number of particles N. It can be compared with the result in [14] where, in the contest of the derivation of incompressible Navier-Stokes equations from stochastic lattice gas, it is shown that the probability of violating the incompressibility condition is of the order e^{-N^2}.

Referring to [1] for a general discussion, we emphasize that, due to the micro-canonical constraint, at the kinetic level the energy cannot increase and the candidate rate function I is different from the one in [6, 8].

1.1. The model

Given $N \geq 2$, a *configuration* is defined by N energies in $\mathbb{N}$. The configuration space is therefore given by $\Sigma_N = \mathbb{N}^N$. Elements of Σ_N are denoted by $\boldsymbol{\varepsilon} := (\varepsilon_1, .., \varepsilon_N)$ and we denote by $\Sigma_{N,E}$ the configuration space with total energy $E \in \mathbb{N}$, i.e.,

$$\Sigma_{N,E} := \{\boldsymbol{\varepsilon} \in \mathbb{N}^N : \sum_{i=1}^{N} \varepsilon_i = E\}.$$

The microscopic dynamics is defining by choosing at random a pair $\{i, j\}$ and redistributing uniformly the corresponding energies. Therefore we consider the Markov processes on Σ_N whose generator acts on bounded functions $f : \Sigma_N \to \mathbb{R}$ as

$$\mathcal{L}_N f = \frac{1}{N} \sum_{\{i,j\}} L_{ij} f,$$

where the sum is carried over the unordered pairs $\{i, j\} \subset \{1, .., N\}$, $i \neq j$, and

$$L_{ij} f(\boldsymbol{\varepsilon}) = \frac{1}{\varepsilon_i + \varepsilon_j + 1} \sum_{\ell=0}^{\varepsilon_i+\varepsilon_j} [f(T_{ij}^{\ell}\boldsymbol{\varepsilon}) - f(\boldsymbol{\varepsilon})], \tag{1.1}$$

in which

$$(T_{ij}^{\ell}\boldsymbol{\varepsilon})_k := \begin{cases} \ell & \text{if } k = i \\ \varepsilon_i + \varepsilon_j - \ell & \text{if } k = j \\ \varepsilon_k & \text{otherwise.} \end{cases}$$

Observe that, for each $E > 0$, the dynamics preserves $\Sigma_{N,E}$. Moreover, it is ergodic when restricted to $\Sigma_{N,E}$ and reversible with respect to the uniform measure on $\Sigma_{N,E}$.

We denote by $(\boldsymbol{\varepsilon}(t))_{t\geq 0}$ the continuous time Markov chain generated by $\mathcal{L}_N$. In particular, the path $\boldsymbol{\varepsilon}(\cdot)$ is piecewise constant, and the transition probability of its jumps is given in (1.1). Fix hereafter $T > 0$. Given a probability ν on $\Sigma_{N,E}$ we denote by $\mathbb{P}^N_\nu$ the law of this chain on the time interval $[0, T]$, when the initial datum is sampled according to ν. Observe that $\mathbb{P}^N_\nu$ is a probability on the Skorokhod space $D([0, T]; \Sigma_{N,E})$. As usual if $\nu = \delta_{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon} \in \Sigma_{N,E}$, the corresponding law is simply denoted by $\mathbb{P}^N_{\boldsymbol{\varepsilon}}$. We refer to [9] for a gentle introduction to continuous time Markov chains.

1.2. Empirical observables

Given $e \in (0, +\infty)$, we denote by $\mathcal{P}_e(\mathbb{N})$ the set of probability measures π on $\mathbb{N}$ with mean bounded by e, i.e., such that $\sum_\varepsilon \varepsilon\pi(\varepsilon) \leq e$. We consider $\mathcal{P}_e(\mathbb{N})$ as a closed subset of the space of probability measure on $\mathbb{N}$ equipped with the weak topology. Then $\mathcal{P}_e(\mathbb{N})$ endowed with the relative topology is a compact Polish space. Indeed, $\mathcal{P}_e(\mathbb{N})$ is the weak closure of the set of probabilities on $\mathbb{N}$ with mean e. The *empirical measure* records the energy of the particles forgetting their labels, for $E = \lfloor N e\rfloor$ it is defined as the map $\pi^N : \Sigma_{N,E} \to \mathcal{P}_e(\mathbb{N})$ given by

$$\pi^N(\boldsymbol{\varepsilon}) := \frac{1}{N}\sum_{i=1}^N \delta_{\varepsilon_i}. \tag{1.2}$$

Let $D([0, T]; \mathcal{P}_e(\mathbb{N}))$ the set of $\mathcal{P}_e(\mathbb{N})$-valued cádlág paths endowed with the Skorokhod topology and the corresponding Borel σ-algebra. With a slight abuse of notation we denote also by π^N the map from $D([0, T]; \Sigma_{N,E})$ to $D([0, T]; \mathcal{P}_e(\mathbb{N}))$ defined by $\pi^N_t(\boldsymbol{\varepsilon}) := \pi^N(\boldsymbol{\varepsilon}(t))$, $t \in [0, T]$.

We also introduce the *empirical flow* that records the collisions of the particles forgetting their labels. To this end, we denote by $\mathcal{M}$ the subset of the finite measures Q on $[0, T]\times\mathbb{N}^2\times\mathbb{N}^2$ that satisfy $Q(\mathrm{d}t; \varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = Q(\mathrm{d}t; \varepsilon_*, \varepsilon, \varepsilon', \varepsilon'_*) = Q(\mathrm{d}t; \varepsilon, \varepsilon_*, \varepsilon'_*, \varepsilon')$. We endow $\mathcal{M}$ with the weak* topology and the associated Borel σ-algebra. The empirical flow is the map $Q^N : D([0, T]; \Sigma_{N,E}) \to \mathcal{M}$ defined by

$$Q^N(\boldsymbol{\varepsilon})(F) := \frac{1}{N}\sum_{\{i,j\}}\sum_{k\geq 1} F(\tau^{i,j}_k; \varepsilon_i(\tau^{i,j}_k-), \varepsilon_j(\tau^{i,j}_k-), \varepsilon_i(\tau^{i,j}_k), \varepsilon_j(\tau^{i,j}_k)) \tag{1.3}$$

where $F : [0, T] \times \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{R}$ is continuous, bounded, and $F(t; \varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = F(t; \varepsilon_*, \varepsilon, \varepsilon', \varepsilon'_*) = F(t; \varepsilon, \varepsilon_*, \varepsilon'_*, \varepsilon')$, while $(\tau^{i,j}_k)_{k\geq 1}$ are the jump times of the pair $(\varepsilon_i, \varepsilon_j)$. Here, $\varepsilon_i(t-) = \lim_{s\uparrow t} \varepsilon_i(s)$. In view of the conservation of the energy, the measure $Q^N(\mathrm{d}t; \cdot)$ is supported on $\mathcal{E} := \{\varepsilon+\varepsilon_* = \varepsilon'+\varepsilon'_*\} \subset \mathbb{N}^2\times\mathbb{N}^2$.

For each $\boldsymbol{\varepsilon} \in \Sigma_{N,E}$, with $\mathbb{P}^N_{\boldsymbol{\varepsilon}}$ probability one, the pair (π^N, Q^N) satisfies the following balance equation that express the conservation of probability. For each $\phi : [0, T] \times \mathbb{N} \to \mathbb{R}$ bounded and continuously differentiable with respect to time

$$\begin{aligned} &\pi^N_T(\phi_T) - \pi^N_0(\phi_0) - \int_0^T \mathrm{d}t\, \pi^N_t(\partial_t\phi_t) \\ &\quad + \int_0^T \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} Q^N(\mathrm{d}t; \varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*)[\phi_t(\varepsilon) + \phi_t(\varepsilon_*) - \phi_t(\varepsilon') - \phi_t(\varepsilon'_*)] = 0. \end{aligned} \tag{1.4}$$

1.3. Law of large numbers

Fix $m \in \mathcal{P}(\mathbb{N})$ and assume one of the following condition: m is a point mass or the support of m does not generate a proper sub-lattice of $\mathbb{Z}$. Note that in the second case m satisfies the condition for the local central limit theorem for i.i.d. lattice random variables, see [13, §VII.1]. For $\gamma \in \mathbb{R}$ we set

$$Z_\gamma = Z_\gamma(m) := \sum_{\varepsilon} m(\varepsilon) e^{\gamma\varepsilon}. \tag{1.5}$$

We assume that there exists $\gamma^* \in (0, +\infty]$ such that $Z_\gamma < +\infty$ for $\gamma \in (0, \gamma^*)$ and $Z_\gamma \uparrow +\infty$ for $\gamma \uparrow \gamma^*$. For $e \in (0, +\infty)$, we then define the probability $\mu_{N,e}$ on $\Sigma_{N,\lfloor Ne \rfloor}$ by considering i.i.d. m-distributed energies and conditioning to the total energy, i.e.,

$$\mu_{N,e} := m^{\otimes N}\Big(\cdot \,\Big|\, \sum_{i=1}^{N} \varepsilon_i = \lfloor Ne \rfloor \Big), \tag{1.6}$$

that will be chosen as the initial distribution of the microscopic dynamics. In the case of point mass, we require that $\{e\}$ is exactly the support of m. Observe that, by the equivalence of the ensembles, as $N \to +\infty$ the one-marginal of $\mu_{N,e}$ converges to the probability m_e given by

$$m_e(\varepsilon) := \frac{e^{\gamma_e \varepsilon} m(\varepsilon)}{Z_{\gamma_e}}, \text{ where } \gamma_e < \gamma^* \text{ is such that } \sum_{\varepsilon} \varepsilon m_e(\varepsilon) = e. \tag{1.7}$$

Denoting by B the collision kernel in (1.1), i.e.,

$$B(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = \frac{1}{\varepsilon + \varepsilon_* + 1} \mathbb{1}_{\{\varepsilon + \varepsilon_* = \varepsilon' + \varepsilon'_*\}} \mathbb{1}_{\big\{\{\varepsilon, \varepsilon_*\} \neq \{\varepsilon', \varepsilon'_*\}\big\}}, \tag{1.8}$$

the law of the large numbers for the empirical measure is described by the following discrete homogeneous Boltzmann equation

$$\partial_t f_t(\varepsilon) = \sum_{\varepsilon_*, \varepsilon', \varepsilon'_*} B(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \big[f_t(\varepsilon') f_t(\varepsilon'_*) - f_t(\varepsilon) f_t(\varepsilon_*) \big]. \tag{1.9}$$

More precisely, in probability with respect to $\mathbb{P}^N_{\mu_{N,e}}$, the empirical path $(\pi^N_t)_{t \in [0,T]}$ converges to $(f_t)_{t \in [0,T]}$ where f_t is the unique solution of the Cauchy problem associated to (1.9) with initial datum $f_0 = m_e$. As the proof of this statement can be achieved by adapting the chaos propagation arguments in [16], we omit the details. The law of the large numbers of the empirical flow Q^N reads

$$Q^N(\mathrm{d}t; \varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \longrightarrow \frac{1}{2} \mathrm{d}t\, f_t(\varepsilon) f_t(\varepsilon_*) B(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*), \tag{1.10}$$

where the convergence is in probability with respect to $\mathbb{P}^N_{\mu_{N,e}}$. We refer to Lemma 3.3 below for the proof.

In the general contest of homogeneous Boltzmann equations, uniqueness of the Cauchy problem associated to (1.9) holds for paths f_t that conserves the energy, see e.g., [12]. However in the present case, since the $\sup_{\varepsilon, \varepsilon_*} \sum_{\varepsilon', \varepsilon'_*} B(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*)$ is bounded, Gronwall's inequality implies the uniqueness without assuming the energy conservation, see e.g., Lemma 4.1 in [2]. In particular, by uniqueness, for this model Lu and Wennberg like solutions do not exist. We finally observe that (1.9) admits a one-parameter family of stationary solutions given by $f_{\mathrm{stat}}(\varepsilon) = p(1-p)^{\varepsilon}$, $p \in (0, 1]$.

1.4. The candidate rate function

For $e \in (0,+\infty)$, let $\mathcal{S}_e$ be the (closed) subset of $D([0,T];\mathcal{P}_e(\mathbb{N}))\times\mathcal{M}$ given by elements (π,Q) that satisfies the balance equation

$$\begin{aligned}\pi_T(\phi_T)-\pi_0(\phi_0)-\int_0^T \mathrm{d}t\,\pi_t(\partial_t\phi_t)\\ +\int_0^T \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} Q(\mathrm{d}t;\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)[\phi_t(\varepsilon)+\phi_t(\varepsilon_*)-\phi_t(\varepsilon')-\phi_t(\varepsilon'_*)]=0\end{aligned} \tag{1.11}$$

for each $\phi:[0,T]\times\mathbb{N}\to\mathbb{R}$ bounded and continuously differentiable in t. We consider $\mathcal{S}_e$ endowed with the relative topology and the corresponding Borel σ-algebra.

For $\pi\in D([0,T];\mathcal{P}_e(\mathbb{N}))$ let Q^π be the measure defined by

$$Q^\pi(\mathrm{d}t;\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*) := \frac{1}{2}\,\mathrm{d}t\,\pi_t(\varepsilon)\pi_t(\varepsilon_*)B(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*), \tag{1.12}$$

where B is the collision kernel in (1.8). Observe that $Q^\pi(\mathrm{d}t,\cdot)$ is supported on $\mathcal{E}$. Let $\mathcal{S}_e^{\mathrm{ac}}$ be the subset of $\mathcal{S}_e$ given by the elements (π,Q) such that $\pi\in C([0,T];\mathcal{P}_e(\mathbb{N}))$ and $Q\ll Q^\pi$. The dynamical rate function $J\colon \mathcal{S}_e\to[0,+\infty]$ is defined by

$$J(\pi,Q) := \begin{cases}\displaystyle\int_0^T \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \mathrm{d}Q^\pi\Big[\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}\log\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}-\frac{\mathrm{d}Q}{\mathrm{d}Q^\pi}+1\Big] & \text{if } (\pi,Q)\in\mathcal{S}_e^{\mathrm{ac}}\\ +\infty & \text{otherwise}\end{cases} \tag{1.13}$$

Given two probabilities μ_1,μ_2, the relative entropy $\mathrm{Ent}(\mu_2|\mu_1)$ is defined as $\mathrm{Ent}(\mu_2|\mu_1) = \int d\mu_1\rho\log\rho$, where $d\mu_2=\rho\,d\mu_1$, understanding that $\mathrm{Ent}(\mu_2|\mu_1)=+\infty$ if μ_2 is not absolutely continuous with respect to μ_1. Let $H_e\colon\mathcal{P}_e(\mathbb{N})\to[0,+\infty]$ be defined by

$$H_e(\pi) = \mathrm{Ent}(\pi|m_e)+(\gamma^*-\gamma_e)\Big[e-\sum_\varepsilon \varepsilon\,\pi(\varepsilon)\Big], \tag{1.14}$$

where m_e and γ_e are as in (1.7). When m is the point mass on e, $H_e(\pi)$ is zero when $\pi=\delta_e$ and $+\infty$ otherwise. The candidate large deviation rate function is given by

$$I(\pi,Q) := H_e(\pi_0)+J(\pi,Q). \tag{1.15}$$

As discussed in [1], the (static) large deviations of the empirical measure with respect to the probability $\mu_{N,e}$ are described by the rate function H_e, where the second term on the r.h.s. of (1.14) is the cost of having energy less than e. Note that if $\gamma^*=+\infty$, then $H_e(\pi)$ is finite only if the energy of π is e. A key ingredient in the proof is the local central limit theorem for the sum of independent m_e distributed random variables.

Denote by $\hat{\mathcal{S}}_e$ the subset of $\mathcal{S}$ given by the pair (π,Q) such that

$$\int_0^T \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} Q(\mathrm{d}t,\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)(\varepsilon+\varepsilon_*)<+\infty.$$

If $(\pi, Q) \in \hat{\mathcal{S}}_e$, the balance equation (1.11) implies that the path π_t conserves the energy.

As already mentioned, a proof of a large deviations principle for the pair (π^N, Q^N) with matching upper and lower has not been yet achieved. The analysis in [2, 6] implies however the large deviation upper bound with rate I with a matching lower bound on the set $\hat{\mathcal{S}}_e$. The precise statement is the following.

Theorem 1.1. *Fix $e \in (0, +\infty)$ and let $\mu_{N,e}$ be the family of probabilities on $\Sigma_{N,\lfloor Ne \rfloor}$ defined in* (1.6). *The family $\{\mathbb{P}^N_{\mu_{N,e}} \circ (\pi^N, Q^N)^{-1}\}$ satisfies a large deviations upper bound with good rate function $I \colon \mathcal{S}_e \to [0, +\infty]$, namely I has compact level sets and for each closed $C \subset \mathcal{S}$*

$$\varlimsup_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu_{N,e}}\Big((\pi^N, Q^N) \in C\Big) \le - \inf_{(\pi,Q)\in C} I(\pi, Q). \tag{1.16}$$

Moreover, for each open $O \subset \mathcal{S}_e$

$$\varliminf_{N\to+\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu_{N,e}}\Big((\pi^N, Q^N) \in O\Big) \ge - \inf_{(\pi,Q)\in O\cap\hat{\mathcal{S}}_e} I(\pi, Q). \tag{1.17}$$

Referring to [2, 6] for comments on the technicalities involved in the lower bound, we now turn to the novel point of the present analysis, that is the construction of paths (π, Q) – with π not energy conserving – whose probability is precisely of the order $\exp\{-NI(\pi, Q)\}$. Since these paths do not belong to $\hat{\mathcal{S}}_e$, this result provides insights on the large deviations properties of Kac's walk not covered by (1.17). As the large deviations upper bound is already covered by (1.16), we focus on the matching lower bound.

Theorem 1.2. *Fix $e \in (0, +\infty)$ and let $\mu_{N,e}$ be the family of probabilities on $\Sigma_{N,\lfloor Ne \rfloor}$ defined in* (1.6). *For each $t^* \in (0, T)$ there exists a path $(\bar{f}, \bar{Q})$, satisfying $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon = e$ for $t \in [0, t^*)$ and $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon < e$ for $t \in [t^*, T]$, such that $I(\bar{f}, \bar{Q}) < +\infty$ and*

$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}^N_{\mu_{N,e}}((\pi^N, Q^N) \in O) \ge -I(\bar{f}, \bar{Q}), \tag{1.18}$$

for any open neighborhood $O \ni (\bar{f}, \bar{Q})$.

We will provide a self-contained proof of this statement that do not rely on Theorem 1.1. In the argument we take advantage of the fact that the energies are in $\mathbb{N}$. However we expect that the strategy can be extended also to the continuous case. In Section 2 we construct a path $(\bar{f}, \bar{Q})$ satisfying the above requirements, i.e., with evaporating energy for $t > t^*$ and such that $I(\bar{f}, \bar{Q}) < +\infty$. The lower bound (1.18) is then proven in Section 3. For the sake of concreteness, the proposed path $(\bar{f}, \bar{Q})$ has zero energy for $t \in (t_*, T]$.

2. Perturbed Boltzmann equation

Fix $t^* \in (0, T)$. In order to construct a path $(\bar{f}, \bar{Q})$, satisfying $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon = e$ for $t \in [0, t^*)$ and $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon < e$ for $t \in (t^*, T]$, we start by considering a solution to a perturbed Boltzmann equation, namely a Boltzmann equation with a suitable modified collision kernel.

Consider the collision kernel $\tilde{B}$ given by

$$\tilde{B}(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)=\frac{1}{2}\delta_{\varepsilon,\varepsilon_*}\delta_{\varepsilon+\varepsilon_*,\varepsilon'+\varepsilon'_*}[\delta_{\varepsilon',\varepsilon+\varepsilon_*}+\delta_{\varepsilon'_*,\varepsilon+\varepsilon_*}]\mathbb{1}_{\{\{\varepsilon,\varepsilon_*\}\neq\{\varepsilon',\varepsilon'_*\}\}}, \tag{2.1}$$

that describes a scenario in which only particles with the same energy collide, and in each collision the whole energy is transferred to a single particle. The Cauchy problem for the corresponding modified homogeneous Boltzmann equation reads

$$\begin{cases}\partial_t f_t(\varepsilon)=\sum_{\varepsilon_*,\varepsilon',\varepsilon'_*}[\tilde{B}(\varepsilon',\varepsilon'_*,\varepsilon,\varepsilon_*)f_t(\varepsilon')f_t(\varepsilon'_*)-\tilde{B}(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)f_t(\varepsilon)f_t(\varepsilon_*)],\\ f_0(\varepsilon)=m(\varepsilon).\end{cases} \tag{2.2}$$

Proposition 2.1. *Assume that m has energy e, and let f be the unique solution to* (2.2)*. Then its energy is conserved, i.e., for any $t\in[0,+\infty)$ $\sum_{\varepsilon\geq 0}f_t(\varepsilon)\varepsilon=e$, while f_t weakly converges to δ_0, as $t\to+\infty$. Moreover, for every $t\geq 0$,*

$$\begin{aligned} &i)\ \ f_t(\varepsilon)\leq\frac{2}{1+t},\quad \textit{for } \varepsilon\geq 1,\\ &ii)\ \ \sum_{\varepsilon\geq 1}f_t(\varepsilon)\leq\frac{c}{\sqrt{1+t}},\\ &iii)\ \ \sum_{\varepsilon\geq 1}f_t(\varepsilon)\log\varepsilon\leq c\frac{1}{\sqrt{1+t}}(1+\log(1+t))\end{aligned} \tag{2.3}$$

where $c=c(e)$ does not depend on t and the initial datum m.

Proof. We prove Eq (2.3), from which the convergence of f_t follows. The modified Boltzmann equation reads as

$$\begin{aligned} &\dot{f}_t(0)=\frac{1}{2}\sum_{\varepsilon\geq 1}f_t(\varepsilon)^2,\\ &\dot{f}_t(\varepsilon)=-f_t(\varepsilon)^2 && \text{for } \varepsilon\geq 1 \text{ odd},\\ &\dot{f}_t(\varepsilon)=\frac{1}{2}f_t(\varepsilon/2)^2-f_t(\varepsilon)^2 && \text{for } \varepsilon\geq 2 \text{ even}.\end{aligned} \tag{2.4}$$

Note that the equation for $f_t(\varepsilon)$ involves only $f_t(\varepsilon')$ with $\varepsilon'\leq\varepsilon$, then the system has global and unique solution. If ε is odd,

$$f_t(\varepsilon)=\frac{f_0(\varepsilon)}{1+tf_0(\varepsilon)}\leq\frac{1}{1+t}.$$

If ε is even, set $\xi_t(\varepsilon)=(1+t)f_t(\varepsilon)$. Let T_0 be the first time t such that $\xi_t(\varepsilon')=2$ for some $\varepsilon'\leq\varepsilon$. The time T_0 is strictly positive, since $\xi_0(\varepsilon')\leq 1$ for any ε'. For $t<T_0$ it holds

$$\dot{\xi}_t(\varepsilon)=\frac{1}{1+t}\left(\xi_t(\varepsilon)-\xi_t^2(\varepsilon)+\frac{1}{2}\xi_t^2(\varepsilon/2)\right)<\xi_t(\varepsilon)-\xi_t^2(\varepsilon)+2\leq 3(2-\xi_t(\varepsilon)),$$

then $T_0=+\infty$, and this concludes the proof of $i)$ in (2.3).

In order to prove $ii)$ and $iii)$ we first note that

$$\sum_0^{2^n}\varepsilon f_t(\varepsilon)=\sum_0^{2^n}\varepsilon f_0(\varepsilon)-\int_0^t \mathrm{d}s\sum_{2^{n-1}+1}^{2^n}\varepsilon f_s(\varepsilon)^2, \tag{2.5}$$

and then $\sum_0^{+\infty} \varepsilon f_t(\varepsilon) \le e$. Inequality *ii*) and *iii*) follows by using this fact and the Chebyshev's inequality. We conclude the proof by noticing that the energy is in fact conserved, since $f_t(\varepsilon) \le e/\varepsilon$ and then for any $h \ge 1$,

$$\sum_h^{+\infty} \varepsilon f_t(\varepsilon)^2 \le e \max_{\varepsilon\ge h} f_t(\varepsilon) \le \frac{e^2}{h},$$

which assures that the right-hand-side of Eq (2.5), is vanishing as $n \to +\infty$. □

For the modified Boltzmann equation with rate in (2.1) the energy vanishes for dispersion to infinity as $t \to +\infty$. We reparametrize the time so that this happens at a finite time. Fixed $t^* \in (0, T)$, let $\alpha\colon [0, t^*) \to [0, +\infty)$ given by $\alpha(t) = \frac{t}{1-t/t^*}$. Letting f the solution to (2.2), set

$$\bar{f}_t(\varepsilon) = \begin{cases} f_{\alpha(t)}(\varepsilon) & t \in [0, t^*) \\ \delta_{\varepsilon,0} & t \in [t^*, T], \end{cases} \tag{2.6}$$

which satisfies the homogeneous Boltzmann equation with time dependent collision kernel

$$\bar{B}_t(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = \begin{cases} \dot{\alpha}(t)\tilde{B}(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) & t \in [0, t^*) \\ 0 & t \in [t^*, T]. \end{cases} \tag{2.7}$$

We define the corresponding flux $d\bar{Q} = \mathrm{d}t\, \bar{q}_t$, where

$$\bar{q}_t(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = \frac{1}{2}\bar{f}_t(\varepsilon)\bar{f}_t(\varepsilon_*)\bar{B}_t(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*), \tag{2.8}$$

so that the pair $(\bar{f}, \bar{Q})$ satisfies the balance equation (1.11). Observe that, by construction, $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon = e$ for $t \in [0, t^*)$ and $\sum_\varepsilon \bar{f}_t(\varepsilon)\varepsilon = 0$ for $t \in [t^*, T]$. We now show that the pair $(\bar{f}, \bar{Q})$ is such that $I(\bar{f}, \bar{Q}) < +\infty$. Since $\bar{f}_0 = m$, it is enough to show $J(\bar{f}, \bar{Q}) < +\infty$. This is stated in the next Proposition.

Proposition 2.2. *For $(\bar{f}, \bar{Q})$ defined above the dynamical rate function $J(\bar{f}, \bar{Q})$ is finite.*

Proof. Since $(\bar{f}, \bar{Q}) \in \mathcal{S}_e^{\mathrm{ac}}$, the dynamical rate function defined in Eq (1.13) is given by

$$\frac{1}{2}\int_0^{t^*} \mathrm{d}t \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \bar{f}_t(\varepsilon)\bar{f}_t(\varepsilon_*)\bar{B}_t\left(\log\frac{\bar{B}_t}{B} - 1\right) + \frac{1}{2}\int_0^T \bar{f}_t(\varepsilon)\bar{f}_t(\varepsilon_*)B. \tag{2.9}$$

For $t < t^*$ we have that

$$\bar{B}_t\left(\log\frac{\bar{B}_t}{B} - 1\right) = \dot{\alpha}\tilde{B}\left(\log\dot{\alpha} + \log\frac{1+2\varepsilon}{2} - 1\right).$$

Since $\log\dot{\alpha} = 2\log(1 + \alpha/t^*)$, the first integral is

$$\frac{1}{2}\int_0^{+\infty} d\alpha \sum_{\varepsilon\ge1} f_\alpha^2(\varepsilon)\left(2\log\left(1 + \frac{\alpha}{t^*}\right) + \log\frac{1+2\varepsilon}{2} - 1\right).$$

Using (2.3) we bound this term by

$$c\int_0^{+\infty} \frac{1}{(1+\alpha)^{3/2}}(1 + \log(1+\alpha)) < +\infty,$$

where c depend only on e and t^*.

The second integral in Eq (2.9) is

$$\frac{1}{2}\int_0^{t^*} \mathrm{d}t \left(\sum_{\varepsilon \geq 1} f_{\alpha(t)} \right)^2 + \frac{1}{2}(T - t^*) \leq \frac{T}{2},$$

which completes the proof. □

3. Large deviations lower bound

In order to explain the strategy to prove (1.18), we recall some basic facts on the large deviations lower bound. Let $\{P_n\}$ be a sequence of probabilities on a topological space $\mathcal{X}$. Fix $x \in \mathcal{X}$ and a open neighborhood $\mathcal{O} \ni x$. To obtain a lower bound for $P_n(\mathcal{O})$ we modify the probability P_n so that x becomes the typical behavior. If we are able to do so by paying – as measured by the relative entropy with respect to P_n – not too much then we obtain a good lower bound. The precise statement is summarized in the next lemma, see e.g., [11] for its proof.

Lemma 3.1. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of probabilities on a completely regular topological space $\mathcal{X}$ and fix $x \in \mathcal{X}$. Assume that there exists a sequence $\{P_n^x\}$ weakly convergent to δ_x and such that*

$$\varlimsup_{n\to\infty} \frac{1}{n} \operatorname{Ent}\left(P_n^x \middle| P_n\right) \leq I(x) \tag{3.1}$$

for some $I\colon \mathcal{X} \to [0, +\infty]$. Then for any open neighborhood $\mathcal{O} \ni x$

$$\varliminf_{n\to\infty} \frac{1}{n} \log P_n(\mathcal{O}) \geq -I(x).$$

In most of the applications, and indeed also in our case, the strategy suggested by the above lemma is implemented together with a density argument. The family of perturbed probabilities P_n^x is not constructed for the point x itself but rather for an approximation x_k; if the function I is continuous along the sequence x_k then this will do as well. We emphasize that in typical infinite dimensional applications – as in the present case – the rate function I is only lower semicontinuous so the sequence x_k has to be properly chosen. We summarize the argument in the next statement which is deduced from Lemma 3.1 by a straightforward diagonal argument.

Lemma 3.2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of probabilities on a completely regular topological space $\mathcal{X}$, fix $x \in \mathcal{X}$, and a sequence $x_k \to x$. Assume that there exists $I\colon \mathcal{X} \to [0, +\infty]$ meeting the following conditions:*

(i) for each $k \in \mathbb{N}$ there exists a family $\{P_n^{x_k}\}_{n\in\mathbb{N}}$ satisfying the conditions in Lemma 3.1;
(ii) $\varlimsup_k I(x_k) \leq I(x)$.

Then, for any open neighborhoods $\mathcal{O} \ni x$

$$\varliminf_{n\to\infty} \frac{1}{n} \log P_n(\mathcal{O}) \geq -I(x).$$

To implement condition *(i)* and *(ii)* in the previous lemma, for $0 < \delta < t_*$, define the pair $(\bar f^\delta, \bar Q^\delta)$ by

$$\bar f_t^\delta(\varepsilon) = \begin{cases} \bar f_t(\varepsilon) & t \in [0, t_* - \delta) \\ \bar f_{t_*-\delta}(\varepsilon) & t \in [t^* - \delta, T], \end{cases} \tag{3.2}$$

and $d\bar Q^\delta = \mathrm{d}t\, \bar q_t^\delta$ with $\bar q_t^\delta = \bar q_t \mathbb{1}_{[0,t_*-\delta)}(t)$. Let $\mu_{N,e}$ as in the statement of Theorem 1.2, and denote by $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$ the law of the microscopic dynamics with the perturbed collision kernel $\bar B^\delta = \bar B \mathbb{1}_{[0,t_*-\delta)}(t)$, $\bar B$ in (2.7). Then the following two Lemmata imply the large deviation lower bound (1.18).

Lemma 3.3. *For each $\delta \in (0, t_*)$ as $N \to +\infty$ the pair (π^N, Q^N) converges in $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$ probability to $(\bar f^\delta, \bar Q^\delta)$. Furthermore,*

$$\lim_{N\to+\infty} \frac{1}{N} \mathrm{Ent}(\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}} | \mathbb{P}^N_{\mu_{N,e}}) = I(\bar f^\delta, \bar Q^\delta). \tag{3.3}$$

Lemma 3.4. *As $\delta \downarrow 0$ we have $(\bar f^\delta, \bar Q^\delta) \to (\bar f, \bar Q)$ and $I(\bar f^\delta, \bar Q^\delta) \to I(\bar f, \bar Q)$.*

Proof of Lemma 3.3. By definition of $\bar B^\delta$, $\sup_t \sup_{\varepsilon,\varepsilon'} \sum_{\varepsilon',\varepsilon'_*} \bar B_t^\delta(\varepsilon, \varepsilon', \varepsilon'_*) \le c_\delta$. Therefore, by classical chaos propagation argument, π^N converges in $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$ probability to $\bar f^\delta$. To deduce the convergence of the empirical flow, it is enough to observe that for each bounded $F_t(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*)$

$$\begin{aligned} M_t^F := &\int_0^t \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} Q^N(\mathrm{d}s, \varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) F_s(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \\ &- \frac{1}{2} \int_0^t \mathrm{d}s \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \pi_s^N(\varepsilon) \pi_s^N(\varepsilon_*) \bar B_s^\delta(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) F_s(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \\ &+ \frac{1}{2} \frac{1}{N} \int_0^t \mathrm{d}s \sum_{\varepsilon,\varepsilon',\varepsilon'_*} \pi_s^N(\varepsilon) \bar B_s^\delta(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*) F_s(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*) \end{aligned} \tag{3.4}$$

is a $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$ martingale with predictable quadratic variation

$$\begin{aligned} \langle M^F \rangle_t = &\frac{1}{2} \frac{1}{N} \int_0^t \mathrm{d}s \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \pi_s^N(\varepsilon) \pi_s^N(\varepsilon_*) \bar B_s^\delta(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) F_s^2(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) \\ &- \frac{1}{2} \frac{1}{N^2} \int_0^t \mathrm{d}s \sum_{\varepsilon,\varepsilon',\varepsilon'_*} \pi_s^N(\varepsilon) \bar B_s^\delta(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*) F_s^2(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*). \end{aligned}$$

Set $F_t^\delta(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) = \log(\bar B_t^\delta / B)$. By standard Markov chain computation, the relative entropy of $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$ with respect to $\mathbb{P}^N_{\mu_{N,e}}$ is given by

$$\begin{aligned} &\frac{1}{N} \mathrm{Ent}(\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}} | \mathbb{P}^N_{\mu_{N,e}}) \\ &= \bar{\mathbb{E}}^{N,\delta}_{\mu_{N,e}} \Big(Q^N(F^\delta) - \frac{1}{2} \int_0^T \mathrm{d}t \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \pi_t^N(\varepsilon) \pi_t^N(\varepsilon_*) [\bar B_t^\delta(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*) - B(\varepsilon, \varepsilon_*, \varepsilon', \varepsilon'_*)] \\ &+ \frac{1}{N} \frac{1}{2} \int_0^T \mathrm{d}t \sum_{\varepsilon,\varepsilon',\varepsilon'_*} \pi_t^N(\varepsilon) [\bar B_t^\delta(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*) - B(\varepsilon, \varepsilon, \varepsilon', \varepsilon'_*)] \Big). \end{aligned} \tag{3.5}$$

Since $\sup_t \sup_{\varepsilon,\varepsilon'} \sum_{\varepsilon',\varepsilon'_*} \bar{B}^\delta_t(\varepsilon,\varepsilon',\varepsilon'_*) \le c_\delta$, by the law of large numbers

$$\begin{aligned}&\lim_{N\to\infty} \bar{\mathbb{E}}^{N,\delta}_{\mu_{N,e}}\Big(\frac{1}{2}\int_0^T \mathrm{d}t \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \pi^N_t(\varepsilon)\pi^N_t(\varepsilon_*)[\bar{B}^\delta_t(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*) - B(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)]\Big)\\ &= \frac{1}{2}\int_0^T \mathrm{d}t \sum_{\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*} \bar{f}^\delta_t(\varepsilon)\bar{f}^\delta_t(\varepsilon_*)[\bar{B}^\delta_t(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*) - B(\varepsilon,\varepsilon_*,\varepsilon',\varepsilon'_*)],\end{aligned}$$

while the last term on the right hand side of (3.5) vanishes as N diverges. Again, by the law of large numbers, in order to prove $\bar{\mathbb{E}}^{N,\delta}_{\mu_{N,e}}\big(Q^N(F^\delta)\big) \to \bar{Q}^\delta(F^\delta)$ it is enough to show the uniform integrability of $Q^N(F^\delta)$ with respect to $\bar{\mathbb{P}}^{N,\delta}_{\mu_{N,e}}$. By exploiting the martingale decomposition (3.4), since $|F^\delta| \le c_\delta(1 + \log(1+\varepsilon+\varepsilon_*))$, a direct computation yields $\bar{\mathbb{E}}^{N,\delta}_{\mu_{N,e}}\big(Q^N(F^\delta)^2\big) \le c_\delta$, which implies the requested uniform integrability. Observing that $H_e(\bar{f}^\delta_0) = 0$, and recalling (1.13) and (1.15), the proof is concluded. □

Proof of Lemma 3.4. The convergence of $(\bar{f}^\delta, \bar{Q}^\delta)$ to $(\bar{f}, \bar{Q})$ follows from (3.2), the continuity of $t \mapsto \bar{f}_t$, and the integrability of $\bar{q}_t$. The convergence of the rate function is achieved by the arguments in the proof of Proposition 2.2 and dominated convergence.

□

Conflict of interest

The authors declare no conflict of interest.

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