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Stochastic Allen–Cahn equation with mobility

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Abstract. We introduce a class of stochastic Allen–Cahn equations with a mobility coefficient and colored noise. For initial data with finite free energy, we analyze the corresponding Cauchy problem on the d-dimensional torus in the time interval $[0, T]$. Assuming that $d \leq 3$ and that the potential has quartic growth, we prove existence and uniqueness of the solution as a process u in L^2 with continuous paths, satisfying almost surely the regularity properties $u \in C([0, T]; H^1)$ and $u \in L^2([0, T]; H^2)$.

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1. Introduction

The analysis of stochastic perturbations of the Allen–Cahn equation, due to their relevance both from a theoretical and applied viewpoint, has been a main topic in the development of the theory of stochastic partial differential equations. We consider the case in which the space variable belongs to the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. The typical setting is the following. Fix a smooth double well potential $W : \mathbb{R} \to \mathbb{R}$ and a filtered probability space equipped with a cylindrical Wiener process α. A class of stochastic perturbations of the Allen–Cahn equation is then given by

$$\mathrm{d}u_t = \big(\Delta u_t - W'(u_t)\big)\mathrm{d}t + \sqrt{2} j * \mathrm{d}\alpha_t. \tag{1.1}$$

Here, the unknown $u = u_t(x)$, $(t, x) \in [0, T] \times \mathbb{T}^d$, $T > 0$, is real-valued and it represents the local order parameter, Δ is the Laplacian, $j = j(x) : \mathbb{T}^d \to \mathbb{R}$, and $*$ denotes convolution in the space variable. The case of perturbation by space-time white noise is formally recovered when j is the Dirac's delta function.

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The so-called semigroup approach [9] to the analysis of the stochastic Allen–Cahn consists in writing the Cauchy problem with initial datum $\bar{u}_0$ associated to (1.1) in the mild form, i.e.,

$$u_t = e^{t\Delta}\bar{u}_0 - \int_0^t e^{(t-s)\Delta} W'(u_s)\,\mathrm{d}s + \sqrt{2}\int_0^t e^{(t-s)\Delta} j * \mathrm{d}\alpha_s, \tag{1.2}$$

where $e^{t\Delta}$ denotes the heat semigroup. In the one-dimensional case with $j = \delta$ or in $d > 1$ with j smooth enough, the last term on the right-hand side of (1.2) is, with probability one, a process in $C(\mathbb{T}^d)$ with continuos paths. By a fixed point argument in $C([0, T]; C(\mathbb{T}^d))$, it is then possible to prove existence and uniqueness to (1.2), for almost all realizations of the noise, see, e.g., [6], where a more general setting is considered. When W' is Lipschitz, this approach applies also to the case in which the state space is $L^2(\mathbb{T}^d)$ instead of $C(\mathbb{T}^d)$, and the same holds even when W' has polynomial growth, relying on the one-side Liptschitz property of W' [7,9].

Considering still the case with W' Lipschitz and with the same restrictions on j, the stochastic Allen–Cahn Eq. (1.1) can be also analyzed using the so-called variational approach [23,30]. This approach relies on the embeddings $H^1(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset H^{-1}(\mathbb{T}^d)$, the Cauchy problem associated to (1.1) is then understood as the following equality in $H^{-1}(\mathbb{T}^d)$,

$$u_t = \bar{u}_0 + \int_0^t \big[\Delta u_s - W'(u_s)\big]\,\mathrm{d}s + \sqrt{2} j * \alpha_t. \tag{1.3}$$

The main step for existence is an Itô's formula for the map $u \mapsto \|u\|^2_{L^2(\mathbb{T}^d)}$, which yields the a priori bounds needed to construct the solution u, by compactness arguments, as a process in $L^2(\mathbb{T}^d)$ with continuous paths and $u \in L^2([0, T]; H^1(\mathbb{T}^d))$ with probability one. More recently, in [25] the variational approach has been extended to the case of W' with some polynomial growth again in view of the one-side Liptschitz property.

Approximation of the Allen–Cahn Eq. (1.1) by time discretization has been considered in [22], in terms of the backward Euler scheme; indeed, as the time step goes to zero, this method recovers the unique solution discussed above. Similar time and space discretization of (1.1) were previously investigated first in, e.g., [15] under Lipschitz assumption on the nonlinear term and extended in [18] when W' has polynomial growth.

In the case of perturbation by space-time white noise, $j = \delta$, and $d > 1$, the last term in the right-hand side of (1.2) is, with probability one, only a distribution and the well-posedness of the stochastic perturbation of the Allen–Cahn equation becomes a major issue. In particular, to make sense of the equation a proper renormalization of the non linear term W' is needed. In dimension $d = 2$, when W is a polynomial, this renormalization amounts to the Wick ordering [2,8,19]. In dimension $d = 3$, the renormalization of the non linearity is more involved; for a quartic potential W, a local existence and uniqueness result is proven in [14], and it has been extended in [28] to arbitrary time intervals.

Regarding the choice of the random forcing term in (1.1), we would like to make the following model remark. The choice of the space-time white noise has the doubtless appeal of simplicity and universality, and it is really mandatory when (1.1) is used in the framework of stochastic quantization or to model dynamical critical fluctuations [16]. In the latter case, the potential W is not arbitrary but the quartic potential. Indeed, as shown in [4,12] for $d = 1$ and in [27] for $d = 2$, with these choices (1.1) describes the asymptotic of the fluctuations at the critical point for a Glauber dynamics with local mean field interaction. On the other hand, if we regard (1.1) as a phenomenological model for phase segregation and interface dynamics, the choice of a noise with nonzero spatial correlation length, i.e., a smooth j, is not unsound since we are going to look at the order parameter on larger space scales. Analogously, any reasonable double well potential W will yield essentially the same limiting behavior.

The deterministic Allen–Cahn equation, i.e., (1.1) with $j = 0$, can be viewed as the L^2-gradient flow of the van der Waals free energy functional,

$$\mathcal{F}(u) := \int \Big[\frac{1}{2}|\nabla u|^2 + W(u)\Big]\,\mathrm{d}x. \tag{1.4}$$

Correspondingly, in the case when j is the Dirac's delta function, the process u is (informally) reversible with respect to the (informal) probability measure $P(\mathcal{D}u) \propto \exp\big\{-\mathcal{F}(u)\big\}\mathcal{D}u$.

With respect to the setting described above, in this paper we analyze a stochastic Allen–Cahn equation in which we introduce a mobility coefficient, that is,

$$\mathrm{d}u_t = \sigma(u_t)\big(\Delta u_t - W'(u_t)\big)\mathrm{d}t + \sqrt{2\sigma(u_t)}\, j * \mathrm{d}\alpha_t, \tag{1.5}$$

where the *mobility* $\sigma\colon \mathbb{R} \to \mathbb{R}_+$ is smooth, bounded, and uniformly strictly positive. Moreover, W is convex at infinity with at most quartic growth. In terms of gradient flows, (1.5) with $j = 0$ is the gradient flow of $\mathcal{F}$ in $L^2(\sigma(u)^{-1}\mathrm{d}x)$. Finally, the choice of the random forcing term in (1.5) is suggested by the case of constant mobility. Indeed, when σ is constant and j is the Dirac's delta function, the process u is still (informally) reversible with respect to the (informal) probability $P(\mathcal{D}u) \propto \exp\big\{-\mathcal{F}(u)\big\}\mathcal{D}u$ regardless of the specific value of σ. In the physical literature, see e.g., [16, Sect. IV.A.1] or [33, Sect. II.7.3], this choice is usually referred to as the Onsager's prescription.

A motivation for the introduction of the mobility in the Allen–Cahn equation relies in the analysis of the corresponding sharp interface limits. For instance, as well known, for suitably prepared initial data, in this singular limit the deterministic Allen–Cahn equation (with constant mobility) converges to the motion by mean curvature, see e.g. [11,17]. As discussed in [34, Sect. 4], this approximation to motion by mean curvature has the peculiar feature of exhibiting a trivial transport coefficient in the limiting evolution. On the other hand, when a non-constant mobility coefficient is introduced as in (1.5), we expect that the limiting interface evolution is described by motion by mean curvature with a non-trivial transport coefficient satisfying the corresponding Einstein's relation [34, Sect. 3] (see [10,20] for the case of a non-local equation). As far as the stochastic Allen–Cahn equation is considered, a relevant issue is the large deviation asymptotics in such sharp interface limit. In the case of constant mobility, this analysis is carried out in [3], see also the related discussion in [21].

To our knowledge, the stochastic Allen–Cahn equation with mobility has not been discussed in the literature. In this paper, we consider the Cauchy problem associated to (1.5) with initial datum $\bar{u}_0 \in H^1(\mathbb{T}^d)$ when $d \le 3$, the potential W is convex at infinity with at most quartic growth, and j belongs to the Sobolev space $H^1(\mathbb{T}^d)$. We prove the existence and uniqueness of the solution as a process u in $L^2(\mathbb{T}^d)$ with continuous paths satisfying $u \in C([0,T]; H^1(\mathbb{T}^d)) \cap L^2([0,T]; H^2(\mathbb{T}^d))$ almost surely and such that the corresponding norms are random variables whose moments are all finite.

The semigroup approach does not seem to be applicable to equation (1.5), first because it cannot be recasted in a mild form in terms of a linear semigroup (the diffusion term is now nonlinear), but also because the reaction term $-\sigma W'$ no longer satisfies the one-side Lipschitz property. On the other hand, our result seems difficult to obtain by the variational approach discussed above even in the case of constant mobility, see the discussion at the end of Sect. 2.

The restriction $d \le 3$ is connected to the quartic growth of the potential, allowing to control some non-linear terms via Sobolev embeddings. The choice of periodic boundary conditions does simplify computations, but the arguments here presented are robust enough to be adapted to the case of a bounded domain with either Dirichlet or Neumann boundary conditions.

From a technical viewpoint, existence of solutions to (1.5) will be proven by a compactness argument on suitable approximate solutions in the same spirit of the variational approach. More precisely, the approximate solutions are constructed by time discretization of the mobility coefficient and regularizing the nonlinear term. The necessary a-priori bounds are obtained, taking full advantage of the variational structure of the equation, by deriving an Itô's formula for suitable approximations of the map $u \mapsto \mathcal{F}(u)$ defined in (1.4). Uniqueness will be achieved by an H^{-1} estimate inspired by the one in [1] for similar deterministic evolution equations, together with a Yamada-Watanabe type argument.

2. Notation and results

Throughout this paper we shall shorthand $L^p = L^p(\mathbb{T}^d)$, $p \in [1, +\infty]$, and let $H^s = H^s(\mathbb{T}^d)$, $s \in \mathbb{R}$, be the fractional Sobolev space. Moreover, given $T > 0$ we also shorthand $C(L^p) = C([0,T]; L^p)$, $C(H^s) = C([0,T]; H^s)$, and $L^p(H^s) = L^p([0,T]; H^s)$.

We consider the following stochastic partial differential equation,

$$\mathrm{d}u = \sigma(u)\,[\Delta u - W'(u)]\,\mathrm{d}t + \mathrm{d}M, \tag{2.1}$$

where, for $\varphi \in L^2$, $M_t^\varphi := \langle M_t, \varphi\rangle_{L^2}$, $t \ge 0$, is a continuous square integrable martingale with quadratic variation,

$$\left[M^\varphi\right]_t = \int_0^t \int \left[j * (\sqrt{\sigma(u_s)}\,\varphi)\right]^2 \mathrm{d}x\,\mathrm{d}s. \tag{2.2}$$

Here $j \in H^1$ is a fixed function, $*$ denotes the convolution on $\mathbb{T}^d$, and the following conditions on the potential W and the mobility σ are assumed to hold.

Assumption 2.1. (Assumptions on W and σ)

(1) $W \in C^2\big(\mathbb{R};[0,+\infty)\big)$ and W is uniformly convex at infinity, i.e., there exists a constant $C \in (0,+\infty)$ and a compact $K \subset \mathbb{R}$ such that $W''(u) \geq \frac{1}{C}$ for any $u \notin K$.

(2) W has at most growth 4, i.e., there exists a constant $C \in (0,+\infty)$ such that $|W(u)| \leq C(|u|^4+1)$ for any $u \in \mathbb{R}$.

(3) W' has at most growth 3, i.e., there exists a constant $C \in (0,+\infty)$ such that $|W'(u)| \leq C(|u|^3+1)$ for any $u \in \mathbb{R}$.

(4) There exists a constant $C \in (0,+\infty)$ such that $|W''(u)| \leq C(\sqrt{W(u)}+1)$ for any $u \in \mathbb{R}$.

(5) $\sigma \in C^2(\mathbb{R})$, σ is bounded and uniformly strictly positive, i.e., there exists a constant $C \in (0,+\infty)$ such that $\frac{1}{C} \leq \sigma(u) \leq C$ for any $u \in \mathbb{R}$.

(6) σ', σ'' are bounded.

We prove the existence and uniqueness of the Cauchy problem associated to (2.1) with a deterministic initial datum $\bar{u}_0 \in H^1$ in space dimensions $d \leq 3$. To formulate the precise result we introduce two different notions of solution.

Given $T > 0$, we consider $C(L^2) \equiv C([0,T];L^2)$, endowed with the norm topology, the associated Borel σ-algebra $\mathcal{B}$, and the canonical filtration $\mathcal{B}_t$, $t \in [0,T]$. The canonical coordinate on $C(L^2)$ is denote by $u = (u_t)_{t\in[0,T]}$.

Given $\bar{u}_0 \in H^1$, a probability $\mathbb{P}$ on $C(L^2)$ solves the *martingale problem* associated to (2.1) with initial datum $\bar{u}_0$ iff $\mathbb{P}(u_0 = \bar{u}_0) = 1$, $\mathbb{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$, and for each $\psi \in C^\infty([0,T]\times\mathbb{T}^d)$ the process,

$$M_t^\psi := \int u_t\psi_t \,\mathrm{d}x - \int u_0\psi_0 \,\mathrm{d}x - \int_0^t \int \Big[u_s\partial_s\psi_s + \sigma(u_s)\Big(\Delta u_s - W'(u_s)\Big)\psi_s\Big] \,\mathrm{d}x\,\mathrm{d}s \tag{2.3}$$

is a continuous, square integrable $\mathbb{P}$-martingale with quadratic variation,

$$\big[M^\psi\big]_t = 2\int_0^t \int \big[j * \big(\sqrt{\sigma(u_s)}\psi_s\big)\big]^2 \,\mathrm{d}x\,\mathrm{d}s. \tag{2.4}$$

We shall refer to such probability $\mathbb{P}$ as a *martingale solution* to (2.1) with initial datum $\bar{u}_0$. *Uniqueness in law* (or uniqueness of martingale solutions) holds whenever there exists at most one probability on $C(L^2)$ meeting the above requirements.

To introduce the notion of strong solution it is first necessary to construct the martingale in terms of cylindrical Wiener process, whose definition we next recall. A L^2-cylindrical Wiener process on the probability space $(\Omega,\mathcal{G},\mathcal{P})$ is a measurable map $\alpha\colon \Omega \to C(H^{-\bar{s}})$, $\bar{s} > d/2$, such that α_t, $t \in [0,T]$, is a mean zero Gaussian process with covariance,

$$\mathcal{E}\big(\alpha_t(\phi)\alpha_{t'}(\phi')\big) = t\wedge t'\,\langle\phi,\phi'\rangle_{L^2} = t\wedge t'\,\langle\phi,(\mathrm{Id}-\Delta)^{-\bar{s}}\phi'\rangle_{H^{\bar{s}}}, \qquad \phi,\phi' \in H^{\bar{s}},$$

where $\mathcal{E}$ denotes the expectation with respect to $\mathcal{P}$. A L^2-cylindrical Wiener process can be constructed as $\alpha_t = \sum_k \beta_t^k e_k$, where $\{e_k\}$ is an orthonormal basis in L^2 and $\{\beta^k\}$ are independent standard Brownian processes on $\mathbb{R}$. Note that, since the embedding $L^2 \hookrightarrow H^{-s}$ is Hilbert-Schmidt for $s > d/2$, $(\mathrm{Id} - \Delta)^{-\bar{s}}$ is trace-class on $H^{-\bar{s}}$. We refer to [9] for a general overview on infinite dimensional stochastic calculus. We denote by $\{\mathcal{G}_t^\alpha\}$ the filtration generated by α completed with respect to $\mathcal{P}$.

Given $v \in L^2$, we let $B(v)\colon L^2 \to L^2$ be the linear operator defined by $B(v)\psi = \sqrt{2\sigma(v)}\, j * \psi$. Since $j \in H^1$, $B(v)$ is Hilbert-Schmidt, i.e., $\mathrm{Tr}_{L^2}(B(v)B(v)^*) < \infty$.

Given $\bar{u}_0 \in H^1$, a measurable map $u\colon \Omega \to C(L^2)$ is a *strong solution* to (2.1) with initial datum $\bar{u}_0$ iff u is a $\mathcal{G}_t^\alpha$-adapted process, $\mathcal{P}(u_0 = \bar{u}_0) = 1$, $\mathcal{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$, and, for each $\psi \in C^\infty\big([0,T]\times\mathbb{T}^d\big)$ and $t \in [0,T]$, the following equality holds $\mathcal{P}$-a.s.,

$$\langle u_t, \psi_t\rangle_{L^2} = \langle u_0, \psi_0\rangle_{L^2} + \int_0^t \langle u_s, \partial_s\psi_s\rangle_{L^2}\,\mathrm{d}s + \int_0^t \langle \sigma(u_s)(\Delta u_s - W'(u_s)), \psi_s\rangle_{L^2}\,\mathrm{d}s + \int_0^t \langle \psi_s, B(u_s)\,\mathrm{d}\alpha_s\rangle_{L^2}, \tag{2.5}$$

where the last term is understood as an Itô stochastic integral, see [9]. Note that (2.5) corresponds to (1.5) tested with the function ψ. *Uniqueness of strong solutions* holds if any two such solutions u, u' satisfy $\mathcal{P}(u_t = u'_t \ \ \forall\, t \in [0,T]) = 1$.

It is worthwhile to observe that the requirement that strong solutions are adapted to the filtration generated by α means that they can be obtained as non-anticipative functions of α.

In the analysis of (2.1) two specific functionals play an essential role, the aforementioned van der Waals' free energy functional $\mathcal{F}\colon L^2 \to [0,+\infty]$, defined by

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int \Big[\frac{1}{2}|\nabla u|^2 + W(u)\Big]\,\mathrm{d}x & \text{if } u \in H^1, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.6}$$

and the Wilmore functional $\mathcal{W}\colon L^2 \to [0,+\infty]$, defined by

$$\mathcal{W}(u) := \begin{cases} \displaystyle\int \sigma(u)\Big[\Delta u - W'(u)\Big]^2\,\mathrm{d}x & \text{if } u \in H^2. \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Observe that since W has at most quartic growth and $d \le 3$, by Sobolev embedding, $u \in H^1$ implies $W(u) \in L^1$, and, more precisely, $\mathcal{F}(u) \le C(1 + \|u\|_{H^1}^4)$. Similarly, since W' has at most cubic growth, again by Sobolev embedding, if $u \in H^2$ then $W'(u) \in L^2$ and $\mathcal{W}(u) \le C(1 + \|u\|_{H^2}^2 + \|u\|_{H^1}^6)$.

Theorem 2.2. *Given $\bar{u}_0 \in H^1$, there exists a unique martingale solution $\mathbb{P}$ to (2.1). Moreover, $\mathbb{P}(u \in C(H^1)) = 1$ and for $p \in [1,\infty)$ there exists $C = C(\bar{u}_0, T, p) > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \mathcal{F}(u_t) + \int_0^T \mathcal{W}(u_t)\,\mathrm{d}t\Big)^p \le C. \tag{2.8}$$

In addition, given a probability space $(\Omega, \mathcal{G}, \mathcal{P})$ *equipped with a* L^2*-cylindrical Wiener process* α*, there exists a unique strong solution* u *to* (2.1) *with initial condition* $\bar{u}_0$*. The law of* u *is the martingale solution* $\mathbb{P}$*.*

In view of the bounds on $\mathcal{F}$ and $\mathcal{W}$ discussed above, we notice that if $\mathbb{P}$ is a martingale solution then the integrand in (2.8) is $\mathbb{P}$-a.s. finite; the estimate (2.8) states that its moments are finite. We remark that this bound relies on the assumption that the function j in (2.2) belongs to H^1. In particular, in one dimension, the case of space-time white noise is not covered by the previous theorem. In the case of constant mobility the corresponding solution u exists, e.g., in $C([0,T]; C(\mathbb{T}))$, but the H^1 norm of u_t is infinite almost surely for each $t \in [0,T]$.

The proof of Theorem 2.2 is structured through the following steps, in the same spirit of [26]. The existence of a martingale solution is obtained in Sect. 4 by means of compactness estimates on the laws of a sequence of adapted processes in $C(L^2)$. In order to handle the mobility, these processes are constructed by introducing a time discretization and solving in each time interval a suitable semilinear approximated versions of (2.1) obtained by freezing the mobility and regularizing the reaction term. The actual construction of these approximated processes requires an existence result for semilinear equations in $C(H^1)$, which is the content of Sect. 3. To prove compactness, the key ingredient is the apriori estimate in Lemma 4.2 which basically states that the bound (2.8) holds uniformly in the approximation parameters. This estimate relies on the variational structure of (2.1) and its proof is achieved by applying Itô's formula to a suitable approximation of $\mathcal{F}$. To prove uniqueness of martingale solutions, in Sect. 5 we introduce the notion of weak solution to (2.1); by a martingale representation lemma we show that martingale solutions produce weak solutions. Then, after proving the regularity properties of solutions, we show pathwise uniqueness of weak solutions via H^{-1} estimates. Finally, by adapting the argument in the Yamada-Watanabe theorem, we obtain the existence and uniqueness result as stated in Theorem 2.2. Some generation results on a class of C_0-semigroups, needed to the theory developed in Sect. 3, are stated and proved in Appendix A by applying the Lumer-Phillips theorem.

As stated before, the Allen–Cahn equation with mobility does not appear to have been considered in the mathematical literature and it does not seem directly analyzable by the variational method. For instance, the one-side Lipschitz condition (see, e.g., [24, Condition (H3)]) fails for the Gelfand triple $H^1(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset H^{-1}(\mathbb{T}^d)$ even in the case of W with quadratic growth. On the other hand, in the case of constant mobility the method applies, as shown in [25] and in the subsequent paper [24], with a dimensional dependent growth condition on the reaction term. In the one dimensional case the cubic growth is covered, however, Theorem 2.2 provides better regularity properties of the solution u, in particular, $u \in L^2([0,T]; H^2)$ almost surely. In principle, the latter regularity property could be deduced by working with the Gelfand triple $H^2(\mathbb{T}^d) \subset H^1(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$, but this would require strong restrictions on the nonlinearity.

Still in the case of constant mobility, an abstract existence result for stochastic partial differential equation of gradient type is proven in [13], using an approximation argument relying on apriori bounds analogous to the ones in the present paper. When applied to the Allen–Cahn equation, see [13, Rem. 4.9], the regularity properties are slightly weaker than the ones in Theorem 2.2. It would be interesting to generalize the approach of [13] to cover the case of nonconstant mobility.

3. An auxiliary semilinear equation

In this section, we provide an existence result for a semilinear equation that will be used to construct an approximation of the stochastic Allen–Cahn equation. The arguments below follow the semigroup approach in [9], it is however possible to obtain the same result by the variational approach in [30] choosing the Gelfand triple $H^2 \subset H^1 \subset L^2$.

Recall that $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P})$ is a standard filtered probability space equipped with a cylindrical Wiener process $\alpha\colon \Omega \to C(H^{-\bar{s}})$, $\bar{s} > d/2$, such that $\mathcal{G}_t = \mathcal{G}_t^\alpha$ is the filtration generated by α completed with respect to $\mathcal{P}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be globally Lipschitz. Fix a subinterval $[t_0, t_1] \subset [0, T]$, a $\mathcal{G}_{t_0}$-measurable random variable $w\colon \Omega \to H^1$ and a $\mathcal{G}_{t_0}$-measurable random variable $v\colon \Omega \to H^2$. Let $\eta > 0$ and $R_\eta = (\mathrm{Id} - \eta\Delta)^{-1}$. Consider the following Cauchy problem on the time interval $[t_0, t_1]$,

$$\begin{cases} \mathrm{d}u_t = \sigma(v)\big[\Delta u_t + R_\eta f(R_\eta u_t)\big]\mathrm{d}t + \sqrt{2\sigma(v)}\, j * \mathrm{d}\alpha_t, \\ u_{t_0} = w. \end{cases} \tag{3.1}$$

We say that u is a *strong* solution to (3.1) if $u\colon \Omega \to C([t_0, t_1]; H^1)$ is $\mathcal{G}_t$-adapted, $\mathcal{P}(u_{t_0} = w) = 1$, $\mathcal{P}(u \in L^2([t_0, t_1]; H^2)) = 1$, and, for each $\psi \in C^\infty\big([t_0, t_1] \times \mathbb{T}^d\big)$ and $t \in [0, T]$, the following equality holds $\mathcal{P}$-a.s.,

$$\begin{aligned} \langle u_t, \psi_t\rangle_{L^2} &= \langle w, \psi_{t_0}\rangle_{L^2} + \int_{t_0}^t \langle u_s, \partial_s \psi_s\rangle_{L^2}\, \mathrm{d}s \\ &\quad + \int_{t_0}^t \langle \sigma(v)(\Delta u_s + R_\eta f(R_\eta u_s)), \psi_s\rangle_{L^2}\, \mathrm{d}s + \int_{t_0}^t \langle \psi_s, B(v)\, \mathrm{d}\alpha_s\rangle_{L^2}, \end{aligned} \tag{3.2}$$

where we recall $B(v)\colon L^2 \to L^2$ is defined by $B(v)\psi = \sqrt{2\sigma(v)}\, j * \psi$.

Proposition 3.1. *Assume the initial datum w in (3.1) satisfies $\mathcal{E}(\|w\|_{H^1}^2) < \infty$. Then, the Cauchy problem (3.1) has a strong solution u. Moreover, there exists $C > 0$ depending only on η, $\mathrm{Lip}(f)$, and $\mathcal{E}(\|w\|_{H^1}^2)$ such that*

$$\mathcal{E}(\|u\|_{C([t_0,t_1];H^1)}^2 + \|u\|_{L^2([t_0,t_1];H^2)}^2) \le C. \tag{3.3}$$

Furthermore, if $\mathcal{E}(\|w\|_{H^1}^{2p}) < \infty$ for some $p > 1$ then there exists $C > 0$ depending only on η, $\mathrm{Lip}(f)$, $\mathcal{E}(\|w\|_{H^1}^{2p})$, and p such that

$$\mathcal{E}(\|u\|_{C([t_0,t_1];H^1)}^{2p}) \le C. \tag{3.4}$$

Proof. By Lemma A.1, the operator $A = \sigma(v)\Delta$ generates a C_0-semigroup $S(t)$ on the Hilbert space H^1. As in Lemma A.1, $\sigma(v) \in H^2$ and it is a bounded multiplier on H^1. Moreover, the function f induces a Lipschitz map $F_\eta\colon H^1 \to H^1$ given by $F_\eta(u) := \sigma(v)R_\eta f(R_\eta u)$, in view of the simple estimate,

$$\begin{aligned}\|F_\eta(u_1) - F_\eta(u_2)\|_{H^1} &\leq C\|\sigma(v)\|_{H^2}\|R_\eta\|_{L^2\to H^1}\|f(R_\eta u_1) - f(R_\eta u_2)\|_{L^2}\\ &\leq C\|\sigma(v)\|_{H^2}\|R_\eta\|_{L^2\to H^1}\operatorname{Lip}(f)\|u_1 - u_2\|_{L^2}.\end{aligned}$$

Finally, by a direct computation, the operator $\tilde{B}\colon L^2 \to H^1$ defined by $\tilde{B}\psi = \sqrt{2\sigma(v)}j * \psi$, $\psi \in L^2$, is an Hilbert-Schmidt operator with norm bound $\|\tilde{B}\|_{HS}^2 \leq C\|\sqrt{\sigma(v)}\|_{H^2}^2\|j\|_{H^1}^2$.

In view of the previous statements, we can apply [9, Thm. 7.4] and deduce the existence of a unique $\mathcal{G}_t$-progressively measurable map $u\colon \Omega \to C([t_0,t_1];H^1)$ satisfying the mild formulation of (3.1), i.e.,

$$u_t = S(t-t_0)w + \int_{t_0}^t S(t-s)F_\eta(u_s)\,\mathrm{d}s + \int_{t_0}^t S(t-s)\tilde{B}\,\mathrm{d}\alpha_s, \tag{3.5}$$

and the estimate $\sup_{t\in[t_0,t_1]}\mathcal{E}(\|u_t\|_{H^1}^2) \leq C(1+\mathcal{E}(\|w\|_{H^1}^2))$.

In order to obtain the bound (3.3) we would like to apply Itô's formula to $\Phi(u) := \frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x$. However, since Φ is not C^2 on L^2, to accomplish this step we first introduce a suitable approximation scheme. For $\delta > 0$ let A_δ be as in Lemma A.1, and consider the following linear Cauchy problems,

$$\begin{cases}\mathrm{d}z_t = \big[A_\delta z_t + F_\eta(u_t)\big]\mathrm{d}t + \tilde{B}\mathrm{d}\alpha_t,\\ z_{t_0} = w,\end{cases} \tag{3.6}$$

where u is the unique solution to (3.5). By Lemma A.1, A_δ generates a C_0-semigroup S_δ and there exists constants m_0 and C such that,

$$\int_{t_0}^{t_1}\|S_\delta(r-t_0)\tilde{B}\|_{HS}^2\,\mathrm{d}r \leq C(t_1-t_0)e^{m_0(t_1-t_0)}\|\sqrt{\sigma(v)}\|_{H^2}^2\|j\|_{H^1}^2.$$

Therefore, the process u_t^δ defined by

$$u_t^\delta := S_\delta(t-t_0)w + \int_{t_0}^t S_\delta(t-s)F_\eta(u_s)\,\mathrm{d}s + \int_{t_0}^t S_\delta(t-s)\tilde{B}\,\mathrm{d}\alpha_s \tag{3.7}$$

is a $\mathcal{P}$-a.s. well defined H^1-valued with continuous trajectories and $\mathcal{G}_t$-progressively measurable. By Fubini's Theorem (see [9, Thm. 4.18] for the stochastic case) a direct computation shows that u^δ solves (3.6) in the sense that, $\mathcal{P}$-a.s.,

$$u_t^\delta = w + \int_{t_0}^t\big[A_\delta u_s^\delta + F_\eta(u_s)\big]\,\mathrm{d}s + \tilde{B}\alpha_t, \qquad t\in[t_0,t_1]. \tag{3.8}$$

Observe that, for each $t \in [t_0,t_1]$,

$$\mathcal{E}\Big(\Big\|\int_{t_0}^t\big[S_\delta(t-s) - S(t-s)\big]\tilde{B}\,\mathrm{d}\alpha_s\Big\|_{H^1}^2\Big) = \int_{t_0}^t\Big\|\big[S_\delta(t-s) - S(t-s)\big]\tilde{B}\Big\|_{HS}^2\,\mathrm{d}s.$$

Combining the previous identity with Lemma A.1, items (2) and (3), equation (3.7) and dominated convergence easily imply that for each $t \in [t_0, t_1]$ $u^\delta_t \to u_t$ in $L^2(\Omega; H^1)$ and, in addition, $u^\delta \to u$ in $L^2(\Omega; L^2([t_0, t_1] \times \mathbb{T}^d))$ as $\delta \to 0$.

Let $\Phi^\delta \colon L^2 \to \mathbb{R}$ be defined by

$$\Phi^\delta(u) := \int \frac{1}{2} |\nabla R_\delta u|^2 \, \mathrm{d}x = -\frac{1}{2} \langle u, R_\delta \Delta R_\delta u \rangle_{L^2},$$

so that for $u \in H^1$ we have $\Phi^\delta(u) \to \int \frac{1}{2} |\nabla u|^2 \, \mathrm{d}x = \Phi(u)$ as $\delta \to 0$.

Since Φ^δ is C^2 with locally bounded and uniformly continuous first and second derivatives, we can apply Itô's formula, see, e.g., [9, Thm. 4.17]. Then, in view of (3.6), we get,

$$\begin{aligned} &\Phi^\delta(u^\delta_t) + \int_{t_0}^t \int \sigma(R_\delta v) \left| R_\delta \Delta R_\delta u^\delta_s \right|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\quad = \Phi^\delta(w) + \int_{t_0}^t \int R_\delta(-\Delta) R_\delta u^\delta_s F_\eta(u_s) \, \mathrm{d}x \, \mathrm{d}s \\ &\qquad + \frac{t - t_0}{2} \operatorname{Tr}_{L^2} \left(R_\delta(-\Delta) R_\delta B B^* \right) + \int_{t_0}^t \langle R_\delta(-\Delta) R_\delta u^\delta_s, B \, \mathrm{d}\alpha_s \rangle_{L^2}, \end{aligned} \tag{3.9}$$

where $B := B(v) = \mathrm{Id}_{H^1 \to L^2} \tilde{B}$.

We next estimate separately the terms on the right-hand side of (3.9). By Young's inequality, for each $\zeta > 0$ there exists $C_\zeta > 0$ such that

$$\begin{aligned} \int_{t_0}^t \int R_\delta(-\Delta) R_\delta u^\delta_s F_\eta(u_s) \, \mathrm{d}x \, \mathrm{d}s &\le \zeta \int_{t_0}^t \int \left| R_\delta \Delta R_\delta u^\delta_s \right|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\quad + C_\zeta \|\sigma\|_\infty \left(|f(0)|^2 + \mathrm{Lip}(f)^2 \int_{t_0}^t \|u_s\|^2_{L^2} \, \mathrm{d}s \right). \end{aligned}$$

Clearly, if $\{e_\ell\} \subset L^2$ is an orthonormal basis we have,

$$\operatorname{Tr}_{L^2} \left(R_\delta(-\Delta) R_\delta B B^* \right) = \sum_\ell \|R_\delta \nabla B e_\ell\|^2_{L^2} \le C \|\sqrt{\sigma(v)}\|^2_{H^2} \|j\|^2_{H^1}.$$

Let N_t, $t \in [t_0, t_1]$, be the continuous martingale $N_t = \int_{t_0}^t \langle R_\delta(-\Delta) R_\delta u^\delta_s, B \, \mathrm{d}\alpha_s \rangle_{L^2}$. Since $B^* \psi = j * (\sqrt{2\sigma(v)} \psi)$, the quadratic variation of N can be estimated as follows,

$$[N]_t = \int_{t_0}^t \left\| B^* R_\delta(-\Delta) R_\delta u^\delta_s \right\|^2_{L^2} \, \mathrm{d}s \le C \|\sigma\|_\infty \|j\|^2_{L^1} \int_{t_0}^t \int \left| R_\delta \Delta R_\delta u^\delta_s \right|^2 \, \mathrm{d}x \, \mathrm{d}s.$$

By taking the supremum for $t \in [t_0, t_1]$ in (3.9), using again Young's inequality, taking the expectation, and gathering the above bounds together with the L^2-Doob's inequality,

$$\mathcal{E} \Big(\sup_{t \in [t_0, t_1]} N_t \Big)^2 \le 4 \, \mathcal{E}([N]_{t_1}),$$

we conclude that there exists $C > 0$ such that

$$\mathcal{E}\left(\sup_{t\in[t_0,t_1]}\Phi^\delta(u_t^\delta)+\int_{t_0}^{t_1}\int\left|R_\delta\Delta R_\delta u_s^\delta\right|^2\,\mathrm{d}x\,\mathrm{d}s\right)\leq C\Big(\mathcal{E}(\|w\|_{H^1}^2)+\|\sigma\|_\infty\|j\|_{L^1}^2$$
$$+\ \|\sigma\|_\infty\Big(|f(0)|^2+\mathrm{Lip}(f)^2\int_{t_0}^{t_1}\mathcal{E}(\|u_s\|_{L^2}^2)\,\mathrm{d}s\Big)+(t_1-t_0)\|\sqrt{\sigma(v)}\|_{H^2}^2\|j\|_{H^1}^2\Big).$$

Since $\sup_{t\in[t_0,t_1]}\mathcal{E}(\|u_t\|_{H^1}^2)\leq C(1+\mathcal{E}(\|w\|_{H^1}^2))$ we finally get,

$$\mathcal{E}\left(\sup_{t\in[t_0,t_1]}\Phi^\delta(u_t^\delta)+\int_{t_0}^{t_1}\int\left|R_\delta\Delta R_\delta u_s^\delta\right|^2\,\mathrm{d}x\,\mathrm{d}s\right)\leq C\left(1+\mathcal{E}(\|w\|_{H^1}^2)\right), \quad (3.10)$$

for some $C = C(t_1 - t_0, \sigma, \|v\|_{H^2}, \mathrm{Lip}(f)) > 0$.

Since $u \in C([t_0, t_1]; H^1)$, $\mathcal{P}$-a.s. we can take a countable dense set $\mathcal{S} \subset [t_0, t_1]$ and a subsequence still denoted by $\delta \to 0$ such that, $\mathcal{P}$-a.s., $u_s^\delta \to u_s$, $s \in \mathcal{S}$, in H^1 as $\delta \to 0$. Thus, as $\Phi^\delta \to \Phi$ pointwise in H^1, we get, $\mathcal{P}$-a.s., $\Phi(u_s) \leq \underline{\lim}_{\delta\to 0}\Phi^\delta(u_s^\delta)$ for all $s \in \mathcal{S}$. Then, the continuity $t \mapsto u_t$ implies that, $\mathcal{P}$-a.s., $\sup_{t\in[t_0,t_1]}\Phi(u_t) \leq \underline{\lim}_{\delta\to 0}\sup_{t\in[t_0,t_1]}\Phi^\delta(u_t^\delta)$. By Fatou's' Lemma and (3.10) we conclude $\mathcal{E}(\|u\|^2_{C([t_0,t_1];H^1)}) \leq C\left(1 + \mathcal{E}(\|w\|_{H^1}^2)\right)$.

Again by Fatou's Lemma and (3.10) we have,

$$\mathcal{E}\left(\underline{\lim}_{\delta\to 0}\|\Delta R_\delta R_\delta u^\delta\|^2_{L^2([t_0,t_1]\times\mathbb{T}^d)}\right)\leq C\left(1+\mathcal{E}(\|w\|_{H^1}^2)\right).$$

In particular, $\mathcal{P}$-a.s. we have $\underline{\lim}_{\delta\to 0}\|\Delta R_\delta R_\delta u^\delta\|^2_{L^2([t_0,t_1]\times\mathbb{T}^d)} < \infty$. As $R_\delta R_\delta u^\delta \to u$ in $L^2(\Omega\times[t_0,t_1]\times\mathbb{T}^d)$, by elliptic regularity and lower semicontinuity we get $u \in L^2(\Omega \times [t_0, t_1]; H^2)$ and $\mathcal{E}(\|u\|^2_{L^2(H^2)}) \leq C\left(1 + \mathcal{E}(\|w\|_{H^1}^2)\right)$.

Next, we show that u satisfies (3.2). Fix $\psi \in C^\infty([t_0, t_1]\times\mathbb{T}^d)$ and consider the function $\Psi\colon [t_0, t_1] \times L^2 \to \mathbb{R}$ given by $\Psi(t, u) := \langle\psi_t, u_t\rangle_{L^2}$. Clearly Ψ is C^2 with locally uniformly continuous first and second derivatives, hence Itô's formula and (3.6) give, for any $t \in [t_0, t_1]$,

$$\langle u_t^\delta,\psi_t\rangle_{L^2}=\langle w,\psi_{t_0}\rangle_{L^2}+\int_{t_0}^t\Big[\langle u_s^\delta,\partial_s\psi_s\rangle_{L^2}+\langle A_\delta u_s^\delta+F_\eta(u_s),\psi_s\rangle_{L^2}\Big]\,\mathrm{d}s$$
$$+\int_{t_0}^t\langle\psi_s,B(v)\mathrm{d}\alpha_s\rangle_{L^2}\qquad\mathcal{P}\text{-a.s.} \quad (3.11)$$

Recalling that $u^\delta \to u$ in $L^2(\Omega \times [t_0, t_1] \times \mathbb{T}^d)$ and $u_t^\delta \to u_t$ in $L^2(\Omega; L^2)$, up to subsequences we have $u^\delta \to u$ in $L^2([t_0, t_1] \times \mathbb{T}^d)$ and $u_t^\delta \to u_t$ in L^2 $\mathcal{P}$-a.s. In order to take the limit as $\delta \to 0$ in (3.11), it remains to show that $\mathcal{P}$-a.s. we have $A_\delta u^\delta \rightharpoonup \sigma(v)\Delta u$ in $L^2([t_0, t_1] \times \mathbb{T}^d)$. To this end, notice that for $\mathcal{P}$-a.s. $\omega \in \Omega$ there exists a subsequence depending on ω such that $\Delta R_\delta R_\delta u^\delta \rightharpoonup \Delta u$ in $L^2([t_0, t_1] \times \mathbb{T}^d)$. Since $R_\delta v \to v$ in H^2, by Sobolev embedding we have $\sigma(R_\delta v) \to \sigma(v)$ uniformly, hence the desired statement follows.

Finally, the bound (3.4) is the content of [9, Thm. 7.4, item (iii)]. $\square$

4. Existence of martingale solutions

In this section we prove the following existence result.

Theorem 4.1. *Given $\bar{u}_0 \in H^1$, there exists a martingale solution $\mathbb{P}$ to (2.1) with initial condition $\bar{u}_0$. Furthermore, for any $p \in [1, \infty)$ there exists $C > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u_t\|_{H^1}^{2p} + \|u\|_{L^2(H^2)}^{2p} \right) \le C. \tag{4.1}$$

The martingale solution will be obtained as a weak limit point of an approximating sequence of probabilities on $C(L^2)$, that are the laws of a sequence of processes recursively defined according to the following scheme.

Let $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P})$ be a standard filtered probability space equipped with a L^2-cylindrical Wiener process $\alpha \colon \Omega \to C(H^{-\bar{s}})$, $\bar{s} > d/2$, such that $\mathcal{G}_t = \mathcal{G}_t^\alpha$ is the filtration generated by α completed with respect to $\mathcal{P}$. Given $\ell > 0$ let also $W_\ell \colon \mathbb{R} \to \mathbb{R}$ be the C^2 function defined by

$$W_\ell(u) := \begin{cases} W(u) & \text{if } |u| \le \ell, \\ W(\ell) + W'(\ell)(|u| - \ell) + \frac{1}{2}W''(\ell)(|u| - \ell)^2 & \text{if } |u| > \ell. \end{cases} \tag{4.2}$$

Observe that, for any ℓ large enough, the function W_ℓ has quadratic growth at infinity both from above and below. Moreover, W_ℓ' is globally Lipschitz.

Fix $\eta > 0$ and a smooth approximation $\imath_n$ of the Dirac δ-function with $\|\imath_n\|_{L^1} = 1$. Given $n \in \mathbb{N}$, consider the partition $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ with $t_{i+1}^n - t_i^n = T/n$ for $i = 0, \ldots, n-1$. In each time step of this partition, we recursively construct a sequence of $\mathcal{G}_t$-adapted continuous processes u^n and a sequence of $\mathcal{G}_t$-adapted processes v^n which is constant on each time interval $[t_i^n, t_{i+1}^n)$ as follows. Define,

$$v_t^n := \imath_n * \bar{u}_0 \in H^2(\mathbb{T}^d) \quad \text{for} \quad t \in [t_0^n, t_1^n) \tag{4.3}$$

and set $\sigma_0^n = \sigma(v_{t_0^n}^n)$. According to Proposition 3.1 we define u_t^n, $t \in [t_0^n, t_1^n)$, as a solution to

$$\begin{cases} \mathrm{d}u_t^n = \sigma_0^n\big(\Delta u_t^n - R_\eta W_\ell'(R_\eta u_t^n)\big)\mathrm{d}t + \sqrt{2\sigma_0^n}\, j * \mathrm{d}\alpha_t, \\ u_0^n = \bar{u}_0. \end{cases} \tag{4.4}$$

By induction, for $1 \le i \le n-1$ we define,

$$v_t^n := \frac{1}{t_i^n - t_{i-1}^n} \int_{t_{i-1}^n}^{t_i^n} u_s^n \, \mathrm{d}s, \qquad t \in [t_i^n, t_{i+1}^n) \tag{4.5}$$

and $\sigma_i^n := \sigma(v_{t_i^n}^n)$. Again by Proposition 3.1, we let u_t^n, $t \in [t_i^n, t_{i+1}^n)$, be the solution to

$$\begin{cases} \mathrm{d}u_t^n = \sigma_i^n\big(\Delta u_t^n - R_\eta W_\ell'(R_\eta u_t^n)\big)\mathrm{d}t + \sqrt{2\sigma_i^n}\, j * \mathrm{d}\alpha_t, \\ u_{t_i^n}^n = \lim_{s\uparrow t_i^n} u_s^n. \end{cases} \tag{4.6}$$

We finally set $u_{t_n^n}^n = \lim_{s\uparrow t_n^n} u_s^n$. Notice that, by recursively using Proposition 3.1 and (4.5), we get $\mathcal{P}$-a.s. $v^n \in L^\infty(H^2)$ and $u^n \in C(H^1) \cap L^2(H^2)$.

Note that, although not indicated in the notation, the process u^n also depends on η and ℓ.

The proof of Theorem 4.1 is split into three lemmata.

Lemma 4.2. (A priori bounds) *Let u^n be the process constructed by solving (4.4)–(4.6) and $\mathcal{F}_{\ell,\eta}\colon H^1 \to \mathbb{R}$ be the functional defined by*

$$\mathcal{F}_{\ell,\eta}(u) = \int \frac{1}{2}|\nabla u|^2 + W_\ell(R_\eta u)\,\mathrm{d}x. \tag{4.7}$$

Then, for any $p \in [1, +\infty)$,

$$\mathcal{E}\left(\sup_{t\in[0,T]} \mathcal{F}_{\ell,\eta}(u^n_t)\right)^p + \frac{1}{2}\mathcal{E}\left(\int_0^T \int \sigma(v^n_t)\big(\Delta u^n_t - R_\eta W'_\ell(R_\eta u^n_t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t\right)^p \le C, \tag{4.8}$$

where $C > 0$ depends only on $\|\bar{u}_0\|_{H^1}$, T, and p but is independent of n, ℓ, and η.

Proof. The lemma is essentially achieved by applying Itô's formula to $\mathcal{F}_{\ell,\eta}$, however we need to introduce a suitable approximation scheme to actually carry out the computation. Given $\delta > 0$, let $\mathcal{F}^\delta_{\ell,\eta}\colon L^2 \to \mathbb{R}$ be defined by

$$\mathcal{F}^\delta_{\ell,\eta}(u) = \int \frac{1}{2}|\nabla R_\delta u|^2 + W_\ell(R_\eta u)\,\mathrm{d}x. \tag{4.9}$$

By straightforward computations, $\mathcal{F}^\delta_{\ell,\eta}$ is C^2 with locally bounded and uniformly continuous first derivative $\big(D\mathcal{F}^\delta_{\ell,\eta}\big)_u \in L^2$ and second derivative $\big(D^2\mathcal{F}^\delta_{\ell,\eta}\big)_u\colon L^2 \to L^2$ given by

$$\big(D\mathcal{F}^\delta_{\ell,\eta}\big)_u = R_\delta \Delta R_\delta u - R_\eta W'_\ell(R_\eta u),\ \big(D^2\mathcal{F}^\delta_{\ell,\eta}\big)_u = R_\delta(-\Delta)R_\delta + R_\eta W''_\ell(R_\eta u)R_\eta.$$

Hence, by Itô's formula, for each $t \in [t^n_i, t^n_{i+1}]$ we have,

$$\begin{aligned}\mathcal{F}^\delta_{\ell,\eta}(u^n_t) &+ \int_{t^n_i}^t \int \sigma^n_i \left(R_\delta \Delta R_\delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\right)\left(\Delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\right)\mathrm{d}x\,\mathrm{d}s \\ &= \mathcal{F}^\delta_{\ell,\eta}(u^n_{t^n_i}) + \frac{1}{2}\int_{t^n_i}^t \mathrm{Tr}_{L^2}\left(B^*_{n,i}\left[R_\delta(-\Delta)R_\delta + R_\eta W''_\ell(R_\eta u^n_s)R_\eta\right]B_{n,i}\right)\mathrm{d}s + N^{n,i,\delta}_t,\end{aligned} \tag{4.10}$$

where $B_{n,i}\colon L^2 \to L^2$ and is defined as $B_{n,i}\psi = \sqrt{2\sigma^n_i}\, j * \psi$ and $N^{n,i,\delta}$, $t \in [t^n_i, t^n_{i+1}]$, is the martingale

$$N^{n,i,\delta}_t = \int_{t^n_i}^t \langle R_\delta(-\Delta)R_\delta u^n_s + R_\eta W'_\ell(R_\eta u^n_s), B_{n,i}\,\mathrm{d}\alpha_s\rangle_{L^2}. \tag{4.11}$$

Letting $\{e_k\}$, $k \in \mathbb{Z}^d$, be the standard orthonormal Fourier basis in L^2, we bound the trace terms as follows,

$$\begin{aligned}
&\mathrm{Tr}_{L^2}\left(B^*_{n,i} R_\delta(-\Delta) R_\delta B_{n,i}\right) = \sum_k \|R_\delta \nabla B_{n,i} e_k\|^2_{L^2} \le \sum_k \|\nabla \left(\sqrt{2\sigma^n_i} j * e_k\right)\|^2_{L^2} \\
&\quad \le 4 \sum_k \left(\|(\nabla\sqrt{\sigma^n_i})\, j * e_k\|^2_{L^2} + \|\sqrt{\sigma^n_i}\,(\nabla j) * e_k\|^2_{L^2}\right) \\
&\quad \le C \sum_k \left(\frac{\|\sigma'\|^2_\infty}{4\inf\sigma} |\widehat{j}(k)|^2 \|\nabla v^n_{t^n_i}\|^2_{L^2} + \|\sigma\|_\infty |\widehat{\nabla j}(k)|^2\right) \le C(\sigma)\|j\|^2_{H^1}(1 + \|\nabla v^n_{t^n_i}\|^2_{L^2})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Tr}_{L^2}\left(B^*_{n,i} R_\eta W''_\ell(R_\eta u^n_s) R_\eta B_{n,i}\right) = \sum_k \int |R_\eta B_{n,i} e_k|^2 W''_\ell(R_\eta u^n_s)\,\mathrm{d}x \\
&\quad \le \sum_k \|R_\eta B_{n,i} e_k\|^2_{L^\infty} \int |W''_\ell(R_\eta u^n_s)|\,\mathrm{d}x \le \|\sigma\|_\infty \|j\|^2_{L^2} \int |W''_\ell(R_\eta u^n_s)|\,\mathrm{d}x,
\end{aligned}$$

where we used that $\|j * e_k\|_{L^\infty} \le |\widehat{j}(k)|$ and $\|\nabla j * e_k\|_{L^\infty} \le |\widehat{\nabla j}(k)|$.

As $u^n \in L^2(\Omega; L^2([t^n_i, t^n_{i+1}]; H^2))$ then $R_\delta(-\Delta)R_\delta u^n - (-\Delta)u^n \to 0$ in $L^2(\Omega \times [t^n_i, t^n_{i+1}] \times \mathbb{T}^d)$. This implies $N^{n,i,\delta}_t \to N^{n,i}_t$ in $L^2(\Omega)$, where $N^{n,i}$ is the martingale

$$N^{n,i}_t = \int_{t^n_i}^t \langle -\Delta u^n_s + R_\eta W'_\ell(R_\eta u^n_s), B_{n,i}\,\mathrm{d}\alpha_s\rangle_{L^2}\,. \tag{4.12}$$

Indeed,

$$\mathcal{E}\left(N^{n,i,\delta}_t - N^{n,i}_t\right)^2 = \mathcal{E}\int_{t^n_i}^t \|B_{n,i}\left[R_\delta(-\Delta)R_\delta u^n_s - (-\Delta)u^n_s\right]\|^2_{L^2}\,\mathrm{d}s \overset{\delta\to 0}{\longrightarrow} 0.$$

Using that $\mathcal{P}$-a.s. $u^n \in C([t^n_i, t^n_{i+1}]; H^1) \cap L^2([t^n_i, t^n_{i+1}]; H^2)$ we can take the limit as $\delta \to 0$ in the first three terms of (4.10) by dominated convergence. As $N^{n,i,\delta}_t \to N^{n,i}_t$ $\mathcal{P}$-a.s. for a suitable subsequence, combining with the previous bound on the trace terms we finally get, for each $t \in [t^n_i, t^n_{i+1}]$,

$$\begin{aligned}
&\mathcal{F}_{\ell,\eta}(u^n_t) + \int_{t^n_i}^t \int \sigma^n_i \left(\Delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\right)^2 \mathrm{d}x\,\mathrm{d}s \\
&\quad \le \mathcal{F}_{\ell,\eta}(u^n_{t^n_i}) + C(\sigma)\|j\|^2_{H^1} \int_{t^n_i}^t \int \left(1 + |\nabla v^n_s|^2 + |W''_\ell(R_\eta u^n_s)|\right)\mathrm{d}x\,\mathrm{d}s + N^{n,i}_t,
\end{aligned} \tag{4.13}$$

where we used that v^n is constant in the time interval $[t^n_i, t^n_{i+1})$.

Fix $t \in [0,T]$ and let $i_n(t)$ be such that $t \in [t^n_{i_n(t)}, t^n_{i_n(t)+1})$, by summing (4.13) in all the time intervals $[t^n_j, t^n_{j+1})$, $j \le i_n(t)$, we deduce,

$$\begin{aligned}
&\mathcal{F}_{\ell,\eta}(u^n_t) + \int_0^t \int \sigma(v^n_s)\left(\Delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\right)^2 \mathrm{d}x\,\mathrm{d}s \\
&\quad \le \mathcal{F}_{\ell,\eta}(\bar{u}_0) + C(\sigma)\|j\|^2_{H^1} \int_0^t \int \left(1 + |\nabla v^n_s|^2 + |W''_\ell(R_\eta u^n_s)|\right)\mathrm{d}x\,\mathrm{d}s + N^n_t,
\end{aligned} \tag{4.14}$$

where N^n is the continuous $\mathcal{P}$-martingale $N^n_t = \sum_{j<i_n(t)} N^{n,j}_{t^n_{j+1}} + N^{n,i_n(t)}_t$. In particular, the quadratic variation of N^n is

$$[N^n]_t = 2\int_0^t \int \big\{ j * \big[\sqrt{\sigma(v^n_s)}\big(-\Delta u^n_s + R_\eta W'_\ell(R_\eta u^n_s)\big)\big]\big\}^2 \,\mathrm{d}x\,\mathrm{d}s. \tag{4.15}$$

By the assumptions on W and the definition of W_ℓ, (4.2), there exists $C>0$ independent of ℓ such that $|W''_\ell(\cdot)| \le C(1+W_\ell(\cdot))$. Moreover, for each $s\in[t^n_i, t^n_{i+1})$ with $i\ge 1$ we have,

$$\|\nabla v^n_s\|^2_{L^2} \le \frac{1}{t^n_i - t^n_{i-1}} \int_{t^n_{i-1}}^{t^n_i} \|\nabla u^n_{s'}\|^2_{L^2}\,\mathrm{d}s' \le \sup_{s'\le s} \|\nabla u^n_{s'}\|^2_{L^2}.$$

Since $v^n_0 \equiv i_n * \bar u_0$, the previous bound yields $\|\nabla v^n_s\|_{L^2} \le \sup_{0\le s'\le s}\|\nabla u^n_{s'}\|_{L^2}$ for any $0\le s\le T$. Thus, combining the two estimates above we have,

$$\int \big(1+|\nabla v^n_s|^2 + |W''_\ell(R_\eta u^n_s)|\big)\,\mathrm{d}x \le C\big(1+\sup_{s'\le s}\mathcal{F}_{\ell,\eta}(u^n_{s'})\big),$$

hence, taking the supremum over time in (4.14) we obtain,

$$\begin{aligned}\sup_{s\le t}\mathcal{F}_{\ell,\eta}(u^n_s) &+ \int_0^t\int \sigma(v^n_s)\big(\Delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\big)^2 \mathrm{d}x\,\mathrm{d}s \\ &\le 2\Big\{\mathcal{F}_{\ell,\eta}(\bar u_0) + C\int_0^t 1+\sup_{s'\le s}\mathcal{F}_{\ell,\eta}(u^n_{s'})\,\mathrm{d}s + \sup_{s\le t} N^n_s\Big\}.\end{aligned} \tag{4.16}$$

Given $p\ge 1$, the previous inequality implies,

$$\begin{aligned}\Big(\sup_{s\le t}\mathcal{F}_{\ell,\eta}(u^n_s)\Big)^p &+ \Big(\int_0^t\int \sigma(v^n_s)\big(\Delta u^n_s - R_\eta W'_\ell(R_\eta u^n_s)\big)^2 \mathrm{d}x\,\mathrm{d}s\Big)^p \\ &\le C\Big\{ (\mathcal{F}_{\ell,\eta}(\bar u_0))^p + \int_0^t 1 + \Big(\sup_{s'\le s}\mathcal{F}_{\ell,\eta}(u^n_{s'})\Big)^p\,\mathrm{d}s + \Big(\sup_{s\le t}N^n_s\Big)^p\Big\},\end{aligned} \tag{4.17}$$

for some $C=C_p>0$.

By Young's and BDG's inequalities (see, e.g., [31] for the latter) there exists a constant $C=C_p>0$ such that for each $\gamma>0$ we have,

$$\begin{aligned}\mathcal{E}\Big(\Big(\sup_{s\le t}N^n_s\Big)^p\Big) &\le \frac{\gamma}{2}\mathcal{E}\Big(\Big(\sup_{s\le t}N^n_s\Big)^{2p}\Big) + \frac{1}{2\gamma} \le 2C\gamma\,\mathcal{E}([N^n]^p_t) + \frac{1}{2\gamma} \\ &\le 4C\gamma\,\mathcal{E}\Big(\Big(\int_0^t\int \sigma(v^n_s)\big(-\Delta u^n_s + R_\eta W'_\ell(R_\eta u^n_s)\big)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^p\Big) + \frac{1}{2\gamma},\end{aligned}$$

where we used (4.15). Choosing $\gamma > 0$ small enough and taking the expectation in (4.17) we have,

$$\mathcal{E}\left(\sup_{s\le t}\mathcal{F}_{\ell,\eta}(u_s^n)\right)^p + \frac{1}{2}\mathcal{E}\left(\left(\int_0^t\int\sigma(v_s^n)\big(\Delta u_s^n - R_\eta W_\ell'(R_\eta u_s^n)\big)^2\mathrm{d}x\,\mathrm{d}s\right)^p\right)$$
$$\le C\Big\{ (\mathcal{F}_{\ell,\eta}(\bar u_0))^p + \int_0^t 1 + \mathcal{E}\left(\sup_{s'\le s}\mathcal{F}_{\ell,\eta}(u_{s'}^n)\right)^p \mathrm{d}s\Big\} \tag{4.18}$$

As $\mathcal{F}_{\ell,\eta}(\cdot) \le C(1+\|\cdot\|_{H^1}^4)$ for some C independent of ℓ and η, applying recursively (3.3) and (3.4) on each time interval $[t_i^n, t_{i+1}^n]$ we get $\mathcal{E}\left(\sup_{t\in[0,T]}\mathcal{F}_{\ell,\eta}(u_t^n)\right)^p < \infty$. Thus, the bound (4.8) follows from (4.18) by Gronwall's inequality. □

Lemma 4.3. (Tightness of the approximating sequence) *Let $\mathbb{P}_{\ell,\eta}^n$ be the law of the process u^n constructed by solving (4.4)–(4.6). Then $(\mathbb{P}_{\ell,\eta}^n)$ is a tight family of probabilities on $C(L^2)$.*

Proof. In view of compact embedding $H^1 \hookrightarrow L^2$, a sufficient condition for a subset A of $C(L^2)$ to be precompact is that

$$\sup_{u\in A}\sup_{0\le t\le T}\|u_t\|_{H^1} < +\infty, \qquad \lim_{\delta\to 0}\sup_{u\in A}\omega(u;\delta) = 0, \tag{4.19}$$

where $\omega(u;\delta)$ is the modulus of continuity of the element $u \in C(L^2)$, i.e.,

$$\omega(u;\delta) := \sup_{\substack{t,s\in[0,T]\\ |t-s|\le\delta}}\|u_t - u_s\|_{L^2}. \tag{4.20}$$

The family $(\mathbb{P}_{\ell,\eta}^n)$ is tight if the following conditions are fulfilled.

(i) For each $\zeta > 0$ there exists $a > 0$ such that

$$\mathbb{P}_{\ell,\eta}^n\Big(\sup_{0\le t\le T}\|u_t\|_{H^1}^2 > a\Big) \le \zeta \qquad \forall\, n,\ell,\eta.$$

(ii) For each $\varepsilon > 0$ and $\zeta > 0$ there exists $\delta \in (0,T)$ such that

$$\mathbb{P}_{\ell,\eta}^n\big(\omega(u;\delta) > \varepsilon\big) \le \zeta \qquad \forall\, n,\ell,\eta.$$

Indeed, if (i) and (ii) are verified, given any $\zeta > 0$ we can find $a > 0$ and δ_k, $k \in \mathbb{N}_0$, such that

$$\mathbb{P}_{\ell,\eta}^n\Big(\sup_{0\le t\le T}\|u_t\|_{H^1}^2 \le a\Big) > 1 - \frac{\zeta}{2}, \qquad \mathbb{P}_{\ell,\eta}^n\Big(\omega(u;\delta_k) \le \frac{1}{k}\Big) > 1 - \frac{\zeta}{2^{k+1}}.$$

Therefore, the closure K_ζ of the set

$$\Big\{u\colon \sup_{0\le t\le T}\|u_t\|_{H^1}^2 \le a,\ \ \omega(u;\delta_k) \le \frac{1}{k}\ \ \forall\, k \in \mathbb{N}_0\Big\}$$

is compact in view of (4.19) and has probability $\mathbb{P}_{\ell,\eta}^n(K_\zeta) > 1 - \zeta$.

Now, we claim that, for any $p \ge 1$,

$$\sup_{\ell,\eta,n}\mathcal{E}\Big(\sup_{0\le t\le T}\|u_t^n\|_{H^1}^{2p}\Big) < \infty \tag{4.21}$$

and, for each $p > 1$,

$$\lim_{\delta\to 0} \sup_{\ell,\eta,n} \sup_{s\in[0,T-\delta]} \frac{1}{\delta}\mathcal{E}\Big(\sup_{t\in[s,s+\delta]} \|u_t - u_s\|_{L^2}^{2p} \Big) = 0. \tag{4.22}$$

By Chebyshev's inequality, (4.21) implies (i) and, by a simple inclusion of events, see, e.g., [5, Eq. (8.9)], and again Chebyshev's inequality, (4.22) implies (ii).

The estimate (4.21) is a direct consequence of (4.8) since it can be easily checked that, in view of the assumptions on W, there exists $C > 0$ such that $\|u\|_{H^1}^2 \le C\big(1 + \mathcal{F}_{\ell,\eta}(u)\big)$ for any ℓ, η.

To prove (4.22) we observe that by (4.4)–(4.6) and Itô's formula, for each $s \in [0, T-\delta]$ and $t \in [s, s+\delta]$,

$$\|u_t^n - u_s^n\|_{L^2}^2 = A_t^{s,n} + R_t^{s,n} + M_t^{s,n}, \tag{4.23}$$

where

$$A_t^{s,n} := 2\int_s^t \int \sigma(v_r^n)\big[\Delta u_r^n - R_\eta W_\ell'(R_\eta u_r^n)\big](u_r^n - u_s^n)\,\mathrm{d}x\,\mathrm{d}r,$$

and

$$R_t^{s,n} = \int_s^t \int \Big(j * \sqrt{\sigma(v_r^n)}\Big)^2 \mathrm{d}x\,\mathrm{d}r \le \|\sigma\|_\infty \|j\|_{L^2}^2\, \delta; \tag{4.24}$$

finally, $M_t^{s,n,\phi}$, $t \in [s, s+\delta]$, is a continuous square integrable $\mathcal{P}$-martingale with quadratic variation,

$$\begin{aligned}\big[M^{s,n}\big]_t &= 4\int_s^t \int \Big[j * \Big(\sqrt{\sigma(v_r^n)}(u_r^n - u_s^n)\Big)\Big]^2 \mathrm{d}x\,\mathrm{d}r \\ &\le 4\|\sigma\|_\infty \sup_{r\in[s,s+\delta]} \|u_r^n - u_s^n\|_{L^2}^2\, \delta,\end{aligned} \tag{4.25}$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}|A_t^{s,n}| &\le 2\|\sigma\|_\infty \Big(\int_s^t \|u_r^n - u_s^n\|_{L^2}^2\,\mathrm{d}r\Big)^{\frac12} \\ &\qquad \times \Big(\int_s^t \int \sigma(v_r^n)\big(\Delta u_r^n - R_\eta W_\ell'(R_\eta u_r^n)\big)^2 \mathrm{d}x\,\mathrm{d}r\Big)^{\frac12} \\ &\le 2\,\delta^{\frac12}\,\|\sigma\|_\infty \sup_{r\in[s,s+\delta]} \|u_r^n - u_s^n\|_{L^2} \\ &\qquad \times \Big(\int_0^T \int \sigma(v_r^n)\big(\Delta u_r^n - R_\eta W_\ell'(R_\eta u_t^n)\big)^2 \mathrm{d}x\,\mathrm{d}r\Big)^{\frac12},\end{aligned}$$

so that, by Young's inequality, there exists $C > 0$ such that

$$|A_t^{s,n}| \le \frac12 \sup_{r\in[s,s+\delta]} \|u_r^n - u_s^n\|_{L^2}^2 + C\delta \int_0^T \int \sigma(v_r^n)\big(\Delta u_r^n - R_\eta W_\ell'(R_\eta u_t^n)\big)^2 \mathrm{d}x\,\mathrm{d}r.$$

Therefore, taking the supremum for $t \in [s, s+\delta]$ in (4.23) we deduce,

$$\sup_{t\in[s,s+\delta]} \|u_t^n - u_s^n\|_{L^2}^2 \le 2 \sup_{t\in[s,s+\delta]} M_t^{s,n} + 2 \sup_{t\in[s,s+\delta]} R_t^{s,n} + 2C\delta \int_0^T \int \sigma(v_r^n)\big(\Delta u_r^n - R_\eta W_\ell'(R_\eta u_t^n)\big)^2 \mathrm{d}x\, \mathrm{d}r. \tag{4.26}$$

By BDG inequality, see, e.g., [31], for any $p > 1$ there exists $C = C_p$ such that

$$\mathcal{E}\Big(\sup_{t\in[s,s+\delta]} (M_t^{s,n})^p\Big) \le C\,\mathcal{E}\big([M^{s,n}]_{s+\delta}^{p/2}\big) \le \frac{1}{2^{p+1}}\mathcal{E}\Big(\sup_{r\in[s,s+\delta]} \|u_r^n - u_s^n\|_{L^2}^{2p}\Big) + C\delta^p,$$

where we used the bound (4.25) and Young's inequality in the second step. By taking the p-th power and then the expectation value in (4.26), the last bound, together with (4.8) and (4.24) implies the claim (4.22). □

Lemma 4.4. (Properties of the cluster points) *Let $\mathbb{P}$ be a cluster point of the sequence $(\mathbb{P}_{\ell,\eta}^n)$. Then $\mathbb{P}$ is a martingale solution to* (2.1) *with initial condition $\bar{u}_0$. Furthermore, $\mathbb{P}$ satisfies the bound* (4.1).

Proof. Let $\mathbb{P}$ be a cluster point of the sequence $(\mathbb{P}_{\ell,\eta}^n)$, so that, passing to a subsequence, $\mathbb{P}_{\ell,\eta}^n \to \mathbb{P}$ weakly.

We start by proving the estimate (4.1). In view of the assumptions on W and the definition (4.2) of W_ℓ, there is $C > 0$ such that $\|u\|_{H^1}^2 \le C(1+\mathcal{F}_{\ell,\eta}(u))$ and, by Sobolev embedding, $\|R_\eta W_\ell'(R_\eta u)\|_{L^2}^2 \le C(1+\|u\|_{H^1}^6)$ for any $u \in H^1$, where $C > 0$ is independent of ℓ and η.

Hence, the bound (4.8) combined with Calderon-Zygmund inequality readily implies, for any $p \ge 1$,

$$\mathbb{E}_{\ell,\eta}^n \left(\sup_{t\in[0,T]} \|u_t\|_{H^1}^{2p} + \|u\|_{L^2(H^2)}^{2p}\right) \le C, \tag{4.27}$$

where $C > 0$ depends only on $\|\bar{u}_0\|_{H^1}$, T and p but is independent of n, ℓ, and η. Since both the norms in (4.27) are lower semicontinuous under $C(L^2)$-convergence, by Portmanteau's Theorem we infer that for any $p \ge 1$ the bound (4.1) holds with the same constant $C > 0$ in (4.27).

Now, we show that $\mathbb{P}$ is a martingale solution to (2.1) with initial condition $\bar{u}_0$. By construction, $\mathbb{P}_{\ell,\eta}^n(u_0 = \bar{u}_0) = 1$ for any n, ℓ, η so that $\mathbb{P}(u_0 = \bar{u}_0) = 1$. Furthermore, by (4.1), $\mathbb{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$.

It remains to prove that for any $\psi \in C^\infty([0,T] \times \mathbb{T}^d)$ the process M^ψ as defined in (2.3) is a continuous square integrable $\mathbb{P}$-martingale with quadratic variation as in (2.4).

Fix $0 \le s < t \le T$, $\psi \in C^\infty([0,T] \times \mathbb{T}^d)$, and for $u \in L^\infty(H^1) \cap L^2(H^2)$ let

$$G_{n,\ell,\eta}(u) := \langle u_t, \psi_t\rangle_{L^2} - \langle u_s, \psi_s\rangle_{L^2} - \int_s^t \big\{\langle u_r, \partial_r\psi_r\rangle_{L^2} + \langle \sigma(v_r^n)\big[\Delta u_r - R_\eta W_\ell'(R_\eta u_r)\big], \psi_r\rangle_{L^2}\big\}\, \mathrm{d}r, \tag{4.28}$$

with v^n the average of u defined as in (4.3), (4.5). Similarly, let

$$G(u) := \langle u_t, \psi_t, \rangle_{L^2} - \langle u_s, \psi_s \rangle_{L^2} - \int_s^t \big\{ \langle u_r, \partial_r \psi_r \rangle_{L^2} + \langle \sigma(u_r)\big[\Delta u_r - W'(u_r)\big], \psi_r \rangle_{L^2} \big\} \, \mathrm{d}r, \tag{4.29}$$

and, for $\delta > 0$ and $u \in C(L^2)$, we define the regularized version of (4.29) as

$$G^\delta(u) = G(R_\delta u). \tag{4.30}$$

Observe that the function defined in (4.30) is continuous on $C(L^2)$ since $R_\delta : L^2 \to H^2$ continuously. Furthermore, by Sobolev and Hölder inequalities and since R_η contracts any L^p norm,

$$|G_{n,\ell,\eta}(u)| + |G(u)| + |G^\delta(u)| \le C(1 + \|u\|_{L^2(H^2)} + \|u\|^3_{L^\infty(H^1)}), \tag{4.31}$$

where $C > 0$ does not depend on n, ℓ, η, and δ. As a consequence, by (4.27) and (4.1) we get, for any $p \ge 1$,

$$\mathbb{E}^n_{\ell,\eta} \left(|G^\delta|^p + |G_{n,\ell,\eta}|^p \right) + \mathbb{E} \left(|G|^p + |G^\delta|^p \right) \le C, \tag{4.32}$$

where $C > 0$ does not depend on n, ℓ, η, and δ.

In view of (4.3)–(4.6), for any $F \colon C(L^2) \to \mathbb{R}$ continuous, bounded, and measurable with respect to the canonical filtration at time s we have,

$$\mathbb{E}^n_{\ell,\eta}(F(u) G_{n,\ell,\eta}(u)) = 0 \tag{4.33}$$

and

$$\mathbb{E}^n_{\ell,\eta} \left(F(u) \left[G_{n,\ell,\eta}(u)^2 - \int_s^t \int \left(j * (\sqrt{2\sigma(v^n_r)} \psi_r) \right)^2 \mathrm{d}x \, \mathrm{d}r \right] \right) = 0. \tag{4.34}$$

We would like to pass to the limit in (4.33)–(4.34) as $n, \ell \to +\infty$ and $\eta \to 0$ in order to conclude that

$$\mathbb{E}(F(u) G(u)) = 0 \tag{4.35}$$

and

$$\mathbb{E} \left(F(u) \left[G(u)^2 - \int_s^t \int \left(j * (\sqrt{2\sigma(u_r)} \psi_r) \right)^2 \mathrm{d}x \, \mathrm{d}r \right] \right) = 0, \tag{4.36}$$

which, by the arbitrariness of F and $0 \le s < t \le T$, shows that M^ψ as defined in (2.3) is a continuous $\mathbb{P}$-martingale with quadratic variation as in (2.4).

In order to prove (4.35) and (4.36) we use an approximation scheme based on (4.30). We fix a decreasing sequence $\delta_k \searrow 0$ and, for each $a > 0$ we define $\mathcal{D}_a$ as the closure in $C(L^2)$ of the following set,

$$\bigcap_{k \in \mathbb{N}} \Big\{ u \in L^\infty(H^1) \cap L^2(H^2) \colon \|u\|^2_{L^\infty(H^1)} + \|u\|^2_{L^2(H^2)} \le a, \;\; \omega(u; \delta_k) < \frac{1}{k} \Big\}, \tag{4.37}$$

where ω is defined in (4.20).

We observe that in view of (4.19) the set $\mathcal{D}_a$ is compact for any $a > 0$. Moreover, by Lemma 4.2 and the argument in the proof of Lemma 4.3, we can choose $\delta_k \searrow 0$ such that

$$\lim_{a \to +\infty} \sup_{n,\ell,\eta} \mathbb{P}^n_{\ell,\eta}(\mathcal{D}^{\mathrm{c}}_a) = 0, \qquad \lim_{a \to +\infty} \mathbb{P}(\mathcal{D}^{\mathrm{c}}_a) = 0. \tag{4.38}$$

We claim that, for each $a > 0$,

$$\lim_{\delta\to 0}\overline{\lim_{n,\ell,\eta}}\sup_{u\in\mathcal{D}_a}|G_{n,\ell,\eta}(u)-G^\delta(u)|=0,\qquad \lim_{\delta\to 0}\sup_{u\in\mathcal{D}_a}|G(u)-G^\delta(u)|=0. \tag{4.39}$$

Postponing the proof of this claim we first derive (4.35). We write,

$$\mathbb{E}^n_{\ell,\eta}(FG_{n,\ell,\eta})=\mathbb{E}^n_{\ell,\eta}(FG^\delta)+\mathbb{E}^n_{\ell,\eta}(1\!\!1_{\mathcal{D}_a}F(G_{n,\ell,\eta}-G^\delta))+\mathbb{E}^n_{\ell,\eta}(1\!\!1_{\mathcal{D}_a^c}F(G_{n,\ell,\eta}-G^\delta)). \tag{4.40}$$

Since F is bounded, by (4.39), for any $a > 0$,

$$\lim_{\delta\to 0}\overline{\lim_{n,\ell,\eta}}\,\mathbb{E}^n_{\ell,\eta}(1\!\!1_{\mathcal{D}_a}|F(G_{n,\ell,\eta}-G^\delta)|)=0$$

and, in view of (4.32), (4.38), and Chebyshev's inequality,

$$\lim_{a\to\infty}\overline{\lim_{\delta\to 0}}\,\overline{\lim_{n,\ell,\eta}}\,\mathbb{E}^n_{\ell,\eta}(1\!\!1_{\mathcal{D}_a^c}|F(G_{n,\ell,\eta}-G^\delta)|)=0,$$

hence, by (4.33) and (4.40),

$$\lim_{\delta\to 0}\lim_{n,\ell,\eta}\mathbb{E}^n_{\ell,\eta}(F(u)G^\delta(u))=0.$$

Since $G^\delta\colon C(L^2)\to\mathbb{R}$ is continuous and satisfies (4.32) we get,

$$0=\lim_{\delta\to 0}\mathbb{E}(FG^\delta)=\mathbb{E}(FG)+\lim_{\delta\to 0}\mathbb{E}(F(G^\delta-G)).$$

Finally, writing

$$\mathbb{E}(F(G^\delta-G))=\mathbb{E}(1\!\!1_{\mathcal{D}_a}F(G^\delta-G))+\mathbb{E}(1\!\!1_{\mathcal{D}_a^c}F(G^\delta-G)),$$

by using (4.32), (4.38), and (4.39) as before we obtain (4.35).

In order to prove (4.39), first notice that for $u\in C(L^2)$ and v^n the average of u defined as in (4.3), (4.5) we have,

$$\begin{aligned}\|u-v^n\|_{L^\infty(L^2)}&\le\sup_{0\le t<\frac{T}{n}}\|u_t-\imath_n*u_0\|_{L^2}\vee\max_{i=1,\dots,n-1}\sup_{t\in[t^n_i,t^n_{i+1})}\left\|u_t-\frac{n}{T}\int_{t^n_{i-1}}^{t^n_i}u_s\,\mathrm{d}s\right\|_{L^2}\\&\le\omega(u;2T/n)+\|\imath_n*u_0-u_0\|_{L^2}.\end{aligned}$$

Thus, by definition of $\mathcal{D}_a$, for each $a > 0$,

$$\lim_{n\to\infty}\sup_{u\in\mathcal{D}_a}\|u-v^n\|_{L^\infty(L^2)}=0. \tag{4.41}$$

We define

$$\begin{aligned}G_{\ell,\eta}(u)&:=\int u_t\psi_t\,\mathrm{d}x-\int u_s\psi_s\,\mathrm{d}x\\&\quad+\int_s^t\int\big\{u_r\partial_r\psi_r+\sigma(u_r)\big[\Delta u_r-R_\eta W'_\ell(R_\eta u_r)\big]\psi_r\big\}\,\mathrm{d}x\,\mathrm{d}r.\end{aligned} \tag{4.42}$$

Since σ is Lipschitz and by Sobolev embedding $\|R_\eta W'_\ell(R_\eta u)\|^2_{L^2}\le C(1+\|u\|^6_{H^1})$ for any $u\in H^1$, where $C>0$ is independent of ℓ and η, by definition of $\mathcal{D}_a$ and (4.41) we easily obtain,

$$\begin{aligned}&\overline{\lim_{n,\ell,\eta}}\sup_{u\in\mathcal{D}_a}|G_{n,\ell,\eta}(u)-G_{\ell,\eta}(u)|\\&\quad\le\overline{\lim_{n,\ell,\eta}}\sup_{u\in\mathcal{D}_a}C\|u-v^n\|_{L^\infty(L^2)}(1+\|u\|_{L^2(H^2)}+\|u\|^3_{L^\infty(H^1)})=0.\end{aligned} \tag{4.43}$$

We are going to show that

$$\overline{\lim_{\ell,\eta}} \sup_{u\in\mathcal{D}_a} |G_{\ell,\eta}(u) - G(u)| = 0, \qquad \lim_{\delta\to 0} \sup_{u\in\mathcal{D}_a} |G(u) - G^\delta(u)| = 0, \tag{4.44}$$

which clearly imply (4.39) by (4.43).

Since ψ is bounded and R_η contracts also the L^1-norm we estimate,

$$\begin{aligned} |G_{\ell,\eta}(u) - G(u)| \leq & C\int_s^t \int \big|W'(u_r) - R_\eta W'_\ell(R_\eta u_r)\big|\, \mathrm{d}x\, \mathrm{d}r, \\ \leq & C\int_s^t \int \big|W'(u_r) - R_\eta W'(u_r)\big| + \big|W'(u_r) - W'_\ell(u_r)\big|\, \mathrm{d}x\, \mathrm{d}r \\ & + C\int_s^t \int \big|W'_\ell(u_r) - W'_\ell(R_\eta u_r)\big|\, \mathrm{d}x\, \mathrm{d}r = I + II + III. \end{aligned} \tag{4.45}$$

Notice that $\|R_\eta v - v\|^2_{L^2} \leq \eta \|v\|^2_{H^1}$ for any $v \in H^1$ and, by Sobolev and Holder inequalities, for any $v \in H^2$ we have,

$$\|W'(v)\|_{H^1} \leq C\|1 + |v|^3\|_{H^1} \leq C(1 + \|v\|^3_{H^1} + \|v\|_{H^2}\|v\|^2_{H^1}),$$

hence,

$$I \leq C\int_s^t \|W'(u_r) - R_\eta W'(u_r)\|_{L^2}\, \mathrm{d}r \leq C\eta \int_s^t (1 + \|u_r\|^3_{H^1} + \|u_r\|_{H^2}\|u_r\|^2_{H^1})\, \mathrm{d}r. \tag{4.46}$$

On the other hand, by the assumptions on W and the definition (4.2) of W_ℓ, we have $|W'(u)| + |W'_\ell(u)| \leq C(1 + |u|^3)$ for a $C > 0$ independent of ℓ. Combining Cauchy-Schwartz, Sobolev, Chebyschev, and Young inequalities we get,

$$\begin{aligned} II &\leq C\int_s^t \int_{|u_r|>l} (1 + |u_r|^3)\, \mathrm{d}r \leq C\int_s^t (1 + \|u_r\|^3_{L^6}) |\{|u_r| > l\}|^{1/2}\, \mathrm{d}r \\ &\leq C\int_s^t (1 + \|u_r\|^3_{H^1}) \|u_r\|^{1/2}_{L^2} \ell^{-1/2}\, \mathrm{d}r \leq C\ell^{-1/2}(1 + \|u\|^4_{L^\infty(H^1)}). \end{aligned} \tag{4.47}$$

Finally, noticing that $|W'_\ell(u) - W'_\ell(u')| \leq C(1 + |u|^2 + |u'|^2)|u - u'|$ for an absolute constant $C > 0$ independent of τ, τ', and ℓ, by Cauchy-Schwartz, Holder, and Sobolev inequalities, arguing as above we get,

$$\begin{aligned} III &\leq C\int_s^t \int (1 + |u_r|^2 + |R_\eta u_r|^2)|u_r - R_\eta u_r|\, \mathrm{d}x\, \mathrm{d}r \\ &\leq C\int_s^t (1 + \|u_r\|^2_{L^4}) \|u_r - R_\eta u_r\|_{L^2}\, \mathrm{d}r \\ &\leq C\eta^{1/2} \int_s^t (1 + \|u_r\|^2_{H^1}) \|u_r\|_{H^1}\, \mathrm{d}r \leq C\eta^{1/2}(1 + \|u\|^3_{L^\infty(H^1)}). \end{aligned} \tag{4.48}$$

Combining (4.45)–(4.48) the first claim in (4.44) follows. The proof of the second claim in (4.44) is entirely similar. Indeed, first recall that σ and ψ are bounded and that R_δ and Δ commute on H^2. Thus, in view of (4.29) and (4.30),

$$\begin{aligned} |G^\delta(u) - G(u)| &\le C \left(\|u_t - R_\delta u_t\|_{L^1} + \|u_s - R_\delta u_s\|_{L^1} + \int_s^t \|u_r - R_\delta u_r\|_{L^1} \, \mathrm{d}r \right) \\ &\quad + C \int_s^t \|W'(u_r) - W'(R_\delta u_r)\|_{L^1} \, \mathrm{d}r \\ &\quad + \left| \int_s^t \int (R_\delta \sigma(R_\delta u_r)\Delta u_r - \sigma(u_r)\Delta u_r)\, \psi_r \, \mathrm{d}x \, \mathrm{d}r \right| = I' + II' + III'. \end{aligned} \tag{4.49}$$

Since $\|R_\delta v - v\|_{L^2}^2 \le \delta \|v\|_{H^1}^2$ for any $v \in H^1$, we have,

$$I' \le C\|u - R_\delta u\|_{L^\infty(L^2)} \le C\delta^{1/2}\|u\|_{L^\infty(H^1)}, \tag{4.50}$$

and arguing as in (4.47) we also obtain,

$$II' \le C\delta^{1/2}(1 + \|u\|_{L^\infty(H^1)}^3). \tag{4.51}$$

Combining (4.49)–(4.51), the second claim in (4.44) follows once we prove that, for each $a > 0$,

$$\lim_{\delta \to 0} \sup_{u \in \mathcal{D}_a} \left| \int_s^t \langle R_\delta \sigma(R_\delta u_r)\Delta u_r - \sigma(u_r)\Delta u_r, \psi_r \rangle_{L^2} \, \mathrm{d}r \right| = 0. \tag{4.52}$$

We argue by contradiction and suppose that (4.52) fails. Then, there exists $a > 0$, $\rho > 0$, $\delta_k \to 0$, and a sequence $\{u^{(k)}\} \subset \mathcal{D}_a$ such that, for each $k \ge 1$,

$$\left| \int_s^t \langle R_{\delta_k} \sigma(R_{\delta_k} u_r^k)\Delta u_r^k - \sigma(u_r^k)\Delta u_r^k, \psi_r \rangle_{L^2} \mathrm{d}r \right| \ge \rho > 0. \tag{4.53}$$

It is easy to check that $\{u^k\} \subset \mathcal{D}_a \subset C(L^2)$ is equicontinuous and $\{u_t^k\} \subset H^1$ is equibounded; in view of the compact embedding $H^1 \hookrightarrow L^2$ we can apply the Ascoli-Arzelà theorem to infer that, up to subsequences, $u^k \to u \in C(L^2)$ as $k \to \infty$. Moreover, by standard lower semicontinuity argument is easy to check that $u \in \mathcal{D}_a$ and in addition $\Delta u^k \rightharpoonup \Delta u$ in $L^2([0,T] \times \mathbb{T}^d)$ as $k \to \infty$.

Since $u^k \to u$ in $C(L^2)$ and σ is bounded and continuous we have $R_{\delta_k} u^k \to u$, $\sigma(u^k) \to \sigma(u)$, $\sigma(R_{\delta_k} u^k) \to \sigma(u)$ and $R_{\delta_k}\sigma(R_{\delta_k} u^k) \to \sigma(u)$ in $C(L^2)$ and in turn in $L^2([0,T] \times \mathbb{T}^d)$ as $k \to \infty$. As ψ is bounded and smooth and $\Delta u^k \rightharpoonup \Delta u$ in $L^2([0,T] \times \mathbb{T}^d)$, as $k \to \infty$ we have,

$$\lim_{k \to \infty} \int_s^t \langle R_{\delta_k} \sigma(R_{\delta_k} u_r^k)\Delta u_r^k - \sigma(u_r^k)\Delta u_r^k, \psi_r \rangle_{L^2} \, \mathrm{d}r = 0,$$

which contradicts (4.53) and proves (4.52).

To deduce (4.36) from (4.34) we first notice that

$$\begin{aligned} &\lim_{n,\ell,\eta} \mathbb{E}_{\ell,\eta}^n \left(F(u) \int_s^t \int \left(j * (\sqrt{2\sigma(v_r^n)}\psi_r) \right)^2 \mathrm{d}x \, \mathrm{d}r \right) \\ &\quad = \mathbb{E} \left(F(u) \int_s^t \int \left(j * (\sqrt{2\sigma(u_r)}\psi_r) \right)^2 \mathrm{d}x \, \mathrm{d}r \right). \end{aligned} \tag{4.54}$$

Indeed, as $u \mapsto F(u) \int_s^t \int \big(j * (\sqrt{2\sigma(u_r)}\psi_r)\big)^2 \, dx \, dr$ is bounded and continuous, restricting the expectations to $\mathcal{D}_a$ and its complement the conclusion follows from (4.38) and (4.41).

Finally, arguing as in the proof of (4.35) and using (4.32) for some $p > 2$ we have,

$$\lim_{n,\ell,\eta} \mathbb{E}^n_{\ell,\eta} \left(F(u) G_{n,\ell,\eta}(u)^2\right) = \mathbb{E}\left(F(u) G(u)^2\right),$$

which together with (4.34) and (4.54) yields (4.36). □

5. Uniqueness results and strong existence

In this section we conclude the proof of Theorem 2.2. To connect the notions of martingale and strong solutions we first introduce the notion of weak solution.

A pair $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$, where $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P})$ is a standard filtered probability space and (u, α) are $\mathcal{G}_t$-adapted processes, is a *weak solution* to (2.1) with initial datum $\bar{u}_0$ iff

- (i) $\alpha \colon \Omega \to C(H^{-\bar{s}})$, $\bar{s} > d/2$, is a L^2-cylindrical Wiener process with respect to $\mathcal{G}_t$, i.e., it is a L^2-cylindrical Wiener process and its increments $\alpha_t - \alpha_s$ are independent of $\mathcal{G}_s$ for $0 \le s < t \in [0, T]$;
- (ii) $u \colon \Omega \to C(L^2)$, $\mathcal{P}(u_0 = \bar{u}_0) = 1$, and $\mathcal{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$;
- (iii) for each $\psi \in C^\infty\big([0, T] \times \mathbb{T}^d\big)$ and $t \in [0, T]$, the identity (2.5) holds $\mathcal{P}$-a.s.

Pathwise uniqueness of weak solutions holds if whenever $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ and $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u', \alpha'))$ are two weak solutions on the same filtered space with $\alpha = \alpha'$ then $\mathcal{P}(u_t = u'_t \;\; \forall t \in [0, T]) = 1$.

We remark that if a weak solution $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ is such that u is $\mathcal{G}^\alpha_t$-adapted (recall that $\mathcal{G}^\alpha_t$ denotes the filtration generated by α completed with respect to $\mathcal{P}$) then the map $u \colon \Omega \to C(L^2)$ is a strong solution on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$ equipped with the cylindrical Wiener process α.

By a martingale representation lemma, we first show that existence of weak solutions can be deduced from the existence of martingale solutions.

Lemma 5.1. *Given $\bar{u}_0 \in H^1$, let $\mathbb{P}$ be a martingale solution to (2.1) with initial condition $\bar{u}_0$. There exists a weak solution $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ to (2.1) such that $\mathcal{P} \circ u^{-1} = \mathbb{P}$.*

Proof. Let $\mathbb{P}$ be a martingale solution. Recall (2.3) and let $\{e_k\}$, $k \in \mathbb{Z}^d$, be an orthonormal basis in L^2. We claim that the process $M = (M_t)_{t \in [0,T]}$ defined by $M_t := \sum_k M^{e_k}_t e_k$ is a L_2-valued, continuous square integrable $\mathbb{P}$-martingale with quadratic variation,

$$[M]_t = \int_0^t B(u_s) B(u_s)^* \, ds, \tag{5.1}$$

where we recall $B(u) \colon L^2 \to L^2$ is the Hilbert-Schmidt operator given by $B(u)\psi = \sqrt{2\sigma(u)}\, j * \psi$.

The martingale property of M is obvious and (5.1) is a direct consequence of (2.4). Since $\|B(u)\|_{HS}$ is bounded uniformly w.r.t. $u \in L^2$, M is square integrable. Moreover, by (2.3),

$$M_t = u_t - u_0 - \int_0^t \sigma(u_s)\Big(\Delta u_s - W'(u_s)\Big)\,\mathrm{d}s \quad \forall\, t \in [0,T] \qquad \mathbb{P}\text{-a.s.},$$

where the identity has to be understood between elements of L^2. Since $\mathbb{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$ we deduce the $\mathbb{P}$-a.s. continuity of M.

In view of the previous claim, we can apply the representation theorem [9, Thm. 8.2] and deduce the existence of an enlargement of the filtered probability space $(C(L^2), \mathcal{B}, \mathcal{B}_t, \mathbb{P})$, denoted by $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P})$, equipped with a cylindrical Wiener process $\alpha\colon \Omega \to C(H^{-\bar{s}})$, $\bar{s} > d/2$, and a $\mathcal{G}_t$-progressively measurable map $u\colon \Omega \to C(L^2)$ such that $\mathcal{P} \circ u^{-1} = \mathbb{P}$ and $M_t = \int_0^t B(u_s)\,\mathrm{d}\alpha_s$. In particular, $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ is a weak solution to (2.1) with initial condition $\bar{u}_0$. □

By the previous lemma and Itô formula we next show the continuity property of the trajectories for martingale solutions.

Lemma 5.2. *Given $\bar{u}_0 \in H^1$, let $\mathbb{P}$ be a martingale solution to* (2.1) *with initial condition $\bar{u}_0$. Then, $\mathbb{P}(u \in C(H^1)) = 1$.*

Proof. Since $\mathbb{P}(u \in C(L^2) \cap L^\infty(H^1)) = 1$, we already know that $\mathbb{P}$-a.s. the trajectories are H^1-weak continuous, so we have only to show the $\mathbb{P}$-a.s. continuity of the real-valued process $t \mapsto \|u_t\|_{H^1}$. To this end, let $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ be the weak solution associated to $\mathbb{P}$ as constructed in Lemma 5.1. We shall prove the $\mathcal{P}$-a.s. continuity of the map $t \mapsto \mathcal{F}(u_t)$, where $\mathcal{F}\colon H^1 \to \mathbb{R}$ is the functional,

$$\mathcal{F}(u) = \int \frac{1}{2}|\nabla u|^2 + W(u)\,\mathrm{d}x. \tag{5.2}$$

Note indeed the map $[0,T] \ni t \mapsto \int W(u_t)\,\mathrm{d}x$ is $\mathcal{P}$-a.s. continuous since $\mathcal{P}(u \in C(L^2) \cap L^\infty(H^1)) = 1$.

In order to apply Itô's formula to $\mathcal{F}$, we proceed by approximation as in the proof of Lemma 4.2. Given $\delta > 0$ and $\ell > 0$, let $\mathcal{F}_\ell^\delta\colon L^2 \to \mathbb{R}$ be the regularized version of $\mathcal{F}$ defined by

$$\mathcal{F}_\ell^\delta(u) = \int \frac{1}{2}|\nabla R_\delta u|^2 + W_\ell(R_\delta u)\,\mathrm{d}x, \tag{5.3}$$

where, as usual, $R_\delta = (\mathrm{Id} - \delta\Delta)^{-1}$ and W_ℓ is defined in (4.2). Since $\mathcal{F}_\ell^\delta$ is C^2 with locally uniformly continuous first and second derivatives, we can apply Itô's formula and deduce,

$$\begin{aligned}
&\mathcal{F}_\ell^\delta(u_t) + \int_0^t \int \sigma(u_s)(\Delta u_s - W'(u_s))(R_\delta \Delta R_\delta u_s^n - R_\eta W_\ell'(R_\delta u_s^n))\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \mathcal{F}_\ell^\delta(u_0) + \frac{1}{2}\int_0^t \mathrm{Tr}_{L^2}(B(u_s)^*\left[R_\delta(-\Delta)R_\delta + R_\delta W_\ell''(R_\delta u_s)R_\delta\right] B(u_s))\,\mathrm{d}s \\
&\qquad + \int_0^t \langle R_\delta(-\Delta)R_\delta u_s + R_\delta W_\ell'(R_\delta u_s), B(u_s)\,\mathrm{d}\alpha_s\rangle_{L^2} \quad \forall\, t \in [0,T] \quad \mathcal{P}\text{-a.s.}.
\end{aligned}$$

Recall that, by definition of martingale solution, $\mathbb{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$. Given $\kappa > 0$ let

$$\tau_\kappa := \inf\left\{t \in [0,T]\colon \int_0^t \|u_s\|_{H^2}^2 \,\mathrm{d}s > \kappa\right\},$$

setting $\tau_\kappa = T$ if the set in the right-hand side is empty. Note that $\tau_\kappa \uparrow T$ as $\kappa \to \infty$ $\mathbb{P}$-a.s. By stopping at τ_κ the Itô 's formula above, straightforward estimates (similar to those in the proof of Lemma 4.2) allow to take the limit $\delta \to 0$ and $\ell \to \infty$. By taking afterwards the limit as $\kappa \to \infty$ we finally get,

$$\begin{aligned}
&\mathcal{F}(u_t) + \int_0^t \int \sigma(u_s)(\Delta u_s - W'(u_s))^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \mathcal{F}(u_0) + \frac{1}{2}\int_0^t \mathrm{Tr}_{L^2}(B(u_s)^*[-\Delta + W''(u_s)]B(u_s))\,\mathrm{d}s \\
&\qquad + \int_0^t \langle -\Delta u_s + W'(u_s), B(u_s)\,\mathrm{d}\alpha_s\rangle_{L^2} \quad \forall\, t \in [0,T] \qquad \mathcal{P}\text{-a.s.}
\end{aligned}$$

Since $\mathbb{P}(u \in L^\infty(H^1) \cap L^2(H^2)) = 1$, $\mathcal{P}$-a.s. the second term on the left-hand side is $\mathcal{P}$-a.s. continuous and the trace in the second term on the right-hand side is $\mathcal{P}$-a.s. bounded uniformly in time. Indeed, arguing as in (4.10)–(4.14) we deduce the inequality,

$$\begin{aligned}
\mathrm{Tr}_{L^2}(B(u_s)^*[-\Delta + W''(u_s)]B(u_s)) &\le C_\sigma \|j\|_{H^1}^2 \int (1 + |\nabla u_s|^2 + |W''(u_s)|)\,\mathrm{d}x \\
&\le C_\sigma \|j\|_{H^1}^2 (1 + \mathcal{F}(u_s)).
\end{aligned}$$

Combining these facts with the a.s. continuity of the stochastic integral we get the $\mathcal{P}$-a.s. continuity of the map $t \mapsto \mathcal{F}(u_t)$. □

By means of an H^{-1} estimate inspired by [1], we next prove pathwise uniqueness of weak solutions.

Proposition 5.3. *Let $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ and $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (v, \alpha'))$ be two weak solutions to (2.1) with initial condition $\bar{u}_0 \in H^1$ defined on the same filtered space. If $\alpha' = \alpha$ then $\mathcal{P}(u_t = v_t \;\forall\, t \in [0,T]) = 1$.*

Proof. We observe that by $\mathcal{P}$-a.s. continuity it is enough to show that $\mathcal{P}(u_t = v_t) = 1$ for any $t \in [0,T]$. For $\kappa > \kappa_0 := 2\|\bar{u}_0\|_{H^1}$ we introduce the stopping time $\tau_\kappa\colon \Omega \to \mathbb{R}_+$ defined by

$$\tau_\kappa := \inf\{t \in [0,T]\colon \|u_t\|_{H^1} + \|v_t\|_{H^1} > \kappa\},$$

setting $\tau_\kappa = T$ if the set in the right-hand side is empty. Note that, $\mathcal{P}$-a.s. the H^1-continuity yields $\tau_\kappa > 0$ and $\tau_\kappa \uparrow T$ as $\kappa \to \infty$. Therefore, it is enough to prove that $\mathcal{P}(u_{t\wedge\tau_\kappa} = v_{t\wedge\tau_\kappa}) = 1$ for any $\kappa > \kappa_0$ and $t \in [0,T]$. This will be achieved by showing that the real random variable $\Psi_t\colon \Omega \to \mathbb{R}$ defined by

$$\Psi_t := \frac{1}{2}\big\|h(u_{t\wedge\tau_\kappa}) - h(v_{t\wedge\tau_\kappa})\big\|_{H^{-1}}^2, \qquad h(u) := \int_0^u \frac{1}{\sigma(r)}\,\mathrm{d}r,$$

is $\mathcal{P}$-a.s. vanishing for any $\kappa > \kappa_0$ and $t \in [0,T]$. To this purpose, we estimate the evolution in time of Ψ_t via Itô's calculus.

Since the function $\Psi\colon L^2\times L^2\to\mathbb{R}$ defined by $\Psi(u,v):=\frac{1}{2}\big\|h(u)-h(v)\big\|^2_{H^{-1}}$ is not twice differentiable, we proceed by approximation. We introduce the regularized version of Ψ defined by $\Psi^\delta(u,v):=\frac{1}{2}\big\|h(R_\delta u)-h(R_\delta v)\big\|^2_{H^{-1}}$. Since $h\colon\mathbb{R}\to\mathbb{R}$ is C^3, $R_\delta\colon L^2\to H^2$ is bounded, and H^2 is compactly embedded in $C(\mathbb{T}^d)$, it is easy to show that Ψ^δ is C^2. Setting $f^\delta(u,v):=R_1(h(R_\delta u)-h(R_\delta v))$, the first derivative $(D\Psi^\delta)_{u,v}\in L^2\times L^2$ is given by

$$(D\Psi^\delta)_{u,v}=R_\delta\begin{pmatrix}h'(R_\delta u)f^\delta(u,v)\\ h'(R_\delta v)f^\delta(u,v)\end{pmatrix},$$

and the second derivative $(D^2\Psi^\delta)_{u,v}\colon L^2\times L^2\to L^2\times L^2$ reads,

$$\begin{aligned}(D^2\Psi^\delta)_{u,v}=R_\delta&\begin{pmatrix}h''(R_\delta u)f^\delta(u,v) & 0\\ 0 & -h''(R_\delta v)f^\delta(u,v)\end{pmatrix}R_\delta\\ &+R_\delta\begin{pmatrix}h'(R_\delta u)R_1h'(R_\delta u) & -h'(R_\delta u)R_1h'(R_\delta v)\\ -h'(R_\delta v)R_1h'(R_\delta u) & h'(R_\delta v)R_1h'(R_\delta v)\end{pmatrix}R_\delta.\end{aligned}$$

As h is Lipschitz we have $\|(D\Psi^\delta)_{u,v}\|_{L^2\times L^2}\le C(\|u\|_{L^2}+\|v\|_{L^2})$ hence the first derivative is bounded on bounded subsets of $L^2\times L^2$. As $R_\delta\colon L^2\hookrightarrow C(\mathbb{T}^d)$ is compact, $L^2\ni u\mapsto h'(R_\delta u)\in C(\mathbb{T}^d)$ is uniformly continuous on bounded subset, hence the uniform continuity of $D\Psi^\delta$ on bounded subset follows. Concerning the second derivative, notice that since h' and h'' are bounded and $\|f^\delta(u,v)\|_{L^\infty}\le C(\|u\|_{L^2}+\|v\|_{L^2})$ with a constant independent of δ, for each $\phi_1,\phi_2\in L^2$ we have

$$\left|\left\langle\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},(D^2\Psi^\delta)_{u,v}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right\rangle_{L^2\times L^2}\right|\le C(\|u\|_{L^2}+\|v\|_{L^2}+1)(\|\phi_1\|^2_{L^2}+\|\phi_2\|^2_{L^2}),\tag{5.4}$$

where the constant C does not depend on δ, ϕ_1, and ϕ_2. The same argument used for the first derivative entails that the second derivative $D^2\Psi^\delta$ is uniformly continuous on bounded subsets of $L^2\times L^2$.

By Itô's formula (notice $\Psi^\delta(u_0,v_0)=\Psi^\delta(\bar u_0,\bar u_0)=0$),

$$\begin{aligned}\Psi^\delta(u_t,v_t)=&\int_0^t\left\langle(D\Psi^\delta)_{u_s,v_s},\begin{pmatrix}\sigma(u_s)(\Delta u_s-W'(u_s))\\ \sigma(v_s)(\Delta v_s-W'(v_s))\end{pmatrix}\right\rangle_{L^2\times L^2}\mathrm{d}s\\ &+\int_0^t\mathrm{Tr}_{L^2\times L^2}\big((D^2\Psi^\delta)_{u_s,v_s}\mathbb{B}(u_s,v_s)\mathbb{B}(u_s,v_s)^*\big)\,\mathrm{d}s\\ &+\int_0^t\big\langle(D\Psi^\delta)_{u_s,v_s},\mathbb{B}(u_s,v_s)\,\mathrm{d}\alpha_s\big\rangle_{L^2\times L^2},\end{aligned}\tag{5.5}$$

where $\mathbb{B}$ is the Hilbert-Schmidt operator on $L^2\times L^2$ defined by

$$\mathbb{B}(u_s,v_s)=\begin{pmatrix}B(u_s)\\ B(v_s)\end{pmatrix}.$$

In order to take the limit as $\delta \to 0$ in the previous identinty, notice that for $u, v \in C(L^2)$ we have $\Psi^\delta(u_t, v_t) \to \Psi(u_t, v_t)$ and

$$\lim_{\delta \to 0} (D\Psi^\delta)_{u_t, v_t} = \begin{pmatrix} h'(u_t)R_1(h(u_t) - h(v_t)) \\ h'(v_t)R_1(h(u_t) - h(v_t)) \end{pmatrix},$$

in $L^2 \times L^2$ uniformly for $t \in [0, T]$, as h, h' are Lipschitz and $R_1 \colon L^2 \hookrightarrow L^\infty$. Since $\mathcal{P}$-a.s. $u, v \in C(H^1) \cap L^2(H^2)$, this allows to pass to the limit also in the first term on the r.h.s. of (5.5) by dominated convergence. Moreover, the same uniform convergence together with the computation of the quadratic variation allows to pass to the limit (up to subsequences) in the stochastic integral. Finally, we rewrite the trace term in (5.5) as

$$\begin{aligned} &\mathrm{Tr}_{L^2 \times L^2}\big((D^2\Psi^\delta)_{u_s, v_s}\mathbb{B}(u_s, v_s)\mathbb{B}(u_s, v_s)^*\big) \\ &\qquad = \sum_k \big\langle \mathbb{B}(u_s, v_s)e_k, (D^2\Psi^\delta)_{u_s, v_s}\mathbb{B}(u_s, v_s)e_k \big\rangle_{L^2 \times L^2}, \end{aligned}$$

where $\{e_k\} \subset L^2$ is an orthonormal basis. Since $\mathbb{B}$ is Hilbert-Schmidt and the bound (5.4) holds, in order to take the limit in the trace term by dominated convergence w.r.t. to k it is enough to show that

$$\begin{aligned} \lim_{\delta \to 0} (D^2\Psi^\delta)_{u_s, v_s} &= \begin{pmatrix} h''(u_s)R_1(h(u_s) - h(v_s)) & 0 \\ 0 & -h''(v_s)R_1(h(u_s) - h(v_s)) \end{pmatrix} \\ &\quad + \begin{pmatrix} h'(u_s)R_1h'(u_s) & -h'(u_s)R_1h'(v_s) \\ -h'(v_s)R_1h'(u_s) & h'(v_s)R_1h'(v_s) \end{pmatrix}, \end{aligned} \tag{5.6}$$

in the weak operator topology on $L^2 \times L^2$, uniformly for $s \in [0, T]$, as h, h', h'' are Lipschitz and $R_1 \colon L^2 \hookrightarrow L^\infty$.

By stopping at τ_κ and recalling that $h' \equiv 1/\sigma$, we finally get

$$\begin{aligned} \Psi_t &= \int_0^{t \wedge \tau_\kappa} \big\langle R_1(h(u_s) - h(v_s)), \Delta(u_s - v_s) - W'(u_s) + W'(v_s) \big\rangle_{L^2} \,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^{t \wedge \tau_\kappa} \mathrm{Tr}_{L^2}\Big(\big[R_1(h(u_s) - h(v_s))\big]\big[h''(u_s)B(u_s)B(u_s)^* - h''(v_s)B(v_s)B(v_s)^*\big] \\ &\quad + (B(u_s)^*h'(u_s) - B(v_s)^*h'(v_s))R_1(h'(u_s)B(u_s) - h'(v_s)B(v_s))\Big)\,\mathrm{d}s \\ &\quad + \int_0^{t \wedge \tau_\kappa} \Big\langle R_1[h(u_s) - h(v_s)], \Big[\frac{1}{\sigma(u_s)}B(u_s) - \frac{1}{\sigma(v_s)}B(v_s)\Big]\,\mathrm{d}\alpha_s \Big\rangle_{L^2}. \end{aligned}$$

Let $\{e_k\} \subset L^2$ be the Fourier orthonormal basis and define

$$f_s = R_1(h(u_s) - h(v_s)), \qquad \beta_s := \frac{1}{\sqrt{\sigma(u_s)}} - \frac{1}{\sqrt{\sigma(v_s)}}.$$

By using that $R_1\Delta = -\mathrm{Id} + R_1$, and recalling the definitions of $h(u)$ and $B(u)$, after some simple algebraic computations evaluating the trace by Fourier series the above identity reads,

$$\Psi_t + \int_0^{t \wedge \tau_\kappa} \big\langle h(u_s) - h(v_s), u_s - v_s \big\rangle_{L^2} \,\mathrm{d}s = I_t^1 + I_t^2 + I_t^3 + M_{t \wedge \tau_\kappa}, \tag{5.7}$$

where

$$I_t^1 := \int_0^{t\wedge\tau_\kappa} \big\langle f_s, u_s - v_s - W'(u_s) + W'(v_s)\big\rangle_{L^2}\, \mathrm{d}s,$$

$$I_t^2 := \|j\|_{L^2}^2 \int_0^{t\wedge\tau_\kappa} \int f_s(x) \left(\frac{\sigma'(v_s(x))}{\sigma(v_s(x))} - \frac{\sigma'(u_s(x))}{\sigma(u_s(x))}\right) \mathrm{d}x,$$

$$I_t^3 := \int_0^{t\wedge\tau_\kappa} \sum_{k\in\mathbb{Z}^d} \big\langle \beta_s j * e_k, R_1\beta_s j * e_k\big\rangle_{L^2},$$

and $M_{t\wedge\tau_\kappa}$ is the stochastic integral (the last term in the previous Itô's formula).

By Assumption 2.1, Hölder inequality, and the Sobolev embedding $H^1 \hookrightarrow L^6$ we have,

$$\begin{aligned}
|I_t^1| &\le \int_0^{t\wedge\tau_\kappa} \big\langle |f_s|, |u_s - v_s|(1 + u_s^2 + v_s^2)\big\rangle_{L^2}\, \mathrm{d}s \\
&\le \int_0^{t\wedge\tau_\kappa} \|f_s\|_{L^6}\|u_s - v_s\|_{L^2}\|1 + u_s^2 + v_s^2\|_{L^3}\, \mathrm{d}s \\
&\le C\int_0^{t\wedge\tau_\kappa} \|f_s\|_{H^1}\|u_s - v_s\|_{L^2}\big(1 + \|u_s\|_{H^1}^2 + \|v_s\|_{H^1}^2\big)\, \mathrm{d}s \\
&\le C(1+\kappa^2)\int_0^{t\wedge\tau_\kappa} \sqrt{\Psi_s}\, \|u_s - v_s\|_{L^2}\, \mathrm{d}s,
\end{aligned}$$

where in the last inequality we used that $\|f_s\|_{H^1} = \sqrt{2\Psi_s}$ and $\|u_s\|_{H^1}^2 + \|v_s\|_{H^1}^2 \le 2\kappa^2$ for any $s \le \tau_\kappa$.

To estimate I_t^2 we observe that by Assumption 2.1 there is $C > 0$ for which $\big|\sigma(a)^{-1}\sigma'(a) - \sigma(b)^{-1}\sigma'(b)\big| \le C|a-b|$, so that, from Cauchy-Schwartz inequality and arguing as before,

$$\begin{aligned}
|I_t^2| \le C\|j\|_{L^2}^2 \int_0^{t\wedge\tau_\kappa} \int f_s(x)|u_s(x) - v_s(x)|\, \mathrm{d}x\, \mathrm{d}s \le C\int_0^{t\wedge\tau_\kappa} \|f_s\|_{H^1}\|u_s - v_s\|_{L^2} \\
\le C\int_0^{t\wedge\tau_\kappa} \sqrt{\Psi_s}\, \|u_s - v_s\|_{L^2}\, \mathrm{d}s.
\end{aligned}$$

In view of the definition of $h(u)$ and Assumption 2.1, it is easy to show that, for a suitable $C > 0$,

$$\max\{(a-b)^2; (h(a) - h(b))^2\} \le C(h(a) - h(b))(a-b). \tag{5.8}$$

Therefore, by the previous estimates and Young inequality,

$$|I_t^1| + |I_t^2| \le \frac{1}{2}\int_0^{t\wedge\tau_\kappa} \big\langle h(u_s) - h(v_s), u_s - v_s\big\rangle_{L^2}\, \mathrm{d}s + C(1+\kappa^2)\int_0^{t\wedge\tau_\kappa} \Psi_s\, \mathrm{d}s. \tag{5.9}$$

To estimate I_t^3, we first notice that there is $C > 0$ such that, for any $f \in H^1$ and $g \in H^{-1/2}$,

$$\|fg\|_{H^{-1}} \le C\|f\|_{H^1}\|g\|_{H^{-1/2}}. \tag{5.10}$$

Indeed, by Fourier expansion and Parseval identity,

$$\begin{aligned}\|fg\|_{H^{-1}}^2 &= \sum_{k\in\mathbb{Z}^d} \frac{\big|\widehat{fg}(k)\big|^2}{1+|k|^2} = \sum_{k\in\mathbb{Z}^d} \frac{1}{1+|k|^2}\bigg|\frac{1}{(2\pi)^{d/2}}\sum_{k'\in\mathbb{Z}^d}\widehat{f}(k-k')\widehat{g}(k')\bigg|^2\\ &\le \sum_{k\in\mathbb{Z}^d}\frac{1}{1+|k|^2}\bigg(\sum_{k'\in\mathbb{Z}^d}|\widehat{f}(k-k')||\widehat{g}(h)|\bigg)^2\\ &\le \sum_{k\in\mathbb{Z}^d}\frac{1}{1+|k|^2}\sum_{k'\in\mathbb{Z}^d}|\widehat{f}(k-k')|^2(1+|k-k'|^2)\sum_{k'\in\mathbb{Z}^d}\frac{|\widehat{g}(k')|^2}{1+|k-k'|^2}\\ &= \|f\|_{H^1}^2\sum_{k'\in\mathbb{Z}^d}|\widehat{g}(k')|^2\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|^2)(1+|k-k'|^2)} \le C\|f\|_{H^1}^2\|g\|_{H^{-1/2}}^2,\end{aligned}$$

where we used that there is $C>0$ such that, for $d=1,2,3$,

$$\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|^2)(1+|k-k'|^2)} \le \frac{C}{\sqrt{1+|k'|^2}}.$$

By (5.10) and standard interpolation,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\big\langle \beta_s j*e_k, R_1\beta_s j*e_k\big\rangle_{L^2} = \sum_{k\in\mathbb{Z}^d}\|\beta_s j*e_k\|_{H^{-1}}^2\\ &\le C\sum_{k\in\mathbb{Z}^d}\|j*e_k\|_{H^1}^2\|\beta_s\|_{H^{-1/2}}^2 \le C\|j\|_{H^1}^2\|\beta_s\|_{H^{-1/2}}^2 \le C\|\beta_s\|_{L^2}\|\beta_s\|_{H^{-1}}.\end{aligned} \tag{5.11}$$

By the definition of $h(\cdot)$ and Assumption 2.1 it is straightforward to verify that $|\beta_s| \le C|h(u_s)-h(v_s)|$. Moreover, we claim that $\gamma_s := (h(u_s)-h(v_s))^{-1}\beta_s \in L^\infty\cap H^1$ and that, for a suitable $C>0$, $\|\gamma_s\|_{H^1} \le C(1+\|u_s\|_{H^1}+\|v_s\|_{H^1})$. To see this, notice that the function $\tilde\sigma(r) := \sigma(h^{-1}(r))^{-1/2}$ is C^2 with bounded derivatives, and satisfies

$$\gamma_s = \int_0^1 \tilde\sigma'(h(v_s(x))+\lambda(h(u_s(x))-h(v_s(x))))\,\mathrm{d}\lambda,$$

from which the claim follows.

By using (5.10), for any $s\le\tau_\kappa$,

$$\big\|\beta_s\big\|_{H^{-1}} \le C\big\|\gamma_s\big\|_{H^1}\big\|h(u_s)-h(v_s)\big\|_{H^{-1/2}} \le C(1+2\kappa)\big\|h(u_s)-h(v_s)\big\|_{H^{-1/2}},$$

hence, by (5.11), interpolation and Young inequality, for any $\varepsilon>0$,

$$\begin{aligned}|I_t^3| &\le C(1+2\kappa)\int_0^{t\wedge\tau_\kappa}\big\|h(u_s)-h(v_s)\big\|_{L^2}\big\|h(u_s)-h(v_s)\big\|_{H^{-1/2}}\,\mathrm{d}s\\ &\le C(1+2\kappa)\int_0^{t\wedge\tau_\kappa}\big\|h(u_s)-h(v_s)\big\|_{L^2}^{3/2}\big\|h(u_s)-h(v_s)\big\|_{H^{-1}}^{1/2}\,\mathrm{d}s\\ &\le \varepsilon\int_0^{t\wedge\tau_\kappa}\big\|h(u_s)-h(v_s)\big\|_{L^2}^2\,\mathrm{d}s + C_\varepsilon(1+\kappa^2)\int_0^{t\wedge\tau_\kappa}\big\|h(u_s)-h(v_s)\big\|_{H^{-1}}^2\,\mathrm{d}s.\end{aligned}$$

Since by (5.8) $\|h(u_s) - h(v_s)\|_{L^2}^2 \le \langle h(u_s) - h(v_s), u_s - v_s \rangle_{L^2}$, by choosing ε small enough we conclude that there is $C > 0$ such that,

$$|I_t^3| \le \frac{1}{2} \int_0^{t\wedge\tau_\kappa} \langle h(u_s) - h(v_s), u_s - v_s \rangle_{L^2} \, \mathrm{d}s + C(1+\kappa^2) \int_0^{t\wedge\tau_\kappa} \Psi_s \, \mathrm{d}s. \quad (5.12)$$

By (5.7), (5.9), and (5.12) we get,

$$\Psi_t \le C \int_0^{t\wedge\tau_\kappa} \Psi_s \, \mathrm{d}s + M_{t\wedge\tau_\kappa} \le C \int_0^t \Psi_s \, \mathrm{d}s + M_{t\wedge\tau_\kappa}.$$

By the optional stopping theorem $\mathcal{E}(M_{t\wedge\tau_\kappa}) = \mathcal{E}(M_0) = 0$; therefore, by taking the expectation in both sides and applying Gronwall's inequality we conclude that $\mathcal{E}(\Psi_t) = 0$, and therefore $\mathcal{P}$-a.s. $\Psi_t = 0$. □

By the Yamada-Watanabe argument [36] (see also [31,32]) we next deduce uniqueness of the law of weak solutions.

Proposition 5.4. *Given $\bar{u}_0 \in H^1$, the following holds.*

1. *The law $\mathbb{Q} = \mathcal{P} \circ (u, \alpha)^{-1}$ on $C(L^2) \times C(H^{-\bar{s}})$ is the same for any weak solution $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ to (2.1) with initial datum $\bar{u}_0$.*
2. *There exists a Borel map $\Theta \colon C(H^{-\bar{s}}) \to C(L^2)$, $\mathcal{B}_t(C(H^{-\bar{s}}))/\mathcal{B}_t$ measurable, and such that for any weak solution $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ we have $u = \Theta \circ \alpha$ $\mathcal{P}$-a.s.*

Proof. The proof can be easily achieved by adapting the argument in [31] for finite dimensional diffusions. However, for the reader's convenience, we present the complete strategy.
1. Fix $\bar{u}_0 \in H^1$ and let $((\Omega^i, \mathcal{G}^i, \mathcal{G}_t^i, \mathcal{P}^i), (u^i, \alpha^i))$, $i = 1, 2$, be two weak solutions to (2.1) with initial condition $\bar{u}_0$. To take advantage of the pathwise uniqueness proved in Proposition 5.3, we need to bring them on a same filtered probability space.

Denote by $\mathbb{P}^*$ the law of the cylindrical Wiener process on $C(H^{-\bar{s}})$. Set also $\mathbb{Q}^i = \mathcal{P}^i \circ (u^i, \alpha^i)^{-1}$ be the probabilities on $C(L^2) \times C(H^{-\bar{s}})$ induced by the pair (u^i, α^i). As the spaces involved are Polish, these probabilities can be disintegrated w.r.t. $\mathbb{P}^*$ so that

$$\mathbb{Q}^i(\mathrm{d}v, \mathrm{d}v') = \mathbb{Q}^i_{v'}(\mathrm{d}v) \, \mathbb{P}^*(\mathrm{d}v'), \quad i = 1, 2,$$

where $\mathbb{P}^*$-a.s. $\mathbb{Q}^i_{v'}$ is a probability on $C(L^2)$. Moreover, $v' \mapsto \mathbb{Q}^i_{v'}(A)$ is a Borel map for any $A \in \mathcal{B}(C(L^2))$. In the sequel, we need the following result, which is a straightforward adaptation to the present context of [31, Chap. 4, Lemma (1.6)].

Lemma 5.5. *If $A \in \mathcal{B}_t(C(L^2))$, $t \in [0, T]$, the map $w^3 \mapsto \mathbb{Q}^i_{w^3}(A)$ is $\mathcal{B}_t(C(H^{-\bar{s}}))$-measurable up to a negligible set.*

Consider now the product space $\boldsymbol{W} := C(L^2) \times C(L^2) \times C(H^{-\bar{s}})$, whose elements are denoted by $w = (w^1, w^2, w^3)$. On $\boldsymbol{W}$ we define the probability measure,

$$\Pi(\mathrm{d}w^1, \mathrm{d}w^2, \mathrm{d}w^3) := \mathbb{Q}^1_{w^3}(\mathrm{d}w^1) \mathbb{Q}^2_{w^3}(\mathrm{d}w^2) \mathbb{P}^*(\mathrm{d}w^3),$$

and we endow $\boldsymbol{W}$ with the filtration $\mathcal{G}_t$ defined as the completion with respect to Π of the canonical filtration $\mathcal{B}_t(\boldsymbol{W})$.

We now claim that, for $i = 1, 2$, $((\boldsymbol{W}, \mathcal{G}, \mathcal{G}_t, \Pi), (w^i, w^3))$ are weak solutions to (2.1) with initial condition $\bar{u}_0$ on the same filtered space, and such that $\mathbb{P}^i = \mathcal{P}^i \circ (u^i)^{-1}$ is the law of w^i. The latter assertion, which is immediate by construction, clearly implies condition ii) in the definition of weak solution, hence it remains to verify conditions i) and iii).

To prove that w^3 is a L^2-cylindrical Wiener process with respect to $\mathcal{G}_t$, we only need to check that for any $0 \le s < t \in [0, T]$ the process $w^3_t - w^3_s$ is independent of $\mathcal{G}_s$. This property follows by noticing that, letting $A_1, A_2 \in \mathcal{B}_s(C(L^2))$, $B \in \mathcal{B}_s(C(H^{-\bar{s}}))$, by Lemma 5.5, for any $\psi \in C^\infty(\mathbb{T}^d)$,

$$\begin{aligned}
&\mathbb{E}^\Pi \big[\exp \big(\mathrm{i}\langle \psi, w^3_t - w^3_s \rangle \big) \mathbb{I}_{w^1 \in A_1} \mathbb{I}_{w^2 \in A_2} \mathbb{I}_{w^3 \in B} \big] \\
&\quad = \int_B \exp \big(\mathrm{i}\langle \psi, w^3_t - w^3_s \rangle \big) \mathbb{Q}^1_{w^3}(A_1) \mathbb{Q}^2_{w^3}(A_2) \mathbb{P}^*(\mathrm{d}w^3) \\
&\quad = \int_B \bigg[\int \exp \big(\mathrm{i}\langle \psi, \bar{w}_t - \bar{w}_s \rangle \big) \, \mathbb{P}^*_{t,w^3}(\mathrm{d}\bar{w}) \bigg] \mathbb{Q}^1_{w^3}(A_1) \mathbb{Q}^2_{w^3}(A_2) \mathbb{P}^*(\mathrm{d}w^3) \\
&\quad = \exp \big(-(t-s) \|\psi\|^2_{L^2} \big) \int_B \mathbb{Q}^1_{w^3}(A_1) \mathbb{Q}^2_{w^3}(A_2) \mathbb{P}^*(\mathrm{d}w^3) \\
&\quad = \exp \big(-(t-s) \|\psi\|^2_{L^2} \big) \, \Pi(A_1 \times A_2 \times B),
\end{aligned}$$

where $\mathbb{P}^*_{t,w^3}$ is a regular version of the conditional probability $\mathbb{P}^*(\,\cdot\,| w^3_s, s \in [0, t])$.

Let $\psi \in C^\infty([0, T] \times \mathbb{T}^d)$ and $t \in [0, T]$. Since $\mathcal{P}^i$-a.s.,

$$\begin{aligned}
\langle u^i_t, \psi_t \rangle_{L^2} &= \langle \bar{u}_0, \psi_0 \rangle_{L^2} + \int_0^t \langle u^i_s, \partial_s \psi_s \rangle_{L^2} \, \mathrm{d}s \\
&\quad + \int_0^t \langle \sigma(u^i_s)(\Delta u^i_s - W'(u^i_s)), \psi_s \rangle_{L^2} \, \mathrm{d}s + \int_0^t \langle \psi_s, B(u^i_s) \, \mathrm{d}\alpha^i_s \rangle_{L^2},
\end{aligned}$$

then, as follows from, e.g., [31, Chap. 4, Ex. (5.16)], Π-a.s.,

$$\begin{aligned}
\langle w^i_t, \psi_t \rangle_{L^2} &= \langle \bar{u}_0, \psi_0 \rangle_{L^2} + \int_0^t \langle w^i_s, \partial_s \psi_s \rangle_{L^2} \, \mathrm{d}s \\
&\quad + \int_0^t \langle \sigma(w^i_s)(\Delta w^i_s - W'(w^i_s)), \psi_s \rangle_{L^2} \, \mathrm{d}s + \int_0^t \langle \psi_s, B(w^i_s) \, \mathrm{d}w^3_s \rangle_{L^2},
\end{aligned}$$

which is property iii) in the definition of weak solutions.

By the pathwise uniqueness in Proposition 5.3, $\Pi((w^1_t, w^3_t) = (w^2_t, w^3_t) \ \forall\, t \in [0, T]) = 1$, which implies $\mathbb{Q}^1 = \mathbb{Q}^2$, i.e., the uniqueness of the law of weak solutions.

2. Let $((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ be a weak solution of (2.1) with initial condition $\bar{u}_0$, whose existence is ensured by Lemma 5.1. We apply the previous construction with $((\Omega^i, \mathcal{G}^i, \mathcal{G}^i_t, \mathcal{P}^i), (u^i, \alpha^i)) = ((\Omega, \mathcal{G}, \mathcal{G}_t, \mathcal{P}), (u, \alpha))$ for $i = 1, 2$. Thus, for

$$\begin{aligned}
\mathcal{P} \circ (u, \alpha)^{-1} = \mathbb{Q}, \qquad \mathbb{Q}(\mathrm{d}w^i, \mathrm{d}w^3) = \mathbb{Q}_{w^3}(\mathrm{d}w^i) \, \mathbb{P}^*(\mathrm{d}w^3), \quad i = 1, 2, \\
\Pi(\mathrm{d}w^1, \mathrm{d}w^2, \mathrm{d}w^3) = \mathbb{Q}_{w^3}(\mathrm{d}w^1) \mathbb{Q}_{w^3}(\mathrm{d}w^2) \mathbb{P}^*(\mathrm{d}w^3),
\end{aligned}$$

pathwise uniqueness yields $\Pi(w^1_t = w^2_t \ \forall\, t \in [0,T]) = 1$. As a consequence, the processes w^1 and w^2 are simultaneously equal and independent under the measure $\mathbb{Q}_{w^3}(\mathrm{d}w^1)\mathbb{Q}_{w^3}(\mathrm{d}w^2)$ for $\mathbb{P}^*$-a.s w^3. This is possible only if there exists a Borel map $\Theta\colon C(H^{-\bar{s}}) \to C(L^2)$ such that $\mathbb{Q}_{w^3} = \delta_{\Theta(w^3)}$ for $\mathbb{P}^*$-a.s w^3. Furthermore, in view of Lemma 5.5, Θ is $\mathcal{B}_t(C(H^{-\bar{s}}))/\mathcal{B}_t(C(L^2))$ measurable. Therefore, $\mathbb{Q}(\mathrm{d}w^1, \mathrm{d}w^3) = \delta_{\Theta(w^3)}(\mathrm{d}w^1)\,\mathbb{P}^*(\mathrm{d}w^3)$, whence $u = \Theta \circ \alpha$ $\mathcal{P}$-a.s. □

Proof of Theorem 2.2. By Theorem 4.1, for each initial datum $\bar{u}_0 \in H^1$ there exists a martingale solution $\mathbb{P}$ to (2.1) satisfying (4.1), which implies (2.8) in view of the growth assumptions of the potential W. Moreover, $\mathbb{P}(u \in C(H^1)) = 1$ in view of Lemma 5.2. By Lemma 5.1 and item a) of Proposition 5.4, the uniqueness of the martingale solution $\mathbb{P}$ follows.

The pathwise uniqueness proved in Proposition 5.3 clearly implies the uniqueness of strong solutions. Therefore, we are left with the proof of existence of strong solutions.

Given $\bar{u}_0 \in H^1$ and a probability space $(\Omega, \mathcal{G}, \mathcal{P})$ equipped with a cylindrical Wiener process α, we claim that the process $u = \Theta \circ \alpha$, with Θ as given in item b) of Proposition 5.4, is a strong solution to (2.1) with initial datum $\bar{u}_0$. To show this, we observe that the law of (u, α) is equal to the law $\mathbb{Q}$ of weak solutions, uniquely determined according to Proposition 5.4. In addition, as seen in the proof of that proposition, denoting by (w^1, w^2) the elements of $C(L^2) \times C(H^{-\bar{s}})$ and by $\tilde{\mathcal{G}}$ [resp. $\tilde{\mathcal{G}}_t$] the σ-algebra $\mathcal{B}(C(L^2) \times C(H^{-\bar{s}}))$ [resp. filtration $\mathcal{B}_t(C(L^2) \times C(H^{-\bar{s}}))$] completed under $\mathbb{Q}$, the pair $((C(L^2) \times C(H^{-\bar{s}}), \tilde{\mathcal{G}}, \tilde{\mathcal{G}}_t, \mathbb{Q}), (w^1, w^2))$ is a weak solution to (2.1) with initial datum $\bar{u}_0$. Therefore, setting $\mathcal{G}_t = (u, \alpha)^{-1}(\tilde{\mathcal{G}}_t)$, the same reasoning as in the proof of Proposition 5.4, based on [31, Chap. 4, Ex. (5.16)], implies that the pair $((\Omega, \mathcal{G}_t, \mathcal{G}, \mathcal{P}), (u, \alpha))$ is a weak solution to (2.1) with initial datum $\bar{u}_0$. Since Θ is $\mathcal{B}_t(C(H^{-\bar{s}}))/\mathcal{B}_t(C(L^2))$ measurable, the process u is $\mathcal{G}^\alpha_t$-adapted, hence u is a strong solution. □

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Appendix A. A class of C_0-semigroups

In this appendix we prove the following lemma, concerning the generation of C_0-semigroups on the Sobolev space H^1. Here we set $\mathcal{H} := H^1$ and denote the norm and inner product in $\mathcal{H}$ simply by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$.

Lemma A.1. *Let $\mathcal{H} = H^1$, $d = 2, 3$, $v \in H^2$ and $A\colon H^3 \subset \mathcal{H} \to \mathcal{H}$ defined by $Au = \sigma(v)\Delta u$. Then the following holds.*

(1) A is closed, densely defined, and it generates a C_0-semigroup $S(t)$, $t \ge 0$, on $\mathcal{H}$ satisfying $\|S(t)\| \le e^{m_0 t}$ for any $t \ge 0$ for some $m_0 > 0$ (depending only on σ).

(2) *Given* $\delta > 0$ *let* $R_\delta := (\mathrm{Id} - \delta\Delta)^{-1}$ *and* $A_\delta \colon \mathcal{H} \to \mathcal{H}$ *be defined by* $A_\delta u := \sigma(R_\delta v) R_\delta \Delta R_\delta u$*. Then* A_δ *is a bounded (indeed compact) operator on* $\mathcal{H}$ *and generates a uniformly continuous (semi)group of linear operators* $S_\delta(t)$*,* $t \geq 0$*. Moreover,* $\|S_\delta(t)\| \leq e^{m_0 t}$ *for any* $\delta > 0$ *small enough (depending only on* $\sigma(v)$*) and any* $t \geq 0$*.*

(3) *Consider the linear operator* $\lim_{\delta\to 0} A_\delta$ *defined on* $\{u \in \mathcal{H} \colon \exists \lim_{\delta \to 0} A_\delta u\}$ *as* $(\lim_{\delta\to 0} A_\delta)u := \lim_{\delta\to 0} A_\delta u$*. Then* $\lim_{\delta\to 0} A_\delta = A$ *as unbounded operators and* $S_\delta(t)u \to S(t)u$ *in* $\mathcal{H}$ *for every* $u \in \mathcal{H}$ *and every* $t \geq 0$*.*

Proof. (1) The operator A is densely defined and closed. Indeed, let $Au_n = f_n \to f$ in $\mathcal{H}$ and $\{u_n\} \subset H^3$, $u_n \to u$ in $\mathcal{H}$. Since $v \in H^2$ and $d \leq 3$, it is easy to check that multiplication by $\sigma(v)^{-1}$ is a bounded operator on $\mathcal{H}$, hence $\Delta u_n = \sigma(v)^{-1} f_n \to \sigma(v)^{-1} f$ in $\mathcal{H}$. By elliptic regularity $\{u_n\} \subset H^3$ is bounded, hence $u \in H^3$. In addition, $u_n \to u$ in H^2 and therefore $Au = f$ in L^2 and in turn in $\mathcal{H}$.

The rest of statement (1) follows from the Lumer-Phillips theorem [29] once we prove that there exists $m_0 > 0$ such that

a) $\|((m+m_0)\mathrm{Id} - A)u\| \geq m\|u\|$ for any $m > 0$ and $u \in H^3$;

b) $A - (m+m_0)\mathrm{Id}$ is surjective for each $m > 0$.

To prove a) it is clearly enough to show that $\langle (m_0 \mathrm{Id} - A)u, u\rangle \geq 0$ for some $m_0 > 0$ and any $u \in H^3$. We have,

$$
\begin{aligned}
\langle (m_0 \mathrm{Id} - A)u, u\rangle &= m_0\|u\|^2 - \int \sigma(v) u \Delta u \,\mathrm{d}x + \int \sigma(v) \left(\Delta u\right)^2 \,\mathrm{d}x \\
&\geq m_0 \|u\|^2 + (\inf \sigma) \int (\Delta u)^2 \,\mathrm{d}x - (\sup \sigma) \int |u||\Delta u| \,\mathrm{d}x \\
&\geq \left(m_0 - \frac{(\sup \sigma)^2}{2(\inf \sigma)} \right) \|u\|^2 + \frac{1}{2} (\inf \sigma) \int (\Delta u)^2 \,\mathrm{d}x,
\end{aligned}
$$

where we used Young's inequality in the last step. Claim a) follows for m_0 large enough.

To prove b), for $\lambda \in [0,1]$ we consider the family of bounded operators $A^\lambda \colon H^3 \to \mathcal{H}$ defined by $A^\lambda u = -(m+m_0)u + (\lambda\sigma(v) + 1 - \lambda)\,\Delta u$. Notice that $[0,1] \ni \lambda \to A^\lambda \in \mathcal{L}(H^3;\mathcal{H})$ is norm continuous and $A^0 = -(m+m_0)\mathrm{Id} + \Delta$ is surjective (actually a Banach space isomorphism). Thus, by the continuity method, claim b) follows once we prove that there exists $c > 0$ such that $\|u\|_{H^3} \leq c\|A^\lambda u\|$ for any $u \in H^3$ and $\lambda \in [0,1]$. Arguing as in the last displayed formula we get,

$$
\left(m + m_0 - \frac{(\max\{1, \sup \sigma\})^2}{2 \min\{(1, \inf \sigma)\}} \right) \|u\|^2 + \frac{1}{2} \min\{1, \inf \sigma\} \int (\Delta u)^2 \,\mathrm{d}x \leq \|A^\lambda u\| \|u\|,
$$

and, by Young's inequality, $\|u\| \leq c\|A^\lambda u\|$ for a suitable $c > 0$ depending only on m_0, m and σ.

Finally, notice that $\Delta u = (\lambda\sigma(v) + 1 - \lambda)^{-1} \left(A^\lambda u + (m+m_0)u\right) \in \mathcal{H}$, which together with the previous inequality gives $\|\Delta u\| \leq C(m_0, m, \sigma, v)\|A^\lambda u + (m+m_0)u\| \leq C'(m_0, m, \sigma, v)\|A^\lambda u\|$ and the conclusion follows by elliptic regularity.

(2) First we notice that $R_\delta\colon H^s \to H^{s+2}$ is continuous (a linear isomorphism), $\Delta R_\delta u = R_\delta \Delta u$, $\|R_\delta u\|_{H^s} \le \|u\|_{H^s}$, and $R_\delta u \to u$ in H^s as $\delta \to 0$ for any $u \in H^s$ and $s \in \mathbb{R}$. Note also that $R_\delta v \in H^4 \subset C^2$ for $d = 2, 3$, hence $\sigma(R_\delta v) \in C^2$. Since $R_\delta \Delta R_\delta\colon \mathcal{H} \to H^3$ is bounded, $H^3 \hookrightarrow H^2$ is compact, and multiplication by $\sigma(R_\delta v)$ is bounded on H^2 we see that $A_\delta\colon \mathcal{H} \to H^2$ is compact, hence $A_\delta\colon \mathcal{H} \to \mathcal{H}$ is a compact operator. Therefore, A_δ generates a uniformly continuous (semi)group of linear operators $S_\delta(t)$, $t \ge 0$. As in part (1) above, in view of the Lumer-Phillips theorem, the exponential estimate follows once we prove that there exists $m_0 > 0$ such that, for any $\delta > 0$ small enough,

a) $\|((m + m_0)\mathrm{Id} - A_\delta)u\| \ge m\|u\|$ for any $m > 0$ and $u \in H^3$;
b) $A_\delta - (m + m_0)\mathrm{Id}$ is surjective for each $m > 0$.

To prove a), again it is enough to check that there exists $m_0 > 0$ such that for any $\delta > 0$ small enough $\langle (m_0\mathrm{Id} - A_\delta)u, u\rangle \ge 0$ for any $u \in H^3$. Integrating by parts,

$$\langle (m_0\mathrm{Id} - A_\delta)u, u\rangle = m_0\|u\|^2 + I_\delta + II_\delta,$$

where

$$I_\delta := -\int \sigma(R_\delta v) u R_\delta \Delta R_\delta u \,\mathrm{d}x, \qquad II_\delta := \int \Delta u\, \sigma(R_\delta v) R_\delta \Delta R_\delta u \,\mathrm{d}x\,.$$

Now, using Sobolev inequality,

$$\begin{aligned} I_\delta &= \int \big(\sigma(R_\delta v)\nabla u \nabla R_\delta^2 u + u\sigma'(R_\delta v)\nabla R_\delta v \nabla R_\delta^2 u\big)\,\mathrm{d}x \\ &\ge -\left(\sup\sigma + C\|\sigma'\|_\infty \|v\|_{H^2}\right)\|u\|^2, \end{aligned}$$

which leads to chose $m_0 = \sup\sigma + C\|\sigma'\|_\infty\|v\|_{H^2}$. On the other hand, as $\mathrm{Id} = (\mathrm{Id} - \delta\Delta)R_\delta$ we have,

$$\begin{aligned} II_\delta &= \int \sigma(R_\delta v)|\Delta R_\delta u|^2\,\mathrm{d}x + \int \Delta R_\delta u\big(\mathrm{Id} - \delta\Delta\big)\left[\sigma(R_\delta v), R_\delta\right]\Delta R_\delta u\,\mathrm{d}x \\ &\ge \Big(\inf\sigma - \big\|\big(\mathrm{Id} - \delta\Delta\big)\left[\sigma(R_\delta v), R_\delta\right]\big\|_{L_0^2 \to L^2}\Big)\int |\Delta R_\delta u|^2\,\mathrm{d}x, \end{aligned}$$

since $\Delta R_\delta u \in L_0^2$, the closed subspace of functions with zero average, and claim a) follows once we show that $\big\|\big(\mathrm{Id} - \delta\Delta\big)\left[\sigma(R_\delta v), R_\delta\right]\big\|_{L_0^2\to L^2} = o(1)$ as $\delta \to 0$.

To estimate the commutator, notice that for $w \in L_0^2$, $g = R_\delta w$ and $f = R_\delta(\sigma(R_\delta v)w)$, we have $f, g \in H^2$, $g - \delta\Delta g = w$, $f - \delta\Delta f = \sigma(R_\delta v)w$ and

$$\begin{aligned} \big(\mathrm{Id} - \delta\Delta\big)\left[\sigma(R_\delta v), R_\delta\right]w &= \big(\mathrm{Id} - \delta\Delta\big)(-f + \sigma(R_\delta v)g) \\ &= -\delta\big(g\Delta\sigma(R_\delta v) + 2\nabla g\nabla\sigma(R_\delta v)\big) \\ &= -\delta\big[\sigma'(R_\delta v)gR_\delta\Delta v + \sigma''(R_\delta v)g|\nabla R_\delta v|^2 + 2\sigma'(R_\delta v)\nabla g\nabla R_\delta v\big]. \end{aligned}$$

If $\varphi \in L_0^2$ and $\psi = R_\delta\varphi \in H^2$ solves $\psi - \delta\Delta\psi = \varphi$, then $\int \psi^2 + \delta|\nabla\psi|^2 \le \|\varphi\|_{L^2}\|\psi\|_{L^2}$, whence $\|\nabla\psi\|_{L^2} \le \delta^{-1/2}\|\varphi\|_{L^2}$ and, by Hölder and Sobolev embedding $H^1 \hookrightarrow L^6$, we have $\|\psi\|_{L^p} \le C\delta^{(6-3p)/4p}\|\varphi\|_{L^2}$ for any $2 \le p \le 6$. Analogously, $\int \delta|\nabla\psi|^2 + \delta^2|\Delta\psi|^2 \le \delta\|\varphi\|_{L^2}\|\Delta\psi\|_{L^2}$, whence $\|\Delta\psi\|_{L^2} \le$

$\delta^{-1}\|\varphi\|_{L^2}$ and by Holder, Sobolev and Calderon-Zygmund inequalities we have $\|\nabla\psi\|_{L^p} \le C\delta^{(6-5p)/4p}\|\varphi\|_{L^2}$ for any $2 \le p \le 6$. Applying these estimates to $\varphi = w$ and $\varphi = \Delta v \in L^2_0$ we have,

$$\left\|\left(\mathrm{Id} - \delta\Delta\right)\left[\sigma(R_\delta v), R_\delta\right] w\right\|_{L^2} \le \delta\|\sigma\|_{C^2}\left(\|g\|_{L^4}\|R_\delta \Delta v\|_{L^4} + \|g\|_{L^6}\|\nabla R_\delta v\|^2_{L^6}\right.$$
$$\left. + \|\nabla g\|_{L^4}\|\nabla R_\delta v\|_{L^4}\right) \le C\delta^{1/8}\|\sigma\|_{C^2}(1 + \|v\|^2_{H^2})\|w\|_{L^2},$$

so that $\left\|\left(\mathrm{Id} - \delta\Delta\right)\left[\sigma(R_\delta v), R_\delta\right]\right\|_{L^2_0 \to L^2} \le C\delta^{1/8}$ and the claim follows.

To prove b), notice that A_δ is a compact operator, hence $A_\delta - (m+m_0)\mathrm{Id}$ is Fredholm of index zero. Since it is injective by part a), then it is surjective.

(3) Concerning the first statement, we notice that $\lim_{\delta\to 0} A_\delta u = Au$ for any $u \in H^3$ because $R_\delta \Delta R_\delta u \to \Delta u$ in H^1, $\sigma(R_\delta v) \to \sigma(v)$ in H^2 and the product is jointly continuous for $d = 2, 3$. Conversely, suppose $\lim_{\delta\to 0} A_\delta u$ exists for some $u \in \mathcal{H}$, we claim that $u \in H^3$ and the limit is Au. To see this, notice that $A_\delta u$ is bounded in $\mathcal{H}$, hence $R_\delta \Delta R_\delta u = \sigma(R_\delta v)^{-1} A_\delta u$ is also bounded in $\mathcal{H}$, which implies $\Delta u \in \mathcal{H}$, where the Laplacian is taken in the sense of distributions. Then $u \in H^3$ by elliptic regularity and the conclusion follows from the initial observation.

To finish the proof it is enough to apply [35, Thm. 5.2] to infer convergence of semigroups from convergence of the corresponding resolvent operators at some common point. Fix $m_0 > 0$ as in part (1) and (2) above and $m > 0$ so that both $(m + m_0)\mathrm{Id} - A$ and $(m + m_0)\mathrm{Id} - A_\delta$ are injective on their respective domains and onto. We claim that $u_\delta := \left((m + m_0)\mathrm{Id} - A_\delta\right)^{-1} f \to \left((m + m_0)\mathrm{Id} - A\right)^{-1} f =: u$ as $\delta \to 0$ for any $f \in \mathcal{H}$. Indeed, by definition $(m + m_0)u_\delta - A_\delta u_\delta = f$ and in view of the dissipativity inequality (part (2) of the proof, claim a)), we have $m\|u_\delta\| \le \|f\|$ but indeed even $\|\Delta R_\delta u_\delta\|_{L^2} \le C(m, m_0, \|v\|_{H^2})\|f\|$ for $\delta > 0$ small enough. Thus, by elliptic regularity $R_\delta u_\delta$ is bounded in H^2 and, up to subsequences, $R_\delta u_\delta \to \bar{u}$ strongly in H^1 and weakly in H^2 as $\delta \to 0$ for some $\bar{u} \in H^2$ possibly depending on the subsequence. Observe that $u_\delta - R_\delta u_\delta = \delta\Delta R_\delta u_\delta \to 0$ in L^2, hence $u_\delta \rightharpoonup \bar{u}$ weakly in H^1. Since $\Delta R_\delta R_\delta u_\delta = \sigma(R_\delta v)^{-1}\left((m + m_0)u_\delta - f\right)$ is also bounded in $\mathcal{H}$, hence $R_\delta R_\delta u_\delta$ is bounded in H^3 by elliptic regularity, $R_\delta R_\delta u_\delta \to \bar{u}$ in H^1, which in turn gives $\bar{u} \in H^3$ and $R_\delta \Delta R_\delta u_\delta \rightharpoonup \Delta u$ weakly in H^1.

Since $H^2 \hookrightarrow C^0$ for $d = 2, 3$ and $\sigma(R_\delta v) \to \sigma(v)$ uniformly, taking L^2 scalar product of the equation for u_δ with some $g \in L^2$, as $\delta \to 0$ we get $\langle (m + m_0)\bar{u} - A\bar{u}, g\rangle_{L^2} = \langle f, g\rangle_{L^2}$, which in turn gives $(m + m_0)\bar{u} - A\bar{u} = f$ because g is arbitrary. By injectivity, $\bar{u} = u$ is independent of the chosen subsequence and so far we get $u_\delta \rightharpoonup u$ weakly in $\mathcal{H}$ and the proof is complete once we show that $\|u_\delta\| \to \|u\|$ as $\delta \to 0$. In order to conclude, we argue as in part (2) above and and we write,

$$(m_0 + m)\|u_\delta\|^2 + \int \sigma(R_\delta v)|\Delta R_\delta u_\delta|^2 \,\mathrm{d}x = \langle f, u_\delta\rangle$$
$$+ \int \sigma(R_\delta v) u_\delta R_\delta \Delta R_\delta u \,\mathrm{d}x - \int \Delta R_\delta u_\delta \left(\mathrm{Id} - \delta\Delta\right)\left[\sigma(R_\delta v), R_\delta\right] \Delta R_\delta u_\delta \,\mathrm{d}x.$$

Since $\left\| \left(\mathrm{Id} - \delta\Delta \right) \left[\sigma(R_\delta v), R_\delta \right] \right\|_{L_0^2 \to L^2} = o(1)$ as $\delta \to 0$, the previous convergence properties yields,

$$\varlimsup_{\delta\to 0} \Big[(m_0+m)\|u_\delta\|^2 + \int \sigma(R_\delta v)|\Delta R_\delta u_\delta|^2 \,\mathrm{d}x \Big] = \langle f, u_\delta\rangle + \int \sigma(v) u \Delta u \,\mathrm{d}x$$
$$= (m_0+m)\|u\|^2 + \int \sigma(v)|\Delta u|^2 \,\mathrm{d}x.$$

On the other hand, by L^2-weak lower semicontinuity,

$$\varlimsup_{\delta\to 0}(m_0+m)\|u_\delta\|^2 + \int \sigma(v)|\Delta u|^2 \,\mathrm{d}x \le \varlimsup_{\delta\to 0}(m_0+m)\|u_\delta\|^2$$
$$+ \varliminf_{\delta\to 0} \int \sigma(R_\delta v)|\Delta R_\delta u_\delta|^2 \,\mathrm{d}x \le \varlimsup_{\delta\to 0} \big[(m_0+m)\|u_\delta\|^2 + \int \sigma(R_\delta v)|\Delta R_\delta u_\delta|^2 \,\mathrm{d}x \big],$$

hence $\varlimsup_{\delta\to 0} \|u_\delta\|^2 \le \|u\|^2$ and therefore $\|u_\delta\| \to \|u\|$ as $\delta \to 0$ as claimed. □

References

[1] Alt, H.W., Luckhaus, S.: Quasilinear elliptic-parabolic differential equations. Math. Z. **183**, 311–341 (1983)

[2] Albeverio, S., Röckner, M.: Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Probab. Theory Related Fields **89**, 347–386 (1991)

[3] Bertini, L., Buttà, P., Pisante, A.: Stochastic Allen–Cahn approximation of the mean curvature flow: large deviations upper bound. Arch. Ration. Mech. Anal. **224**, 659–707 (2017)

[4] Bertini, L., Presutti, E., Rüdiger, B., Saada, E.: Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic PDE. Teor. Veroyatnost. i Primenen. **38**, 689–741 (1993); translation. Theory Probab. Appl. **38**, 586–629 (1993)

[5] Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)

[6] Cerrai, S.: Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. Probab. Theory Related Fields **125**, 271–304 (2003)

[7] Cerrai, S., Debussche, A.: Large deviations for the dynamic Φ_d^{2n} model. https://arxiv.org/pdf/1705.00541.pdf

[8] Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. **31**, 1900–1916 (2003)

[9] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions Encyclopedia of Mathematics and Its Applications, vol. 44. Cambridge University Press, Cambridge (1992)

[10] De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Motion by curvature by scaling nonlocal evolution equations. J. Statist. Phys. **73**, 543–570 (1993)

[11] Evans, L.C., Soner, H.M., Souganidis, P.E.: Phase transitions and generalized motion by mean curvature. Comm. Pure Appl. Math. **45**, 1097–1123 (1992)

[12] Fritz, J., Rüdiger, B.: Time dependent critical fluctuations of a one-dimensional local mean field model. Probab. Theory Related Fields **103**, 381–407 (1995)

[13] Gess, B.: Strong solutions for stochastic partial differential equations of gradient type. J. Funct. Anal. **263**, 2355–2383 (2012)

[14] Hairer, M.: A theory of regularity structures. Invent. Math. **198**, 269–504 (2014)

[15] Hausenblas, E.: Approximation for semilinear stochastic evolution equations. Potential Anal. **18**, 141–186 (2003)

[16] Hohenberg, P.C., Halperin, B.I.: Theory of dynamic critical phenomena. Rev. Mod. Phys. **49**, 435–479 (1977)

[17] Ilmanen, T.: Convergence of the Allen–Cahn equation to the Brakkes motion by mean curvature. J. Diff. Geom. **31**, 417–461 (1993)

[18] Jentzen, A.: Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Anal. **31**, 375–404 (2009)

[19] Jona-Lasinio, G., Mitter, P.K.: On the stochastic quantization of field theory. Commun. Math. Phys. **101**, 409–436 (1985)

[20] Katsoulakis, M.A., Souganidis, P.E.: Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics. Commun. Math. Phys. **169**, 61–97 (1995)

[21] Kohn, R., Otto, F., Reznikoff, M.G., Vanden-Eijnden, E.: Action minimization and sharp-interface limits for the stochastic Allen–Cahn equation. Commun. Pure Appl. Math. **60**, 393–438 (2007)

[22] Kovács, M., Larsson, S., Lindgren, F.: On the backward Euler approximation of the stochastic Allen–Cahn equation. J. Appl. Probab. **52**, 323–338 (2015)

[23] Krylov, N.V., Rozovskii, B.L.: Stochastic Evolution Equations. Plenum Publishing Corp., 1981; Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki **14**, 71–146 (1979)

[24] Liu, W.: Well-posedness of stochastic partial differential equations with Lyapunov condition. J. Differ. Equ. **255**, 572–592 (2013)

[25] Liu, W., Rckner, M.: SPDE in Hilbert space with locally monotone coefficients. J. Funct. Anal. **259**, 2902–2922 (2010)

[26] Mariani, M.: Large deviations principles for stochastic scalar conservation laws. Probab. Theory Related Fields **147**, 607–648 (2010)

[27] Mourrat J.-C., Weber, H.: Convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2. Preprint 2014, arXiv:1410.1179

[28] Mourrat J.-C., Weber, H.: The dynamic Φ^4_3 model comes down from infinity. Preprint (2017), arxiv:1601.01234

[29] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44. Springer, New York (1983)

[30] Prévôt, C., Röckner, M.: A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, vol. 1905. Springer, Berlin (2007)

[31] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edn. Springer, Berlin (1999)

[32] Röckner, M., Schmuland, B., Zhang, X.: Yamada–Watanabe theorem for stochastic evolution equations in infinite dimensions. Condens. Matter Phys. **11**, 247–259 (2008)

[33] Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, Berlin (1991)

[34] Spohn, H.: Interface motion in models with stochastic dynamics. J. Stat. Phys. **71**, 1081–1132 (1993)

[35] Trotter, H.F.: Approximation of semi-groups of operators. Pacific J. Math. **8**, 887–919 (1958)

[36] Yamada, T., Watanabe, S.: On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. **11**, 155–167 (1971)

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