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CONVERGENCE OF THE ONE-DIMENSIONAL CAHN–HILLIARD EQUATION*

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Abstract. We consider the Cahn–Hilliard equation in one space dimension with scaling parameter ε, i.e., $u_t = (W'(u) - \varepsilon^2 u_{xx})_{xx}$, where W is a nonconvex potential. In the limit $\varepsilon \downarrow 0$, under the assumption that the initial data are energetically well prepared, we show the convergence to a Stefan problem. The proof is based on variational methods and exploits the gradient flow structure of the Cahn–Hilliard equation.

Key words. Cahn–Hilliard equation, Gamma-convergence, forward-backward equations

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1. Introduction.

In this paper we are interested in the convergence of solutions $u_\varepsilon = u_\varepsilon(\cdot, \cdot, \overline{u}_\varepsilon)$ to the equation

$$\begin{cases} \mathrm{u}_t = \big(W'(\mathrm{u}) - \varepsilon^2 \mathrm{u}_{xx}\big)_{xx} & \text{in } (0, +\infty) \times \mathbb{T}, \\ \mathrm{u} = \overline{u}_\varepsilon & \text{on } \{0\} \times \mathbb{T} \end{cases} \tag{1.1}$$

as $\varepsilon \downarrow 0$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. Here ε is a spatial scale parameter and W is a rather general smooth potential. Our analysis covers, in particular, the choice of the double-well potential

$$W(\xi) = \frac{(1-\xi^2)^2}{4}, \qquad \xi \in \mathbb{R}, \tag{1.2}$$

corresponding to the Cahn–Hilliard equation. We refer the reader, for instance, to [5, 8] for the physical motivations leading to (1.1) in relation with the theory of phase transitions, and to [18, 2, 6] for some mathematical results and connections with the Stefan problem [14].

Equation (1.1) can be seen as the gradient flow, in the H^{-1}-topology, of the Allen–Cahn-type functional

$$F_\varepsilon(v) = \int_{\mathbb{T}} \left(\varepsilon^2 \frac{v_x^2}{2} + W(v) \right) dx, \tag{1.3}$$

where the scalar field v represents the local order parameter. The gradient flow structure of (1.1) allows us to look at the convergence of the functions u_ε in a purely variational way, at least under the assumption of energetically well-prepared initial data.

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The main difficulty in studying the limit of u_ε is due to the fact that when the function W is nonconvex, (1.1) is forward-backward parabolic for $\varepsilon = 0$. Looking at (1.1), it is rather natural to expect a limit equation related to the H^{-1}-gradient flow of the functional

$$F(v) = \int_{\mathbb{T}} W(v)\, dx. \tag{1.4}$$

However, when W is nonconvex, the functional F is not convex and not lower semicontinuous with respect to the H^{-1}-topology, and the gradient flow dynamics is not well-posed. The lower semicontinuous envelope of F is given by

$$F^{**}(v) = \int_{\mathbb{T}} W^{**}(v)\, dx,$$

where W^{**} denotes the convex envelope of W. It is not difficult to prove (see Proposition A.1) that F^{**} is the Γ-limit of the functionals F_ε as $\varepsilon \downarrow 0$, with respect to the H^{-1}-topology.

In this paper we prove that the solutions u_ε to (1.1) converge to the gradient flow of F^{**}, as $\varepsilon \downarrow 0$, under a suitable assumption on the initial data $\overline{u}_\varepsilon$. Our main result can be informally stated as follows (see Theorem 3.2 for the precise statement). Let $\overline{u}$ be such that $F^{**}(\overline{u}) < +\infty$, take a sequence $(\overline{u}_\varepsilon)$ of initial data satisfying $F_\varepsilon(\overline{u}_\varepsilon) < +\infty$, converging to $\overline{u}$ in $H^{-1}(\mathbb{T})$ such that

$$\int_{\mathbb{T}} \overline{u}_\varepsilon\, dx = \int_{\mathbb{T}} \overline{u}\, dx$$

and

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon(\overline{u}_\varepsilon) = F^{**}(\overline{u}). \tag{1.5}$$

Then the solution $u_\varepsilon(\cdot,\cdot,\overline{u}_\varepsilon)$ of (1.1) converges to the H^{-1}-gradient flow of F^{**}, namely, to the solution u of

$$\begin{cases} \partial_t u = \big(W^{**\,\prime}(u)\big)_{xx} & \text{in } (0,+\infty)\times\mathbb{T}, \\ u = \overline{u} & \text{on } \{0\}\times\mathbb{T}, \end{cases} \tag{1.6}$$

which, for W nonconvex, is the weak formulation of the Stefan problem [14].

Some comments concerning hypothesis (1.5) are in order, related to the so-called wrinkling phenomenon. Given $\overline{u} \in H^{-1}(\mathbb{T})$, define

$$\Sigma_G := \{\xi \in \mathbb{R} : W(\xi) > W^{**}(\xi)\}, \qquad \Sigma_L := \{\xi \in \mathbb{R} : W''(\xi) < 0\} \tag{1.7}$$

and

$$\Sigma_G(\overline{u}) := \{x \in \mathbb{T} : \overline{u}(x) \in \Sigma_G\}, \qquad \Sigma_L(\overline{u}) := \{x \in \mathbb{T} : \overline{u}(x) \in \Sigma_L\}.$$

We call $\Sigma_G(\overline{u})$ the global unstable set of $\overline{u}$, and $\Sigma_L(\overline{u})$ the local unstable set of $\overline{u}$. Numerical simulations performed in [3] (see also [11]) show a quick formation of oscillations, and these microstructures seem to generically appear only in $\Sigma_L(\overline{u})$, instead of on the whole of $\Sigma_G(\overline{u})$. In addition, superimposing on $\overline{u}$ a microstructure in a region $\Sigma \subseteq \Sigma_G(\overline{u}) \setminus \Sigma_L(\overline{u})$ leads to a numerical solution which seems to depend on the choice of Σ. These simulations show an instability of solutions $u_\varepsilon(\cdot,\cdot,\overline{u})$ with respect to $\overline{u}$. In particular, if we take two sequences $(\widetilde{u}_\varepsilon)$, $(\widehat{u}_\varepsilon)$ of initial data both approximating $\overline{u}$ and corresponding to two different choices of Σ, in general one may expect that

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(\cdot,\cdot,(\widetilde{u}_\varepsilon)) \neq \lim_{\varepsilon\downarrow 0} u_\varepsilon(\cdot,\cdot,(\widehat{u}_\varepsilon)).$$

Hypothesis (1.5) can thus be interpreted as an energetically well-prepared assumption on the initial data $\overline{u}_\varepsilon$, corresponding to the choice of the above-mentioned region $\Sigma = \Sigma_G(\overline{u})\backslash\Sigma_L(\overline{u})$. It is worth to remark that, in view of the Γ-limit $F_\varepsilon \to F^{**}$ stated above, given any $\overline{u} \in H^{-1}$, there exists a sequence $(\overline{u}_\varepsilon)$ converging to $\overline{u}$ and satisfying (1.5).

The proof of our main result is entirely variational, and it is worthwhile to observe that we never directly use (1.1). The main point, indeed, is to derive sufficient information on a sequence (v_ε) of functions (independent of time) satisfying the uniform bound

$$\sup_{\varepsilon\in(0,1]} \left\{ F_\varepsilon(v_\varepsilon) + \int_{\mathbb{T}} \left[\left(W'(v_\varepsilon) - \varepsilon^2 v_{\varepsilon\, xx}\right)_x \right]^2 dx \right\} < +\infty. \tag{1.8}$$

We follow an idea formalized by Sandier and Serfaty in [17] (see also [16]), where it is shown that the convergence of the gradient flows of a sequence of functionals $\mathcal{F}_\varepsilon : H \to [0,+\infty]$, where H is a Hilbert space, to the gradient flow of $\mathcal{F} := \Gamma - \lim \mathcal{F}_\varepsilon$ is basically a consequence of the Γ-convergence of the sequence of the slopes of the gradients $|\nabla\mathcal{F}_\varepsilon|$ of $\mathcal{F}_\varepsilon$ to the slope of the gradient $|\nabla\mathcal{F}|$ of $\mathcal{F}$. More precisely, it suffices to show the Γ-liminf inequality

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} \ |\nabla\mathcal{F}_\varepsilon| \ \geq \ |\nabla\mathcal{F}|. \tag{1.9}$$

The above inequality, in our setting, is the content of Theorem 3.3. We then obtain the corresponding convergence of the gradient flows of F_ε in Theorem 3.2. The main difficulty in the proof is contained in Lemma 5.1, where a careful analysis of the regions where the functions v_ε oscillate is performed.

We mention that the same method proposed in [17] has been successfully applied in [12, 13] to show the convergence, in all space dimensions, of solutions to the rescaled Cahn–Hilliard equation

$$\begin{cases} u_t = \Delta\big(\varepsilon^{-1}W'(u) - \varepsilon\Delta u\big), \\ u(0,\cdot) = \overline{u}_\varepsilon \end{cases}$$

under suitable simplifying assumptions, in particular those related to the validity of the analogue of (1.9).

We observe that (1.1) is not the only way to regularize the ill-posed gradient flow equation of the functional (1.4): other regularizations have been considered in the literature; see, for instance, [15, 7, 10, 9, 19]. In particular, in [7] an implicit variational scheme is proposed for the functional (1.4) which converges to (1.6) as the discretization parameter tends to zero. Due to the high instability of the problem, different regularizations could in principle lead to different limiting solutions.

2. Notation. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus of side length 1, and let dx be the Lebesgue measure on $\mathbb{T}$. For $m \in \mathbb{R}$, let

$$\mathcal{H}^{-1}_m(\mathbb{T}) := \big\{ v \in H^{-1}(\mathbb{T}) \, : \, \langle v, 1\rangle = m \big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the $H^{-1}(\mathbb{T})$-$H^1(\mathbb{T})$ duality. $\mathcal{H}^{-1}_m(\mathbb{T})$ is a closed affine subspace of $H^{-1}(\mathbb{T})$, which will be considered equipped with the induced metric. The linear space associated with $\mathcal{H}^{-1}_m(\mathbb{T})$ is the homogeneous negative Sobolev space

$$\dot{H}^{-1}(\mathbb{T}) \sim \mathcal{H}^{-1}_0(\mathbb{T}).$$

In the following, we denote by $\|\cdot\|_{-1}$ the Hilbert norm on $\dot{H}^{-1}(\mathbb{T})$, namely,

$$\|v\|^2_{-1} := \|v\|^2_{\dot{H}^{-1}(\mathbb{T})} = \sup_{\varphi \in H^1(\mathbb{T})} \left\{ 2\langle v, \varphi \rangle - \|\varphi_x\|^2_{L^2(\mathbb{T})} \right\}, \tag{2.1}$$

and we understand $\|v\|_{-1} := +\infty$ if $v \notin \dot{H}^{-1}(\mathbb{T})$.

Throughout the paper, we use the term sequence also to denote families labeled by the continuous positive parameter ε. A subsequence of (f_ε) is a sequence (f_{ε_h}) with $\varepsilon_h \downarrow 0$ as $h \to +\infty$.

2.1. Assumptions on $\boldsymbol{W}$. In what follows we assume that W is a function in $\mathcal{C}^2(\mathbb{R}; [0, +\infty))$ satisfying the following properties:

(i) there exists a constant $C > 0$ such that

$$|W'(\xi)| \leq C(1 + W(\xi)), \qquad \xi \in \mathbb{R}, \tag{2.2}$$

and

$$\lim_{|\xi| \to +\infty} W(\xi) = +\infty;$$

(ii) W is not affine in any interval of $\mathbb{R}$;

(iii) the global unstable set Σ_G of W, as defined in (1.7), is a bounded open set, consisting of a finite number of open connected components, denoted by

$$\Sigma_1, \ldots, \Sigma_\ell.$$

For the standard double-well potential (1.2) one has $\ell = 1$ and $\Sigma_G = \Sigma_1 = (-1, 1)$.

2.2. The functionals $\boldsymbol{F_\varepsilon, F^{**}, |\nabla F_\varepsilon|, |\nabla F^{**}|}$. For any $\varepsilon \in (0, 1]$ we indicate by

$$F_\varepsilon : \mathcal{H}^{-1}_m(\mathbb{T}) \to [0, +\infty]$$

the functional defined as

$$F_\varepsilon(v) := \begin{cases} \displaystyle\int_{\mathbb{T}} \left(\varepsilon^2 \frac{(v_x)^2}{2} + W(v) \right) dx & \text{if } v_x \in L^2(\mathbb{T}) \text{ and } W(v) \in L^1(\mathbb{T}), \\ +\infty & \text{elsewhere,} \end{cases}$$

and by

$$F^{**} : \mathcal{H}^{-1}_m(\mathbb{T}) \to [0, +\infty]$$

the functional defined as

$$F^{**}(v) := \begin{cases} \displaystyle\int_{\mathbb{T}} W^{**}(v)\ dx & \text{if } W^{**}(v) \in L^1(\mathbb{T}), \\ +\infty & \text{elsewhere.} \end{cases}$$

It is clear that F^{**} is a convex functional.

We denote by

$$|\nabla F_\varepsilon|\colon \mathcal{H}_m^{-1}(\mathbb{T}) \to [0,+\infty]$$

the functional defined as

$$|\nabla F_\varepsilon|(v) := \begin{cases} \|\,(W'(v) - \varepsilon^2 v_{xx})_x\|_{L^2(\mathbb{T})} & \text{if } F_\varepsilon(v) < +\infty \text{ and} \\ & (W'(v) - \varepsilon^2 v_{xx})_x \in L^2(\mathbb{T}), \\ +\infty & \text{elsewhere,} \end{cases}$$

and by

$$|\nabla F^{**}| : \mathcal{H}_m^{-1}(\mathbb{T}) \to [0,+\infty]$$

the functional defined as

$$|\nabla F^{**}|(v) := \begin{cases} \|(W^{**\prime}(v))_x\|_{L^2(\mathbb{T})} & \text{if } F^{**}(v) < +\infty \text{ and } (W^{**\prime}(v))_x \in L^2(\mathbb{T}), \\ +\infty & \text{elsewhere.} \end{cases}$$

3. Statement of the main result. Given $\varepsilon \in (0,1]$ and $\overline{u}_\varepsilon \in \mathcal{H}_m^{-1}(\mathbb{T})$ such that

$$F_\varepsilon(\overline{u}_\varepsilon) < +\infty,$$

we let $u_\varepsilon \in \mathcal{C}^\infty((0,+\infty)\times\mathbb{T}) \cap \mathcal{C}^0([0,+\infty);\mathcal{H}_m^{-1}(\mathbb{T}))$ be the solution to the Cauchy problem

$$\begin{cases} \mathrm{u}_t = \Big(W'(\mathrm{u}) - \varepsilon^2 \mathrm{u}_{xx}\Big)_{xx} & \text{in } \ (0,+\infty)\times\mathbb{T}, \\ \mathrm{u} \ = \overline{u}_\varepsilon & \text{on } \{0\}\times\mathbb{T}. \end{cases} \tag{3.1}$$

We notice that u_ε is the gradient flow of F_ε in $\mathcal{H}_m^{-1}(\mathbb{T})$ starting at $\overline{u}_\varepsilon$ in the sense of [1], that is, it satisfies the following:

- $u_\varepsilon \in AC^2\left([0,+\infty);\mathcal{H}_m^{-1}(\mathbb{T})\right)$, where $AC^2\left([0,+\infty);\mathcal{H}_m^{-1}(\mathbb{T})\right)$ denotes the space of absolutely continuous curves from $[0,+\infty)$ to $\mathcal{H}_m^{-1}(\mathbb{T})$ having derivative in $L^2((0,+\infty))$;
- $(0,+\infty) \ni t \mapsto |\nabla F_\varepsilon|(u_\varepsilon(t))$ belongs to $L^2((0,+\infty))$;
- for all $t \geq 0$

$$F_\varepsilon(\overline{u}_\varepsilon) = F_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}\int_0^t \|\partial_t u_\varepsilon(s)\|_{-1}^2\,ds + \frac{1}{2}\int_0^t |\nabla F_\varepsilon|^2(u_\varepsilon(s))\,ds. \tag{3.2}$$

A differential characterization of the gradient flow of F^{**} in $\mathcal{H}_m^{-1}(\mathbb{T})$ is more delicate, as regularity issues appear. Indeed, the function W^{**} is just of class $\mathcal{C}^{1,1}(\mathbb{R})$, and not of class $\mathcal{C}^2(\mathbb{R})$. Yet it is possible to see that $|\nabla F^{**}|$ is a strong upper gradient for F^{**} in the sense of [1, Definition 1.2.1], so that from the general theory of maximal monotone operators (see, for instance, [4, Theorem 3.2]) one gets the following result.

Proposition 3.1 (gradient flow of F^{**}). *Let $\overline{u} \in \mathcal{H}_m^{-1}(\mathbb{T})$ be such that*

$$F^{**}(\overline{u}) < +\infty.$$

Then there exists a unique gradient flow solution u of F^{**} starting at $\overline{u}$, which satisfies

- $u \in AC^2\left([0,+\infty);\mathcal{H}_m^{-1}(\mathbb{T})\right)$;
- $(0,+\infty) \ni t \mapsto |\nabla F^{**}|(u(t))$ *belongs to* $L^2((0,+\infty))$;
- *for all* $t \geq 0$

$$F^{**}(\overline{u}) = F^{**}(u(t)) + \frac{1}{2}\int_0^t \|\partial_t u(s)\|_{-1}^2\,ds + \frac{1}{2}\int_0^t |\nabla F^{**}|^2(u(s))\,ds. \tag{3.3}$$

Note that u solves (1.6) in the sense of distributions.

We are now in the position to state the main result of this paper.

THEOREM 3.2 (convergence of solutions). *Let* $\overline{u}_\varepsilon, \overline{u} \in \mathcal{H}_m^{-1}(\mathbb{T})$ *be such that*

$$F_\varepsilon(\overline{u}_\varepsilon) < +\infty, \qquad F^{**}(\overline{u}) < +\infty.$$

Suppose that

$$\lim_{\varepsilon\downarrow 0} \overline{u}_\varepsilon = \overline{u} \qquad \text{in } \mathcal{H}_m^{-1}(\mathbb{T}) \tag{3.4}$$

and

$$\lim_{\varepsilon\downarrow 0}\ F_\varepsilon(\overline{u}_\varepsilon) = F^{**}(\overline{u}). \tag{3.5}$$

Then for any $T > 0$,

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon = u \quad \text{in } \mathcal{C}^0([0,T];\mathcal{H}_m^{-1}(\mathbb{T})) \tag{3.6}$$

and

$$\lim_{\varepsilon\downarrow 0}\int_0^T \Big(|\nabla F_\varepsilon|(u_\varepsilon(t)) - |\nabla F^{**}|(u(t))\Big)^2\,dt = 0.$$

In particular

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon(u_\varepsilon(t)) = F^{**}(u(t)), \qquad t \geq 0.$$

As already mentioned, following [17], the main ingredient to prove Theorem 3.2 is the following (time-independent) result, which concerns the Γ-limit of the slope in $\mathcal{H}_m^{-1}(\mathbb{T})$ of the functionals F_ε.

THEOREM 3.3 (Γ-liminf of $(|\nabla F_\varepsilon|)$). *Let* $v \in \mathcal{H}_m^{-1}(\mathbb{T})$ *and let* (v_ε) *be a sequence in* $\mathcal{H}_m^{-1}(\mathbb{T})$ *such that*

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon = v \qquad \text{in } \mathcal{H}_m^{-1}(\mathbb{T}) \tag{3.7}$$

and

$$\sup_{\varepsilon\in(0,1]} F_\varepsilon(v_\varepsilon) < +\infty. \tag{3.8}$$

Then

$$\liminf_{\varepsilon\downarrow 0} |\nabla F_\varepsilon|(v_\varepsilon) \geq |\nabla F^{**}|(v). \tag{3.9}$$

We expect a full Γ-convergence result to hold for $(|\nabla F_\varepsilon|)$; however, such result is not needed in order to prove Theorem 3.2.

4. Proof of Theorem 3.3: Preliminary lemmas. We first introduce some regularity remarks for fixed $\varepsilon > 0$ that will be used in the following to establish uniform estimates.

Remark 4.1. We have

$$F_\varepsilon(v) < +\infty \Rightarrow v \in L^\infty(\mathbb{T}).$$

Indeed, for $x_1, x_2 \in \mathbb{T}$,

$$|v(x_1) - v(x_2)| \le \int_{\mathbb{T}} |v_x|\ dx \le \left(\int_{\mathbb{T}} (v_x)^2\ dx\right)^{1/2} < +\infty.$$

Hence, recalling that $\int_{\mathbb{T}} v\ dx = m$, it follows $v \in L^\infty(\mathbb{T})$.

Definition 4.2 (the function $e_\varepsilon(v)$). *If v belongs to the domain of $|\nabla F_\varepsilon|$, we set*

$$e_\varepsilon(v) := W'(v) - \varepsilon^2 v_{xx}.$$

Remark 4.3. We have

$$|\nabla F_\varepsilon|(v) < +\infty \Rightarrow v \in H^3(\mathbb{T}).$$

In particular, if $|\nabla F_\varepsilon(v)| < +\infty$, then

$$|\nabla F_\varepsilon|(v) = \|\ (W'(v) - \varepsilon^2 v_{xx})_{xx}\|_{-1} = \sup_{\varphi \in H^1(\mathbb{T})} \left\{2\langle e_\varepsilon(v)_{xx}, \varphi\rangle - \|\varphi_x\|^2_{L^2(\mathbb{T})}\right\}. \tag{4.1}$$

Indeed, remembering Remark 4.1, we have $v \in L^\infty(\mathbb{T})$. Hence, from the assumption $F_\varepsilon(v) < +\infty$ it follows that

$$W'(v)_x = W''(v) v_x \in L^2(\mathbb{T}). \tag{4.2}$$

From (4.2) and the assumption $|\nabla F_\varepsilon|(v) < +\infty$, we obtain $v_{xxx} \in L^2(\mathbb{T})$ and therefore $v \in H^3(\mathbb{T})$.

Such a regularity allows integration by parts in the expression obtained of $\|\ (W'(v) - \varepsilon^2 v_{xx})_{xx}\|_{-1}$ from the rightmost equality in (2.1), namely, (4.1) holds.

We next establish uniform bounds to be used for the proof of Theorem 3.3.

Lemma 4.4 (uniform L^∞-bound). *Let $v_\varepsilon \in \mathcal{H}^{-1}_m(\mathbb{T})$ be such that*

$$\sup_{\varepsilon \in (0,1]} (F_\varepsilon(v_\varepsilon) + |\nabla F_\varepsilon|(v_\varepsilon)) < +\infty. \tag{4.3}$$

Then

$$\sup_{\varepsilon \in (0,1]} \|v_\varepsilon\|_{L^\infty(\mathbb{T})} < +\infty. \tag{4.4}$$

Moreover (v_ε) admits a converging subsequence in $\mathcal{H}^{-1}_m(\mathbb{T})$.

Proof. From Remark 4.3 we have $v_\varepsilon \in H^3(\mathbb{T})$ and $e_\varepsilon(v_\varepsilon) \in H^1(\mathbb{T})$. Moreover (4.3) guarantees

$$\sup_{\varepsilon \in (0,1]} |\nabla F_\varepsilon|(v_\varepsilon) = \sup_{\varepsilon \in (0,1]} \|e_\varepsilon(v_\varepsilon)_x\|_{L^2(\mathbb{T})} < +\infty. \tag{4.5}$$

We claim that

$$\sup_{\varepsilon \in (0,1]} \|e_\varepsilon(v_\varepsilon)\|_{L^\infty(\mathbb{T})} < +\infty. \tag{4.6}$$

Using assumption (2.2) on W and the periodicity of v_ε, it follows that

$$\left|\int_{\mathbb{T}} e_\varepsilon(v_\varepsilon)\ dx\right| = \left|\int_{\mathbb{T}} W'(v_\varepsilon)\ dx\right| \leq C\int_{\mathbb{T}}(1+W(v_\varepsilon))\ dx,$$

hence from (4.3)

$$\sup_{\varepsilon\in(0,1]}\left|\int_{\mathbb{T}} e_\varepsilon(v_\varepsilon)\ dx\right| < +\infty.$$

From this estimate and (4.5), claim (4.6) follows.

Let us now show that

$$\sup_{\varepsilon\in(0,1]}\|W'(v_\varepsilon)\|_{L^\infty(\mathbb{T})} < +\infty. \tag{4.7}$$

Since W' is monotone increasing out of a compact set (see section 2.1), to show (4.7) it is enough to check that

$$\sup_{\varepsilon\in(0,1]} W'(v_\varepsilon(x_\varepsilon^+)) < +\infty, \qquad \sup_{\varepsilon\in(0,1]}(-W'(v_\varepsilon(x_\varepsilon^-))) < +\infty, \tag{4.8}$$

where $x_\varepsilon^\pm \in \mathbb{T}$ are such that

$$v_\varepsilon(x_\varepsilon^+) = \max\{v_\varepsilon(x) : x\in\mathbb{T}\}, \qquad v_\varepsilon(x_\varepsilon^-) = \min\{v_\varepsilon(x) : x\in\mathbb{T}\}.$$

We have, using $v_{\varepsilon xx}(x_\varepsilon^+) \leq 0$ and $v_{\varepsilon xx}(x_\varepsilon^-) \geq 0$,

$$\|e_\varepsilon(v_\varepsilon)\|_{L^\infty(\mathbb{T})} \geq e_\varepsilon(v_\varepsilon(x_\varepsilon^+)) \geq W'(v_\varepsilon(x_\varepsilon^+))$$

and

$$-\|e_\varepsilon(v_\varepsilon)\|_{L^\infty(\mathbb{T})} \leq e_\varepsilon(v_\varepsilon(x_\varepsilon^-)) \leq W'(v_\varepsilon(x_\varepsilon^-)).$$

Therefore, thanks to (4.6), (4.8) is proven, and (4.4) follows. The last assertion follows from the compact embedding of $L^\infty(\mathbb{T})$ in $H^{-1}(\mathbb{T})$. $\square$

In the next lemma we introduce a parametrized family μ of probability measures, associated with suitable sequences (v_ε), the so-called Young measures. Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$. For $\lambda\in\mathcal{P}(\mathbb{R})$ we let $\mathrm{spt}(\lambda)$ be the support of λ; moreover, if f is a continuous function on $\mathbb{R}$, we let $\lambda(f) = \int_{\mathbb{R}} f\ d\lambda$. If $\lambda : \mathbb{T}\ni x\mapsto\lambda_x\in\mathcal{P}(\mathbb{R})$ is a parametrized family of probability measures, by $\lambda(f)$ we mean the function $\mathbb{T}\ni x\mapsto\lambda_x(f)\in\mathbb{R}$.

LEMMA 4.5 (the measure μ). *Let $v\in\mathcal{H}_m^{-1}(\mathbb{T})$ and let $(v_\varepsilon)\subset\mathcal{H}_m^{-1}(\mathbb{T})$ be a sequence such that*

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon = v \qquad \text{in } \mathcal{H}_m^{-1}(\mathbb{T}) \tag{4.9}$$

and satisfying (4.3). *Then there exists a measurable map*

$$\mu : \mathbb{T}\ni x\mapsto\mu_x\in\mathcal{P}(\mathbb{R})$$

for which the following properties hold:

(a) *there exists a constant $M>0$ such that*

$$\mathrm{spt}(\mu_x)\subseteq[-M,M] \qquad \text{for a.e. } x\in\mathbb{T};$$

(b) $v = \mu(\imath)$, *where* $\imath$ *is the identity map on* $\mathbb{R}$*;*
(c) *there exists a subsequence* (v_{ε_k}) *such that*

$$\lim_{k\to+\infty} \int_{\mathbb{T}} f(v_{\varepsilon_k})\ \varphi\ dx = \int_{\mathbb{T}} \mu(f)\ \varphi\ dx, \qquad f \in \mathcal{C}^0(\mathbb{R}),\ \varphi \in L^1(\mathbb{T});$$

(d) $\mu(W') \in H^1(\mathbb{T})$*, and*

$$\lim_{k\to+\infty} e_{\varepsilon_k}(v_{\varepsilon_k}) = \mu(W') \quad \textit{weakly in } H^1(\mathbb{T}) \quad \textit{and strongly in } L^2(\mathbb{T}).$$

Proof. By Lemma 4.4 we have

$$M := \sup_{\varepsilon\in(0,1]} \|v_\varepsilon\|_{L^\infty(\mathbb{T})} < +\infty. \tag{4.10}$$

Therefore there exists a (not relabeled) subsequence such that $\delta_{v_\varepsilon(x)} \otimes dx$ converges to $\mu_x \otimes dx$ weakly* in the space of measures on $\mathbb{T} \times \mathbb{R}$, where $\mu_x \in \mathcal{P}(\mathbb{R})$ for almost every $x \in \mathbb{T}$, and hence (c) holds for all continuous φ. The sequence $(f(v_\varepsilon))$ being bounded in $L^\infty(\mathbb{T})$, the convergence holds for any $\varphi \in L^1(\mathbb{T})$, and this proves (c).

Since all measures $\delta_{v_\varepsilon(x)}$ have support in $[-M, M]$, also μ_x has support in $[-M, M]$, which gives (a). Assertion (b) follows by taking $f = \imath$ in (c).

From Remark 4.3 and the proof of Lemma 4.4, it follows that the sequence $(e_\varepsilon(v_\varepsilon))$ is bounded in $L^2(\mathbb{T})$. The uniform bound (4.3) then implies

$$\sup_{\varepsilon\in(0,1]} \|e_\varepsilon(v_\varepsilon)\|_{H^1(\mathbb{T})} < +\infty. \tag{4.11}$$

Hence there exists a (not relabeled) subsequence along which the functions $e_\varepsilon(v_\varepsilon)$ converge weakly in $H^1(\mathbb{T})$ and strongly in $L^2(\mathbb{T})$. On the other hand, $e_\varepsilon(u_\varepsilon)$ converges to $W'(v)$ in the sense of distributions on $\mathbb{T}$. By uniqueness of the limit, assertion (d) follows. ∎

The meaning of the next proposition is better illustrated by the subsequent Corollary 4.7, where the assumptions allow one, roughly speaking, to locally choose $l = W'$.

PROPOSITION 4.6. *Let* (v_ε) *and* μ *be as in Lemma 4.5. Let* $l \in \mathcal{C}^0(\mathbb{R})$ *be nondecreasing. Then*

$$\mu(lW') \le \mu(l)\mu(W') < +\infty. \tag{4.12}$$

Proof. Since l is continuous, from Lemma 4.4 it follows that the sequence $(l(v_\varepsilon))$ is bounded in $L^\infty(\mathbb{T})$. Using Lemma 4.5 (c), possibly passing to a (not relabeled) subsequence, we have that $l(v_\varepsilon)$ converge to $\mu(l)$ weakly* in $L^\infty(\mathbb{T})$ and strongly in $H^{-1}(\mathbb{T})$. Then

$$\begin{aligned}
&\int_{\mathbb{T}} |l(v_\varepsilon)e_\varepsilon(v_\varepsilon) - \mu(l)\mu(W')|\ dx \\
&\quad \le \int_{\mathbb{T}} |(l(v_\varepsilon) - \mu(l))e_\varepsilon(v_\varepsilon)|\ dx + \int_{\mathbb{T}} |\mu(l)(e_\varepsilon(v_\varepsilon) - \mu(W')|\ dx \\
&\quad \le \|l(v_\varepsilon) - \mu(l)\|_{H^{-1}(\mathbb{T})}\|e_\varepsilon(v_\varepsilon)\|_{H^1(\mathbb{T})} + \|\mu(l)\|_{L^2(\mathbb{T})}\|e_\varepsilon(v_\varepsilon) - \mu(W')\|_{L^2(\mathbb{T})}.
\end{aligned}$$

Hence, recalling (4.11) and Lemma 4.5 (d), it follows that $l(v_\varepsilon)e_\varepsilon(v_\varepsilon)$ converge to $\mu(l)\mu(W')$ in $L^1(\mathbb{T})$ as $\varepsilon \downarrow 0$.

On the other hand, for all $\varphi \in \mathcal{C}^1(\mathbb{T};[0,+\infty))$, integrating by parts and using the fact that l is nondecreasing,

$$
\begin{aligned}
&\int_{\mathbb{T}} l(v_\varepsilon)\ e_\varepsilon(v_\varepsilon)\ \varphi\ dx \\
&= \int_{\mathbb{T}} l(v_\varepsilon)\ W'(v_\varepsilon)\ \varphi\ dx + \varepsilon^2 \int_{\mathbb{T}} l'(v_\varepsilon)\ (v_{\varepsilon x})^2\ \varphi\ dx + \varepsilon^2 \int_{\mathbb{T}} l(v_\varepsilon)\ v_{\varepsilon x}\ \varphi_x\ dx \\
&\geq \int_{\mathbb{T}} l(v_\varepsilon)\ W'(v_\varepsilon)\varphi\, dx + \varepsilon^2 \int_{\mathbb{T}} l(v_\varepsilon)\ v_{\varepsilon x}\ \varphi_x\ dx.
\end{aligned} \tag{4.13}
$$

From the uniform bound (4.3) and Cauchy–Schwarz's inequality, it follows that the last term on the right-hand side of (4.13) vanishes as $\varepsilon \downarrow 0$. On the other hand, applying Lemma 4.5 (c) with the choice $f = lW'$, we deduce that

$$\int_{\mathbb{T}} l(v_\varepsilon)\ W'(v_\varepsilon)\ \varphi\ dx \to \int_{\mathbb{T}} \mu(lW')\ \varphi\ dx.$$

We conclude

$$\int_{\mathbb{T}} \mu(l)\ \mu(W')\ \varphi\ dx \geq \int_{\mathbb{T}} \mu(lW')\ \varphi\ dx. \qquad \square$$

As a consequence of Proposition 4.6 we have the following result, which, roughly speaking, says that the oscillations of a sequence (v_ε) satisfying (4.3), if contained in a connected component of $\mathbb{R} \setminus \Sigma_L$, namely, in an interval where W' is monotone, are damped down. This result should be considered together with Lemma 5.3 of section 5, which gives further information on $\mu_x(W^{**\prime})$.

Corollary 4.7 (support of μ_x, I). *Let μ be as in Lemma 4.5. For almost every $x \in \mathbb{T}$ for which $\operatorname{spt}(\mu_x)$ is contained in a connected component of $\mathbb{R} \setminus \Sigma_L$, we have that μ_x is a Dirac delta.*

Proof. Since the intervals where W' is strictly monotone are at most countable, we can fix an interval I where W' is strictly increasing, and suppose that there exists a set $A \subseteq \mathbb{T}$ of positive measure so that for almost every $x \in A$ the support of μ_x is contained in I. Choose now a nondecreasing continuous function l so that $l = W'$ in I. Then from (4.12) it follows that

$$\mu_x(W'^2) \leq (\mu_x(W'))^2 \qquad \textit{for a.e. } x \in A,$$

which is a reverse Cauchy–Schwarz inequality. It follows that W' is constant μ_x-almost everywhere in A, and the thesis follows recalling that, by assumption, W is not affine in any interval. $\square$

5. Localization of oscillations. The information gained from the results of the previous section, and in particular from Corollary 4.7, are not enough to conclude the proof of Theorem 3.3. Our aim now (see Lemma 5.3) is to prove that for almost every $x \in \mathbb{T}$, either μ_x is a Dirac delta or its support is contained in the closure of a connected component of Σ_G. The following result, heavily relying on the one-dimensional setting, is the crucial step toward the proof of this assertion.

For any $\rho > 0$ define

$$\Sigma_G^\rho := \{\xi \in \mathbb{R} : \operatorname{dist}(\xi, \Sigma_G) < \rho\}.$$

LEMMA 5.1 (localization of oscillations, I). *Let* $v_\varepsilon \in \mathcal{H}_m^{-1}(\mathbb{T})$ *and let* $c \in (0, +\infty)$ *be such that*

$$F_\varepsilon(v_\varepsilon) + |\nabla F_\varepsilon|(v_\varepsilon) \leq c, \qquad \varepsilon \in (0, 1]. \tag{5.1}$$

For any $\eta > 0$ *there exists* $\delta = \delta(\eta, c) > 0$*, depending on* η *and* c*, but independent of* ε*, such that for any pair* $x_\varepsilon \in \mathbb{T}$*,* $y_\varepsilon \in \mathbb{T}$ *of points satisfying the properties*

(i) $0 < y_\varepsilon - x_\varepsilon \leq \delta$,

(ii) $v_{\varepsilon x}(x_\varepsilon) = v_{\varepsilon x}(y_\varepsilon) = 0$,

we have either

$$v_\varepsilon(z) \in \Sigma_G^\eta, \qquad z \in [x_\varepsilon, y_\varepsilon], \tag{5.2}$$

or

$$|v_\varepsilon(y_\varepsilon) - v_\varepsilon(x_\varepsilon)| < \eta. \tag{5.3}$$

Remark 5.2. Before proving Lemma 5.1, some comments are in order. First of all remember that (5.1) implies (see Remark 4.3) that $v_\varepsilon \in H^3(\mathbb{T})$, and therefore v_ε are Hölder continuous (in particular uniformly continuous). This fact, provided we assume $0 < y_\varepsilon - x_\varepsilon \leq \delta$, does not imply inequality (5.3), since η is required not to depend on ε. The second observation concerns the meaning of Lemma 5.1: this lemma states, roughly speaking, that between two stationary points the functions v_ε either have a small oscillation or must be close to the set Σ_G of the ε-independent quantity η. In some sense, if v_ε have a sufficiently large excursion between two critical points, their values cannot lie inside the region where W is convex. Finally, the qualitative behavior of δ in dependence of η is explicit to a certain extent; see (5.18) below.

Proof. Fix $\eta > 0$, and let $x_\varepsilon, y_\varepsilon \in \mathbb{T}$ be such that $0 < y_\varepsilon - x_\varepsilon$ and $v_{\varepsilon x}(x_\varepsilon) = v_{\varepsilon x}(y_\varepsilon) = 0$. For simplicity of notation, throughout the proof we skip the dependence on ε of x_ε and y_ε, and thus we set $x = x_\varepsilon$ and $y = y_\varepsilon$.

Take a point

$$z \in [x, y].$$

We have

$$\begin{aligned} \int_x^z e_\varepsilon(v_\varepsilon)\ v_{\varepsilon x}\ dx &= \int_x^z \left(W'(v_\varepsilon) - \varepsilon^2 v_{\varepsilon xxx}\right)\ v_{\varepsilon x}\ dx \\ &= W(v_\varepsilon(z)) - W(v_\varepsilon(x)) - \frac{\varepsilon^2}{2}(v_{\varepsilon x}(z))^2 \\ &\leq W(v_\varepsilon(z)) - W(v_\varepsilon(x)), \end{aligned} \tag{5.4}$$

and moreover

$$\int_x^y e_\varepsilon(v_\varepsilon)\ v_{\varepsilon x}\ dx = W(v_\varepsilon(y)) - W(v_\varepsilon(x)). \tag{5.5}$$

On the other hand, integrating by parts we have

$$\int_x^z e_\varepsilon(u_\varepsilon)\ v_{\varepsilon x}\ dx = -\int_x^z e_\varepsilon(v_\varepsilon)_x\ v_\varepsilon\ dx + [e_\varepsilon(v_\varepsilon)\ v_\varepsilon]_x^z.$$

Using (4.4) and (4.5), and recalling assumption (5.1), we have

$$-\int_x^z e_\varepsilon(v_\varepsilon)_x\ v_\varepsilon\ dx = O\left((z - x)^{1/2}\right),$$

where O is independent of ε (while x, y, and hence also z, depend on ε), so that

$$\int_x^z e_\varepsilon(v_\varepsilon)\; v_{\varepsilon\, x}\; dx = [e_\varepsilon(v_\varepsilon)\; v_\varepsilon]_x^z + O\left((z-x)^{1/2}\right). \tag{5.6}$$

On the other hand, using again (4.5), for the boundary term we have

$$\begin{aligned} [e_\varepsilon(v_\varepsilon)\; v_\varepsilon]_x^z &= e_\varepsilon(v_\varepsilon(x))\; [v_\varepsilon]_x^z + v_\varepsilon(z)\; [e_\varepsilon(v_\varepsilon)]_x^z \\ &= e_\varepsilon(v_\varepsilon(x))\; [v_\varepsilon]_x^z + O\left((z-x)^{1/2}\right), \end{aligned} \tag{5.7}$$

where O is (another infinitesimal) still independent of ε. Collecting together (5.4), (5.5), (5.6), and (5.7) we deduce

$$W(v_\varepsilon(z)) \geq W(v_\varepsilon(x)) + e_\varepsilon(v_\varepsilon(x))\big(v_\varepsilon(z) - v_\varepsilon(x)\big) + O\left((z-x)^{1/2}\right), \qquad z \in [x,y], \tag{5.8}$$

and at $z = y$,

$$W(v_\varepsilon(y)) = W(v_\varepsilon(x)) + e_\varepsilon(v_\varepsilon(x))\big(v_\varepsilon(y) - v_\varepsilon(x)\big) + O\left((y-x)^{1/2}\right). \tag{5.9}$$

Assume now

$$|v_\varepsilon(y) - v_\varepsilon(x)| \geq \eta. \tag{5.10}$$

Under this assumption we can rewrite (5.9) as

$$\begin{aligned} e_\varepsilon(v_\varepsilon(x)) &= s(x,y) + O\Big((y-x)^{1/2}(v_\varepsilon(y) - v_\varepsilon(x))^{-1}\Big) \\ &= s(x,y) + O\Big((y-x)^{1/2}/\eta\Big), \end{aligned} \tag{5.11}$$

where

$$s(x,y) := \frac{W(v_\varepsilon(y)) - W(v_\varepsilon(x))}{v_\varepsilon(y) - v_\varepsilon(x)}.$$

From (5.8) and (5.11) we have

$$\begin{aligned} W(v_\varepsilon(z)) &\geq W(v_\varepsilon(x)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) \\ &\quad + O\left((z-x)^{1/2}\right) + O\left((y-x)^{1/2}/\eta\right) \\ &= W(u_\varepsilon(x)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) + O\left((y-x)^{1/2}/\eta\right), \end{aligned} \tag{5.12}$$

where, again, O is independent of ε. Inequality (5.12) says, roughly speaking, that between $v_\varepsilon(z)$ and $v_\varepsilon(x)$, the function W must be concave, where, however, one must take into account the presence of the error term $O((y-x)^{1/2}/\eta)$. For future purposes, it is convenient to rewrite (5.12) in the form

$$W(v_\varepsilon(x)) - W(v_\varepsilon(z)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) \leq O\left((y-x)^{1/2}/\eta\right). \tag{5.13}$$

Without loss of generality, in the remainder of the proof we assume

$$v_\varepsilon(x) \leq v_\varepsilon(y).$$

Recalling Lemma 4.4, we set

$$M := \sup_{\varepsilon\in(0,1]} \|v_\varepsilon\|_{L^\infty(\mathbb{T})} < +\infty.$$

Given $a, b \in \mathbb{R}$, $a < b$, define

$$\psi(a,b) := \max_{c\in[a,b]} \left[W(a) - W(c) + \frac{W(b)-W(a)}{b-a}(c-a) \right].$$

Notice that the positivity of $\psi(a,b)$ measures how much the function W fails to be concave. Observe also that

$$\lim_{b\downarrow a} \psi(a,b) = 0. \tag{5.14}$$

For any $\rho > 0$ let $\mathcal{I}_\rho$ be the family of those intervals $[a,b] \subset \mathbb{R}$ satisfying the following two properties:

- $b - a \geq \rho$;
- $[a,b]$ is not contained in Σ_G^ρ, i.e.,

$$[a,b] \cap \left(\mathbb{R} \setminus \Sigma_G^\rho\right) \neq \emptyset. \tag{5.15}$$

It is convenient to introduce the function $\omega : (0,+\infty) \to [0,+\infty]$ defined as follows:

$$\omega(\rho) := \inf_{[a,b]\subset[-M,M],\ [a,b]\in\mathcal{I}_\rho} \psi(a,b). \tag{5.16}$$

If $\mathcal{I}_\rho = \emptyset$ (namely, if $\rho > 0$ is such that there are no intervals $[a,b]$ contained in $[-M,M]$ with $b-a \geq \rho$ and satisfying (5.15) at the same time), then the infimum on the right-hand side of (5.16) is $+\infty$, so that $\omega(\rho) = +\infty$. On the other hand, possibly increasing the value of M, we can always ensure that $\omega < +\infty$ on $(0,\rho_0)$ for some $\rho_0 > 0$. In what follows we shall assume $\eta < \rho_0$, so that $\omega(\eta) < +\infty$.

Note that if $\omega(\rho) < +\infty$, then the infimum on the right-hand side of (5.16) is a minimum, since $[a,b]$ are constrained to lie in the compact set $[-M,M]$. Moreover, recalling that by assumption W is not affine in any interval, we have

- $\omega(\rho) > 0$;
- if $\rho_1 < \rho_2$, then $\mathcal{I}_{\rho_1} \supseteq \mathcal{I}_{\rho_2}$, and therefore ω is nondecreasing;
- $\lim_{\rho\downarrow 0} \omega(\rho) = 0$ as a consequence of (5.14).

Suppose now that

$$[v_\varepsilon(x), v_\varepsilon(y)] \text{ is not contained in } \Sigma_{\mathrm{G}}^\eta. \tag{5.17}$$

Recalling that ω is positive, choose δ such that

$$O(\delta^{1/2}/\eta) \leq \frac{\omega(\eta)}{2}, \tag{5.18}$$

where O denotes the remainder term appearing in (5.13). From (5.13) it then follows that

$$\max_{z\in[x,y]} \Big(W(v_\varepsilon(x)) - W(v_\varepsilon(z)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) \Big) \leq O(\delta^{1/2}/\eta) \leq \frac{\omega(\eta)}{2}. \tag{5.19}$$

On the other hand, choosing

$$a = v_\varepsilon(x), \qquad b = v_\varepsilon(y)$$

on the right-hand side of (5.16), and remembering (5.10) and (5.17), it follows that

$$\max_{z\in[x,y]} \Big(W(v_\varepsilon(x)) - W(v_\varepsilon(z)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) \Big) \geq \omega(\eta),$$

which contradicts (5.19). We conclude that

$$[v_\varepsilon(x), v_\varepsilon(y)] \subseteq \Sigma_G^\eta. \tag{5.20}$$

Let us now complete the proof of (5.2). If $v_\varepsilon(z) \in [v_\varepsilon(x), v_\varepsilon(y)]$ for any $z \in [x,y]$, from (5.20) we deduce $v_\varepsilon(z) \in \Sigma_G^\eta$, and the proof is concluded. It remains to consider the case when there exists $z \in (x,y)$ such that

$$v_\varepsilon(z) \notin [v_\varepsilon(x), v_\varepsilon(y)].$$

We can assume that $v_\varepsilon(z) > v_\varepsilon(y)$, the case $v_\varepsilon(z) < v_\varepsilon(x)$ being similar. Choose $y' \in [x,y]$ so that $v_\varepsilon(y') = \max_{\tau\in[x,y]} v_\varepsilon(\tau) \geq v_\varepsilon(z)$, and $x' \in [x,y]$ so that $v_\varepsilon(x') = \min_{\tau\in[x,y]} v_\varepsilon(\tau) \leq v_\varepsilon(z)$. Recalling (5.10) we have $|v_\varepsilon(y') - v_\varepsilon(x')| \geq \eta$. Therefore we can apply the previous arguments replacing x with x' and y with y', so that inclusion (5.20) now reads as $[v_\varepsilon(x'), v_\varepsilon(y')] \subseteq \Sigma_G^\eta$. This is precisely inclusion (5.2). ∎

The next lemma says, roughly speaking, that if v_ε asymptotically oscillates (as $\varepsilon \downarrow 0$), then it necessarily does it within the same connected component of Σ_G. We will focus our attention on $W^{**\prime}(v_\varepsilon)$, in view of the applications in section 6.

Lemma 5.3 (support of μ_x, II). *Let v, (v_ε), and μ be as in Lemma* 4.5. *Then one of the following two alternatives holds:*

- *for almost every $x \in \mathbb{T}$ such that $\mu_x(W^{**\prime})$ is not contained in $W^{**\prime}\left(\overline{\Sigma_G}\right)$, then μ_x is a Dirac delta;*
- *for almost every $x \in \mathbb{T}$ such that $\mu_x(W^{**\prime})$ is contained in $W^{**\prime}\left(\overline{\Sigma_G}\right)$, then μ_x is supported on $\overline{\Sigma_G}$.*

Proof. Define

$$w_\varepsilon := W^{**\,\prime}(v_\varepsilon),$$

which, recalling (4.4), is a Lipschitz function on $\mathbb{T}$. We now translate the thesis of Lemma 5.1 for w_ε. For δ as in Lemma 5.1 we set

$$\delta'(\eta) := \delta\left(\frac{\eta}{2L}, c\right), \qquad \eta > 0,$$

where L is the Lipschitz constant of $W^{**\,\prime}$ in $[-M, M]$, and M is as in (4.10). Notice that in the definition of δ' we need $2L$ instead of L to cover the case when (5.2) holds.

If x_ε and y_ε satisfy the assumption of Lemma 5.1 with δ replaced by δ', we have

$$|w_\varepsilon(x_\varepsilon) - w_\varepsilon(y_\varepsilon)| \leq \eta. \tag{5.21}$$

Observe that this is not a uniform continuity condition on w_ε, since the points $x_\varepsilon, y_\varepsilon$ are just critical points of v_ε (and depend on ε), and therefore are not arbitrary points of $\mathbb{T}$.

Possibly replacing δ' with its convex envelope, we can assume that δ' is a nonzero convex function (tending to zero at zero) in a bounded open interval having zero as the left extremum.

From Lemma 4.5 (c) we know that

$$\lim_{\varepsilon\downarrow 0} w_\varepsilon = \mu(W^{**\,\prime}) =: w \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{T}).$$

We now want to pass from a control on critical points to a control on the whole of $\mathbb{T}$. We therefore find it convenient to consider linear interpolations.

Claim. Up to extracting a (not relabeled) subsequence, we have

$$w_\varepsilon \to w \qquad \text{a.e. in } \mathbb{T} \text{ as } \varepsilon \downarrow 0. \tag{5.22}$$

Let $\widehat{w}_\varepsilon \in \mathrm{Lip}(\mathbb{T})$ be such that $\widehat{w}_\varepsilon$ is affine in each maximal open interval of strict monotonicity of v_ε and coincides with w_ε on the boundary of such an interval. Notice that there exists at most a countable number of such intervals.

Let us show that from (5.21) it follows that for all $\eta > 0$ there exists $\delta''(\eta) > 0$ independent of ε such that

$$x \in \mathbb{T},\ y \in \mathbb{T},\ \ |x-y| \le \delta''(\eta) \quad \Rightarrow \quad |\widehat{w}_\varepsilon(x) - \widehat{w}_\varepsilon(y)| \le \eta. \tag{5.23}$$

To prove (5.23) we distinguish two cases.

First case: *x and y belong to the same monotonicity interval I of v_ε*. Assuming without loss of generality that $x < y$, let $x' \le x$ and $y' \ge y$ be such that $I = (x', y')$. Set

$$\lambda := \frac{|x-y|}{|x'-y'|} \in (0,1].$$

By construction and from (5.21) we know

$$|x'-y'| < \delta' \Rightarrow |\widehat{w}_\varepsilon(x') - \widehat{w}_\varepsilon(y')| < \eta.$$

Hence, as $\widehat{w}_\varepsilon$ is affine in I,

$$|x-y| < \lambda\delta' \Rightarrow |\widehat{w}_\varepsilon(x) - \widehat{w}_\varepsilon(y)| < \lambda\eta. \tag{5.24}$$

Since δ' is convex and $\delta'(0) = 0$, we have $\lambda\delta'(\eta) \ge \delta'(\lambda\eta)$, and therefore replacing η by $\lambda\eta$ and using (5.24) we deduce (5.23) with δ'' replaced by δ'.

Second case: *x and y do not belong to the same monotonicity interval of v_ε*. Assuming without loss of generality that $x < y$, let x', y' be such that

- $x \le x' \le y' \le y$;
- x' and y' are critical points of v_ε;
- $\widehat{w}_\varepsilon$ is strictly monotone between x' and y'.

Then the formula

$$|x-y| = |x-x'| + |x'-y'| + |y'-y| < \delta'(e)$$

implies, using the first case in $[x, x']$ and in $[y', y]$, and using (5.21) in $[x', y']$, that

$$|\widehat{w}_\varepsilon(x) - \widehat{w}_\varepsilon(y)| \le |\widehat{w}_\varepsilon(x) - \widehat{w}_\varepsilon(x')| + |\widehat{w}_\varepsilon(x') - \widehat{w}_\varepsilon(y')| + |\widehat{w}_\varepsilon(y') - \widehat{w}_\varepsilon(y)| \le 3\eta.$$

That is,

$$|x-y| < \delta''(\eta) \Rightarrow |\widehat{w}_\varepsilon(x) - \widehat{w}_\varepsilon(y)| \le \eta,$$

where

$$\delta''(\eta) = \delta'(\eta/3).$$

This concludes the proof of (5.23).

From (5.23) it follows that the functions $\widehat{w}_\varepsilon$ are equicontinuous, and by Lemma 4.4 they are also uniformly bounded. We can apply Ascoli–Arzelà's theorem to get that, possibly passing to a (not relabeled) subsequence,

$$\widehat{w}_\varepsilon \to \widehat{w} \text{ uniformly in } \mathbb{T}$$

for some $\widehat{w} \in \mathcal{C}^0(\mathbb{T})$.

For any $n \in \mathbb{N}$ we let $I_1^{\varepsilon,n}, \dots, I_{N_{\varepsilon,n}}^{\varepsilon,n}$ be such that $I_j^{\varepsilon,n} = (a_j^{\varepsilon,n}, b_j^{\varepsilon,n}) \subseteq \mathbb{T}$ is a maximal interval of strict monotonicity of v_ε and

$$\operatorname{osc}(v_\varepsilon; I_j^{\varepsilon,n}) := \sup_{I_j^{\varepsilon,n}} v_\varepsilon - \inf_{I_j^{\varepsilon,n}} v_\varepsilon \in \left(\frac{1}{n}, \frac{1}{n-1}\right], \qquad j = 1, \dots, N_{\varepsilon,n}, \tag{5.25}$$

where $N_{\varepsilon,n} \in \mathbb{N} \cup \{+\infty\}$. Actually, $N_{\varepsilon,n}$ is finite, since from Lemma 5.1 it follows that

$$|I_j^{\varepsilon,n}| = |a_j^{\varepsilon,n} - b_j^{\varepsilon,n}| > \delta(1/n),$$

so that

$$N_{\varepsilon,n} \le \frac{1}{\delta(1/n)}.$$

Up to extracting a further (not relabeled) subsequence, we may assume that

$$N_{\varepsilon,n} = N_n,$$

where N_n depends only on n, and

$$a_j^{\varepsilon,n} \to a_j, \qquad b_j^{\varepsilon,n} \to b_j^n \quad \text{as } \varepsilon \downarrow 0, \qquad j \in \{1, \dots, N_n\}.$$

Let $I^{\varepsilon,n} := \cup_{j=1}^{N_n} I_j^{\varepsilon,n}$ and $I^n := \cup_{j=1}^{N_n} I_j^n$. Notice that from (5.25) it follows that

$$I^n \cap I^m = \emptyset \quad \text{if } n \ne m.$$

For any interval $[a,b] \subset \cup_{n\in\mathbb{N}} I^n$, the functions w_ε are monotone on $[a,b]$ for all $\varepsilon > 0$ small enough. As a consequence, up to a further subsequence,

$$w_\varepsilon \to w \qquad \text{a.e. on } \cup_{n\in\mathbb{N}} I^n \text{ as } \varepsilon \downarrow 0. \tag{5.26}$$

On the other hand, given $n \in \mathbb{N}$ and $x \in \mathbb{T} \setminus (\cup_{m\in\mathbb{N}} \overline{I^m})$, we have $\operatorname{dist}(x, I_n^\varepsilon) \ge c(n) > 0$ for all $\varepsilon > 0$ small enough, so that

$$|w_\varepsilon(x) - \widehat{w}(x)| \le |w_\varepsilon(x) - \widehat{w}_\varepsilon(x)| + |\widehat{w}_\varepsilon(x) - \widehat{w}(x)| \le \frac{1}{n} + \frac{1}{n} \tag{5.27}$$

for $\varepsilon > 0$ small enough. By the arbitrariness of $n \in \mathbb{N}$ we then get $w_\varepsilon \to \widehat{w}$ uniformly on $\mathbb{T} \setminus \cup_{m\in\mathbb{N}} \overline{I^m}$ as $\varepsilon \downarrow 0$. This shows that

$$\widehat{w} = w \qquad \text{in } \mathbb{T} \setminus \bigcup_{m\in\mathbb{N}} \overline{I^m}. \tag{5.28}$$

Then (5.26) and (5.28) conclude the proof of claim (5.22).

Eventually, we show that the claim implies the thesis of the lemma. Indeed, for almost every $x \in \mathbb{T}$ such that $w(x) \notin W^{**\,\prime}(\overline{\Sigma_G})$, by the strict monotonicity of $W^{**\,\prime}$ we have

$$v_\varepsilon(x) \to v(x) \quad \text{as } \varepsilon \downarrow 0,$$

which implies $\mu_x = \delta_{v(x)}$. On the other hand, for almost every $x \in \mathbb{T}$ such that $w(x) \in W^{**\,\prime}(\overline{\Sigma_G})$, we have $\mathrm{dist}(v_\varepsilon(x), \Sigma_G) \to 0$ as $\varepsilon \downarrow 0$, which implies $\mathrm{spt}(\mu_x) \subseteq \overline{\Sigma}_G$. ∎

A useful consequence of Lemma 5.3 is the following.

COROLLARY 5.4. *Under the assumptions of Lemma* 5.3, *we have*

$$\mu_x(W^{**\prime}) = W^{**\prime}(v(x)) \qquad \text{for a.e. } x \in \mathbb{T}.$$

Proof. If $\mu_x(W^{**\prime})$ is not contained in $W^{**\prime}\left(\overline{\Sigma_G}\right)$, then μ_x is a Dirac delta, and the assertion follows. If $\mu_x(W^{**\prime})$ is contained in $W^{**\prime}\left(\overline{\Sigma_G}\right)$, then μ_x is supported on $\overline{\Sigma_G}$, where $W^{**\prime}$ is constant. ∎

We now improve Lemma 5.1 and deduce two corollaries, which will be necessary in the proof of Theorem 3.3. For clarity of exposition, we prefer to state the next lemma separately from Lemma 5.1, even if its proof remains almost unchanged.

LEMMA 5.5 (localization of oscillations, II). *Let $(v_\varepsilon) \subset \mathcal{H}^{-1}_m(\mathbb{T})$ be a sequence of functions satisfying the bound* (5.1). *For any $\eta > 0$ and $C > 0$*

- *there exists $\delta = \delta(\eta, c) > 0$ depending on η and c but independent of ε and C;*
- *there exists $\varepsilon_0 = \varepsilon_0(\eta, c, C) > 0$ depending on η, c, and C,*

such that for any pair $x_\varepsilon \in \mathbb{T}$, $y_\varepsilon \in \mathbb{T}$ of points satisfying the properties

(i) $0 < y_\varepsilon - x_\varepsilon \leq \delta$,

(ii) $|v_{\varepsilon x}(x_\varepsilon)| \leq C$, $|v_{\varepsilon x}(y_\varepsilon)| \leq C$,

we have either

$$v_\varepsilon(z) \in \Sigma^\eta_G, \qquad z \in [x_\varepsilon, y_\varepsilon], \qquad \varepsilon \in (0, \varepsilon_0), \tag{5.29}$$

or

$$|v_\varepsilon(y_\varepsilon) - v_\varepsilon(x_\varepsilon)| < \eta, \qquad \varepsilon \in (0, \varepsilon_0). \tag{5.30}$$

Proof. The proof closely follows the proof of Lemma 5.1. Set $x = x_\varepsilon$ and $y = y_\varepsilon$. In the present situation, inequality (5.4) must be replaced by

$$\int_x^z e_\varepsilon(v_\varepsilon)\; v_{\varepsilon x}\; dx \leq W(v_\varepsilon(z)) - W(v_\varepsilon(x)) + O(\varepsilon^2, C), \tag{5.31}$$

and equality (5.5) by

$$\int_x^y e_\varepsilon(v_\varepsilon)\; v_{\varepsilon x}\; dx = W(v_\varepsilon(y)) - W(v_\varepsilon(x)) + O(\varepsilon^2, C), \tag{5.32}$$

where the term $O(\varepsilon^2, C)$ is actually of the form $O(C^2\varepsilon^2)$. Following the same computations as those of Lemma 5.1 we must now add to the right-hand sides of (5.6), (5.7), (5.8), (5.9), (5.11), (5.12), and (5.13) a remainder term of the form $O(C^2\varepsilon^2)$.

Next we take $\varepsilon_0 > 0$ so that

$$O(C^2\varepsilon^2) \leq \frac{\omega(\eta)}{4}, \qquad \varepsilon \in (0, \varepsilon_0), \tag{5.33}$$

and $\delta > 0$ so that

$$O(\delta^{1/2}/\eta) \le \frac{\omega(\eta)}{4}. \tag{5.34}$$

Then (5.18) transforms into

$$O(C^2\varepsilon^2) + O(\delta^{1/2}/\eta) \le \frac{\omega(\eta)}{2},$$

and (5.19) into

$$\begin{aligned} &\max_{z\in[x,y]} \Big(W(v_\varepsilon(x)) - W(v_\varepsilon(z)) + s(x,y)(v_\varepsilon(z) - v_\varepsilon(x)) \Big) \\ &\le O(\delta^{1/2}/\eta) + O(C^2\varepsilon^2) \le \frac{\omega(\eta)}{2}. \end{aligned} \tag{5.35}$$

Then the assertions of the lemma follow along the same lines of reasoning as in the proof of Lemma 5.1. $\square$

Corollary 5.6. *For any $\eta > 0$ and $c > 0$ there exist $\varepsilon_0 > 0$ and $\delta' > 0$ such that if $(v_\varepsilon) \subset \mathcal{H}^{-1}_m(\mathbb{T})$ is a sequence of functions satisfying*

$$F_\varepsilon(v_\varepsilon) + |\nabla F_\varepsilon|(v_\varepsilon) \le c, \qquad \varepsilon \in (0, \varepsilon_0), \tag{5.36}$$

and $x \in \mathbb{T}$ is such that

$$\operatorname{dist}(v_\varepsilon(x), \Sigma_G) \ge 2\eta, \qquad \varepsilon \in (0, \varepsilon_0),$$

then

$$\operatorname{dist}\big(v_\varepsilon(y), \Sigma_G\big) \ge \eta, \qquad y \in (x - \delta', x + \delta'),\ \varepsilon \in (0, \varepsilon_0).$$

Proof. By Lemma 4.4 there exists $M = M(c)$ such that $\sup_{\varepsilon\in(0,1]} \|v_\varepsilon\|_{L^\infty(\mathbb{T})} \le M$. Letting $\delta = \delta(\eta, c)$ be as in Lemma 5.5, there exist $x_1 \in (x - \delta/2, x - \delta/6)$ and $x_2 \in (x + \delta/6, x + \delta/2)$ such that $|v_{\varepsilon x}(x_1)|, |v_{\varepsilon x}(x_1)| \le C := 6M/\delta$. By Lemma 5.5 there exists ε_0 such that if $\varepsilon \in (0, \varepsilon_0)$, then $|v_\varepsilon(x_1) - v_\varepsilon(x_2)| < \eta$. We now claim that

$$\operatorname{dist}(v_\varepsilon(y), \Sigma_G) \ge \eta \qquad \text{for all } y \in [x_1, x_2], \tag{5.37}$$

which implies the thesis since $(x - \delta', x + \delta') \subset (x_1, x_2)$, with $\delta' = \delta/6$. Indeed, letting y_1 (resp., y_2) be a minimum point (resp., a maximum point) of v_ε on $[x_1, x_2]$, again by Lemma 5.5 we have $|v_\varepsilon(y_1) - v_\varepsilon(y_2)| < \eta$ so that

$$|v_\varepsilon(y) - v_\varepsilon(x)| \le |v_\varepsilon(y_1) - v_\varepsilon(y_2)| < \eta$$

for all $y \in [x_1, x_2]$, which gives (5.37). $\square$

In general we cannot expect the limit function v to be continuous. Nevertheless, we can prove the following results. Recall the definition of $\Sigma_1, \ldots, \Sigma_\ell$ given in section 2.1.

Corollary 5.7. *Let $(v_\varepsilon) \subset \mathcal{H}^{-1}_m(\mathbb{T})$ be a sequence satisfying the uniform bound (4.3) and let $v \in \mathcal{H}^{-1}_m(\mathbb{T})$ be such that*

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon = v \qquad \textit{in } \mathcal{H}^{-1}_m(\mathbb{T}). \tag{5.38}$$

Then the set

$$\Omega := \{x \in \mathbb{T} : v(x) \notin \Sigma_G\}$$

has an open Lebesgue representative, and

$$\operatorname*{ess\,lim}_{\Omega \ni x \to \overline{x} \in \partial\Omega} \operatorname{dist}(v(x), \Sigma_G) = 0. \tag{5.39}$$

Moreover, the sets

$$C_i := \Big\{ x \in \mathbb{T} : v(x) \in \overline{\Sigma}_i \Big\}, \qquad i = 1, \ldots, \ell,$$

have closed Lebesgue representatives and

$$\operatorname{dist}(C_i, C_j) > 0, \qquad i, j = 1, \ldots, \ell, \ i \neq j. \tag{5.40}$$

Proof. Let $x \in \Omega$ be a Lebesgue point of v such that $\operatorname{dist}(v(x), \Sigma_G) \geq 3\eta > 0$. Letting $\delta' > 0$ be as in Corollary 5.6, for all $\varepsilon > 0$ small enough there exists $x_\varepsilon \in (x - \delta'/2, x + \delta'/2)$ such that $|v_\varepsilon(x_\varepsilon) - v(x)| < \eta$, so that $\operatorname{dist}(v_\varepsilon(x_\varepsilon), \Sigma_G) \geq 2\eta$. By Corollary 5.6 it follows that $\operatorname{dist}(v_\varepsilon(y), \Sigma_G) \geq \eta$ for all $y \in (x_\varepsilon - \delta', x_\varepsilon + \delta') \supset (x - \delta'/2, x + \delta'/2)$, which in turn implies

$$\operatorname{dist}(v(y), \Sigma_G) \geq \eta \qquad \text{for all } y \in (x - \delta'/2, x + \delta'/2).$$

It follows that

$$(x - \delta'/2, x + \delta'/2) \subset \Omega$$

and (5.39) holds. The assertion concerning the sets C_i can be proved similarly. Indeed, since $\cup_{i=1}^{\ell} C_i = \mathbb{T} \setminus \Omega$ has a closed representative, it is enough to show (5.40). Assume by contradiction that there exists $\overline{x} \in \overline{C}_i \cap \overline{C}_j$. In this case, in a neighborhood of $\overline{x}$ we can find points x_ε such that $v_\varepsilon(x_\varepsilon) \notin \Sigma_G$ for $\varepsilon > 0$ small enough. Reasoning as above, this implies $v(x_\varepsilon) \in \Omega$, thus leading to a contradiction. ∎

6. Proof of Theorem 3.3. We are now in a position to conclude the proof of Theorem 3.3. Let $v_\varepsilon \to v$ in $\mathcal{H}_m^{-1}(\mathbb{T})$ as $\varepsilon \downarrow 0$, and choose a subsequence $(\varepsilon_k) \subset (0, 1)$ such that

$$\lim_{k \to +\infty} |\nabla F_{\varepsilon_k}|(v_{\varepsilon_k}) = \liminf_{\varepsilon \downarrow 0} |\nabla F_\varepsilon|(v_\varepsilon)$$

and

$$\sup_{k \in \mathbb{N}} \Big(F_{\varepsilon_k}(v_{\varepsilon_k}) + |\nabla F_{\varepsilon_k}|(v_{\varepsilon_k}) \Big) < +\infty.$$

Recalling (4.1) we have

$$\begin{aligned} \lim_{k \to +\infty} |\nabla F_{\varepsilon_k}|(v_{\varepsilon_k}) &= \lim_{k \to +\infty} \sup_{\varphi \in H^1(\mathbb{T})} \int_{\mathbb{T}} \Big(2e_{\varepsilon_k}(v_{\varepsilon_k})_{xx}\ \varphi - (\varphi_x)^2 \Big)\ dx \\ &\geq \sup_{\varphi \in H^1(\mathbb{T})} \limsup_{k \to +\infty} \int_{\mathbb{T}} \big(2e_{\varepsilon_k}(v_{\varepsilon_k})_{xx}\ \varphi - (\varphi_x)^2 \big)\ dx. \end{aligned} \tag{6.1}$$

Since (v_{ε_k}) converges to v in $\mathcal{H}^{-1}_m(\mathbb{T})$ as $k \to +\infty$, we have at our disposal a corresponding measure μ given by Lemma 4.5. Using Lemma 4.5 (d), from (6.1) and (3.7) we have

$$\begin{aligned}\lim_{k\to+\infty} |\nabla F_{\varepsilon_k}|(v_{\varepsilon_k}) &\geq \sup_{\varphi\in H^1(\mathbb{T})} \int_{\mathbb{T}} \left(-2(\mu(W'))_x\ \varphi_x - (\varphi_x)^2\right)\ dx \\ &= \|(\mu(W'))_x\|^2_{L^2(\mathbb{T})}\end{aligned}$$

(recall from Lemma 4.5 (d) that $(\mu(W'))_x \in L^2(\mathbb{T})$).

We now want to show that

$$\|(\mu(W'))_x\|^2_{L^2(\mathbb{T})} \geq \|(W^{**\prime}(v))_x\|^2_{L^2(\mathbb{T})}. \tag{6.2}$$

Let us define

$$\Omega := \left\{x \in \mathbb{T} : v(x) \notin \overline{\Sigma}_G\right\}.$$

In order to prove (6.2), we will show that $W^{**\prime}(v) = \mu(W')$ in Ω, and that $W^{**\prime}(v)$ is constant on the connected components of $\mathbb{T} \setminus \Omega$.

By Corollary 5.6, it follows that Ω has an open Lebesgue representative (still denoted by Ω) and that, for any $i = 1, \ldots, \ell$, the set

$$C_i := \left\{x \in \mathbb{T} : v(x) \in \overline{\Sigma}_i\right\}$$

has a closed Lebesgue representative (still denoted by C_i). Then, by Lemma 5.3,

$$\mu_x(W') = W'(v(x)) = W^{**\prime}(v(x)) \qquad \text{for a.e. } x \in \Omega.$$

Hence, being $\mu_x(W') \in H^1(\mathbb{T})$, we get

$$W^{**\prime}(v) \in H^1(\Omega).$$

In particular $W^{**\prime}(v)$ is uniformly continuous on Ω, and can be continuously extended to $\overline{\Omega}$. Moreover, for all $\overline{x} \in \partial\Omega$, from (5.39) one gets that if $x \in \Omega \to \overline{x}$, then $\mathrm{dist}(v(x), \Sigma_G) \to 0$, and

$$\lim_{\Omega\ni x\to\overline{x},\, x\in\Omega} W^{**\prime}(v(x)) \in W^{**\prime}\left(\overline{\Sigma}_G\right).$$

Recalling (5.40) and the fact that $W^{**\prime}(v)$ is locally constant outside Ω, it follows that

$$W^{**\prime}(v) \in H^1(\mathbb{T}),$$

and in addition

$$\|(W^{**\prime}(v))_x\|_{L^2(\mathbb{T})} = \|(W^{**\prime}(v))_x\|_{L^2(\Omega)}.$$

We then have

$$\begin{aligned}\lim_{k\to+\infty} |\nabla F_{\varepsilon_k}|(v_{\varepsilon_k}) &\geq \|(\mu(W'))_x\|^2_{L^2(\mathbb{T})} \geq \|(\mu(W'))_x\|^2_{L^2(\Omega)} = \|(W^{**\prime}(v))_x\|^2_{L^2(\Omega)} \\ &= \|(W^{**\prime}(v))_x\|^2_{L^2(\mathbb{T})} = |\nabla F^{**}|(v). \qquad \square\end{aligned}$$

7. Proof of Theorem 3.2. With Theorem 3.3 at hand, we can prove our main convergence result, Theorem 3.2. We will use the standard notation $f(t)(x) = f(t,x)$ for a function $f \in \mathcal{C}^0([0,T];\mathbb{T})$.

Since $(F_\varepsilon(u_\varepsilon))$ is bounded by (3.2) in $[0,T] \times \mathbb{T}$, and W has at least linear growth at infinity, the sequence (u_ε) is uniformly bounded in $L^\infty([0,T];L^1(\mathbb{T}))$. Hence (u_ε) is bounded in $L^\infty([0,T];\mathcal{H}^{-1}_m(\mathbb{T}))$ and in particular in $L^2([0,T];\mathcal{H}^{-1}_m(\mathbb{T}))$, since the subspace of all functions in $L^1(\mathbb{T})$ with mean m (compactly) embeds in $\mathcal{H}^{-1}_m(\mathbb{T})$. Using once more (3.2) it follows that

$$(u_\varepsilon) \text{ is uniformly bounded in } H^1([0,T];\mathcal{H}^{-1}_m(\mathbb{T})).$$

Let (u_{ε_k}) be a subsequence weakly converging in $H^1([0,T];\mathcal{H}^{-1}_m(\mathbb{T}))$ to some function w. From Ascoli–Arzelà's theorem in $H^1([0,T];\mathcal{H}^{-1}_m(\mathbb{T}))$, it follows that (u_{ε_k}) has a further (not relabeled) subsequence converging to w in $\mathcal{C}^0([0,T];\mathcal{H}^{-1}_m(\mathbb{T}))$. Hence

$$\lim_{k\to\infty} u_{\varepsilon_k}(t) = w(t) \qquad \text{for all } t \in [0,T] \tag{7.1}$$

and in particular, recalling (3.4),

$$\overline{u} = \lim_{k\to\infty} u_{\varepsilon_k}(0) = w(0). \tag{7.2}$$

We now want to show that $w = u$, and to do this we follow the proof of [17, Theorem 1]. By assumption (3.5), and remembering (3.2), for any $t \in [0,T]$ we have

$$\begin{aligned} \mathrm{I} &:= \lim_{k\to+\infty} \left(F_{\varepsilon_k}(u_{\varepsilon_k}(t)) + \frac{1}{2}\int_0^t \|\partial_t u_{\varepsilon_k}(s)\|^2_{-1}\, ds + \frac{1}{2}\int_0^t |\nabla F_{\varepsilon_k}|^2(u_{\varepsilon_k}(s))\, ds \right) \\ &= \lim_{k\to\infty} F_{\varepsilon_k}(\overline{u}_\varepsilon) = F^{**}(\overline{u}). \end{aligned} \tag{7.3}$$

On the other hand,

$$\begin{aligned} \mathrm{I} \;\geq\; & \liminf_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k}(t)) \\ & + \liminf_{k\to+\infty} \frac{1}{2}\int_0^t \|\partial_t u_{\varepsilon_k}(s)\|^2_{-1}\; ds \\ & + \liminf_{k\to+\infty} \frac{1}{2}\int_0^t |\nabla F_{\varepsilon_k}|^2(u_{\varepsilon_k}(s))\; ds. \end{aligned} \tag{7.4}$$

Applying (7.1) and the lower semicontinuity of F^{**}, it follows that

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u_{\varepsilon_k}(t)) \geq \liminf_{k\to+\infty} F^{**}(u_{\varepsilon_k}(t)) \geq F^{**}(w(t)). \tag{7.5}$$

From Fatou's lemma and Theorem 3.3 we have

$$\liminf_{k\to+\infty} \int_0^t |\nabla F^{**}|^2(u_{\varepsilon_k}(s))\; ds \geq \int_0^t |\nabla F^{**}|^2(w(s))\; ds. \tag{7.6}$$

From the lower semicontinuity of the norm, and using again Fatou's lemma, we have

$$\liminf_{k\to+\infty} \int_0^t \|\partial_t u_{\varepsilon_k}(s)\|^2_{-1}\; ds \geq \int_0^t \|\partial_t w(s)\|^2_{-1}\; ds. \tag{7.7}$$

Collecting together inequalities (7.5), (7.6), and (7.7), from (7.4) and (7.3) we infer

$$F^{**}(\overline{u}) \geq F^{**}(w(t)) + \frac{1}{2}\int_0^t \|\partial_t w(s)\|_{-1}^2 \, ds + \frac{1}{2}\int_0^t |\nabla F^{**}|^2(w(s)) \, ds. \tag{7.8}$$

On the other hand we have, using (7.2),

$$\begin{aligned} &\frac{1}{2}\int_0^t \|\partial_t w(s)\|_{-1}^2 \, ds + \frac{1}{2}\int_0^t |\nabla F^{**}|^2(w(s)) \, ds \geq -\int_0^t \langle w_t, \nabla F^{**}(w)\rangle_{\mathcal{H}^{-1}(\mathbb{T})} \, ds \\ &\quad = -\int_0^t \frac{d}{ds}F^{**}(w(s)) \, ds = F(w(0)) - F(w(t)) = F(\overline{u}) - F(w(t)), \end{aligned} \tag{7.9}$$

which is the reverse inequality of (7.8). Therefore

$$F^{**}(\overline{u}) = F^{**}(w(t)) + \frac{1}{2}\int_0^t \|\partial_t w(s)\|_{-1}^2 \, ds + \frac{1}{2}\int_0^t |\nabla F^{**}|^2(w(s)) \, ds \qquad \text{for all } t \geq 0.$$

Then w is the gradient flow of F^{**} starting from $\overline{u}$, and hence $w = u$. In particular, the whole sequence (u_ε) converges to u, and the proof is concluded. ∎

Appendix A. For completeness, in this appendix we quickly prove here a Γ-convergence result concerning the functionals F_ε. This result is unnecessary for the proof of Theorem 3.2.

PROPOSITION A.1 (Γ-limit of F_ε). *The sequence (F_ε) Γ-converges to F^{**} in $\mathcal{H}_m^{-1}(\mathbb{T})$ as $\varepsilon \downarrow 0$.*

Proof. The functional F^{**} is lower semicontinuous in $\mathcal{H}_m^{-1}(\mathbb{T})$. Since $F_\varepsilon \geq F^{**}$, if $v_\varepsilon \to v$ in $\mathcal{H}_m^{-1}(\mathbb{T})$, then $\liminf_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon) \geq F^{**}(v)$, namely, the Γ-liminf inequality holds.

We now prove the Γ-limsup inequality: Given $v \in \mathcal{H}_m^{-1}(\mathbb{T})$ we have to find a sequence $(v_\varepsilon) \subset \mathcal{H}_m^{-1}(\mathbb{T})$ with

$$v_\varepsilon \to v \quad \text{in } \mathcal{H}_m^{-1}(\mathbb{T}) \tag{A.1}$$

such that

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon) \to F(v) \quad \text{as } \varepsilon \downarrow 0. \tag{A.2}$$

Assume first that v is piecewise constant and takes values in $\mathbb{R} \setminus \Sigma_G$. Then, taking a piecewise linear function $v_\varepsilon \in H^1(\mathbb{T})$ which coincides with v out of a small δ_ε-neighborhood of its jump set, where $\lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\delta_\varepsilon} = 0$, and that keeps the constraint $\fint_{\mathbb{T}} v_\varepsilon \, dx = m$, one gets (A.1) and (A.2).

It is now enough to show that the class of functions v considered above is dense in $\mathcal{H}_m^{-1}(\mathbb{T})$ and with respect to F^{**}, so that the thesis will follow by a standard density argument. Since piecewise constant functions are dense in $\mathcal{H}_m^{-1}(\mathbb{T})$, it is sufficient to show that a piecewise constant function v can be approximated in $\mathcal{H}_m^{-1}(\mathbb{T})$ by piecewise constant functions v_n taking values in $\mathbb{R} \setminus \Sigma_G$ and such that

$$\lim_{n\to+\infty} F^{**}(v_n) = F^{**}(v). \tag{A.3}$$

Let v be piecewise constant. Let $A \subseteq \mathbb{T}$ be an interval where v takes value in (a, b), with (a, b) a connected component of Σ_G. Let $\lambda \in (0, 1)$ be such that $v = \lambda a + (1-\lambda)b$.

We can now take $v_n \in H^{-1}(A)$ such that $v_n \to v$ in $H^{-1}(A)$, $v_n(x) \in \{a, b\}$ for any $x \in A$, and $\int_A v_n\, dx = \int_A v\, dx$. Then

$$F^{**}(v_n, A) := \int_A W^{**}(v_n)\, dx = \lambda W^{**}(a) + (1-\lambda) W^{**}(b) = F^{**}(v, A) = \int_A W^{**}(v)\, dx,$$

since W^{**} is linear on $[a, b]$. We can apply the same argument in the intervals where v takes values in Σ_G, while we keep $v_n = v$ in the rest of the domain. This concludes the proof. ∎

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