Action Functional and Quasi-Potential for the Burgers Equation in a Bounded Interval

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Abstract

Consider the viscous Burgers equation $u_t + f(u)_x = \varepsilon u_{xx}$ on the interval $[0, 1]$ with the inhomogeneous Dirichlet boundary conditions $u(t, 0) = \rho_0$, $u(t, 1) = \rho_1$. The flux f is the function $f(u) = u(1 - u)$, $\varepsilon > 0$ is the viscosity, and the boundary data satisfy $0 < \rho_0 < \rho_1 < 1$. We examine the quasi-potential corresponding to an action functional arising from nonequilibrium statistical mechanical models associated with the above equation. We provide a static variational formula for the quasi-potential and characterize the optimal paths for the dynamical problem. In contrast with previous cases, for small enough viscosity, the variational problem defining the quasi-potential admits more than one minimizer. This phenomenon is interpreted as a nonequilibrium phase transition and corresponds to points where the superdifferential of the quasi-potential is not a singleton. © 2010 Wiley Periodicals, Inc.

1 Introduction

We consider an infinite-dimensional version of the classical Freidlin-Wentzell variational problem for the quasi-potential. We first briefly recall this topic in the context of diffusion processes in $\mathbb{R}^n$ [23]. Let b be a smooth vector field in $\mathbb{R}^n$ and consider the stochastic perturbation of the dynamical systems $\dot{x} = b(x)$ given by

$$\dot{X}_\gamma = b(X_\gamma) + \sqrt{2\gamma}\dot{w},$$

where $\dot{w}$ is a white noise. Under suitable assumptions on b, the process X_γ has a unique invariant measure μ_γ. The Freidlin-Wentzell theory provides a variational

expression for the asymptotics of this invariant measure in the weak noise limit $\gamma \downarrow 0$. To each path $X : (-\infty, 0] \rightarrow \mathbb{R}^n$ associate the *action*

$$(1.1) \qquad I(X) = \frac{1}{4} \int_{-\infty}^{0} |\dot{X}(t) - b(X(t))|^2 \, dt.$$

In what follows we assume that the vector field b has a unique, globally attractive equilibrium point $\bar{x}$. The quasi-potential $V : \mathbb{R}^n \rightarrow [0, +\infty)$ is defined by

$$(1.2) \qquad V(x) = \inf\{I(X), \ X(0) = x, \ X(t) \rightarrow \bar{x} \text{ as } t \rightarrow -\infty\};$$

namely, $V(x)$ is the minimal action to reach x starting from the equilibrium point $\bar{x}$. For each Borel set $B \subset \mathbb{R}^n$ we then have, as $\gamma \downarrow 0$,

$$\mu_\gamma(B) \asymp \exp\{-\gamma^{-1} \inf_{x \in B} V(x)\}.$$

If the vector field b is conservative, namely $b = -\nabla U$ for some $U : \mathbb{R}^n \rightarrow \mathbb{R}$, then $V(x) = U(x) - U(\bar{x})$; i.e., the quasi-potential coincides with the potential. In general, though, there is no simple expression for the quasi-potential.

In this finite-dimensional setting, the quasi-potential V is Lipschitz [23]; in particular, it is a.e. differentiable. As discussed in [14], for some "special" points $x \in \mathbb{R}^n$, the function V might however have "corners," namely, the superdifferential of V might not be a singleton. From a dynamical point of view, for such points x there would exist more than a single minimizer for the variational problem (1.2). We refer to the examples considered in the physical literature [24, 25, 30] for a discussion on the physical interpretation of this lack of uniqueness. In this article we show that this phenomenon occurs for an infinite-dimensional dynamical system. As far as we know, this is the first concrete example in which such a result has been analytically proven.

As shown in [13, 14, 32], if $\mathbb{H}$ denotes the Hamiltonian associated to the action (1.1), then the quasi-potential V is a viscosity solution, the correct PDE formulation of the variational problem (1.2) in the presence of "corners," to the Hamilton-Jacobi equation $\mathbb{H}(x, DV) = 0, x \in \mathbb{R}^n$.

We examine in this article an infinite-dimensional version of the previous variational problem. As the basic dynamical system, we consider the following nonlinear parabolic equation on the interval $[0, 1]$:

$$(1.3) \qquad u_t + f(u)_x = \varepsilon(D(u)u_x)_x$$

with the inhomogeneous Dirichlet boundary condition $u(t, 0) = \rho_0, u(t, 1) = \rho_1$. In the above equation, f is the flux, $D > 0$ the diffusion coefficient, and $\varepsilon > 0$ the viscosity. To introduce the associated action functional, add an external "controlling" field $E = E(t, x)$ to obtain the perturbed equation

$$(1.4) \qquad u_t + f(u)_x + 2\varepsilon(\sigma(u)E)_x = \varepsilon(D(u)u_x)_x,$$

where $\sigma(u) \geq 0$ is the mobility. Denote by u^E the solution of this equation. The action of a path $u : (-\infty, 0] \times [0, 1] \rightarrow \mathbb{R}$ is given by

$$(1.5) \qquad I^\varepsilon(u) = \inf \varepsilon \int_{-\infty}^0 \int_0^1 \sigma(u) E^2 \, dx \, dt,$$

where the infimum is carried over all E such that $u^E = u$. The quasi-potential is then introduced as in the finite-dimensional setting; namely, V_ε is the functional on the set of functions $\rho : [0, 1] \rightarrow \mathbb{R}$ defined by

$$(1.6) \qquad V_\varepsilon(\rho) = \inf\{I^\varepsilon(u) : u(0) = \rho, u(t) \rightarrow \bar{\rho}_\varepsilon \text{ as } t \rightarrow -\infty\},$$

where $\bar{\rho}_\varepsilon$ is the unique stationary solution of (1.3).

Informally, as in the Freidlin-Wentzell theory, one can add a stochastic perturbation to (1.3),

$$(1.7) \qquad u_t^\gamma + f(u^\gamma)_x = \varepsilon(D(u^\gamma)u_x^\gamma)_x + \left(\sqrt{2\,\varepsilon\,\gamma\,\sigma(u^\gamma)}\,\dot{w}_\gamma\right)_x,$$

where $\dot{w}_\gamma$ is a white noise in time that also becomes white in space as $\gamma \downarrow 0$. Then, informally, the finite-dimensional theory carries over to the present setting. In particular, the quasi-potential V_ε in (1.6) describes the asymptotics for the invariant measure of the process u^γ as $\gamma \downarrow 0$. We refer to [31] for an analysis of the large-deviation properties of the stochastic PDE (1.7), with periodic boundary conditions, in the joint limit $\gamma \downarrow 0$ and $\varepsilon \downarrow 0$.

Our main motivation for the analysis of the action functional (1.5) and the quasi-potential (1.6) comes, however, from nonequilibrium statistical mechanics. For a class of interacting particle systems, the so-called stochastic lattice gases, it has been shown that equation (1.3) describes the typical evolution of the empirical density in the diffusive scaling limit [26]. Moreover, the functional I^ε, as in the Freidlin-Wentzell theory, gives the corresponding asymptotic probability of observing deviations from the typical behavior [10, 26, 27].

In the case of equilibrium models, the flux f vanishes and the boundary conditions are equal. For such models, which are analogous to gradient vector fields in the finite-dimensional situation, the quasi-potential (1.6) does not depend on the viscosity ε and coincides with the thermodynamic free energy functional of the underlying microscopic model, whose invariant measure has the standard Gibbs form. If the flux f does not vanish but the boundary data are still equal, $\rho_0 = \rho_1$, the quasi-potential does not depend on f and is equal to that of the corresponding equilibrium model; see [4, 7] for the analogous result in the case of periodic boundary conditions.

On the other hand, if the boundary data are not equal, in general there is no simple expression for the invariant measure of the microscopic dynamics, often called the stationary nonequilibrium state. In order to analyze the behavior of such an invariant measure in the thermodynamic limit, in [5] we introduced the dynamical/variational approach outlined above. In particular, the quasi-potential V_ε plays

an analogous role to the free energy for equilibrium systems. This gives a natural way to extend the notion of thermodynamic potential to nonequilibrium systems.

As has been shown by concrete examples [5, 6, 8, 9, 16, 20], for nonequilibrium models the quasi-potential (1.6) presents peculiar features. While for equilibrium models V_ε is always a convex local functional, i.e., of the form $V_\varepsilon(\rho) = \int_0^1 v(\rho(x))dx$ for some convex real function v, for nonequilibrium models V_ε might be nonlocal and nonconvex. In terms of the underlying microscopic model, the nonlocality of the quasi-potential corresponds to the presence of long-range correlations that are believed to be a generic feature of stationary nonequilibrium states [18]. In these examples, which have the common feature that the diffusion coefficient D is constant and the mobility σ is quadratic, the quasi-potential can be expressed in terms of a one-dimensional boundary value problem.

The main purpose of the present paper is to show, by a concrete example, that for nonequilibrium models the quasi-potential might have "corners," equivalently, that the minimizer for the variational problem (1.6) is not unique. We shall analyze the model defined by a constant diffusion coefficient $D = 1$, while the flux and mobility are given by $f(u) = \sigma(u) = u(1-u)$, $u \in [0, 1]$. Then the parabolic equation (1.3) becomes the viscous Burgers equation and the action functional I^ε in (1.5) can be obtained as the large-deviation rate functional of the so-called weakly asymmetric simple exclusion process [10, 27]. As shown in [11, 21], the quasi-potential (1.6) is also the large-deviation rate function for the invariant measure.

When $\rho_0 > \rho_1$, so that both the boundary conditions and the flux f "push" the density $u \in [0, 1]$ to the right, the behavior of the invariant measure has been discussed in [20] by combinatorial techniques. In particular, in [20] a static variational characterization for the rate function of the invariant measure is derived in terms of a one-dimensional boundary value problem. More recently the same model is analyzed in [8], where it is shown that the quasi-potential (1.6) can be written in terms of the variational expression derived in [20] and the optimal paths for (1.6) are characterized. We emphasize that in this case, i.e., for $\rho_0 > \rho_1$, the quasi-potential has no corners and the minimizer for (1.6) is unique.

In this paper we examine the same model but in the more interesting situation in which $\rho_0 < \rho_1$, so that there is an effective competition between the flux and the boundary conditions. Our main results are summarized as follows. By analyzing the variational problem (1.6), we establish a static variational characterization of V_ε analogous to the one in [8, 20]. We emphasize that in the case discussed here there is no uniqueness for the minimizer of (1.6); this leads to new conceptual features that require a novel approach. In particular, the Hamiltonian formulation discussed in Section 3 gives a natural interpretation of the representations of the quasi-potential obtained in [5, 6, 8, 9, 16, 20]. We then discuss the variational convergence of the quasi-potential V_ε in the inviscid limit $\varepsilon \downarrow 0$. In particular, we show that in this limit we recover the functional derived in [17] in the context of the boundary-driven asymmetric exclusion process. As shown in [17], if we restrict the limiting functional V_0 to constant profiles, V_0 has a corner. Here we

consider V_0 on the whole function space and characterize the profiles ρ where the superdifferential of V_0 is not a singleton. By a perturbation argument with respect to this limiting case, we prove that, for a class of nonconstant profiles ρ, this phenomenon persists when the viscosity ε is small enough. Moreover, for the profiles ρ in which the superdifferential of V_ε is not a singleton, the minimizer of (1.6) is not unique. Finally, we discuss the connection of the quasi-potential V_ε to the Hamilton-Jacobi equation $\mathbb{H}_\varepsilon(\rho, DV) = 0$, where $\mathbb{H}_\varepsilon$ is the Hamiltonian associated with the action (1.5).

In the context of equilibrium statistical mechanics, the existence of more than a single tangent functional to the quasi-potential, which in this case coincides with the free energy functional, is due to the occurrence of phase transitions. We therefore interpret the fact that the superdifferential of V_ε is not a singleton as a nonequilibrium phase transition. In this paper we consider a special model, but the existence of these nonequilibrium phase transitions should be a generic phenomenon.

2 Notation and Results

2.1 Viscous Burgers Equation

Consider the viscous Burgers equation on the interval $[0, 1]$ with inhomogeneous Dirichlet boundary conditions at the endpoints, namely,

$$(2.1) \qquad \begin{cases} u_t + f(u)_x = \varepsilon u_{xx}, \\ u(t, 0) = \rho_0, \; u(t, 1) = \rho_1, \end{cases}$$

where $u = u(t, x)$ is a scalar function; hereafter we denote partial derivatives with subscripts. The *flux* f is the function $f(u) = u(1 - u)$, $\varepsilon > 0$ is the viscosity, and the boundary data, fixed throughout the paper, satisfy $0 < \rho_0 < \rho_1 < 1$.

Simple computations show that the unique stationary solution $\bar{\rho}_\varepsilon$ of the viscous Burgers equation (2.1) can be described as follows. Let $J_0 = J_0(\rho_0, \rho_1) := \min_{r \in [\rho_0, \rho_1]} f(r)$. For each $\varepsilon > 0$ there exists a unique $J_\varepsilon \in (-\infty, J_0)$ such that

$$\int_{\rho_0}^{\rho_1} \frac{\varepsilon}{f(r) - J_\varepsilon} \, dr = 1.$$

The function $\bar{\rho}_\varepsilon$ is then obtained by integrating $f(\bar{\rho}_\varepsilon) - \varepsilon(\bar{\rho}_\varepsilon)_x = J_\varepsilon$ with the boundary condition $\bar{\rho}_\varepsilon(0) = \rho_0$. In particular, the function $\bar{\rho}_\varepsilon$ is strictly increasing. We remark that the constant J_ε can be interpreted as the current maintained by the stationary solution $\bar{\rho}_\varepsilon$. Let $\varphi_i := \log[\rho_i/(1 - \rho_i)] \in \mathbb{R}$, $i = 0, 1$, and set $\varepsilon_0 := 1/(\varphi_1 - \varphi_0)$. Clearly J_ε is increasing as ε decreases and $J_{\varepsilon_0} = 0$; therefore $0 < J_\varepsilon < J_0$ for $0 < \varepsilon < \varepsilon_0$ and $J_\varepsilon < 0$ for $\varepsilon > \varepsilon_0$. A simple computation also shows that $\lim_{\varepsilon \downarrow 0} J_\varepsilon = J_0$.

By standard arguments for parabolic equations (see, e.g., the more sophisticated analysis in [15]), the stationary solution $\bar{\rho}_\varepsilon$ is globally attractive for the flow defined by (2.1). More precisely, fix $\varepsilon > 0$, let $u(t; \rho)$, $t \geq 0$, be the solution to (2.1) with initial condition $u(0, \cdot; \rho) = \rho(\cdot)$, and assume that $\rho : [0, 1] \to \mathbb{R}$ is a

continuous function satisfying the boundary conditions $\rho(0) = \rho_0$, $\rho(1) = \rho_1$. Then $u(t; \rho)$ converges in the C^1-topology to $\bar{\rho}_\varepsilon$ as $t \to \infty$. Furthermore, this convergence is uniform with respect to ρ in a bounded set of $C([0, 1])$ and holds with an exponential rate.

2.2 The Action Functional

To define rigorously the action functional informally introduced in (1.5), we need to introduce some notation. Given $T > 0$, the inner products in $L^2([0, 1])$ and $L^2([-T, 0] \times [0, 1])$ are denoted by $\langle \cdot, \cdot \rangle$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, respectively. We consider the space $L^\infty([0, 1])$ equipped with the weak-* topology and let M be the set

$$M = \{\rho \in L^\infty([0, 1]) : 0 \le \rho \le 1\}$$

equipped with the relative topology. Then M is a compact Polish space, i.e., complete, metrizable, and separable. Recall that, by definition of the weak-* topology, a sequence $\{\rho^n\} \subset M$ converges to ρ in M if and only if $\langle \rho^n, g \rangle \to \langle \rho, g \rangle$ for any $g \in L^1([0, 1])$. For $T > 0$, we let $C([-T, 0]; M)$ be the set of continuous paths $u : [-T, 0] \to M$ equipped with the topology of uniform convergence.

Let $C_0^\infty([-T, 0] \times [0, 1])$ be the space of smooth functions $H : [-T, 0] \times [0, 1] \to \mathbb{R}$ satisfying $H(t, 0) = H(t, 1) = 0$ for $t \in [-T, 0]$ and $H(-T, x) = H(0, x) = 0$ for $x \in [0, 1]$. Given $u \in C([-T, 0]; M)$, let $L_u^\varepsilon : C_0^\infty([-T, 0] \times [0, 1]) \to \mathbb{R}$ be the linear functional defined by

$$L_u^\varepsilon(H) := -\langle\!\langle u, H_t \rangle\!\rangle - \langle\!\langle f(u), H_x \rangle\!\rangle - \varepsilon \langle\!\langle u, H_{xx} \rangle\!\rangle$$

(2.2)
$$+ \varepsilon \int_{-T}^0 [\rho_1 H_x(t, 1) - \rho_0 H_x(t, 0)] dt.$$

If u is a smooth function satisfying the boundary conditions $u(t, 0) = \rho_0$ and $u(t, 1) = \rho_1$ for $t \in [-T, 0]$, then $L_u^\varepsilon(H) = \langle\!\langle u_t + f(u)_x - \varepsilon u_{xx}, H \rangle\!\rangle$. Moreover, the functional L_u^ε vanishes if and only if u is a weak solution to (2.1).

Let $\sigma : [0, 1] \to [0, +\infty)$ be the *mobility* of the system; we assume it is the function defined by $\sigma(a) = a(1 - a)$. Given $u \in C([-T, 0], M)$, let

$$\langle\!\langle u_x, u_x \rangle\!\rangle := \sup_H \{-2\langle\!\langle u, H_x \rangle\!\rangle - \langle\!\langle H, H \rangle\!\rangle\},$$

where the supremum is carried over all smooth functions $H : [-T, 0] \times [0, 1] \to \mathbb{R}$ such that $H(t, 0) = H(t, 1) = 0$, $t \in [-T, 0]$. The action functional $I_{[-T, 0]}^\varepsilon : C([-T, 0]; M) \to [0, +\infty]$ is then defined by

(2.3) $\quad I_{[-T, 0]}^\varepsilon(u) := \begin{cases} \sup_H \{L_u^\varepsilon(H) - \varepsilon \langle\!\langle H_x, \sigma(u) H_x \rangle\!\rangle\} & \text{if } \langle\!\langle u_x, u_x \rangle\!\rangle < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$

where the supremum is carried over all functions H in $C_0^\infty([-T, 0] \times [0, 1])$. Clearly, $I_{[-T, 0]}^\varepsilon(u)$ vanishes if and only if $u \in C([-T, 0]; M)$ admits a square-integrable derivative and is a weak solution to (2.1). We refer to [10] for equivalent definitions of the action functional $I_{[-T, 0]}^\varepsilon$. We remark that in [10] the action functional is defined with the condition $\langle\!\langle u_x, u_x \rangle\!\rangle < +\infty$ replaced by the stronger

condition $\langle\!\langle u_x, \sigma(u)^{-1}u_x \rangle\!\rangle < +\infty$. The argument in [10, lemma 4.9] shows, however, that if the supremum on the right-hand side of (2.3) is finite, these conditions are in fact equivalent.

In order to state the connection of the action functional to the perturbed parabolic problem (1.4), we need a few more definitions. Denote by $C^1([0,1])$ the space of continuously differentiable functions $h : [0,1] \to \mathbb{R}$, and let $C_0^1([0,1]) := \{h \in C^1([0,1]) : h(0) = h(1) = 0\}$. Given a positive bounded measurable function $\gamma : [0,1] \to [0,+\infty)$, let $\mathcal{H}_0^1(\gamma)$ be the Sobolev space induced by $C_0^1([0,1])$ endowed with the inner product

$$\langle h, g \rangle_{1,\gamma} = \int_0^1 h_x g_x \gamma \, dx.$$

To be precise, the induced space $\mathcal{H}_0^1(\gamma)$ is obtained by identifying and completing elements $h \in C_0^1([0,1])$ with respect to the seminorm $\langle h, h \rangle_{1,\gamma}^{1/2}$. When $\gamma = 1$ the space $\mathcal{H}_0^1(\gamma)$ is the standard Sobolev space on $[0,1]$; in this case we denote it simply by $\mathcal{H}_0^1$. Note that in the above equation, as well as below, we drop from the notation the explicit dependence on the integration variable when there is no ambiguity.

Denote by $|\cdot|_{1,\gamma}$ the norm of $\mathcal{H}_0^1(\gamma)$, and let $\mathcal{H}_0^{-1}(\gamma)$ be the dual space of $\mathcal{H}_0^1(\gamma)$. It is equipped with the dual norm $|\cdot|_{-1,\gamma}$ defined by

$$|\ell|_{-1,\gamma}^2 = \sup \{2\langle \ell, h \rangle - |h|_{1,\gamma}^2, \ h \in \mathcal{H}_0^1(\gamma)\},$$

where $\langle \ell, h \rangle$ also stands for the value at h of the linear functional ℓ.

Fix a path u in $C([-T,0], M)$ and denote by $\mathfrak{H}_0^1(\sigma(u))$ the Hilbert space induced by $C_0^\infty([-T,0] \times [0,1])$ endowed with the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{1,\sigma(u)}$ defined by

$$\langle\!\langle H, G \rangle\!\rangle_{1,\sigma(u)} = \int_{-T}^0 \langle H, G \rangle_{1,\sigma(u(t))} dt,$$

and let $\|\cdot\|_{1,\sigma(u)}$ be the associated norm. Let $\mathfrak{H}_0^{-1}(\sigma(u))$ be the dual of $\mathfrak{H}_0^1(\sigma(u))$; it is a Hilbert space equipped with the norm $\|\cdot\|_{-1,\sigma(u)}$ defined by

$$\|L\|_{-1,\sigma(u)}^2 = \sup_H \{2\langle\!\langle L, H \rangle\!\rangle - \|H\|_{1,\sigma(u)}^2\},$$

where the supremum is carried over all functions $H \in C_0^\infty([-T,0] \times [0,1])$, equivalently over all $H \in \mathfrak{H}_0^1(\sigma(u))$, and $\langle\!\langle L, H \rangle\!\rangle$ stands for the value of the linear functional L at H.

The next statement is proven in [10]. A functional $f : X \to (-\infty, +\infty]$ defined on a Polish space X is said to be coercive if all its sublevel sets are precompact: for all $t \in \mathbb{R}$, $\{x : f(x) \le t\}$ is precompact.

THEOREM 2.1 *The functional* $I_{[-T,0]}^\varepsilon : C([-T,0]; M) \to [0,+\infty]$ *is coercive and lower-semicontinuous for each* $\varepsilon > 0$ *and* $T > 0$; *namely, it has compact*

sublevel sets. Moreover, given u such that $I^{\varepsilon}_{[-T,0]}(u) < +\infty$, *there exists a unique* $H = H(u)$ *in* $\mathfrak{H}^1_0(\sigma(u))$ *such that u is a weak solution to*

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} - 2\varepsilon(\sigma(u)H_x)_x, \\ u(t,0) = \rho_0, \ u(t,1) = \rho_1. \end{cases}$$

In this case, the linear functional L^{ε}_u, *as defined in* (2.2), *extends to a linear functional on* $\mathfrak{H}^1_0(\sigma(u))$ *that we denote by* $u_t + f(u)_x - \varepsilon u_{xx}$, *and*

$$I^{\varepsilon}_{[-T,0]}(u) = \varepsilon \|H\|^2_{1,\sigma(u)} = \frac{1}{4\varepsilon}\|u_t + f(u)_x - \varepsilon u_{xx}\|^2_{-1,\sigma(u)}.$$

2.3 The Quasi-Potential

The *quasi-potential* [23] associated with the family of action functionals $I^{\varepsilon}_{[-T,0]}$, $T > 0$, is the functional $V_{\varepsilon} : M \to [0, +\infty]$ defined by

(2.4) $V_{\varepsilon}(\rho) :=$
$$\inf_{T>0} \inf \left\{ I^{\varepsilon}_{[-T,0]}(u) : u \in C([-T,0]; M), \ u(-T) = \bar{\rho}_{\varepsilon}, \ u(0) = \rho \right\},$$

so that $V_{\varepsilon}(\rho)$ measures the minimal cost to reach the function ρ starting from the stationary solution $\bar{\rho}_{\varepsilon}$. In this sense, while $I^{\varepsilon}_{[-T,0]}(u)$ measures how close a path u is to solutions to (2.1), the quasi-potential $V_{\varepsilon}(\rho)$ measures how close ρ is to the stationary solution $\bar{\rho}_{\varepsilon}$.

The previous definition implies that V_{ε} is a *Lyapunov functional* for the Burgers equation (2.1). This is to say that if $u(t; \rho)$, $t \geq 0$, is the solution to (2.1) with initial datum $u(0, \cdot; \rho) = \rho(\cdot) \in M$, then for any $t \geq 0$ we have $V_{\varepsilon}(u(t; \rho)) \leq V_{\varepsilon}(\rho)$. Observe that for each $t \geq 0$ we have $u(t; \rho) \in M$ by the maximum principle. The previous claim is easily proven, recalling that $I^{\varepsilon}_{[-T,0]}$ vanishes on weak solutions to (2.1), by exhibiting a test path for the variational problem (2.4).

It is also convenient to formulate the variational problem (2.4) on paths defined on the semi-infinite time interval $(-\infty, 0]$. To this end we introduce the set

(2.5) $\mathcal{U}(\bar{\rho}_{\varepsilon}) := \left\{ u \in C((-\infty, 0]; M) : \lim_{t \to -\infty} u(t) = \bar{\rho}_{\varepsilon} \right\}$

equipped with the topology of uniform convergence. The family of action functionals $I^{\varepsilon}_{[-T,0]}$, $T > 0$, naturally induces the lower-semicontinuous functional $I^{\varepsilon} : \mathcal{U}(\bar{\rho}_{\varepsilon}) \to [0, +\infty]$ defined by

(2.6) $I^{\varepsilon}(u) := \lim_{T \to \infty} I^{\varepsilon}_{[-T,0]}(u \restriction_{[-T,0]}),$

where $u \restriction_{[-T,0]}$ denotes the restriction of $u \in \mathcal{U}(\bar{\rho}_{\varepsilon})$ to $C([-T,0]; M)$. Note that the above limit always exists, possibly equal to $+\infty$, in view of the monotonicity of $I^{\varepsilon}_{[-T,0]}(u \restriction_{[-T,0]})$ in $T > 0$. Let

(2.7) $\widehat{V}_{\varepsilon}(\rho) := \inf \left\{ I^{\varepsilon}(u) : u \in \mathcal{U}(\bar{\rho}_{\varepsilon}), \ u(0) = \rho \right\}$

and observe that the inequality $\widehat{V}_\varepsilon \le V_\varepsilon$ holds trivially. In the context of diffusion processes in $\mathbb{R}^n$, by the continuity of $\widehat{V}_\varepsilon$, it is easy to show that $V_\varepsilon = \widehat{V}_\varepsilon$ [23]. We shall prove that this identity also holds in the present setting where, as shown below, $\widehat{V}_\varepsilon$ is not continuous but only lower-semicontinuous.

2.4 Characterization of the Quasi-Potential

The analysis of the quasi-potential in the case $\rho_0 > \rho_1$ and in the case $\rho_0 < \rho_1$, $\varepsilon \ge \varepsilon_0$ has been considered in [8]. Here we discuss the more interesting case in which $\rho_0 < \rho_1$ and $\varepsilon \in (0, \varepsilon_0)$. The first main result of this article states that the quasi-potential V_ε can be expressed in terms of a static variational problem.

Given $m > 0$, denote by $\mathcal{P}_m([0,1])$ the set of positive Borel measures on $[0,1]$ with total mass equal to m. Recall that $\varphi_i = \log[\rho_i/(1-\rho_i)]$, $i = 0, 1$, $\varphi_0 < \varphi_1$, and set

$$(2.8) \qquad \mathcal{F} := \{\varphi : \varphi(x) = \varphi_0 + \mu([0,x]) \text{ for some } \mu \in \mathcal{P}_{\varphi_1 - \varphi_0}([0,1])\}.$$

Clearly, if φ belongs to $\mathcal{F}$, then φ is an increasing càdlàg function satisfying $\varphi_0 \le \varphi(0)$, $\varphi(1) = \varphi_1$. We consider the set $\mathcal{F}$ equipped with the topology inherited from the weak convergence of measures; namely, a sequence $\{\varphi^n\} \subset \mathcal{F}$ converges to φ in $\mathcal{F}$ if and only if for any continuous function g on $[0,1]$ we have $\int g \, d\varphi^n \to \int g \, d\varphi$. Then $\mathcal{F}$ is a compact Polish space. Moreover, $\varphi^n \to \varphi$ in $\mathcal{F}$ implies $\varphi^n \to \varphi$ a.e.

Let $s : \mathbb{R} \to (-\infty, +\infty]$ be the convex function defined by

$$(2.9) \qquad s(a) := \begin{cases} a \log a + (1-a) \log(1-a) & \text{if } a \in [0,1], \\ +\infty & \text{otherwise}, \end{cases}$$

and observe that for $a \in (0,1)$ we have $s''(a)\,\sigma(a) = 1$. Let $\mathcal{G}_\varepsilon : M \times \mathcal{F} \to (-\infty, +\infty]$ be the functional defined by

$$(2.10) \qquad \mathcal{G}_\varepsilon(\rho, \varphi) = \int_0^1 [s(\rho) + s(\varepsilon\varphi_x) + (1-\rho)\varphi - \log(1 + e^\varphi)]dx,$$

where we understand that $\mathcal{G}_\varepsilon(\rho, \varphi) = +\infty$ unless the measure $d\varphi$ is absolutely continuous with respect to the Lebesgue measure, and its density, denoted by φ_x, is bounded by ε^{-1}. By the convexity of s, the functional $\mathcal{G}_\varepsilon$ is lower-semicontinuous.

We shall connect the quasi-potential $V_\varepsilon(\rho)$ to the minimum of the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$ over $\mathcal{F}$. Fix ρ in M and consider the Euler-Lagrange equation associated with the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$:

$$(2.11) \qquad \begin{cases} \frac{\varepsilon\varphi_{xx}}{\varphi_x(1-\varepsilon\varphi_x)} - \frac{1}{1+e^\varphi} + \rho = 0, \\ \varphi(0) = \varphi_0, \ \varphi(1) = \varphi_1. \end{cases}$$

Since this equation is not really meaningful for $\varphi \in \mathcal{F}$, we formulate it as a fixed point condition for a suitable operator.

Denote by $K_{\rho,\varepsilon} = K(\varepsilon, \varphi_0, \varphi_1, \rho) : \mathcal{F} \to \mathcal{F}$ the integral operator

$$(2.12) \quad (K_{\rho,\varepsilon}\varphi)(x) = \varphi_0 + \frac{1}{\varepsilon} \int_0^x \frac{A \exp\{\frac{1}{\varepsilon} \int_0^y [(1 + e^\varphi)^{-1} - \rho] dz\}}{1 + A \exp\{\frac{1}{\varepsilon} \int_0^y [(1 + e^\varphi)^{-1} - \rho] dz\}} dy,$$

where $A = A(\varepsilon, \rho, \varphi) \in (0, \infty)$ is chosen so that $(K_{\rho,\varepsilon}\varphi)(1) = \varphi_1$. We prove in Section 4 that such a choice is always possible and unique. Moreover, if φ is a fixed point of $K_{\rho,\varepsilon}$, then φ is in $C^1([0,1])$, it has a Lipschitz derivative φ_x satisfying $0 < \varepsilon \varphi_x < 1$, and φ solves (2.11) a.e. We adopt, in particular, the following terminology: a function $\varphi \in \mathcal{F}$ is said to be a solution of (2.11) if it belongs to $\mathcal{P}_\varepsilon(\rho)$, the set of fixed points of $K_{\rho,\varepsilon}$.

Recall that $\bar{\rho}_\varepsilon$ is the stationary solution to (2.1). Let S_ε^o, $S_\varepsilon : M \to \mathbb{R}$ be the functionals defined by

$$(2.13) \qquad S_\varepsilon^o(\rho) := \inf\{\mathcal{G}_\varepsilon(\rho, \varphi), \varphi \in \mathcal{F}\}, \quad S_\varepsilon(\rho) = S_\varepsilon^o(\rho) - S_\varepsilon^o(\bar{\rho}_\varepsilon),$$

and note that the infimum is achieved by the lower semicontinuity of $\mathcal{G}_\varepsilon(\rho, \cdot)$ and by the compactness of $\mathcal{F}$. Given $\rho \in M$, we denote by $\mathcal{F}_\varepsilon(\rho) \subset \mathcal{F}$ the collection of minimizers for the previous variational problem, i.e.,

$$(2.14) \qquad\qquad \mathcal{F}_\varepsilon(\rho) := \arg\inf\{\mathcal{G}_\varepsilon(\rho, \varphi), \varphi \in \mathcal{F}\},$$

and observe that $\mathcal{F}_\varepsilon(\rho)$ is a nonempty compact subset of $\mathcal{F}$.

THEOREM 2.2 *Fix $\varphi_0 < \varphi_1$, $\varepsilon \in (0, \varepsilon_0)$, and $\rho \in M$. Then any minimizer of $\mathcal{G}_\varepsilon(\rho, \cdot)$ solves (2.11), i.e.,*

$$\mathcal{F}_\varepsilon(\rho) \subset \mathcal{P}_\varepsilon(\rho).$$

Moreover, the functional S_ε is lower-semicontinuous on M. Finally, if the sequence $\{\rho^n\} \subset M$ converges to ρ strongly in $L^1([0,1])$, then $S_\varepsilon(\rho^n) \to S_\varepsilon(\rho)$.

The connection between the quasi-potential and the functional (2.13) is established by the following theorem, which is the first main result of this paper.

THEOREM 2.3 *For each $\rho_0 < \rho_1$ and $\varepsilon \in (0, \varepsilon_0)$, we have $V_\varepsilon = \widehat{V}_\varepsilon = S_\varepsilon$.*

We remark that while the identity $V_\varepsilon = \widehat{V}_\varepsilon$ holds under general conditions, the characterization of the quasi-potential in terms of the static variational problem (2.13) depends crucially on the specific form of the flux f and the mobility σ, namely, $f(a) = \sigma(a) = a(1 - a)$, $a \in [0,1]$.

In the proof of Theorem 2.3, we actually describe some optimal paths for the variational problem (2.7). Fix ρ in M, let $\varphi \in \mathcal{F}_\varepsilon(\rho)$, and denote by $F = F(t, x)$, $(t, x) \in [0, +\infty) \times [0, 1]$, the solution of the viscous Burgers equation (2.1) with initial condition $e^\varphi/(1 + e^\varphi)$. Set $\psi = s'(F)$ and define $v = v(t, x)$ by

$$v = \frac{1}{1 + e^\psi} - \frac{\varepsilon \psi_{xx}}{\psi_x(1 - \varepsilon \psi_x)}.$$

We prove that an optimal path u for the variational problem (2.7) is the path v reversed in time, i.e., $u(t) = v(-t)$. This construction shows that to each $\varphi \in \mathcal{F}_\varepsilon(\rho)$

there is associated a path $u \in \mathcal{U}(\bar{\rho}_\varepsilon)$ that is a minimizer for the variational problem (2.7). If $\mathcal{F}_\varepsilon(\rho)$ is not a singleton, to different elements in $\mathcal{F}_\varepsilon(\rho)$ are associated different minimizers for (2.7), and there is no uniqueness of the minimizer for (2.7). Unfortunately, we are not able to prove that any minimizer of (2.7) can be obtained from the previous construction; in particular, we cannot deduce uniqueness of the minimizer for (2.7) from the uniqueness of the minimizer for $\mathcal{G}_\varepsilon(\rho, \cdot)$. We refer, however, to the heuristic argument presented below, in the context of the Hamiltonian formalism, which suggests that the minimizers for (2.7) are indeed in a one-to-one correspondence with $\mathcal{F}_\varepsilon(\rho)$.

Theorem 2.3 implies that the minimum of S_ε^o is achieved at $\bar{\rho}_\varepsilon$ or, equivalently, that S_ε is a positive functional. This can also be shown by direct computations. Indeed, it is enough to observe that, for a fixed $\varphi \in \mathcal{F}$, the strict convexity of the map $\rho \mapsto \mathcal{G}_\varepsilon(\rho, \varphi)$ implies that the infimum over ρ is uniquely attained for $\varphi = s'(\rho)$. A straightforward computation then shows that the functional $\varphi \mapsto \mathcal{G}_\varepsilon(s'^{-1}(\varphi), \varphi)$ has a unique critical point, which is a global minimum, achieved at $s'(\bar{\rho}_\varepsilon)$.

2.5 Uniqueness/Nonuniqueness of Optimal Paths

The connection between the quasi-potential V_ε and a "trial" functional like $\mathcal{G}_\varepsilon$ has been established for other action functionals arising as large-deviation rate functionals for a few microscopic stochastic dynamics in the diffusive scaling limit [6, 8, 16, 20]. In contrast to all other cases, the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$ is neither concave nor convex and might have more than a single critical point. Fix $\rho \in M$. The construction presented above actually shows that to each critical point φ for $\mathcal{G}_\varepsilon(\rho, \cdot)$ there is associated a path $u \in \mathcal{U}(\bar{\rho}_\varepsilon)$ that is a critical point for the variational problem (2.7). It is therefore natural to investigate whether the sets $\mathcal{P}_\varepsilon(\rho)$ and $\mathcal{F}_\varepsilon(\rho)$ are singletons. Note that, as proven in [9], in the case $\rho_0 > \rho_1$ these sets coincide and are a singleton for all $\rho \in M$.

In this direction, our next result shows that, under suitable conditions, there exists a unique critical point for $\mathcal{G}_\varepsilon(\rho, \cdot)$.

THEOREM 2.4 *The following statements hold*:

(i) *Fix $\varphi_0 < \varphi_1$. There exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for any $\varepsilon \in (\varepsilon_1, \varepsilon_0)$ the set $\mathcal{P}_\varepsilon(\rho)$ is a singleton for any $\rho \in M$.*
(ii) *Fix $\varepsilon > 0$. There exists $\delta \in (0, \varepsilon^{-1})$ such that for any $0 < \varphi_1 - \varphi_0 < \delta$ the set $\mathcal{P}_\varepsilon(\rho)$ is a singleton for any $\rho \in M$.*
(iii) *Fix $\varphi_0 < \varphi_1$ and $\varepsilon \in (0, \varepsilon_0)$. If $\rho \in M$ is in $C^1([0, 1])$ and strictly increasing, then the set $\mathcal{P}_\varepsilon(\rho)$ is a singleton.*

We remark that while the first two results are based on a standard perturbation argument and are quite natural from a statistical mechanics point of view, the third one somehow depends on the global structure of the functional $\mathcal{G}_\varepsilon$. Recalling that the stationary solution $\bar{\rho}_\varepsilon$ is smooth and strictly increasing, the third statement

implies that the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$ has a unique critical point when ρ lies in a C^1-neighborhood of $\bar{\rho}_\varepsilon$.

As discussed above, a most striking feature of the model examined here is that there can be more than a single minimizer for $\mathcal{G}_\varepsilon(\rho, \cdot)$. The next result states that this phenomenon does indeed occur.

THEOREM 2.5 *Fix $\varphi_0 < \varphi_1$. There exists $\varepsilon_2 \in (0, \varepsilon_0)$ such that the following statement holds. For each $\varepsilon \in (0, \varepsilon_2)$ there exist functions $\rho \in M$ such that $\mathcal{F}_\varepsilon(\rho)$ is not a singleton.*

As we show in Proposition 4.6, at the points ρ where $\mathcal{F}_\varepsilon(\rho)$ is not a singleton the quasi-potential admits more than one Gâteaux superdifferential. The proof of the above theorem is based on a perturbation argument with respect to the limiting case $\varepsilon = 0$ that we discuss next.

2.6 The Inviscid Limit

It is well known, see, e.g., [34, chap. 15], that in the inviscid limit $\varepsilon \downarrow 0$, the solution to the Cauchy problem associated with (2.1) converges to the entropy solution of the Cauchy problem associated with the inviscid Burgers equation $u_t + f(u)_x = 0$ with the Bardos-Le Roux-Nédélec boundary conditions [2]. We also mention that in the case of the Burgers equation on $\mathbb{R}$, the variational convergence as $\varepsilon \downarrow 0$ of the action functional $I^\varepsilon_{[-T,0]}$ for a fixed $T > 0$ is discussed in [3]. Referring to [8] for the case $\rho_0 > \rho_1$, we here discuss the variational convergence of the quasi-potential V_ε as $\varepsilon \downarrow 0$.

In the inviscid limit $\varepsilon = 0$, the stationary solutions $\bar{\rho}$ are easily described by considering the propagation of shocks for the inviscid Burgers equation on $\mathbb{R}$. If $1 - (\rho_1 + \rho_0) > 0$, an entropic shock from ρ_0 to ρ_1 travels to the right so that $\bar{\rho} = \rho_0$, while $\bar{\rho} = \rho_1$ if $1 - (\rho_0 + \rho_1) < 0$. In the case $1 - (\rho_0 + \rho_1) = 0$ there is a one-parameter family of stationary entropic solutions that corresponds to a stationary shock that can be placed anywhere in $[0, 1]$. Equivalently, a stationary entropic solution $\bar{\rho}$ satisfies $f(\bar{\rho}) = \min_{r \in [\rho_0, \rho_1]} f(r)$. For $1 - (\rho_0 + \rho_1) \neq 0$ it is not difficult to check that, as $\varepsilon \downarrow 0$, the stationary solution $\bar{\rho}_\varepsilon$ converges strongly in $L^1([0, 1])$ to the constant function equal to $\bar{\rho}$. In the case $1 - (\rho_0 + \rho_1) = 0$, $\bar{\rho}_\varepsilon$ converges to the stationary solution of the inviscid Burgers equation with a shock placed at $x = \frac{1}{2}$.

Recall (2.10) and let $\mathcal{G} : M \times \mathcal{F} \to \mathbb{R}$ be the lower-semicontinuous functional defined by

$$(2.15) \qquad \mathcal{G}(\rho, \varphi) := \int_0^1 [s(\rho) + (1 - \rho)\varphi - \log(1 + e^\varphi)]dx.$$

Let also $S^o, S : M \to \mathbb{R}$ be the functionals defined by

$$(2.16) \qquad S^o(\rho) := \inf\{\mathcal{G}(\rho, \varphi), \varphi \in \mathcal{F}\}, \quad S(\rho) := S^o(\rho) - S^o(\bar{\rho}),$$

and observe that the infimum is achieved by the lower semicontinuity of $\mathcal{G}(\rho, \cdot)$ and the compactness of $\mathcal{F}$. As discussed before, for $\rho_0 + \rho_1 \neq 1$ there exists a unique

stationary entropic solution $\bar{\rho}$ of the inviscid Burgers equation. On the other hand, if $\rho_0 + \rho_1 = 1$ there exists a one-parameter family of stationary entropic solutions $\{\bar{\rho}_\alpha, \alpha \in [0, 1]\}$; however, it is simple to check that $S^o(\bar{\rho}_\alpha)$ is in fact independent of α. This shows the functional S is well-defined.

As we show in Proposition 6.3 below, we may restrict the infimum in (2.16) to functions $\varphi \in \mathcal{F}$ that are step functions in the sense that φ jumps from φ_0 to φ_1 at a single point in $[0, 1]$. In view of this result, simple computations show that S coincides with the functional derived in [17] within the context of the boundary-driven asymmetric exclusion process.

In the inviscid limit $\varepsilon \downarrow 0$ we expect the functional S_ε to converge to S. From a variational point of view, the natural notion of convergence is the so-called Γ-convergence that we next recall; see, e.g., [12]. Let X be a Polish space. A sequence of functionals $F_n : X \to (-\infty, +\infty]$ is said to Γ-*converge* to a functional $F : X \to (-\infty, +\infty]$ if the following two conditions hold for each $x \in X$: there exists a sequence $x_n \to x$ such that $\limsup_n F_n(x_n) \le F(x)$ (Γ-lim sup *inequality*), and for any sequence $x_n \to x$ we have $\liminf_n F_n(x_n) \ge F(x)$ (Γ-lim inf *inequality*).

THEOREM 2.6 *Let* $S_\varepsilon, S : M \to [0, \infty)$ *be defined as in* (2.13) *and* (2.16)*, respectively. The family of functionals* $\{S_\varepsilon\}_{\varepsilon > 0}$ Γ*-converges to* S *in* M *as* $\varepsilon \downarrow 0$*. In particular, the functional* $S : M \to [0, +\infty)$ *is lower-semicontinuous.*

Given $\rho \in M$, we also expect that the minimizers of $\mathcal{G}_\varepsilon(\rho, \cdot)$ converge, as $\varepsilon \downarrow 0$, to a minimizer of $\mathcal{G}(\rho, \cdot)$. The precise statement is the following.

THEOREM 2.7 *Fix* $\rho \in M$ *and let* $\varphi_\varepsilon \in \mathcal{F}_\varepsilon(\rho)$*. If* $\varepsilon_n \downarrow 0$ *is a sequence such that* $\varphi_{\varepsilon_n} \to \varphi$ *for some* $\varphi \in \mathcal{F}$*, then* φ *is a minimizer for* $\mathcal{G}(\rho, \cdot)$*.*

Notation warning. Apart from when we discuss, in Section 6, the inviscid limit, the parameter $\varepsilon > 0$ is kept fixed. We therefore drop from most of the notation the explicit dependence of the functionals on ε.

3 Hamiltonian Picture

As will be clear in the following discussion, the variational problem (2.7) is naturally described within the Hamiltonian formalism. Accordingly, Theorem 2.3 reflects a peculiar geometric structure of the underlying phase space. In this section we present this picture heuristically, but we emphasize that the actual proofs are logically independent from it.

The functional $I_{[-T, 0]}$ in (2.3) is the action functional corresponding to the Lagrangian

$$\mathbb{L}(u, u_t) = \frac{1}{4\varepsilon} |u_t + f(u)_x - \varepsilon u_{xx}|^2_{-1, \sigma(u)}.$$

The associated Hamiltonian, obtained by a Legendre transform, is given by

(3.1) $$\mathbb{H}(\rho, h) = \varepsilon \langle h_x, \sigma(\rho) h_x \rangle + \langle \varepsilon \rho_{xx} - f(\rho)_x, h \rangle,$$

where $\rho : [0, 1] \to [0, 1]$ satisfies $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$ and $h : [0, 1] \to \mathbb{R}$ is the momentum, satisfying the boundary conditions $h(0) = h(1) = 0$. The canonical equations associated with the Hamiltonian $\mathbb{H}$ are

$$(3.2) \qquad \begin{cases} u_t = \varepsilon u_{xx} - f(u)_x - 2\varepsilon(\sigma(u)H_x)_x, \\ H_t = -\varepsilon H_{xx} - f'(u)H_x - \varepsilon\sigma'(u)H_x^2, \end{cases}$$

with the boundary conditions $u(t, 0) = \rho_0, u(t, 1) = \rho_1$, and $H(t, 0) = H(t, 1) = 0$. It is not clear whether the above equations do define, even locally, a flow, and the discussion here will be kept at an informal level.

As follows from the exponential attractiveness of $\bar{\rho}_\varepsilon$ for the flow defined by (2.1), $(\bar{\rho}_\varepsilon, 0)$ is a hyperbolic fixed point of the Hamilton flow (3.2). Denote by $\mathcal{M}_s$ and $\mathcal{M}_u$ the associated stable and unstable manifolds. Of course, $\mathcal{M}_s, \mathcal{M}_u \subset \{(\rho, h) : \mathbb{H}(\rho, h) = \mathbb{H}(\bar{\rho}_\varepsilon, 0) = 0\}$. Each point $(\rho, 0)$ is driven by the flow to $(\bar{\rho}_\varepsilon, 0)$ as $t \to +\infty$, and therefore $\mathcal{M}_s \supset \{(\rho, h) : h = 0\}$.

Recall the Poincaré-Cartan theorem (see, e.g., [1, sec. 44]), which states that the integral of the symplectic 1-form $\langle h, d\rho \rangle$ along any closed path in the phase space is preserved by the Hamiltonian flow. This implies that the stable and unstable manifolds are *Lagrangian*; namely, for any closed path that lies either in $\mathcal{M}_s$ or in $\mathcal{M}_u$,

$$\oint \langle H, du \rangle = 0.$$

We can therefore define the *prepotential* $W_\varepsilon : \mathcal{M}_u \to \mathbb{R}$ by

$$(3.3) \qquad W_\varepsilon(\rho, h) = \int_\Gamma \langle H, du \rangle,$$

where the integral is carried over a path Γ in $\mathcal{M}_u$ that connects $(\bar{\rho}_\varepsilon, 0)$ to (ρ, h). Recalling (2.7), the connection between the quasi-potential and the prepotential is given by

$$(3.4) \qquad \widehat{V}_\varepsilon(\rho) = \inf\{W_\varepsilon(\rho, h), h : (\rho, h) \in \mathcal{M}_u\}.$$

In a finite-dimensional framework, this result is proven in [13, 14]. For the reader's convenience, we sketch the basic argument.

Denote by $U_\varepsilon(\rho)$ the right-hand side of (3.4). By means of compactness arguments, one shows the existence of a path $u \in \mathcal{U}(\bar{\rho}_\varepsilon)$ satisfying $u(0) = \rho$ and such that

$$\widehat{V}_\varepsilon(\rho) = I(u) = \int_{-\infty}^0 \mathbb{L}(u, u_t)dt.$$

Since u is an extremal path, it satisfies the Euler-Lagrange equation or, equivalently, the pair (u, H), where $H = \delta\mathbb{L}/\delta u_t$ stands for the conjugate momentum, solves the canonical equations (3.2). One then shows that the trajectory (u, H) lies in the unstable manifold $\mathcal{M}_u$; while $u(t) \to \bar{\rho}_\varepsilon$ as $t \to -\infty$ follows from the

definition, some efforts are required to show that also $H(t) \to 0$ as $t \to -\infty$. By Legendre duality, the inclusion $\mathcal{M}_u \subset \{\mathbb{H} = 0\}$ and (3.3)

$$\widehat{V}_\varepsilon(\rho) = I(u) = \int_{-\infty}^0 \mathbb{L}(u, u_t)dt = \int_{-\infty}^0 [\langle H, u_t \rangle - \mathbb{H}(u, H)]dt =$$
$$\int_{-\infty}^0 \langle H, u_t \rangle dt = W_\varepsilon(\rho, H(0)) \geq U_\varepsilon(\rho).$$

The proof of the reverse inequality is simple. Fix ρ and choose h to be a minimizer for the right-hand side of (3.4). Since (ρ, h) belongs to $\mathcal{M}_u$, we only need to follow the Hamiltonian flow (3.2) to obtain a path (u, H) such that $u(0) = \rho$, $H(0) = h$, and $(u(t), H(t)) \to (\bar{\rho}_\varepsilon, 0)$ as $t \to -\infty$. The previous computations now give

$$\widehat{V}_\varepsilon(\rho) \leq I(u) = W_\varepsilon(\rho, h) = U_\varepsilon(\rho).$$

The above argument actually shows that any minimizer u for the variational problem (2.7) is a solution to the canonical equations (3.2) with $u(0) = \rho$ and $H(0) = h$ where h is a minimizer for the right-hand side of (3.4).

In a neighborhood of the fixed point $(\bar{\rho}_\varepsilon, 0)$, the unstable manifold $\mathcal{M}_u$ can be written as a graph; namely, it has the form $\mathcal{M}_u = \{(\rho, h) : h = m_u(\rho)\}$ for some map m_u. In this case, the infimum on the right-hand side of (3.4) is trivial and $\widehat{V}_\varepsilon(\rho) = W_\varepsilon(\rho, m_u(\rho))$. In general, though, this is not true globally. Given $\rho \in M$, to each h such that $(\rho, h) \in \mathcal{M}_u$, there corresponds a critical point for the variational problem (2.7). It may happen, for special ρ, that the variational problem on the right-hand side of (3.4) admits more than a single minimizer. In this case there is also more than one minimizer for the variational problem (2.7). In particular, as will be clearer in the following, Theorem 2.4 implies that, for either ε close to ε_0 or $\varphi_1 - \varphi_0$ small, the manifold $\mathcal{M}_u$ is globally a graph. On the other hand, when $\mathcal{P}(\rho)$ is not a singleton for some $\rho \in M$ the manifold $\mathcal{M}_u$ is not globally a graph. Finally, Theorem 2.5 implies that for ε small enough there exist functions ρ such that the minimizer for the right-hand side of (3.4) is not unique.

In view of (3.4), to prove Theorem 2.3 heuristically we need to replace W_ε by $\mathcal{G}_\varepsilon$ on the right-hand side of (3.4). It is convenient to perform the symplectic change of variables $(\rho, h) \mapsto (\varphi, \pi)$ given by

(3.5)
$$\begin{cases} \varphi = s'(\rho) - h, \\ \pi = \rho. \end{cases}$$

In the new variables (φ, π) the Hamiltonian reads

(3.6)
$$\widetilde{\mathbb{H}}(\varphi, \pi) = \mathbb{H}(\pi, s'(\pi) - \varphi)$$
$$= \varepsilon \langle \varphi_x, \sigma(\pi)\varphi_x \rangle - \langle \varepsilon \pi_x + \sigma(\pi), \varphi_x \rangle + \rho_1 - \rho_0,$$

where we used that $s'(\pi) - \varphi$ vanishes at the boundary, $\sigma(\pi)s''(\pi) = 1$, $f = \sigma$, and $\rho(0) = \rho_0$, $\rho(1) = \rho_1$. The corresponding canonical equations are

$$
\text{(3.7)} \qquad
\begin{cases}
\Phi_t = \dfrac{\delta \widetilde{\mathbb{H}}}{\delta \Pi} = \varepsilon \Phi_{xx} - \sigma'(\Pi)\Phi_x(1 - \varepsilon \Phi_x), \\[2mm]
\Pi_t = -\dfrac{\delta \widetilde{\mathbb{H}}}{\delta \Phi} = -\varepsilon \Pi_{xx} - \sigma(\Pi)_x + 2\varepsilon(\sigma(\Pi)\Phi_x)_x.
\end{cases}
$$

In the new variables the fixed point $(\bar{\rho}_\varepsilon, 0)$ reads $(s'(\bar{\rho}_\varepsilon), \bar{\rho}_\varepsilon)$. The associated stable manifold is $\{(\varphi, \pi) : \varphi = s'(\pi)\}$. Let

$$
\text{(3.8)} \qquad
\Sigma = \left\{ (\varphi, \pi) : \pi = \frac{1}{1 + e^\varphi} - \frac{\varepsilon \varphi_{xx}}{\varphi_x(1 - \varepsilon \varphi_x)} \right\}.
$$

By using that $\sigma(\pi) = \pi(1 - \pi)$, a long and tedious computation that we omit shows that the set Σ is invariant under the flow (3.7).

More precisely, pick a point $(\varphi, \pi) \in \Sigma$ and let Φ be the solution to

$$
\Phi_t = -\varepsilon \Phi_{xx} + \frac{1 - e^\Phi}{1 + e^\Phi}\Phi_x(1 - \varepsilon \Phi_x)
$$

with initial condition $\Phi(0) = \varphi$. Set now

$$
\Pi = \frac{1}{1 + e^\Phi} - \frac{\varepsilon \Phi_{xx}}{\Phi_x(1 - \varepsilon \Phi_x)}
$$

and observe that $\Pi(0) = \pi$ since $(\varphi, \pi) \in \Sigma$. Then (Φ, Π) is a solution to the canonical equations (3.7). Moreover, as we show in Lemmata 5.4 and 5.5, under the Hamiltonian flow any point in Σ converges to the fixed point $(s'(\bar{\rho}_\varepsilon), \bar{\rho}_\varepsilon))$ as $t \to -\infty$. This implies Σ is the unstable manifold. The previous arguments really only show that $\{(\varphi, \pi) : \varphi = s'(\pi)\} \subset \mathcal{M}_s$ and $\Sigma \subset \mathcal{M}_u$. On the other hand, as the tangent spaces to $\{(\varphi, \pi) : \varphi = s'(\pi)\}$ and Σ at $(s'(\bar{\rho}_\varepsilon), \bar{\rho}_\varepsilon)$ span the whole space, we informally claim that the previous inclusions are equalities.

At this point, the informal deduction of Theorem 2.3 will be completed by the computation of the prepotential. This is easily achieved in the new variables (φ, π). We start with the generating function of the symplectic transformation (3.5). Let

$$
F(\rho, \varphi) = \int_0^1 [s(\rho) - \rho, \varphi] dx
$$

be the so-called *free generating function* of (3.5) (see, e.g., [1, sec. 48]) so that

$$
h = \frac{\delta F}{\delta \rho}, \quad \pi = -\frac{\delta F}{\delta \varphi}.
$$

Equivalently,

$$
\langle h, d\rho \rangle - \langle \pi, d\varphi \rangle = dF.
$$

Hence, for any path $\Gamma = \{\gamma(t) : t \in [0, 1]\}$ in the phase space,

$$
\int_\Gamma \langle H, d\Pi \rangle = \int_\Gamma \langle \Pi, d\Phi \rangle + F(\gamma(1)) - F(\gamma(0)).
$$

Assume now that $\Gamma \subset \Sigma$. By (3.8) and since

$$\frac{\varepsilon \varphi_{xx}}{\varphi_x(1 - \varepsilon \varphi_x)} = \varepsilon[s'(\varepsilon \varphi_x)]_x,$$

we have that

$$\int_{\Gamma} \langle \Pi, d\Phi \rangle = \int_0^1 [\Phi(t) - \log(1 + e^{\Phi(t)}) + s(\varepsilon \Phi_x(t))] dx \Big|_{t=0}^{t=1}.$$

Therefore, in view of (2.10), the previous identities imply that

$$\int_{\Gamma} \langle H, du \rangle = \int_{\Gamma} \langle H, d\Pi \rangle = \mathscr{G}_\varepsilon(\Pi(1), \Phi(1)) - \mathscr{G}_\varepsilon(\Pi(0), \Phi(0)).$$

Hence, by (3.3), $W_\varepsilon(\rho, h) = \mathscr{G}_\varepsilon(\rho, \varphi) - \mathscr{G}_\varepsilon(\bar{\rho}_\varepsilon, s'(\bar{\rho}_\varepsilon))$, where h and φ are related by (3.5). As stated above, $S_\varepsilon^o(\bar{\rho}_\varepsilon) = \mathscr{G}_\varepsilon(\bar{\rho}_\varepsilon, s'(\bar{\rho}_\varepsilon))$ and therefore, in view of (3.4),

$$\widehat{V}_\varepsilon(\rho) = \inf\{\mathscr{G}_\varepsilon(\rho, \varphi), \varphi : (\varphi, \rho) \in \Sigma\} - S_\varepsilon^o(\bar{\rho}_\varepsilon).$$

Since, by Theorem 2.2, Σ is the set of critical points of the functional $\mathscr{G}_\varepsilon(\pi, \cdot)$, we can drop from the previous formula the condition that (φ, ρ) belongs to the unstable manifold Σ, because this condition will automatically be satisfied by any minimizer. This concludes the heuristic proof of Theorem 2.3.

4   The Static Variational Problem

In this section we analyze the variational problem (2.13). In particular, we prove here Theorems 2.2 and 2.4.

4.1   Critical Points of $\mathscr{G}_\varepsilon(\rho, \cdot)$

Fix $\rho \in M$. Recall definition (2.12) of the operator $K_\rho : \mathscr{F} \to \mathscr{F}$ and that we denote by $\mathscr{P}(\rho) \subset \mathscr{F}$ the set of fixed points of K_ρ. We claim that there exists a unique $A = A(\varphi) \in (0, \infty)$, depending also on φ_0, φ_1, ε, and ρ, such that $(K_\rho \varphi)(1) = \varphi_1$. Note that the integral on the right-hand side of (2.12) evaluated for $x = 1$ is strictly increasing in A, equals 0 when A vanishes, and increases to 1 as $A \uparrow \infty$. Since $\varepsilon(\varphi_1 - \varphi_0) = \varepsilon/\varepsilon_0 \in (0, 1)$ by assumption, the claim follows. By using that $0 \leq \rho \leq 1$ and $\varphi_0 \leq \varphi \leq \varphi_1$, it is also straightforward to check that there exist reals $0 < b_0 < b_1 < \infty$, $b_i = b_i(\varphi_0, \varphi_1, \varepsilon)$, such that for any $\rho \in M$ and $\varphi \in \mathscr{F}$ we have $b_0 \leq A \leq b_1$.

Let M^o be the subset of M given by the smooth functions bounded away from 0 and 1 and satisfying the boundary conditions $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$:

(4.1) $M^o := \{\rho \in C^2([0, 1]) : 0 < \rho < 1, \ \rho(0) = \rho_0, \ \rho(1) = \rho_1\}.$

Recall the notation $\mathcal{H}_0^1(\gamma)$ introduced before Theorem 2.1 and note that for $\rho \in M^o$ the norm in $\mathcal{H}_0^1(\sigma(\rho))$ is in fact equivalent to the norm in the standard Sobolev

space $\mathcal{H}_0^1$. Let the Hamiltonian $\mathbb{H} : M^o \times \mathcal{H}_0^1 \to \mathbb{R}$ be the functional defined in (3.1):

$$(4.2) \qquad \mathbb{H}(\rho, h) := \varepsilon \langle h_x, \sigma(\rho) h_x \rangle - \langle \varepsilon \rho_x - f(\rho), h_x \rangle.$$

PROPOSITION 4.1 *Fix $\varphi_0 < \varphi_1$, $\varepsilon \in (0, \varepsilon_0)$, and $\rho \in M$.*

(i) *The set $\mathcal{P}(\rho)$ is not empty.*

(ii) *If $\varphi \in \mathcal{P}(\rho)$ then $\varphi \in C^1([0, 1])$. Moreover, there exist $\delta \in (0, \frac{1}{2})$ and $C \in (0, \infty)$ independent of ρ such that any $\varphi \in \mathcal{P}(\rho)$ satisfies $\delta \le \varepsilon \varphi_x \le 1 - \delta$, and*

$$|\varphi_x(x) - \varphi_x(y)| \le C|x - y| \quad \forall x, y \in [0, 1].$$

Finally, if $\varphi \in \mathcal{P}(\rho)$ then it solves (2.11) a.e.

(iii) *If ρ belongs to M^o, then any $\varphi \in \mathcal{P}(\rho)$ belongs to $C^2([0, 1])$ and satisfies $\mathbb{H}(\rho, s'(\rho) - \varphi) = 0$.*

PROOF.

(i) It is simple to check that $K_\rho : \mathcal{F} \to \mathcal{F}$ is continuous. By the convexity and compactness of $\mathcal{F}$, the statement follows from Schauder's fixed point theorem.

(ii) Let $\varphi \in \mathcal{P}(\rho)$. The statement $\varphi \in C^1([0, 1])$ follows immediately from the definition of K_ρ. By using that $0 < b_0 \le A \le b_1 < \infty$, it is simple to check that there exists $\delta > 0$ such that $\delta \le \varepsilon \varphi_x \le 1 - \delta$ as well as the Lipschitz bound on φ_x. In view of these bounds, we can rewrite the equation $K_\rho \varphi = \varphi$ as

$$\log \frac{\varepsilon \varphi_x(x)}{1 - \varepsilon \varphi_x(x)} = \log A + \frac{1}{\varepsilon} \int_0^x [(1 + e^{\varphi(y)})^{-1} - \rho(y)] dy.$$

Since φ_x is a.e. differentiable, the above identity implies that φ satisfies the differential equation in (2.11) a.e.

(iii) The first statement is trivial. To prove the second, observe that if $\rho \in M^o$, then $s'(\rho) - \varphi$ vanishes at the boundary. Since $f = \sigma$, an integration by parts shows that $\mathbb{H}(\rho, s'(\rho) - \varphi) = 0$ is equivalent to

$$\langle \rho_x, (1 - \varepsilon \varphi_x) \rangle - \langle \sigma(\rho), \varphi_x(1 - \varepsilon \varphi_x) \rangle = 0.$$

To eliminate from this equation any derivative of ρ, we need to integrate by parts the first term. To avoid boundary terms, we add and subtract $e^\varphi / [1 + e^\varphi]$ from ρ and then integrate by parts. After these steps the previous equation becomes

$$\left\langle \rho - \frac{e^\varphi}{1 + e^\varphi}, \varepsilon \varphi_{xx} + \varphi_x(1 - \varepsilon \varphi_x) \left(\rho - \frac{1}{1 + e^\varphi} \right) \right\rangle = 0,$$

where we used that $\sigma(a) = a(1 - a)$, which implies that

$$\sigma(b) - \sigma(a) = (b - a)(1 - a - b).$$

To conclude the proof, it is now enough to recall that, in view of item (ii), φ solves (2.11). $\square$

PROOF OF THEOREM 2.4.

(i) Recall that $\varepsilon_0(\varphi_1 - \varphi_0) = 1$. Let $\widetilde{\mathcal{F}} := \{\varphi \in C^1([0,1]) : \varphi(0) = \varphi_0$, $\varphi(1) = \varphi_1$, $0 \le \varepsilon\varphi_x \le 1\}$, and observe that $\widetilde{\mathcal{F}} \subset \mathcal{F}$. Fix $\rho \in M$ and consider the integrodifferential operator $\mathcal{K}_\rho^{(1)}$ on $\widetilde{\mathcal{F}}$ defined by

$$(\mathcal{K}_\rho^{(1)}\varphi)(x) := \varphi_0 + \frac{x}{\varepsilon} - \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_0}\right)\frac{\int_0^x \exp\{\int_0^y \mathcal{R}^{(1)}(\rho,\varphi;z)dz\},dy}{\int_0^1 \exp\{\int_0^y \mathcal{R}^{(1)}(\rho,\varphi;z)dz\}dy},$$

where

$$\mathcal{R}^{(1)}(\rho,\varphi;x) := \varepsilon^{-1}\left[\rho(x) - \frac{1}{1+e^{\varphi(x)}}\right]\varphi_x(x),$$

which is informally obtained by multiplying (2.11) by φ_x and integrating the resulting equation. It is simple to check that if $\varepsilon_0 - \varepsilon$ is small enough, then $\mathcal{K}_\rho^{(1)} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$.

Let $\varphi \in \mathcal{P}(\rho)$. By Proposition 4.1, φ solves (2.11) a.e., and therefore it is also a fixed point of $\mathcal{K}_\rho^{(1)}$. Consider now the set $\widetilde{\mathcal{F}}$ endowed with the distance $d(\varphi,\psi) := \sup_x |\varphi_x(x) - \psi_x(x)|$. It is simple to show that, provided $\varepsilon_0 - \varepsilon$ is small enough, the operator $\mathcal{K}_\rho^{(1)}$ is a contraction with respect to this distance. Namely, there exists $\alpha \in (0,1)$ such that $d(\mathcal{K}_\rho^{(1)}\varphi, \mathcal{K}_\rho^{(1)}\psi) \le \alpha\, d(\varphi,\psi)$. This yields uniqueness of the fixed point of $\mathcal{K}_\rho^{(1)}$ and concludes the proof.

(ii) The proof is achieved by the same argument as for the previous item by considering the integrodifferential operator $\mathcal{K}_\rho^{(2)}$ on $\widetilde{\mathcal{F}}$ defined by

$$(\mathcal{K}_\rho^{(2)}\varphi)(x) := \varphi_0 + (\varphi_1 - \varphi_0)\frac{\int_0^x \exp\{\int_0^y \mathcal{R}^{(2)}(\rho,\varphi;z)dz\}dy}{\int_0^1 \exp\{\int_0^y \mathcal{R}^{(2)}(\rho,\varphi;z)dz\}\,dy},$$

where

$$\mathcal{R}^{(2)}(\rho,\varphi;x) := \varepsilon^{-1}\left[\frac{1}{1+e^{\varphi(x)}} - \rho(x)\right][1 - \varepsilon\varphi_x(x)],$$

which is informally obtained multiplying (2.11) by $1 - \varepsilon\varphi_x$ and integrating the resulting equation.

(iii) Fix a strictly increasing function $\rho \in M \cap C^1([0,1])$ and let $\varphi \in \mathcal{P}(\rho)$. We shall show that the quadratic form given by the second variation of the functional $\mathcal{G}_\varepsilon(\rho,\cdot)$ evaluated at φ is uniformly elliptic. This implies uniqueness of the critical point.

The second variation of $\mathcal{G}_\varepsilon(\rho,\cdot)$ evaluated at φ is the quadratic form

$$(4.3) \qquad \left\langle h, \frac{\delta^2}{\delta\varphi^2}\mathcal{G}_\varepsilon(\rho,\varphi)h\right\rangle = \int_0^1 \left[\frac{\varepsilon h_x^2}{\varphi_x(1 - \varepsilon\varphi_x)} - \frac{e^\varphi h^2}{(1+e^\varphi)^2}\right]dx$$

defined on functions h in $\mathcal{H}_0^1$. Let $\psi := \varphi_x$. Since ρ and φ are smooth, by differentiating (2.11) we deduce that

$$\left(\frac{\varepsilon \psi_x}{\varphi_x(1 - \varepsilon \varphi_x)} \right)_x + \frac{e^\varphi \psi}{(1 + e^\varphi)^2} = -\rho_x.$$

Performing the change of variables $h = \psi g$ in (4.3), which is legal because ψ is smooth and strictly positive, a two-line computation based on the previous identity shows that

$$\left\langle (\psi\, g), \frac{\delta^2}{\delta \varphi^2} \mathcal{G}_\varepsilon(\rho, \varphi)(\psi\, g) \right\rangle = \int_0^1 \left[\frac{\varepsilon \psi^2}{\varphi_x[1 - \varepsilon \varphi_x]} g_x^2 + \psi \rho_x g^2 \right] dx.$$

By Proposition 4.1(ii) and the hypothesis $\rho_x > 0$, we deduce that there exists a constant $c > 0$ depending on ρ but independent of $\varphi \in \mathcal{P}(\rho)$ such that

$$\left\langle h, \frac{\delta^2}{\delta \varphi^2} \mathcal{G}_\varepsilon(\rho, \varphi) h \right\rangle \geq c \langle h, h \rangle,$$

which concludes the proof. $\square$

4.2 Minimizers of $\mathcal{G}_\varepsilon(\rho, \cdot)$

Here we analyze the minimizers of the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$.

LEMMA 4.2 *Let $\rho \in M$ be smooth. Then any minimizer of $\mathcal{G}_\varepsilon(\rho, \cdot)$ is a fixed point of K_ρ, i.e.,*

$$\mathcal{F}(\rho) \subset \mathcal{P}(\rho).$$

PROOF. To show that any minimizer of the variational problem (2.13) is a fixed point of K_ρ, we use a dynamical argument. Let $\varphi \in \mathcal{F}$ be such that $0 \leq \varepsilon \varphi_x \leq 1$ and consider the evolution equation

(4.4)
$$\begin{cases} v_t = \varepsilon v_{xx} + v_x(1 - \varepsilon v_x)[\rho - (1 + e^v)^{-1}], \\ v(t, 0) = \varphi_0, \quad v(t, 1) = \varphi_1, \\ v(0, \cdot) = \varphi(\cdot). \end{cases}$$

Since ρ is smooth, by classical results on uniformly parabolic equations, the solution v is smooth in $(0, \infty) \times [0, 1]$.

The functional $\mathcal{G}_\varepsilon(\rho, \cdot)$ is a Lyapunov functional for the evolution (4.4). Indeed, for $t > 0$ we have

$$\frac{d}{dt} \mathcal{G}_\varepsilon(\rho, v(t)) = -\int_0^1 v_x(1 - \varepsilon v_x) \left[\frac{\varepsilon v_{xx}}{v_x(1 - \varepsilon v_x)} + \rho - \frac{1}{1 + e^v} \right]^2 dx,$$

which shows that $\mathcal{G}_\varepsilon(\rho, v(t)) \leq \mathcal{G}_\varepsilon(\rho, \varphi)$ provided φ is smooth. By a standard approximation argument, we then extend this inequality to any $\varphi \in \mathcal{F}$.

Let $F = \rho - (1 + e^v)^{-1}$ and $w = v_x$. Since $v_t(t, 0) = v_t(t, 1) = 0, t > 0$, a simple computation shows that w solves

$$\begin{cases} w_t = \varepsilon w_{xx} + [w(1 - \varepsilon w)F]_x \\ \varepsilon w_x(t, i) + w(t, i)[1 - \varepsilon w(t, i)]F(t, i) = 0, \quad i = 0, 1, \\ w(0, \cdot) = \varphi_x(\cdot). \end{cases}$$

Since F_x is bounded on compact subsets of $(0, \infty) \times [0, 1]$ and φ_x is identically equal to neither 0 nor ε^{-1}, [33, theorem 3.7] and remark (ii) that follows it imply that $0 < \varepsilon w(t, x) < 1$ for any $(t, x) \in (0, \infty) \times [0, 1]$. This proves that $0 < \varepsilon v_x < 1$ in $(0, \infty) \times [0, 1]$.

Now let $\varphi \in \mathcal{F}(\rho)$; i.e., φ is a minimizer for $\mathcal{G}_\varepsilon(\rho, \cdot)$. We deduce $\mathcal{G}_\varepsilon(\rho, v(t)) = \mathcal{G}_\varepsilon(\rho, \varphi)$; namely, that for each $t \geq 0$ the function $v(t)$ is a minimizer for $\mathcal{G}_\varepsilon(\rho, \cdot)$. Since, for $t > 0$, the function $v(t)$ is smooth and $0 < \varepsilon v_x(t) < 1$, it satisfies the Euler-Lagrange equation (2.11). In particular, $v(t)$ is a fixed point of K_ρ, that is, $v(t) \in \mathcal{P}(\rho)$. Since $v(t)$ converges to φ strongly in $L^1([0, 1])$ as $t \downarrow 0$, by taking the limit $t \downarrow 0$ in the equation $K_\rho v(t) = v(t)$, we conclude that $\varphi \in \mathcal{P}(\rho)$. $\square$

Since $\bar{\rho}_\varepsilon$ is smooth and strictly increasing, Theorem 2.4(iii) implies that $\mathcal{P}(\bar{\rho}_\varepsilon)$ is a singleton. A simple computation shows that $s'(\bar{\rho}_\varepsilon)$ solves the differential equation (2.11) for $\rho = \bar{\rho}_\varepsilon$. By the previous lemma, $s'(\bar{\rho}_\varepsilon)$ is the unique minimizer of $\mathcal{G}_\varepsilon(\bar{\rho}_\varepsilon, \cdot)$. Hence,

$$(4.5) \qquad S_\varepsilon^o(\bar{\rho}_\varepsilon) = \inf_{\varphi \in \mathcal{F}} \mathcal{G}_\varepsilon(\bar{\rho}_\varepsilon, \varphi) = \mathcal{G}_\varepsilon(\bar{\rho}_\varepsilon, s'(\bar{\rho}_\varepsilon)).$$

The previous lemma proves the first statement in Theorem 2.2 for smooth functions ρ. The proof of the general result is based on the following estimate.

LEMMA 4.3 *Fix $\varphi_0 < \varphi_1$ and $\varepsilon \in (0, \varepsilon_0)$. There exists $\delta \in (0, \frac{1}{2})$ such that for any $\rho \in M$ and any $\varphi \in \mathcal{F}(\rho)$ we have $\delta \leq \varepsilon \varphi_x \leq 1 - \delta$ a.e.*

The most direct approach to proving this lemma would be by contradiction. Assuming the existence of a minimizer φ with derivative not bounded away from 0 and ε^{-1}, we would need to construct a suitable function ψ with derivative bounded away from 0 and ε^{-1} such that $\mathcal{G}_\varepsilon(\rho, \psi) < \mathcal{G}_\varepsilon(\rho, \varphi)$. However, our attempts in this direction have not been successful, and we shall prove Lemma 4.3 by an indirect route based on a geometric characterization of the minimizers, results from convex analysis, and Lemma 4.2, which implies the statement for smooth ρ. Postponing this proof, we now prove Theorem 2.2.

PROOF OF THEOREM 2.2: THE INCLUSION $\mathcal{F}(\rho) \subset \mathcal{P}(\rho)$. Let $\rho \in M$ and $\varphi \in \mathcal{F}(\rho)$. In view of Lemma 4.3, we easily deduce that φ satisfies the Euler-Lagrange equation (2.11) weakly; namely, that for any $\lambda \in C_0^\infty([0, 1])$

$$\langle \varepsilon s'(\varepsilon \varphi_x), \lambda_x \rangle + \left\langle -\rho + \frac{1}{1 + e^\varphi}, \lambda \right\rangle = 0.$$

We deduce there exists a constant C such that for a.e. $x \in [0, 1]$

$$\varepsilon s'(\varepsilon \varphi_x(x)) = C + \int_0^x \left[\frac{1}{1 + e^\varphi} - \rho \right] dy.$$

Recalling (2.12) and that $A = A(\varphi)$ is chosen so that $(K_\rho \varphi)(1) = \varphi_1$, it is straightforward to check that $K_\rho \varphi = \varphi$. □

To conclude the proof of Theorem 2.2, it remains to prove the continuity properties of the functional S_ε. To this end we consider the space $L^1([0, 1])$ endowed with the weak topology and, recalling (2.9), we let $\Lambda : L^1([0, 1]) \to (-\infty, +\infty]$ be the lower-semicontinuous functional defined by

$$(4.6) \qquad \Lambda(\varphi) := \begin{cases} \int_0^1 [s(\varepsilon \varphi_x) + \varphi - \log(1 + e^\varphi)]dx & \text{if } \varphi \in \mathcal{F}, \\ +\infty & \text{otherwise}, \end{cases}$$

where we understand that $\Lambda(\varphi) = +\infty$ unless φ is absolutely continuous, and the density of the measure $d\varphi$, denoted by φ_x, satisfies $0 \le \varepsilon \varphi_x \le 1$ a.e. Note that the set $\{\varphi : \Lambda(\varphi) < +\infty\}$ is compact in $L^1([0, 1])$. Recalling that we consider $L^\infty([0, 1])$ endowed with the weak-* topology, the Legendre transform of Λ is defined as the function $\Lambda^* : L^\infty([0, 1]) \to \mathbb{R}$ given by

$$(4.7) \qquad \Lambda^*(\rho) := \sup_\varphi \{\langle \rho, \varphi \rangle - \Lambda(\varphi)\},$$

where the supremum is carried over all functions φ in $L^1([0, 1])$. Recalling (2.9), we also let $\mathcal{S} : L^\infty([0, 1]) \to (-\infty, +\infty]$ be the functional defined by

$$(4.8) \qquad \mathcal{S}(\rho) := \int_0^1 s(\rho)dx.$$

In view of (2.13), the previous definitions imply

$$(4.9) \qquad S_\varepsilon^o = \mathcal{S} - \Lambda^*,$$

where we understand that S_ε^o, defined in (2.13), has been extended to a functional on $L^\infty([0, 1])$ by setting $S_\varepsilon^o(\rho) = +\infty$ for $\rho \notin M$.

LEMMA 4.4 *Fix* $\varphi_0 < \varphi_1$, $\varepsilon \in (0, \varepsilon_0)$, *and a not empty closed subset* K *of* $\mathcal{F}$. *Let* $\Lambda_K^* : M \to \mathbb{R}$ *be the functional defined by*

$$\Lambda_K^*(\rho) := \sup\{\langle \rho, \varphi \rangle - \Lambda(\varphi), \varphi \in K\}.$$

Then Λ_K^* *is continuous.*

PROOF. The continuity of the map $M \ni \rho \mapsto \langle \rho, \varphi \rangle \in \mathbb{R}$ for a given $\varphi \in \mathcal{F}$ implies immediately the lower semicontinuity of Λ_K^*. To prove the upper semicontinuity, fix a sequence $\{\rho^n\} \subset M$ converging to ρ. Since $\Lambda_K^*(\rho^n) < \infty$, $n \ge 1$, there exists a sequence $\{\varphi^n\} \subset K$ such that

$$\lim_{n \to \infty} \Lambda_K^*(\rho^n) = \lim_{n \to \infty} \{\langle \rho^n, \varphi^n \rangle - \Lambda(\varphi^n)\}.$$

Since $\mathcal{F}$ is compact, by taking a subsequence, if necessary, we may assume that $\{\varphi^n\}$ converges in $\mathcal{F}$ to some $\varphi \in K$. As $\rho^n \to \rho$ in M and $\varphi^n \to \varphi$ in $\mathcal{F}$ imply $\langle \rho^n, \varphi^n \rangle \to \langle \rho, \varphi \rangle$ and $\liminf_n \Lambda(\varphi^n) \geq \Lambda(\varphi)$, we deduce

$$\limsup_n \Lambda_K^*(\rho^n) \leq \langle \rho, \varphi \rangle - \Lambda(\varphi) \leq \Lambda_K^*(\rho),$$

which is the desired upper semicontinuity. $\qquad\square$

PROOF OF THEOREM 2.2: CONCLUSION. Recall (4.8); observe that the convexity of s immediately implies the lower semicontinuity of $\mathcal{S}$ on M. On the other hand, $\mathcal{S} : M \to \mathbb{R}$ is clearly continuous with respect to the strong topology of $L^1([0,1])$. In view of the decomposition (4.9), the proof of Theorem 2.2 is then concluded by applying Lemma 4.4 with $K = \mathcal{F}$. $\qquad\square$

In what follows we shall need the following density result. Recall the definition of the set M^o introduced in (4.1).

LEMMA 4.5 *Fix ρ in M. There exist a function φ in $\mathcal{F}(\rho)$, a sequence $\{\rho^n\} \subset M^o$, and a sequence $\{\varphi^n\} \subset \mathcal{F}$, $\varphi^n \in \mathcal{F}(\rho^n)$, such that $\rho^n \to \rho$ strongly in $L^1([0,1])$, $\varphi^n \to \varphi$ in the C^1-topology, and $\mathcal{G}_\varepsilon(\rho^n, \varphi^n) \to \mathcal{G}_\varepsilon(\rho, \varphi)$.*

PROOF. Fix ρ in M and pick a sequence $\{\rho^n\} \subset M^o$ converging to ρ strongly in $L^1([0,1])$. Since $\mathcal{F}$ is compact and $\mathcal{G}_\varepsilon$ is lower-semicontinuous, there exists a sequence $\{\varphi^n\} \subset \mathcal{F}$ such that $\varphi^n \in \mathcal{F}(\rho^n)$. By Lemma 4.2 we also have $\varphi^n \in \mathcal{P}(\rho^n)$. Hence, by item (ii) in Proposition 4.1 and the Ascoli-Arzelà theorem, $\{\varphi^n\}$ is a precompact sequence in $C^1([0,1])$. In particular, by taking if necessary a subsequence, there exists a function φ in $\mathcal{F} \cap C^1([0,1])$ such that $\varphi^n \to \varphi$ in $C^1([0,1])$. The last statement and the choice of $\{\rho^n\}$ imply $\mathcal{G}_\varepsilon(\rho^n, \varphi^n) \to \mathcal{G}_\varepsilon(\rho, \varphi)$.

It remains to show that $\varphi \in \mathcal{F}(\rho)$. By taking the limit $n \to \infty$ in the equation $K_{\rho^n} \varphi^n = \varphi^n$, we readily deduce that $\varphi \in \mathcal{P}(\rho)$. To show that φ belongs to $\mathcal{F}(\rho)$, we proceed as follows. Since $\varphi^n \in \mathcal{F}(\rho^n)$, Lemma 4.4 and the continuity of $\mathcal{S}$ with respect to the strong $L^1([0,1])$ topology imply $\mathcal{G}_\varepsilon(\rho^n, \varphi^n) = S_\varepsilon^o(\rho^n) \to S_\varepsilon^o(\rho)$. Therefore, as $\mathcal{G}_\varepsilon(\rho^n, \varphi^n) \to \mathcal{G}_\varepsilon(\rho, \varphi)$, we deduce that $S_\varepsilon^o(\rho) = \mathcal{G}_\varepsilon(\rho, \varphi)$, i.e., $\varphi \in \mathcal{F}(\rho)$. $\qquad\square$

4.3 Convexity Considerations

Let X and X^* be vector spaces in duality with respect to $\langle \cdot, \cdot \rangle$. We consider X and X^*, respectively, endowed with the weak and weak-* topology. Recall that $f : X \to (-\infty, +\infty]$ is *Gâteaux differentiable* at x if there exists $\ell \in X^*$ such that

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda}[f(x + \lambda v) - f(x) - \lambda \langle \ell, v \rangle] = 0 \quad \text{for any } v \in X.$$

In such a case we denote ℓ by $D_G f(x)$. In general, we define the *Gâteaux subdifferential $D_G^- f(x)$* and *Gâteaux superdifferential $D_G^+ f(x)$* of f at the point x as

the following, possibly empty, convex subsets of X^*:

$$D_G^+ f(x) := \{\ell \in X^* : \limsup_{\lambda \downarrow 0} \tfrac{1}{\lambda}[f(x + \lambda v) - f(x) - \lambda \langle \ell, v \rangle] \leq 0$$
$$\text{for any } v \in X\}$$

(4.10)

$$D_G^- f(x) := \{\ell \in X^* : \liminf_{\lambda \downarrow 0} \tfrac{1}{\lambda}[f(x + \lambda v) - f(x) - \lambda \langle \ell, v \rangle] \geq 0$$
$$\text{for any } v \in X\}$$

in which we understand $D_G^\pm f(x) = \varnothing$ if $f(x) = +\infty$.

In the context of convex analysis, the *subdifferential* of a *convex* function $f : X \to (-\infty, +\infty]$ at a point $x \in X$, here denoted by $\partial f(x)$, is the set of linear functionals $\ell \in X^*$ such that

$$f(y) - f(x) \geq \langle \ell, y - x \rangle \quad \text{for any } y \in X,$$

in which we understand that $\partial f(x) = \varnothing$ if $f(x) = +\infty$. It is well known (see, e.g., [19, prop. 1.5.3]), that if f is convex and Gâteaux-differentiable at x, then $\partial f(x) = \{D_G f(x)\}$. It is also simple also to check that for any convex function $f : X \to (-\infty, +\infty]$, the equality $D_G^- f(x) = \partial f(x)$ holds for all $x \in X$.

Recall from (2.14) that the set $\mathcal{F}(\rho) \subset \mathcal{F}$, $\rho \in M$, represents the minimizers of $\mathcal{G}_\varepsilon(\rho, \cdot)$. In view of definition (4.6), the set $\mathcal{F}(\rho)$ coincides with the maximizers for the variational problem on the right-hand side of (4.7). We consider here such a variational problem for $\rho \in L^\infty([0, 1])$ and still denote by $\mathcal{F}(\rho)$ the set of maximizers, i.e., $\mathcal{F}(\rho) = \arg\sup\{\langle \rho, \varphi \rangle - \Lambda(\varphi), \varphi \in L^1([0, 1])\}$. For each $\rho \in L^\infty([0, 1])$, the set $\mathcal{F}(\rho)$ is a not empty compact subset of $\mathcal{F}$. Given a set A we denote by co A its convex hull.

PROPOSITION 4.6 *The functional* $\Lambda^* : L^\infty([0, 1]) \to \mathbb{R}$ *defined in* (4.7) *is lower-semicontinuous and convex. Moreover, for each* $\rho \in L^\infty([0, 1])$ *we have*

$$D_G^- \Lambda^*(\rho) = \partial \Lambda^*(\rho) = \text{co } \mathcal{F}(\rho),$$

$$D_G^+ \Lambda^*(\rho) = \begin{cases} \{\varphi\} & \text{if } \mathcal{F}(\rho) = \{\varphi\} \text{ for some } \varphi \in \mathcal{F}, \\ \varnothing & \text{otherwise.} \end{cases}$$

In particular, Λ^* *is Gâteaux-differentiable at* ρ *if and only if* $\mathcal{F}(\rho)$ *is a singleton.*

To prove this statement we need the following elementary result from convex analysis. We say that $f : X \to (-\infty, +\infty]$ has *superlinear growth* if and only if for each affine function on X, i.e., a map $X \ni x \mapsto \langle \ell, x \rangle + \alpha \in \mathbb{R}$ for some $\ell \in X^*$ and $\alpha \in \mathbb{R}$, there exists a compact $K = K_{\ell,\alpha} \subset X$ such that $f(x) \geq \langle \ell, x \rangle + \alpha$ for any $x \notin K$. Observe that if f has superlinear growth, then f is coercive.

LEMMA 4.7 *Let* $f : X \to (-\infty, +\infty]$ *be lower-semicontinuous with superlinear growth. Denote by* f^{**} *the convex envelope of* f, *i.e., the largest* convex *and lower-semicontinuous function below* f. *Then*

$$\text{co arg inf}\{f(x) : x \in X\} = \arg\inf\{f^{**}(x) : x \in X\}.$$

PROOF. The inclusion $\operatorname{co} \arg\inf\{f(x) : x \in X\} \subset \arg\inf\{f^{**}(x) : x \in X\}$ is trivial; to prove the converse we shall argue by contradiction and assume, with no loss of generality, that $\inf f = 0$. Recall that the *epigraph* of f is the subset of $X \times \mathbb{R}$ given by $\operatorname{epi} f := \{(x,t) : f(x) \leq t\}$. The convex envelope f^{**} is then characterized by the identity $\operatorname{co} \operatorname{epi} f = \operatorname{epi} f^{**}$; see, e.g., [19, prop. 1.3.2].

Assume, by contradiction, that there exists $\bar{x} \in \arg\inf\{f^{**}(x) : x \in X\}$ such that $\bar{x} \notin \operatorname{co} \arg\inf\{f(x) : x \in X\} =: C$. Since f is coercive and lower-semi-continuous, its sublevel sets, and therefore C, are compact. By the Hahn-Banach theorem there exist $\alpha \in \mathbb{R}$ and $\ell \in X^*$ such that $C \subset \{x : \langle \ell, x \rangle > \alpha\}$ and $\bar{x} \in \{x : \langle \ell, x \rangle < \alpha\}$. Since C is compact, we can find an open neighborhood A of C such that $A \subset \{x : \langle \ell, x \rangle > \alpha\}$. Since f has superlinear growth, we can also find a compact K such that for any $x \notin K$ we have $f(x) \geq -\langle \ell, x \rangle + \alpha$. Let now

$$m := \inf\{f(x), x \notin A\} > 0, \quad M := \sup\{|\langle \ell, x \rangle|, x \in K\} < \infty,$$

and set $\lambda := \max\{1, (M + \alpha)/m\} \in [1, \infty)$.

Consider the following half-spaces in $X \times \mathbb{R}$:

$$\Pi_+ := \{(x,t) : \langle \ell, x \rangle + \lambda t \geq \alpha\}, \quad \Pi_- := \{(x,t) : \langle \ell, x \rangle + \lambda t \leq \alpha\},$$

and observe that $(\bar{x}, 0)$ belongs to the interior of Π_-. It is also easy to check that $\operatorname{epi} f \subset \Pi_+$. Therefore, as $\operatorname{co} \operatorname{epi} f$ is the intersection of all the half-spaces containing $\operatorname{epi} f$, we deduce that $(\bar{x}, 0) \notin \operatorname{co} \operatorname{epi} f$, which yields the desired contradiction. $\square$

PROOF OF PROPOSITION 4.6. The lower semicontinuity and convexity of Λ^* follows trivially from its definition. Given $\rho \in L^\infty([0,1]$, set

$$\widetilde{\mathcal{F}}(\rho) := \arg\sup\{\langle \rho, \varphi \rangle - \Lambda^{**}(\varphi), \varphi \in L^1([0,1])\},$$

where, we recall, Λ^{**} denotes the convex envelope of Λ. Because $\{\varphi \in L^1([0,1]) : \Lambda(\varphi) < \infty\}$ is compact, Λ has superlinear growth and Lemma 4.7 yields $\widetilde{\mathcal{F}}(\rho) = \operatorname{co} \mathcal{F}(\rho)$. On the other hand, by, e.g., [19, prop. 1.4.1 and cor. 1.4.1,],

$$\Lambda^*(\rho) = \Lambda^{***}(\rho) := \sup_\varphi\{\langle \rho, \varphi \rangle - \Lambda^{**}(\varphi)\}.$$

Since we are now dealing with Legendre duality between convex functions, by, e.g., [19, prop. 1.5.1 and cor. 1.5.2], $\varphi \in \partial\Lambda^*(\rho)$ if and only if $\rho \in \partial\Lambda^{**}(\varphi)$ if and only if $\varphi \in \widetilde{\mathcal{F}}(\rho)$. This concludes the proof of the equality $\partial\Lambda^*(\rho) = \operatorname{co} \mathcal{F}(\rho)$.

We claim that if $\mathcal{F}(\rho) = \{\varphi\}$ for some $\varphi \in \mathcal{F}$, then $\varphi \in D_G^+\Lambda^*(\rho)$. This statement completes the proof of the proposition. Indeed, if $D_G^+\Lambda^*(\rho) \neq \varnothing$ and $D_G^-\Lambda^*(\rho) \neq \varnothing$, then $\Lambda^*(\rho)$ is necessarily Gâteaux-differentiable at ρ.

To prove the claim, given $v \in L^\infty([0,1])$ and $\lambda > 0$, pick $\varphi_\lambda \in \mathcal{F}(\rho + \lambda v)$. By the very definition of $\Lambda^*(\rho)$,

$$\Lambda^*(\rho + \lambda v) - \Lambda^*(\rho) - \lambda\langle \varphi, v \rangle \leq \lambda\langle \varphi_\lambda - \varphi, v \rangle.$$

The proof will therefore be completed once we show that any element in $\mathcal{F}(\rho + \lambda v)$ converges, as $\lambda \downarrow 0$, weakly in $L^1([0,1])$ to φ. Since we assumed $\mathcal{F}(\rho)$ to be a

singleton, this is a straightforward consequence of the lower semicontinuity of Λ and the compactness of $\mathcal{F}$.   □

The proof of Lemma 4.3 is basically achieved by Lemma 4.2, Proposition 4.6, and the following general result in convex analysis, which is proven in [29]. Let B be a *separable* Banach space and $f : B \to \mathbb{R}$ a continuous convex function. Given $x \in B$ and $\ell \in \partial f(x)$ we say that ℓ is *approximable by unique tangent functionals* if and only if there exists a sequence $\{x_n\} \subset B$ converging (in norm) to x such that $\partial f(x_n) = \{\ell_n\}$ for some $\ell_n \in B^*$ and $\ell_n \to \ell$ in the weak-* topology of B^*. The collection of elements in $\partial f(x)$ approximable by unique tangent functionals is denoted by $\partial_{\mathrm{app}} f(x)$.

THEOREM 4.8 *Let B be a separable Banach space and $f : B \to \mathbb{R}$ be a continuous convex function. Then for each $x \in B$*

$$\partial f(x) = \overline{\mathrm{co}} \, \partial_{\mathrm{app}} f(x),$$

where $\overline{\mathrm{co}}$ denotes the weak- closure of the convex hull.*

As the previous theorem requires working in a separable Banach space, we extend the functional $\Lambda^* : L^\infty([0,1]) \to \mathbb{R}$, as defined in (4.7), to a functional on the space $L^1([0,1])$ endowed with the strong topology. To avoid ambiguities, we shall denote the extended functional by $\widehat{\Lambda}$. In other words, we let $\widehat{\Lambda} : (L^1([0,1]), \text{strong}) \to \mathbb{R}$ be the functional defined by

(4.11) $$\widehat{\Lambda}(\rho) := \sup_{\varphi}\{\langle \rho, \varphi \rangle - \Lambda(\varphi)\}$$

where the supremum is carried over all φ in $L^1([0,1])$ such that $\Lambda(\varphi) < \infty$.

Clearly $\widehat{\Lambda}$ is convex and, by the argument used in Lemma 4.4, continuous with respect to the strong topology of $L^1([0,1]$. We claim that for each $\rho \in L^\infty([0,1])$ we have

(4.12) $$\partial \widehat{\Lambda}(\rho) = \partial \Lambda^*(\rho).$$

By the previous identity we mean that any element in $\partial \Lambda^*(\rho)$, which a priori belongs only to $L^1([0,1])$, belongs also to $L^\infty([0,1])$, and it is an element in $\partial \widehat{\Lambda}(\rho)$. Indeed, on the one hand, it follows from the definition of subdifferentials that $\partial \widehat{\Lambda}(\rho) \subset \partial \Lambda^*(\rho)$ for each $\rho \in L^\infty([0,1])$. The reverse inclusion follows from the definition of subdifferentials, the fact, proven in Proposition 4.6, that $\partial \Lambda^*(\rho) = \mathrm{co}\,\mathcal{F}(\rho)$, and the continuity of $\widehat{\Lambda}$ with respect to the strong topology of $L^1([0,1])$.

PROOF OF LEMMA 4.3. We shall consider the integral operator $K_\rho : \mathcal{F} \to \mathcal{F}$, as defined in (2.12), for $\rho \in L^1([0,1])$ instead of $L^\infty([0,1])$. To avoid ambiguities, denote by $\widehat{\mathcal{P}}(\rho)$ the set of $\varphi \in \mathcal{F}$, which are fixed points of K_ρ, $\rho \in L^1([0,1])$. By the proof of Proposition 4.1, $\widehat{\mathcal{P}}(\rho)$ is not empty, and any $\varphi \in \widehat{\mathcal{P}}(\rho)$ belongs to $C^1([0,1])$. Furthermore, there exist $\delta > 0$ depending on $\varphi_0 < \varphi_1$, $\varepsilon \in (0, \varepsilon_0)$, and $|\rho|_{L^1}$ such that any $\varphi \in \widehat{\mathcal{P}}(\rho)$ satisfies the bound $\delta \leq \varepsilon \varphi_x \leq 1 - \delta$. We stress that δ depends on ρ only via $|\rho|_{L^1}$.

We claim that if $\rho \in L^1([0, 1])$ is such that $\partial\widehat{\Lambda}(\rho) = \{\varphi\}$ for some $\varphi \in L^\infty([0, 1])$, then $\varphi \in \widehat{\mathcal{P}}(\rho)$. Postponing the proof of this claim, we first conclude the proof of the lemma. Fix $\rho \in L^\infty([0, 1])$ and $\varphi \in \partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$. By definition, there exists a sequence $\{\rho^n\} \subset L^1([0, 1])$ converging to ρ strongly in $L^1([0, 1])$ such that $\partial\widehat{\Lambda}(\rho^n) = \{\varphi^n\}$ and $\varphi^n \to \varphi$ weak-* in $L^\infty([0, 1])$. In view of the previous claim, $\varphi^n \in C^1([0, 1])$ and there exists $\delta_n > 0$, depending only on $|\rho^n|_{L^1}$, such that $\delta_n \leq \varepsilon\varphi_x^n \leq 1 - \delta_n$. Since ρ^n converges to ρ in $L^1([0, 1])$, $\delta = \min\{\delta_n : n \geq 1\} > 0$. By duality, it is readily shown that the map $\psi \mapsto |\psi_x|_{L^\infty}$ is lower-semicontinuous with respect to the weak-* convergence in $L^\infty([0, 1])$. Therefore $|\varepsilon\varphi_x|_{L^\infty} \leq 1 - \delta$ and, by the same argument, $|1 - \varepsilon\varphi_x|_{L^\infty} \leq 1 - \delta$; that is, $\delta \leq \varepsilon\varphi_x \leq 1 - \delta$ a.e.

Lemma 4.3 thus holds for φ in $\partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$. Fix now $\rho \in L^\infty([0, 1])$, $|\rho|_{L^\infty} \leq 1$, and $\varphi \in \mathcal{F}(\rho)$. By Proposition 4.6 and by (4.12), φ belongs to $\partial\Lambda^*(\rho) = \partial\widehat{\Lambda}(\rho)$. Hence, by Theorem 4.8 and by the first part of the proof, there exists a sequence $\{\psi^n\}$, ψ^n convex combinations of elements in $\partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$, such that $\delta \leq \varepsilon\psi_x^n \leq 1-\delta$ a.e. for some $\delta > 0$ and ψ^n converges in the weak-* topology of $L^\infty([0, 1])$ to φ. To conclude, it remains to recall the lower semicontinuity of $\psi \mapsto |\psi_x|_{L^\infty}$ with respect to the weak-* convergence in $L^\infty([0, 1])$.

We turn now to the proof of the claim. Pick a sequence of smooth functions $\{\rho^n\}$ converging to ρ strongly in $L^1([0, 1])$ and choose $\varphi^n \in \mathcal{F}(\rho^n)$. By Proposition 4.6 and (4.12), $\varphi^n \in \partial\Lambda^*(\rho^n) = \partial\widehat{\Lambda}(\rho^n)$. On the other hand, by Lemma 4.2, as ρ^n is smooth, $\varphi^n \in \widehat{\mathcal{P}}(\rho^n) = \mathcal{P}(\rho^n)$. In particular, $\{\varphi^n\} \subset \mathcal{F}$. By taking, if necessary, a subsequence, the compactness of $\mathcal{F}$ now yields the existence of $\psi \in \mathcal{F}$ such that $\varphi^n \to \psi$ in $\mathcal{F}$. We can thus take the limit $n \to \infty$ in the equation $\varphi^n = K_{\rho^n}\varphi^n$ and conclude that $\psi \in \widehat{\mathcal{P}}(\rho)$. Finally, by using the definition of subdifferentials, we easily deduce that $\psi \in \partial\widehat{\Lambda}(\rho)$. Since we assumed $\partial\widehat{\Lambda}(\rho) = \{\varphi\}$, we conclude $\psi = \varphi$. $\qquad\square$

5 The Quasi-Potential

Relying on the properties of the static variational problem presented in the previous section, we prove here the first main result, namely, the identity between the quasi-potential V_ε and the functional S_ε. The proof is organized as follows: We first prove the equality $\widehat{V}_\varepsilon = V_\varepsilon$. We then show that the algorithm presented below the statement of Theorem 2.3 provides a legal test path for the variational problem (2.7). By elementary computations, basically the ones described in the context of Hamiltonian formalism, we deduce the inequality $\widehat{V}_\varepsilon \leq S_\varepsilon$. Finally, by using the variational definition (2.3) of the action functional, we prove the inequality $V_\varepsilon \geq S_\varepsilon$ by exhibiting a suitable test function H. In these arguments, all the computations will be performed for smooth paths, and we will use density results to extend the bounds to arbitrary paths.

5.1 The Identity $\widehat{V}_\varepsilon = V_\varepsilon$

We prove in this subsection that the variational problems (2.4) and (2.7) are equivalent. We emphasize that this statement does not depend on the specific form of the flux f and the mobility σ. The argument relies on two results. The first one asserts that if the action of a path u in $\mathcal{U}(\bar{\rho}_\varepsilon)$ is finite, then there exists a sequence $t_n \to -\infty$ such that $u(t_n) - \bar{\rho}_\varepsilon$ converges to 0 in $\mathcal{H}_0^1$; recall the notation for the Sobolev spaces introduced before the statement of Theorem 2.1. The second one asserts that if a function $\rho \in M$ is such that $\rho - \bar{\rho}_\varepsilon$ is small in $\mathcal{H}_0^1$, then there exists a path in a time interval of length 1 that connects $\bar{\rho}_\varepsilon$ to ρ and has small action. In this subsection we also drop the subscript ε in the stationary solution $\bar{\rho}_\varepsilon$ and denote its derivative by $\bar{\rho}_x$.

LEMMA 5.1 *Fix $T > 0$ and a path u in $C([-T, 0], M)$ such that $I_{[-T,0]}(u) < \infty$; then*

$$\frac{\varepsilon}{8} \int_{-T}^0 |u(t) - \bar{\rho}|^2_{\mathcal{H}_0^1}\, dt + \frac{1}{4}|u(0) - \bar{\rho}|^2_{L^2} \leq$$

$$\frac{1}{4}|u(-T) - \bar{\rho}|^2_{L^2} + I_{[-T,0]}(u) + \frac{1}{2\varepsilon} \int_{-T}^0 |u(t) - \bar{\rho}|^2_{L^2}\, dt.$$

PROOF. The proof of this lemma is similar to the one given for lemma 4.9 in [10] or lemma 4.2 in [22]. We thus just sketch the argument.

Fix $T > 0$ and recall the definition of the linear functional L_u introduced in (2.2). By definition of the action functional $I_{[-T,0]}$, for any function H in $C_0^\infty([-T, 0] \times [0, 1])$,

$$L_u(H) - \varepsilon \langle\!\langle H_x, \sigma(u)H_x \rangle\!\rangle \leq I_{[-T,0]}(u).$$

Take $H = (\frac{1}{2})(u - \bar{\rho})$. This function is not smooth and does not satisfy the boundary conditions at $-T$ and 0, but can be approximated by such smooth functions in the norms needed for our purposes. This is presented with all details in the proofs of [10, lemma 4.9] and [22, lemma 4.2]. For instance, to match the boundary conditions at $-T$ and 0, we multiply H by a time-dependent function that vanishes at $-T$ and 0 and is close to the indicator of the interval $[0, T]$.

Integrating by parts and adding to the previous expression $\langle\!\langle f(\bar{\rho})_x - \varepsilon\bar{\rho}_{xx}, H \rangle\!\rangle$, which vanishes, the previous equation implies

$$\frac{1}{4}|u(0) - \bar{\rho}|^2_{L^2} + \frac{\varepsilon}{2}\langle\!\langle u_x - \bar{\rho}_x, u_x - \bar{\rho}_x \rangle\!\rangle - \frac{\varepsilon}{4}\langle\!\langle u_x - \bar{\rho}_x, \sigma(u)[u_x - \bar{\rho}_x] \rangle\!\rangle \leq$$

$$I_{[-T,0]}(u) + \frac{1}{4}|u(-T) - \bar{\rho}|^2_{L^2} + \frac{1}{2}\langle\!\langle f(u) - f(\bar{\rho}), u_x - \bar{\rho}_x \rangle\!\rangle.$$

Since $\sigma(u) \leq \frac{1}{4}$, it remains to apply the Schwarz inequality to bound the last term and to recall that f is Lipschitz with Lipschitz constant 1. $\square$

LEMMA 5.2 *Fix a path u in $\mathcal{U}(\bar{\rho})$ such that $I(u) < \infty$. Then*

$$\lim_{t \to -\infty} |u(t) - \bar{\rho}|_{L^2} = 0 \quad and \quad \lim_{n \to \infty} \int_{-(n+1)}^{-n} |u(t) - \bar{\rho}|_{\mathcal{H}_0^1}^2 \, dt = 0.$$

PROOF. We first show that

(5.1)
$$\lim_{n \to \infty} \int_{-(n+1)}^{-n} \langle u(t) - \bar{\rho}, u(t) - \bar{\rho} \rangle \, dt = 0.$$

Let $\{e_k, k \geq 1\}$ be the complete orthonormal system of $L_2([0,1])$ given by $e_k(x) = \sqrt{2} \sin(k \pi x)$ and set $\theta_k(t) = \langle u(t) - \bar{\rho}, e_k \rangle$, $k \geq 1$. The integral in the previous formula can be written as

$$\sum_{k \geq 1} \int_{-(n+1)}^{-n} \theta_k(t)^2 \, dt \leq \sum_{k=1}^{k_0} \int_{-(n+1)}^{-n} \theta_k(t)^2 \, dt + \frac{1}{k_0^2} \sum_{k \geq 1} \int_{-(n+1)}^{-n} k^2 \theta_k(t)^2 \, dt$$

for any $k_0 \geq 1$. Since e_k is an eigenfunction of the Laplacian with Dirichlet boundary conditions, the second term is equal to the time integral of $(\pi k_0)^{-2} |u(t) - \bar{\rho}|_{\mathcal{H}_0^1}^2$. Therefore, by Lemma 5.1, the last expression is less than or equal to

$$\sum_{k=1}^{k_0} \int_{-(n+1)}^{-n} \langle u(t) - \bar{\rho}, e_k \rangle^2 \, dt + \frac{2}{\pi^2 k_0^2 \varepsilon} |u(-(n+1)) - \bar{\rho}|_{L^2}^2$$

$$+ \frac{1}{\pi^2 k_0^2} \left\{ \frac{8}{\varepsilon} I_{[-(n+1),-n]}(u) + \frac{4}{\varepsilon^2} \int_{-(n+1)}^{-n} |u(t) - \bar{\rho}|_{L^2}^2 \, dt \right\}.$$

Since $u(t) - \bar{\rho}$ is absolutely bounded by 1 and since $I_{[-(n+1),-n]}(u) \leq I(u)$, the previous expression is bounded by

$$\sum_{k=1}^{k_0} \int_{-(n+1)}^{-n} \langle u(t) - \bar{\rho}, e_k \rangle^2 \, dt + \frac{1}{\pi^2 k_0^2} \left\{ \frac{2}{\varepsilon} + \frac{4}{\varepsilon^2} + \frac{8}{\varepsilon} I(u) \right\}.$$

Since $u(t)$ converges weakly in $L^2([0,1])$ to $\bar{\rho}$ as $t \to -\infty$, to conclude the proof of the claim (5.1) it remains to let $n \to \infty$ and then $k_0 \to \infty$.

Since u belongs to $C((-\infty, 0], M)$, the function $t \mapsto |u(t) - \bar{\rho}|_{L^2}$ is lower-semicontinuous. In particular, there exists $s_n \in [-n, -(n-1)]$ such that

$$|u(s_n) - \bar{\rho}|_{L^2}^2 = \min_{-n \leq t \leq -(n-1)} |u(t) - \bar{\rho}|_{L^2}^2.$$

By applying Lemma 5.1 in the time interval $[s_{n+1}, t]$ with $t \in [-n, -(n-1)]$, we deduce

$$\sup_{-n \leq t \leq -(n-1)} |u(t) - \bar{\rho}|_{L^2}^2 \leq$$
$$4 I_{[-(n+1),-(n-1)]}(u) + \left(1 + \frac{2}{\varepsilon}\right) \int_{-(n+1)}^{-(n-1)} |u(t) - \bar{\rho}|_{L^2}^2 \, dt,$$

where we bounded $|u(s_{n+1}) - \bar{\rho}|_{L^2}^2$ by the time integral of $|u(t) - \bar{\rho}|_{L^2}^2$ over the interval $[-(n+1), -n]$. In view of the hypothesis $I(u) < \infty$ and (5.1), the first

statement of the lemma follows from the previous estimate. The second one is a direct consequence of the first and Lemma 5.1.   □

Fix a path u in $\mathcal{U}(\bar{\rho}_\varepsilon)$ such that $I(u) < \infty$. It follows from the previous lemma that there exists a sequence $t_n \to -\infty$ such that

(5.2) $$\lim_n |u(t_n) - \bar{\rho}|_{\mathcal{H}_0^1} = 0.$$

LEMMA 5.3 *Suppose that ρ is a function in M such that*

$$|\rho - \bar{\rho}|_{\mathcal{H}_0^1} \le \frac{1}{2} \min\{\rho_0, 1 - \rho_1\}.$$

Then there exists a constant C_0, depending only on ρ_0 and ρ_1, such that for any $T > 0$

$$\inf\{I_{[-T,0]}(u), u(-T) = \bar{\rho}, u(0) = \rho\} \le C_0\left(T + \tfrac{1}{T}\right)\left(\varepsilon + \tfrac{1}{\varepsilon}\right)|\rho - \bar{\rho}|_{\mathcal{H}_0^1}^2.$$

PROOF. We have to exhibit a path whose action can be estimated by the right-hand side of the inequality appearing in the statement of the lemma. We claim that the straight one $u(t) = [1 + (t/T)]\rho - (t/T)\bar{\rho}$, $t \in [-T, 0]$, fulfills the requirements.

Since $\rho_0 \le \bar{\rho}(x) \le \rho_1$, it follows from the assumption of the lemma and from the elementary estimate between the L^∞-norm and the $\mathcal{H}_0^1$-norm, namely, from the estimate $|h|_{L^\infty}^2 \le \int_0^1 h_x^2\, dx$, that $\rho_0/2 \le u \le \rho_1 + (1 - \rho_1)/2$. In particular, $\sigma(u) \ge c_0 > 0$ for some constant c_0 that depends only on ρ_0 and ρ_1.

Recall definition (2.3) of the action functional $I_{[-T,0]}$. Fix a function H in $C_0^\infty([-T, 0] \times [0, 1])$; we need to estimate three terms to get a bound on $I_{[-T,0]}(u)$. The first one is $\langle\!\langle u_t, H \rangle\!\rangle$. Since $u_t = (\rho - \bar{\rho})/T$, by using the Schwarz inequality and $\langle h, h \rangle \le \langle h_x, h_x \rangle$, we deduce

$$\langle\!\langle u_t, H \rangle\!\rangle \le \frac{\varepsilon c_0}{2}\langle\!\langle H_x, H_x \rangle\!\rangle + \frac{1}{2\varepsilon c_0 T}\,|\rho - \bar{\rho}|_{\mathcal{H}_0^1}^2.$$

The second and third terms are estimated as follows. Since $f(\bar{\rho})_x + \varepsilon\bar{\rho}_{xx} = 0$,

$$\langle\!\langle f(u)_x - \varepsilon u_{xx}, H \rangle\!\rangle = -\langle\!\langle f(u) - f(\bar{\rho}), H_x \rangle\!\rangle + \varepsilon\langle\!\langle u_x - \bar{\rho}_x, H_x \rangle\!\rangle.$$

Since $u - \bar{\rho} = [1 + (t/T)][\rho - \bar{\rho}]$, again by the Schwarz inequality

$$\langle\!\langle f(u)_x - \varepsilon u_{xx}, H \rangle\!\rangle \le \frac{\varepsilon c_0}{2}\langle\!\langle H_x, H_x \rangle\!\rangle + \frac{T}{c_0}\left(\varepsilon + \frac{1}{\varepsilon}\right)|\rho - \bar{\rho}|_{\mathcal{H}_0^1}^2,$$

where we used the fact that f' is absolutely bounded by 1 and again the estimate $\langle h, h \rangle \le \langle h_x, h_x \rangle$. Since $\sigma(u) \ge c_0$, the lemma follows from the previous bounds and (2.3).   □

We are now ready to prove the equivalence between variational problems (2.4) and (2.7).

PROOF OF THEOREM 2.3: THE IDENTITY $\widehat{V}_\varepsilon = V_\varepsilon$. Any path u belonging to $C([-T, 0]; M)$ such that $u(-T) = \bar\rho$ may be extended to a path in $\mathcal{U}(\bar\rho)$ by setting $u(t) = \bar\rho$ for $t \in (-\infty, -T)$. The inequality $\widehat{V}_\varepsilon \le V_\varepsilon$ follows trivially.

To prove the reverse inequality, fix ρ in M such that $\widehat{V}_\varepsilon(\rho) < \infty$. Fix $0 < \delta \le \frac{1}{7}2 \min\{\rho_0, 1 - \rho_1\}$ and let $u^\delta \in \mathcal{U}(\bar\rho)$ be such that $u^\delta(0) = \rho$ and $I(u^\delta) \le \widehat{V}_\varepsilon(\rho) + \delta$. By (5.2), there exists $T > 0$ such that

$$|u^\delta(-T) - \bar\rho|_{\mathcal{H}_0^1} \le \delta.$$

By Lemma 5.3, there exists a path v in $C([-1, 0], M)$ such that $v(-1) = \bar\rho$, $v(0) = u^\delta(-T)$, and $I_{[-1,0]}(v) \le 2C_0(\varepsilon + \varepsilon^{-1})\delta^2$. Consider the path w defined by $w(t) = v(t + T)$, $t \in [-(T + 1), -T]$, $w(t) = u^\delta(t)$, $t \in (-T, 0]$ in $C([-T - 1, 0], M)$. Clearly, $w(-T - 1) = \bar\rho$, $w(0) = \rho$, and

$$I_{[-(T+1),0]}(w) = I_{[-1,0]}(v) + I_{[-T,0]}(u^\delta) \le \widehat{V}_\varepsilon(\rho) + \delta + 2C_0\left(\varepsilon + \tfrac{1}{\varepsilon}\right)\delta^2,$$

which, by the arbitrariness of δ, yields $V_\varepsilon(\rho) \le \widehat{V}_\varepsilon(\rho)$ and concludes the proof. $\square$

5.2 Upper Bound for the Quasi-Potential

Recalling (4.1) and (3.8), the first two lemmata below state that if ρ belongs to M^o and $(\varphi, \rho) \in \Sigma$, then we can construct a solution to the canonical equations (3.7) converging to $(s'(\bar\rho_\varepsilon), \bar\rho_\varepsilon)$ as $t \to -\infty$.

Fix a continuous function $\varphi : [0, 1] \to [\varphi_0, \varphi_1]$ satisfying the boundary conditions $\varphi(0) = \varphi_0$, $\varphi(1) = \varphi_1$, and consider the parabolic equation

(5.3)
$$\begin{cases} \psi_t = \varepsilon\psi_{xx} + \frac{e^\psi - 1}{e^\psi + 1}\psi_x(1 - \varepsilon\psi_x), \\ \psi(t, 0) = \varphi_0, \ \psi(t, 1) = \varphi_1, \\ \psi(0, \cdot) = \varphi(\cdot). \end{cases}$$

A classical solution ψ of (5.3) is a function $\psi : [0, \infty) \times [0, 1] \to \mathbb{R}$ such that ψ is continuous in $[0, \infty) \times [0, 1]$, ψ_t, ψ_x, and ψ_{xx} are continuous in $(0, \infty) \times (0, 1)$, and ψ satisfies the identities (5.3). Let $F : [0, \infty) \times [0, 1] \to [0, 1]$ be a classical solution to the viscous Burgers equation (2.1). A simple computation shows that $\psi = s'(F) = \log[F/(1 - F)]$ is a classical solution to (5.3).

LEMMA 5.4 *Fix an absolutely continuous function $\varphi : [0, 1] \to [\varphi_0, \varphi_1]$ satisfying the boundary conditions $\varphi(0) = \varphi_0$, $\varphi(1) = \varphi_1$, and $0 \le \varepsilon\varphi_x \le 1$. There exists a unique classical solution ψ to (5.3). Moreover, this solution satisfies $0 < \varepsilon\psi_x < 1$ for $(t, x) \in (0, \infty) \times [0, 1]$. Finally, as $t \to +\infty$ the function $\psi(t)$ converges to $s'(\bar\rho_\varepsilon) = \log[\bar\rho_\varepsilon/(1 - \bar\rho_\varepsilon)]$ in the $C^3([0, 1])$ topology uniformly in φ.*

PROOF. We start with existence. Set $F_0 := e^\varphi/[1 + e^\varphi]$. By assumption, the function F_0 is continuous, bounded below by ρ_0, and bounded above by ρ_1, and it satisfies $F_0(0) = \rho_0$, $F_0(1) = \rho_1$. By theorem 4.4 in [28, chap. 6], there exists a unique classical solution, denoted by $F = F(t, x)$, to the viscous Burgers equation (2.1) with initial condition F_0. By the maximum principle, $\rho_0 \le F \le \rho_1$. The

function $\psi : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ given by $\psi := s'(F) = \log[F/(1 - F)]$ is therefore well-defined. An elementary computation that we omit, relying on the special form of the flux given by $f(u) = u(1 - u)$, shows that ψ is a classical solution to (5.3). Uniqueness follows from a similar argument. Indeed, if ψ is a classical solution to (5.3), then $F = e^{\psi}/[1 + e^{\psi}]$ is a classical solution to (2.1). Hence, uniqueness of (5.3) follows from uniqueness of (2.1).

A recursive argument, relying on differentiability properties of solutions of linear parabolic equations, shows that $F(t)$, $F_t(t)$, and therefore $\psi(t)$, $\psi_t(t)$, belong to $C^k([0, 1])$ for any $k \geq 1$. To prove that $0 < \varepsilon \psi_x < 1$ for $(t, x) \in (0, \infty) \times [0, 1]$, let $\gamma := \psi_x$ and observe that γ solves the nonlinear equation with mixed boundary conditions

$$\begin{cases} \gamma_t = \varepsilon \gamma_{xx} + [g\gamma(1 - \varepsilon\gamma)]_x, \\ \varepsilon\gamma_x(t, 0) = -g(t, 0)\gamma(t, 0)(1 - \varepsilon\gamma(t, 0)), \\ \varepsilon\gamma_x(t, 1) = -g(t, 1)\gamma(t, 1)(1 - \varepsilon\gamma(t, 1)), \\ \gamma(0, \cdot) = \varphi_x(\cdot), \end{cases}$$

where $g = [e^{\psi} - 1]/[e^{\psi} + 1]$. Since g_x is bounded on compact subsets of $(0, \infty) \times [0, 1]$, and φ_x is neither identically equal to 0 nor to ε^{-1}, [33, theorem 3.7] and remark (ii) following it imply the result.

Finally, by [15, theorem 4.9], as $t \rightarrow \infty$, the function $F(t)$ converges to $\bar{\rho}_{\varepsilon}$ in the $C^1([0, 1])$ topology, uniformly over $F_0 : [0, 1] \rightarrow [\rho_0, \rho_1]$. By the methods developed there, it is straightforward to prove this statement in the $C^3([0, 1])$ topology. Since $F(t)$ converges to $\bar{\rho}_{\varepsilon}$ in the $C^3([0, 1])$ topology uniformly in F_0, $\psi(t)$ converges to $\log[\bar{\rho}_{\varepsilon}/(1 + \bar{\rho}_{\varepsilon})] = \bar{\varphi}_{\varepsilon}$ in the $C^3([0, 1])$ topology, uniformly in φ. $\qquad \square$

Recalling (4.1), fix ρ in M^o and φ in $\mathcal{P}(\rho)$. Let ψ be the solution to (5.3) and define $v : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ by

$$(5.4) \qquad\qquad v := \frac{1}{1 + e^{\psi}} - \frac{\varepsilon \psi_{xx}}{\psi_x(1 - \varepsilon\psi_x)}.$$

Since φ is smooth, $\psi(t)$ and $v(t)$ belong to $C^k([0, 1])$ for all $t \geq 0$, $k \geq 1$.

LEMMA 5.5 *Function v defined by (5.4) is smooth, solves*

$$(5.5) \qquad \begin{cases} v_t - f(v)_x = \varepsilon v_{xx} - 2\varepsilon[\sigma(v)\psi_x]_x, \\ v(t, 0) = \rho_0, \ v(t, 1) = \rho_1, \\ v(0, \cdot) = \rho(\cdot), \end{cases}$$

and satisfies $0 < v < 1$, $(t, x) \in [0, \infty) \times [0, 1]$. Moreover, as $t \rightarrow +\infty$ the function $v(t)$ converges to $\bar{\rho}_{\varepsilon}$ in the $C^1([0, 1])$ topology, uniformly for ρ in M^o.

PROOF. By using the differential equation in (5.3) and the identity $f(u) = \sigma(u) = u(1 - u)$, a tedious computation shows that v solves the differential equation in (5.5). The boundary conditions in (5.5) follow directly from (5.3) and the definition of v. The initial condition in (5.5) holds, in view of (5.4), because φ is a

fixed point of K_ρ and therefore solves the Euler-Lagrange equation (2.11). Since the boundary conditions are bounded away from 0 and 1, and since ψ_{xx} is bounded on compact subsets of $[0, \infty) \times [0, 1]$, by theorem 3.7 in [33] and remark (ii) following it, $0 < v < 1$, $(t, x) \in [0, \infty) \times [0, 1]$.

By Lemma 5.4, $\psi(t)$ converges in the $C^3([0, 1])$ topology to $\log[\bar{\rho}_\varepsilon / (1 - \bar{\rho}_\varepsilon)]$, uniformly in φ. Therefore, by (5.4), $v(t)$ converges in the $C^1([0, 1])$ topology to $\bar{\rho}_\varepsilon$ uniformly for ρ in M^o. $\qquad\square$

LEMMA 5.6 *Fix a time interval $[T_1, T_2]$, a smooth path $u \in C([T_1, T_2]; M^o)$, and a smooth path $\varphi \in C([T_1, T_2]; \mathcal{F})$ with $0 < \varepsilon\varphi_x < 1$; then*

$$(5.6) \quad \mathscr{G}_\varepsilon(u(T_2), \varphi(T_2)) - \mathscr{G}_\varepsilon(u(T_1), \varphi(T_1)) =$$
$$\int_{T_1}^{T_2} \left\{ \langle s'(u) - \varphi, u_t \rangle - \left\langle \frac{\varepsilon\varphi_{xx}}{\varphi_x(1 - \varepsilon\varphi_x)} - \frac{1}{1 + e^\varphi} + u, \varphi_t \right\rangle \right\} dt.$$

The proof relies on a simple computation, which is omitted. We remark that if the paths u and φ are chosen so that $(\varphi(t), u(t)) \in \Sigma$, $t \in [T_1, T_2]$, then the second term on the right-hand side of (5.6) vanishes. Therefore, the previous lemma provides an explicit expression for the integral of the symplectic 1-form along any path that lies in Σ, showing in particular that the result depends only on the endpoints of the path, namely, that Σ is Lagrangian.

In view of Lemma 5.5, the time reversal of the function v defined in (5.4) can be chosen as a test path for the variational problem (2.7). In the next lemma we compute the action of such a path.

LEMMA 5.7 *Fix a function ρ in M^o and φ in $\mathcal{P}(\rho)$. Let v be the path (5.4) and let $u \in \mathcal{U}(\bar{\rho}_\varepsilon)$ be defined by $u(t) := v(-t)$. Then $I(u) = \mathscr{G}_\varepsilon(\rho, \varphi) - S_\varepsilon^o(\bar{\rho}_\varepsilon)$.*

PROOF. By Lemma 5.5, u is a smooth path in $C((-\infty, 0]; M^o)$, satisfies the final condition $u(0, \cdot) = \rho(\cdot)$, and solves

$$u_t + f(u)_x = -\varepsilon u_{xx} + 2\varepsilon[\sigma(u)\psi_x^*]_x,$$

where $\psi^*(t) = \psi(-t)$, $t \leq 0$. Equivalently, by setting $K = s'(u) - \psi^*$, u solves

$$(5.7) \qquad u_t + f(u)_x = \varepsilon u_{xx} - 2\varepsilon[\sigma(u)K_x]_x.$$

Fix $T > 0$, recall that u satisfies the boundary conditions $u(t, 0) = \rho_0$, $u(t, 1) = \rho_1$, $t \in [-T, 0]$, and observe that K is smooth and satisfies $K(t, 0) = K(t, 1) = 0$, $t \in [-T, 0]$. In particular, $K \in \mathfrak{H}_0^1(\sigma(u))$ and therefore, by Theorem 2.1,

$$I_{[-T,0]}(u) = \varepsilon\|K\|_{1,\sigma(u)}^2 = \varepsilon\langle\!\langle K_x, \sigma(u)K_x \rangle\!\rangle.$$

Recall definition (4.2) of the Hamiltonian $\mathbb{H}$. Multiplying both sides of (5.7) by K and integrating, we get that

$$\langle K, u_t \rangle - \mathbb{H}(u, K) = \varepsilon\langle K_x, \sigma(u)K_x \rangle$$

so that

$$I_{[-T,0]}(u) = \int_{-T}^{0} \{\langle s'(u) - \psi^*, u_t \rangle - \mathbb{H}(u, s'(u) - \psi^*)\} dt.$$

Since v is defined by (5.4), u is given by the same equation with ψ^* replacing ψ. In particular, by Proposition 4.1(iii), $\mathbb{H}(u, s'(u) - \psi^*)$ vanishes. Hence, by Lemma 5.6,

$$(5.8) \qquad I_{[-T,0]}(u) = \int_{-T}^{0} \langle s'(u) - \psi^*, u_t \rangle dt = \mathcal{G}_\varepsilon(\rho, \varphi) - \mathcal{G}_\varepsilon(v(T), \psi(T)).$$

By Lemma 5.4 $\psi(T)$ converges to $s'(\bar\rho_\varepsilon)$ in the $C^3([0, 1])$ topology; by Lemma 5.5 $v(T)$ converges to $\bar\rho_\varepsilon$ in the $C^1([0, 1])$ topology. Therefore $\mathcal{G}_\varepsilon(v(T), \psi(T))$ converges to $\mathcal{G}_\varepsilon(\bar\rho_\varepsilon, s'(\bar\rho_\varepsilon))$, which is equal to $S_\varepsilon^o(\bar\rho_\varepsilon)$ by (4.5). In conclusion,

$$I(u) = \lim_{T \to \infty} I_{[-T,0]}(u) = \mathcal{G}_\varepsilon(\rho, \varphi) - S_\varepsilon^o(\bar\rho_\varepsilon),$$

which proves the lemma. $\qquad\square$

It follows from the previous lemma that if φ is chosen in $\mathcal{F}(\rho)$ (namely, φ is a minimizer for $\mathcal{G}_\varepsilon(\rho, \cdot)$), then $I(u) = S_\varepsilon^o(\rho) - S_\varepsilon^o(\bar\rho_\varepsilon) = S_\varepsilon(\rho)$. In particular, this proves the inequality $\widehat{V}_\varepsilon(\rho) \leq S_\varepsilon(\rho)$ for smooth functions ρ in M^o. By a density argument, we next show this bound holds for any $\rho \in M$.

PROOF OF THEOREM 2.3: THE BOUND $\widehat{V}_\varepsilon \leq S_\varepsilon$. Fix $\rho \in M$ and denote by $\varphi \in \mathcal{F}(\rho)$, $\{\rho^n\} \subset M^o$, and $\{\varphi^n\} \subset \mathcal{F}$ the function and the sequences provided by Lemma 4.5. Denote by ψ the classical solution to (5.3), and define $u : (-\infty, 0] \times [0, 1] \to [0, 1]$ by

$$u(t) = \begin{cases} \rho & \text{if } t = 0, \\ v(-t) & \text{if } t < 0, \end{cases}$$

where v is the function defined in (5.4). Since $\varphi \in C^1([0, 1])$, $\psi(t) \to \varphi$ in the C^1 topology as $t \downarrow 0$. Since by Lemma 4.3 we have $0 < \varepsilon\varphi_x < 1$, it is simple to check that the function $u(t)$ converges to ρ in M as $t \uparrow 0$. Hence, by the convergence of $\psi(t)$ as $t \to +\infty$ stated in Lemma 5.4, the path u belongs to the set $\mathcal{U}(\bar\rho_\varepsilon)$ introduced in (2.5).

Let ψ^n be the solution to (5.3) with φ replaced by φ^n, and let v^n be as defined in (5.4) with ψ replaced by ψ^n. Finally, let $u^n : (-\infty, 0] \times [0, 1] \to [0, 1]$ be defined by $u^n(t) = v^n(-t)$. In view of the continuity with respect to the initial condition of the solution to the viscous Burgers equation (2.1) and the uniformity of the convergence as $t \to +\infty$ stated in Lemma 5.5, the sequence $\{u^n\}$ converges to u in $\mathcal{U}(\bar\rho_\varepsilon)$. The lower semicontinuity of the functional $I : \mathcal{U}(\bar\rho_\varepsilon) \to [0, +\infty]$, together with Lemmata 5.7 and 4.5, now imply

$$I(u) \leq \liminf_n I(u^n) = \liminf_n [\mathcal{G}_\varepsilon(\rho^n, \varphi^n) - S_\varepsilon^o(\bar\rho_\varepsilon)]$$
$$= \mathcal{G}_\varepsilon(\rho, \varphi) - S_\varepsilon^o(\bar\rho_\varepsilon) = S_\varepsilon(\rho),$$

whence $\widehat{V}_\varepsilon(\rho) \leq S_\varepsilon(\rho)$ and the proof is concluded. □

5.3 Lower Bound for the Quasi-Potential

Before carrying out the details, we explain the main idea and the novel difficulty encountered here. Fix $\rho \in M$, $T > 0$, and a path $u \in C([-T, 0]; M)$ such that $u(-T) = \bar{\rho}_\varepsilon$ and $u(0) = \rho$. We need to show $I_{[-T,0]}(u) \geq S_\varepsilon(\rho)$. Assume that the path u is smooth, bounded away from 0 and 1, and satisfies the boundary conditions ρ_0, ρ_1 at the endpoints of $[0, 1]$. By the variational definition (2.3) of the action functional and the definition (4.2) of the Hamiltonian $\mathbb{H}$, for each smooth function $H : [-T, 0] \times [0, 1] \to \mathbb{R}$ vanishing at the boundary, we have

(5.9)
$$I_{[-T,0]}(u) \geq \langle\!\langle u_t + f(u)_x - \varepsilon u_{xx}, H \rangle\!\rangle - \varepsilon \langle\!\langle H_x, \sigma(u) H_x \rangle\!\rangle$$
$$= \int_{-T}^0 [\langle H, u_t \rangle - \mathbb{H}(u, H)] dt.$$

Assume now that for each $t \in [-T, 0]$ there exists a unique solution $\varphi(t)$ to the Euler-Lagrange equation (2.11) with ρ replaced by $u(t)$. Assume furthermore that $\varphi(t)$ is smooth; in this case we may choose above $H = s'(u) - \varphi$. In view of Proposition 4.1(iii) and Lemma 5.6, we then conclude $I_{[-T,0]}(u) \geq S_\varepsilon(\rho)$. If for each $\rho \in M$ the functional $\mathscr{G}(\rho, \cdot)$ has a unique critical point, it is not difficult to turn the previous argument into a proof [6, 8].

On the other hand, in the case discussed here we have to face the lack of uniqueness of (2.11): if we choose the "wrong" φ, the bound we would get by the previous argument would not be sharp. We shall overcome this problem by discretizing the time interval $[-T, 0]$ and choosing a piecewise constant path $\varphi(t)$. As we show, the previous argument then gives the sharp bound provided we choose the optimal φ at the endpoint of each time step.

We start by recalling the following density result, which is proven in theorem 5.1 of [10].

LEMMA 5.8 *Fix $T > 0$ and a path u in $C([-T, 0]; M)$ such that $u(-T) = \bar{\rho}_\varepsilon$ and $I_{[-T,0]}(u) < \infty$. There exists a sequence $\{u^n\} \subset C([-T, 0]; M)$ of smooth functions $u^n : [-T, 0] \times [0, 1] \to (0, 1)$ converging to u in $C([-T, 0]; M)$ such that $u^n(-T, \cdot) = \bar{\rho}_\varepsilon(\cdot)$, $u^n(t, 0) = \rho_0$, $u^n(t, 1) = \rho_1$, $t \in [-T, 0]$, and $I_{[-T,0]}(u^n)$ converges to $I_{[-T,0]}(u)$.*

PROOF OF THEOREM 2.3: THE BOUND $S_\varepsilon \leq V_\varepsilon$. Fix ρ in M, $T > 0$, and a path u in $C([-T, 0]; M)$ such that $u(-T) = \bar{\rho}_\varepsilon$, $u(0) = \rho$. We need to show that $I_{[-T,0]}(u) \geq S_\varepsilon(\rho)$. Assume first that ρ belongs to M^o and that $u : [-T, 0] \times [0, 1] \to [0, 1]$ is a smooth path bounded away from 0 and 1 and satisfies $u(t, 0) = \rho_0$, $u(t, 1) = \rho_1$, $t \in [-T, 0]$. In this case, as we have seen above, (5.9) holds for any H in $\mathfrak{H}_0^1(\sigma(u))$. Consider a partition $[-T, 0) = \bigcup_{k=1}^n [-T_k, -T_{k-1})$ with $T_0 = 0$ and $T_n = T$. For $k = 1, \ldots, n$, choose $\varphi^k \in \mathscr{F}(u(-T_k))$; namely, φ^k is a minimizer for $\mathscr{G}_\varepsilon(u(-T_k), \cdot)$. In view of Lemma 4.2, φ^k also belongs to

$\mathcal{P}(u(-T_k))$. Define the path φ piecewise constant in $[-T, 0]$ by $\varphi(t) = \varphi^k$ for $t \in [-T_k, -T_{k-1})$, $k = 1, \ldots, n$, and $\varphi(0) = \varphi^1$.

Since the path u is smooth, by the definition of φ, $\varphi(t, 0) = \varphi_0$, $\varphi(t, 1) = \varphi_1$, $t \in [-T, 0]$, and $s'(u) - \varphi$ is a smooth function in space, piecewise smooth in time, that vanishes at the endpoints of $[0, 1]$. In particular, $s'(u) - \varphi$ belongs to $\mathfrak{H}_0^1(\sigma(u))$. By choosing $H = s'(u) - \varphi$ in (5.9) we then get

$$I_{[-T,0]}(u) \geq \sum_{k=1}^{n} \int_{-T_k}^{-T_{k-1}} [\langle s'(u) - \varphi^k, u_t \rangle - \mathbb{H}(u, s'(u) - \varphi^k)] dt.$$

By the choice of φ^k and Proposition 4.1(iii), $\mathbb{H}(u(t), s'(u(t)) - \varphi^k)$ vanishes for $t = -T_k$. A simple computation, based on the bounds stated in Proposition 4.1(ii), shows that the map $[-T_k, -T_{k-1}) \ni t \mapsto \mathbb{H}(u(t), s'(u(t)) - \varphi^k)$ is Lipschitz with a Lipschitz constant depending on u but independent of φ^k. Hence

$$\lim_{n \to \infty} \sum_{k=1}^{n} \int_{-T_k}^{-T_{k-1}} \mathbb{H}(u, s'(u) - \varphi^k) dt = 0$$

provided the mesh of the partition vanishes as $n \to \infty$.

On the other hand, because $\varphi(t)$ is constant in the interval $[-T_k, -T_{k-1})$, Lemma 5.6 yields

$$\sum_{k=1}^{n} \int_{-T_k}^{-T_{k-1}} \langle s'(u(t)) - \varphi^k, u_t \rangle dt =$$
$$\sum_{k=1}^{n} [\mathcal{G}_\varepsilon(u(-T_{k-1}), \varphi^k) - \mathcal{G}_\varepsilon(u(-T_k), \varphi^k)].$$

Since $\varphi^{k-1} \in \mathcal{F}(u(-T_{k-1}))$, it follows that $\mathcal{G}_\varepsilon(u(-T_{k-1}), \varphi^k) \geq \mathcal{G}_\varepsilon(u(-T_{k-1}), \varphi^{k-1})$. The previous expression is thus bounded below by the telescopic sum

$$\sum_{k=1}^{n} [\mathcal{G}_\varepsilon(u(-T_{k-1}), \varphi^{k-1}) - \mathcal{G}_\varepsilon(u(-T_k), \varphi^k)] = \mathcal{G}_\varepsilon(u(0), \varphi^1) - \mathcal{G}_\varepsilon(u(-T), \varphi^n).$$

Since $u(-T) = \bar{\rho}_\varepsilon$, by the choice of φ^n, we have $\mathcal{G}_\varepsilon(\bar{\rho}_\varepsilon, \varphi^n) = \inf_\psi \mathcal{G}_\varepsilon(\bar{\rho}_\varepsilon, \psi) = S_\varepsilon^o(\bar{\rho}_\varepsilon)$. On the other hand, since $u(0) = \rho$, we have $\mathcal{G}_\varepsilon(u(0), \varphi^1) = \mathcal{G}_\varepsilon(\rho, \varphi^1) \geq S_\varepsilon^o(\rho)$. By taking the limit $n \to \infty$, the previous bounds imply

$$I_{[-T,0]}(u) \geq S_\varepsilon^o(\rho) - S_\varepsilon^o(\bar{\rho}_\varepsilon) = S_\varepsilon(\rho).$$

Now let $\rho \in M$ be arbitrary and consider an arbitrary path $u \in C([-T, 0]; M)$ such that $u(0) = \rho$ and $u(-T) = \bar{\rho}_\varepsilon$. Since we can assume $I_{[-T,0]}(u) < \infty$, by Lemma 5.8 there exists a sequence of smooth paths u^n bounded away from 0 and 1 that converges to u in $C([-T, 0]; M)$ and such that $\lim_n I_{[-T,0]}(u^n) = I_{[-T,0]}(u)$. The lower semicontinuity of S_ε on M (see Theorem 2.2) and the result for smooth

paths yield

$$I_{[-T,0]}(u) = \lim_n I_{[-T,0]}(u^n) \ \geq \ \liminf_n S_\varepsilon(u^n(0)) \geq S_\varepsilon(u(0)) = S_\varepsilon(\rho),$$

which concludes the proof. $\square$

6 The Inviscid Limit

In this section we discuss the inviscid limit $\varepsilon \downarrow 0$. We first discuss the variational convergence of the functional $\mathscr{G}_\varepsilon$ to $\mathscr{G}$. By analyzing the limiting variational problem (2.16) and using a perturbation argument we then show, provided ε is small enough, that there exists $\rho \in M$ such that $\mathscr{G}_\varepsilon(\rho, \cdot)$ admits at least two minimizers.

6.1 Variational Convergence of $\mathscr{G}_\varepsilon$

By standard properties of Γ-convergence, Theorems 2.6 and 2.7 are corollaries of the following result.

THEOREM 6.1 *Let $\mathscr{G}_\varepsilon : M \times \mathscr{F} \to (-\infty, +\infty]$ be the functional defined in (2.10). As $\varepsilon \downarrow 0$, the family $\{\mathscr{G}_\varepsilon\}_{\varepsilon > 0}$ Γ-converges to the functional $\mathscr{G}$ defined in (2.15).*

PROOF. We start by showing the Γ-$\liminf$ inequality. Fix a sequence $\varepsilon_n \downarrow 0$, $(\rho, \varphi) \in M \times \mathscr{F}$, and a sequence $\{(\rho^n, \varphi^n)\} \subset M \times \mathscr{F}$ converging to (ρ, φ). We need to show $\liminf_n \mathscr{G}_{\varepsilon_n}(\rho^n, \varphi^n) \geq \mathscr{G}(\rho, \varphi)$. The convexity of the real function s trivially implies $\liminf_n \int_0^1 s(\rho^n)dx \geq \int_0^1 s(\rho)dx$. By the Jensen inequality, $\int_0^1 s(\varepsilon_n \varphi_x^n)dx \geq s(\varepsilon_n[\varphi_1 - \varphi_0])$, which vanishes as $\varepsilon_n \downarrow 0$. To conclude the proof, it remains to observe that the convergence of ρ^n to ρ in M and of φ^n to φ in $\mathscr{F}$ implies the convergence of $\int_0^1 [(1 - \rho^n)\varphi^n - \log(1 + e^{\varphi^n})]dx$.

We next show the Γ-$\limsup$ inequality. Fix a sequence $\varepsilon_n \downarrow 0$ and $(\rho, \varphi) \in M \times \mathscr{F}$. We need to exhibit a sequence $\{(\rho^n, \varphi^n)\} \subset M \times \mathscr{F}$ converging to (ρ, φ) such that $\limsup_n \mathscr{G}_{\varepsilon_n}(\rho^n, \varphi^n) \leq \mathscr{G}(\rho, \varphi)$. Consider first the case in which $\varphi \in \mathscr{F}$ is smooth, say C^1, and satisfies $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$. We then claim a *recovering sequence* is simply given by the constant sequence $(\rho^n, \varphi^n) = (\rho, \varphi)$. Indeed, by the smoothness of φ, for such a sequence we have $\int_0^1 s(\varepsilon_n \varphi_x^n)dx \to 0$. The proof is now completed by a density argument. More precisely, it is enough to observe that, given $\varphi \in \mathscr{F}$, there exists a sequence $\{\varphi^k\} \subset \mathscr{F}$ converging to φ such that $\varphi^k \in C^1([0, 1])$, $\varphi^k(0) = \varphi_0, \varphi^k(1) = \varphi_1$, and $\lim_k \mathscr{G}(\rho, \varphi^k) = \mathscr{G}(\rho, \varphi)$. $\square$

The next result is a straightforward consequence of the the previous proof because the sequence used in the Γ-$\limsup$ inequality is constant in ρ.

COROLLARY 6.2 *For every ρ in M, $\mathscr{G}_\varepsilon(\rho, \cdot)$ Γ-converges to $\mathscr{G}(\rho, \cdot)$ as $\varepsilon \downarrow 0$.*

PROOF OF THEOREM 2.7. Because $\mathscr{F}$ is compact, Theorem 2.7 follows from Theorem 6.1 and [12, theorem 1.21]. $\square$

To deduce Theorem 2.6, we only need to "project" Theorem 6.1 to the first variable. For completeness, we detail the proof below.

PROOF OF THEOREM 2.6. We first show that as $\varepsilon \downarrow 0$, the family of functionals $\{S_{\varepsilon}^{o}\}$ Γ-converges to S^{o}. We start by proving the Γ-lim inf inequality. Fix a sequence $\varepsilon_n \downarrow 0$, $\rho \in M$, and a sequence $\{\rho^n\} \subset M$ converging to ρ. In view of the compactness of $\mathcal{F}$ and the lower semicontinuity of $\mathcal{G}_{\varepsilon_n}(\rho^n, \cdot)$, there exists a sequence $\{\varphi^n\} \subset \mathcal{F}$ such that $S_{\varepsilon_n}^{o}(\rho^n) = \mathcal{G}_{\varepsilon_n}(\rho^n, \varphi^n)$. Again by the compactness of $\mathcal{F}$, by taking if necessary a subsequence, there exists $\varphi \in \mathcal{F}$ such that $\varphi^n \to \varphi$. By Theorem 6.1 we then deduce

$$\liminf_{n} S_{\varepsilon_n}^{o}(\rho^n) = \liminf_{n} \mathcal{G}_{\varepsilon_n}(\rho^n, \varphi^n) \geq \mathcal{G}(\rho, \varphi) \geq S^{o}(\rho).$$

We next prove the Γ-lim sup inequality. Fix a sequence $\varepsilon_n \downarrow 0$, $\rho \in M$, and choose the constant sequence $\rho^n = \rho$. By Corollary 6.2, the compactness of $\mathcal{F}$, and [12, theorem 1.21],

$$\lim_{n} S_{\varepsilon_n}^{o}(\rho) = \liminf_{n} \inf_{\varphi} \mathcal{G}_{\varepsilon_n}(\rho, \varphi) = \inf_{\varphi} \mathcal{G}(\rho, \varphi) = S^{o}(\rho).$$

To complete the proof, we need to show that $S_{\varepsilon}^{o}(\bar{\rho}_{\varepsilon}) \to S^{o}(\bar{\rho})$ as $\varepsilon \downarrow 0$. While this statement can be proven by using (4.5) and the explicit expression of $\mathcal{G}_{\varepsilon}$ and $\mathcal{G}$, we next give an argument again based on Theorem 6.1 and the fact that $\bar{\rho}_{\varepsilon}$ converges to $\bar{\rho}$ in $L^1([0,1])$, pointed out just before (2.15). In the case $\rho_0 + \rho_1 = 1$, by the latter statement we mean that $\bar{\rho}_{\varepsilon}$ converges to the stationary entropic solution with a shock placed at $x = \frac{1}{2}$. Since $\bar{\rho}_{\varepsilon} \to \bar{\rho}$ in M, the Γ-lim inf inequality proven above yields $\liminf_{\varepsilon} S_{\varepsilon}^{o}(\bar{\rho}_{\varepsilon}) \geq S^{o}(\bar{\rho})$. To prove the other inequality, fix a smooth $\varphi \in \mathcal{F}$ so that $\int_0^1 s(\varepsilon\varphi_x)dx$ vanishes as $\varepsilon \downarrow 0$. Since $\bar{\rho}_{\varepsilon}$ converges to $\bar{\rho}$ in $L^1([0,1])$, $\int_0^1 s(\bar{\rho}_{\varepsilon})dx$ converges to $\int_0^1 s(\bar{\rho})dx$. Therefore

$$\limsup_{\varepsilon} S_{\varepsilon}^{o}(\bar{\rho}_{\varepsilon}) \leq \limsup_{\varepsilon} \mathcal{G}_{\varepsilon}(\bar{\rho}_{\varepsilon}, \varphi) = \mathcal{G}(\bar{\rho}, \varphi).$$

By optimizing on φ, we then deduce $\limsup_{\varepsilon} S_{\varepsilon}^{o}(\bar{\rho}_{\varepsilon}) \leq S^{o}(\bar{\rho})$, which concludes the proof. $\square$

The variational problem appearing in (2.16) is much simpler than the one in (2.13). In fact, the former can be reduced to a one-dimensional minimum problem. More precisely, we can restrict the infimum in (2.16) to step functions φ. Denote by $\widetilde{\mathcal{G}}(\rho, y)$ the functional $\mathcal{G}(\rho, \varphi)$ defined in (2.15) evaluated at $\varphi = \varphi^{(y)}$, $y \in [0,1]$, where $\varphi^{(y)}(x) := \varphi_0 \mathbb{1}_{[0,y)}(x) + \varphi_1 \mathbb{1}_{[y,1]}(x)$. In other words, let $\widetilde{\mathcal{G}} : M \times [0,1] \to \mathbb{R}$ be the functional defined by

$$\widetilde{\mathcal{G}}(\rho, y) := \int_0^1 s(\rho)dx + \varphi_0 \int_0^y (1-\rho)dx + \varphi_1 \int_y^1 (1-\rho)dx$$
$$- y \log(1 + e^{\varphi_0}) - (1-y) \log(1 + e^{\varphi_1}).$$

PROPOSITION 6.3 *Fix $\rho \in M$. Then,*

$$(6.1) \qquad \inf\{\mathcal{G}(\rho, \varphi), \varphi \in \mathcal{F}\} = \inf\{\widetilde{\mathcal{G}}(\rho, y), y \in [0,1]\}.$$

PROOF. Since the functional $\mathcal{G}(\rho, \cdot)$ is concave and $\mathcal{F}$ is a compact convex set, the infimum on the left-hand side of (6.1) is achieved when φ belongs to the extremal elements of $\mathcal{F}$. We thus have to show that the extremal elements of $\mathcal{F}$ are $\{\varphi^{(y)}, y \in [0, 1]\}$. This is easily proven recalling (2.8) and noticing that, given $m > 0$, the extremal elements of $\mathcal{P}_m([0, 1])$ are $\{m \, \delta_y, y \in [0, 1]\}$, where δ_y is the Dirac measure at y. $\qquad\square$

6.2 Nonuniqueness of the Minimizers

The statements of Theorems 2.2 and 2.5 imply that for ε small enough there exists $\rho \in M$ such that $\mathcal{P}_\varepsilon(\rho)$ is a not a singleton. This result, however, can be directly proven by a simpler argument, presented below, which also shows that there exists points in $\mathcal{P}_\varepsilon(\rho)$ that are not in $\mathcal{F}_\varepsilon(\rho)$.

LEMMA 6.4 *Fix $\varphi_0 < \varphi_1$. There exists $\varepsilon^* \in (0, \varepsilon_0)$ and a smooth function ρ in M such that $K_{\rho, \varepsilon}$ has at least two fixed points for $\varepsilon < \varepsilon^*$.*

PROOF. Let $\varphi \in \mathcal{F}$ be the affine function $\varphi(x) = \varphi_0(1 - x) + \varphi_1 x$, and let $\rho \in M$ be given by $\rho = 1/(1 + e^\varphi)$. Clearly, φ is a fixed point of $K_{\rho, \varepsilon}$. To conclude the proof, it is enough to show that, provided ε is small enough, φ is not a minimizer of $\mathcal{G}_\varepsilon(\rho, \cdot)$. Recall that the second variation of $\mathcal{G}_\varepsilon(\rho, \cdot)$ evaluated at φ is the quadratic form (4.3). For the above choices of ρ and φ, it is easy to check that this quadratic form is not positive semidefinite for ε small enough, which proves the lemma. $\qquad\square$

In order to prove Theorem 2.5, we first analyze the limiting variational problem (2.16) and characterize, for suitable functions ρ, its minimizers. Fix $\varphi_0 < \varphi_1$, set $\bar{\varphi} := (\varphi_0 + \varphi_1)/2$, and let

$$A := 1 - \frac{\log(1 + e^{\varphi_1}) - \log(1 + e^{\varphi_0})}{\varphi_1 - \varphi_0},$$

$$A_+ := 1 - \frac{\log(1 + e^{\bar{\varphi}}) - \log(1 + e^{\varphi_0})}{\bar{\varphi} - \varphi_0},$$

$$A_- := 1 - \frac{\log(1 + e^{\varphi_1}) - \log(1 + e^{\bar{\varphi}})}{\varphi_1 - \bar{\varphi}}.$$

Since the real function $x \mapsto \log(1 + e^x)$ is strictly increasing, strictly convex, and Lipschitz with Lipschitz constant 1, we have $0 < A_- < A < A_+ < 1$. Fix a continuous function $\rho : [0, 1] \to [0, 1]$ satisfying the following condition: There exist three reals $0 \le y_- < y_0 < y_+ \le 1$ such that $\rho(y_0) = \rho(y_\pm) = A$, $\rho(x) < A$ for $x \in [0, y_-) \cup (y_0, y_+)$, $\rho(x) > A$ for $x \in (y_-, y_0) \cup (y_+, 1)$, and $A_- < \rho(x) < A_+$ for $x \in [y_-, y_+]$. Recalling that $\mathcal{P}_m([0, 1])$, $m > 0$, denotes the set of positive Borel measure on $[0, 1]$ with mass m endowed with the topology of weak convergence, let

$$\mathcal{P}_m^-([0, 1]) := \left\{\mu \in \mathcal{P}_m([0, 1]) : \mu([0, y_0]) \ge \tfrac{m}{2}\right\},$$

$$\mathcal{P}_m^+([0, 1]) := \left\{\mu \in \mathcal{P}_m([0, 1]) : \mu([y_0, 1]) \ge \tfrac{m}{2}\right\}.$$

Note that $\mathcal{P}_m^{\pm}([0, 1])$ is a closed convex subset of $\mathcal{P}_m([0, 1])$. Recalling (2.8), let accordingly

$$\mathcal{F}^{\pm} := \{\varphi \in \mathcal{F} : \varphi(x) = \varphi_0 + \mu([0, x]) \text{ for some } \mu \in \mathcal{P}_{\varphi_1 - \varphi_0}^{\pm}([0, 1])\}$$

and observe that $\mathcal{F} = \mathcal{F}^- \cup \mathcal{F}^+$.

LEMMA 6.5 *Fix ρ as above and set $\varphi_{\pm}^* := \varphi_0 \mathbb{1}_{[0, y_{\pm})} + \varphi_1 \mathbb{1}_{[y_{\pm}, 1]}$. Then*

$$\arg\inf\{\mathcal{G}(\rho, \varphi), \varphi \in \mathcal{F}^{\pm}\} = \{\varphi_{\pm}^*\},$$

namely, the infimum of $\mathcal{G}(\rho, \cdot)$ over $\mathcal{F}^{\pm}$ is uniquely achieved at $\varphi_{\pm}^*$. If, further-more, ρ satisfies $\int_{y_-}^{y_+} \rho\, dx = A(y_+ - y_-)$, then $\inf_{\mathcal{F}^-} \mathcal{G}(\rho, \cdot) = \inf_{\mathcal{F}^+} \mathcal{G}(\rho, \cdot)$.

This result implies that if the function ρ satisfies all conditions of the previous lemma then

$$\arg\inf\{\mathcal{G}(\rho, \varphi), \varphi \in \mathcal{F}\} = \{\varphi_-^*, \varphi_+^*\},$$

namely, the functional $\mathcal{G}(\rho, \cdot)$ on $\mathcal{F}$ has exactly two minimizers. In fact, by using Proposition 6.3, this can easily be proven even if the condition $A_- < \rho(x) < A_+$, $x \in [y_-, y_+]$, is dropped. We also mention that, as observed in [17], if ρ is constant and equal to A, then there exists a one-parameter family of minimizers for $\mathcal{G}(\rho, \cdot)$ that is exactly the collection of the extremal elements $\{\varphi^{(y)}, y \in [0, 1]\}$ of $\mathcal{F}$.

PROOF OF LEMMA 6.5. Since $\mathcal{G}(\rho, \cdot)$ is a strictly concave functional and $\mathcal{F}^{\pm}$ is a compact convex set, the infimum can only be achieved at the extremal elements of $\mathcal{F}^{\pm}$. These elements can easily be characterized. Indeed, it is simple to check that the extremal points of $\mathcal{P}_m^-([0, 1])$ are given by $E_1^- \cup E_2^-$ where

$$E_1^- = \{m\delta_x, x \in [0, y_0]\}, \quad E_2^- = \left\{\frac{m}{2}(\delta_x + \delta_{x'}), x \in [0, y_0], x' \in [y_0, 1]\right\}.$$

We denote by $\mathcal{F}_{e,1}^-$ and $\mathcal{F}_{e,2}^-$ the corresponding subsets of $\mathcal{F}^-$ with $m = \varphi_1 - \varphi_0$. Note that $\varphi \in \mathcal{F}_{e,1}^-$ if and only if φ jumps from φ_0 to φ_1 at some point $x \in [0, y_0]$ while $\varphi \in \mathcal{F}_{e,2}^-$ if and only if φ jumps from φ_0 to $\bar{\varphi}$ at some point $x \in [0, y_0]$ and from $\bar{\varphi}$ to φ_1 at some point $x' \in [y_0, 1]$.

We have thus reduced the original problem of the minimum over $\mathcal{F}^-$ to a min-imum problem in one and two real variables. By using that $\rho(x) < A_+$ for $x \in [0, y_0]$ and $\rho(x) > A_-$ for $x \in [y_0, 1]$, elementary computations show that the infimum of $\mathcal{G}(\rho, \cdot)$ over $\mathcal{F}_{e,2}^-$ is achieved when φ has a single jump at y_0. Note that such a φ belongs both to $\mathcal{F}_{e,2}^-$ and $\mathcal{F}_{e,1}^-$. Likewise, by using that $\rho(x) < A$ for $x \in [0, y_-)$ and $\rho(x) > A$ for $x \in (y_-, y_0)$, it is readily seen that the infimum over $\mathcal{F}_{e,1}^-$ is uniquely achieved at φ_-^*.

The argument for $\mathcal{F}^+$ is the same and the last statement follows from a direct computation. $\qquad\square$

By using the variational convergence of $\mathcal{G}_\varepsilon$ to $\mathcal{G}$ and the above lemma, we finally show, provided ε is small enough and ρ is suitably chosen, the functional $\mathcal{G}_\varepsilon(\rho, \cdot)$ admits more than a single minimizer.

PROOF OF THEOREM 2.5. Fix a continuous function $\rho \in M$ bounded away from 0 and 1 satisfying the condition stated above Lemma 6.5 as well as the condition in the last statement of that lemma. Now pick a continuous positive function $\lambda : [0, 1] \to \mathbb{R}_+$ such that supp $\lambda \subset (y_-, y_0)$ and $\int_0^1 dx\, \lambda = 1$. Finally, choose $\delta > 0$ so small that for each $\alpha \in [-\delta, \delta]$ the function $\rho^{(\alpha)} := \rho + \alpha\lambda$ still satisfies the condition stated above Lemma 6.5 with y_0 and $y_\pm$ independent on α. Note, however, that for $\alpha \neq 0$ the functional $\mathcal{G}(\rho^{(\alpha)}, \cdot)$ has a unique minimizer. The proof of the theorem will be accomplished by considering the one-parameter family of functions $\{\rho^{(\alpha)}, |\alpha| \leq \delta\}$ and showing that, for ε small enough, there exists α_0 for which $\mathcal{G}_\varepsilon(\rho^{(\alpha_0)}, \cdot)$ has at least two distinct minimizers.

Given $\varepsilon > 0$, let $g_\varepsilon : [-\delta, \delta] \to \mathbb{R}$ be defined by

$$g_\varepsilon(\alpha) := \inf\{\mathcal{G}_\varepsilon(\rho^{(\alpha)}, \varphi), \varphi \in \mathcal{F}^+\} - \inf\{\mathcal{G}_\varepsilon(\rho^{(\alpha)}, \varphi), \varphi \in \mathcal{F}^-\}$$

and observe that, in view of Lemma 4.4, the function g_ε is continuous. Let int $\mathcal{F}^\pm$ be the *interior* of $\mathcal{F}^\pm$. By Theorem 6.1 and standard properties of Γ-convergence (see, e.g., [12, prop. 1.18]), for each $\alpha \in [-\delta, \delta]$

$$\inf_{\mathcal{F}^\pm} \mathcal{G}(\rho^{(\alpha)}, \cdot) \leq \liminf_{\varepsilon \downarrow 0} \inf_{\mathcal{F}^\pm} \mathcal{G}_\varepsilon(\rho^{(\alpha)}, \cdot)$$

$$\leq \limsup_{\varepsilon \downarrow 0} \inf_{\text{int } \mathcal{F}^\pm} \mathcal{G}_\varepsilon(\rho^{(\alpha)}, \cdot) \leq \inf_{\text{int } \mathcal{F}^\pm} \mathcal{G}(\rho^{(\alpha)}, \cdot);$$

whence, using $\varphi_\pm^* \in \text{int } \mathcal{F}^\pm$ and Lemma 6.5,

$$\lim_{\varepsilon \downarrow 0} \inf_{\varphi \in \mathcal{F}^\pm} \mathcal{G}_\varepsilon(\rho^{(\alpha)}, \varphi) = \mathcal{G}(\rho^{(\alpha)}, \varphi_\pm^*)$$

so that

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(\delta) = \mathcal{G}(\rho^{(\delta)}, \varphi_+^*) - \mathcal{G}(\rho^{(\delta)}, \varphi_-^*)$$

$$= \mathcal{G}(\rho, \varphi_+^*) - \mathcal{G}(\rho, \varphi_-^*) - \delta \int_0^1 \lambda(\varphi_+^* - \varphi_-^*)dx = \delta(\varphi_1 - \varphi_0).$$

In particular, $\varepsilon_1 \in (0, \varepsilon_0)$ exists such that for any $\varepsilon \in (0, \varepsilon_1)$ we have $g_\varepsilon(\delta) > 0$ and, by the same argument, $g_\varepsilon(-\delta) < 0$.

Applying the theorem on the existence of zeros for a continuous function of a real variable, we deduce that for each $\varepsilon \in (0, \varepsilon_1)$ there exists $\alpha_0 \in (-\delta, \delta)$ such that $g_\varepsilon(\alpha_0) = 0$. This implies the existence of at least two distinct minimizers for $\mathcal{G}_\varepsilon(\rho^{(\alpha_0)}, \cdot)$. Note that although the sets $\mathcal{F}^+$ and $\mathcal{F}^-$ are not disjoint, the minimizer of $\mathcal{G}_\varepsilon(\rho^{(\alpha_0)}, \cdot)$ over $\mathcal{F}^-$ cannot coincide with the one over $\mathcal{F}^+$ since they respectively converge to φ_+^* and φ_-^* as $\varepsilon \downarrow 0$. The last statement follows again from Theorem 6.1 and standard properties of Γ-convergence; see, e.g., [12, theorem 1.21]. $\qquad\square$

7 Hamilton-Jacobi Equation

In the context of diffusion processes in $\mathbb{R}^n$, the quasi-potential is connected to a Hamilton-Jacobi equation. More precisely, let I be the action functional corresponding to the Lagrangian $\mathbb{L}(x, \dot{x})$, and denote by $\mathbb{H}(x, p)$ the associated Hamiltonian. Under suitable conditions, the quasi-potential, as defined in (1.2), is then a *viscosity* solution to the Hamilton-Jacobi equation $\mathbb{H}(x, Dv) = 0$ [13, 32]. In an infinite-dimensional setting, the theory of the Hamilton-Jacobi equation, in particular of the stationary Hamilton-Jacobi equation, is much less developed. As Theorem 2.3 gives a somewhat explicit expression for the quasi-potential, the problem discussed here may prove to be a good example for the development of the theory of infinite-dimensional Hamilton-Jacobi equations. In this section, we present some possible formulations of the Hamilton-Jacobi equation that seem apt for the variational problems (2.4) or (2.7).

It is not clear how to introduce a differentiable structure on the set M in such a way that the Hamiltonian $\mathbb{H}$ in (4.2) becomes a well-defined function on the cotangent bundle of M. The formalization of the Hamilton-Jacobi equation is thus a nontrivial issue. In the following we consider two possibilities to circumvent this problem. In the first we simply consider the subset of M given by the smooth functions $\rho : [0, 1] \to [0, 1]$ that are bounded away from 0 and 1 and satisfy the boundary conditions. In the second one we exploit the symplectic transformation (3.5) and deduce that the functional Λ^* (recall (4.7) and (4.9)) solves the Hamilton-Jacobi equation with the Hamiltonian $\widetilde{\mathbb{H}}$ introduced in (3.6).

Recall the definitions (2.13), (4.1), and (4.2). We would like to claim that the functional S_ε^o solves the Hamilton-Jacobi equation $\mathbb{H}(\rho, DU) = 0$ in M^o. In this setting, we formulate the notion of viscosity solutions to such Hamilton-Jacobi equations in terms of Gâteaux subdifferentials; recall (4.10).

THEOREM 7.1 *For each $\rho \in M^o$ we have $D_{\mathrm{G}}^{\pm} S_\varepsilon^o(\rho) \subset \mathcal{H}_0^1$. Moreover, S_ε^o is a Gâteaux viscosity solution to $\mathbb{H}(\rho, DU) = 0$ in M^o; that is, the two following inequalities hold for any $\rho \in M^o$:*

$$\mathbb{H}(\rho, h) \leq 0 \quad \forall h \in D_{\mathrm{G}}^{+} S_\varepsilon^o(\rho),$$
$$\mathbb{H}(\rho, h) \geq 0 \quad \forall h \in D_{\mathrm{G}}^{-} S_\varepsilon^o(\rho).$$

PROOF. Recall the decomposition (4.9) and fix $\rho \in M^o$. It is straightforward to check that $\mathcal{S}$ is Gauteaux-differentiable at ρ and $D_{\mathrm{G}}\mathcal{S}(\rho) = s'(\rho)$. In view of Proposition 4.6 we deduce that $D_{\mathrm{G}}^{+} S_\varepsilon^o(\rho) = s'(\rho) - \mathrm{co}\,\mathcal{F}(\rho)$ and

$$D_{\mathrm{G}}^{-} S_\varepsilon^o(\rho) = \begin{cases} s'(\rho) - \varphi & \text{if } \mathcal{F}(\rho) = \{\varphi\} \text{ for some } \varphi \in \mathcal{F}, \\ \varnothing & \text{otherwise.} \end{cases}$$

By Lemma 4.2 and Proposition 4.1(ii), we then deduce $D_{\mathrm{G}}^{\pm} S_\varepsilon^o(\rho) \subset \mathcal{H}_0^1$.

By Proposition 4.1(iii), $\mathbb{H}(\rho, s'(\rho) - \varphi) = 0$ for any $\rho \in M^o$ and any $\varphi \in \mathcal{P}(\rho)$. In particular, again by Lemma 4.2, this equation holds for any $\rho \in M^o$ and

$\varphi \in \mathcal{F}(\rho)$. The explicit expression of the Gâteaux subdifferentials above and the convexity of $\mathbb{H}(\rho, \cdot)$ on $\mathcal{H}_0^1$ now easily yield the statements. $\qquad \square$

Since in our case $\bar{\rho}_\varepsilon$ is the unique, globally attractive fixed point of the flow defined by (2.1), we can try to characterize the quasi-potential V_ε, as defined in (2.4), as the *maximal* viscosity subsolution of the Hamilton-Jacobi equation $\mathbb{H}(\rho, DU) = 0$ satisfying $U(\bar{\rho}_\varepsilon) = 0$. Our next result, which does not depend on the special form of the flux f and mobility σ, goes in this direction.

THEOREM 7.2 *Let* $U_\varepsilon : M \to \mathbb{R}$ *be a lower-semicontinuous functional such that* $U_\varepsilon(\bar{\rho}_\varepsilon) = 0$ *and* U_ε *satisfies the following condition: For each* $T > 0$ *and each smooth path* $u : [-T, 0] \times [0, 1] \to (0, 1)$ *such that* $u(t, 0) = \rho_0$, $u(t, 1) = \rho_1$, $t \in [-T, 0]$, *consider a partition* $[-T, 0) = \bigcup_{k=1}^n [-T_k, -T_{k-1})$ *with* $T_0 = 0$ *and* $T_n = T$. *Then there are elements* $h^k \in \mathcal{H}_0^1$, $k = 1, \ldots, n$, *that are uniformly bounded in* $\mathcal{H}_0^1$ *and such that*

(7.1) $$\mathbb{H}(u(-T_k), h^k) \leq 0, \quad k = 1, \ldots, n,$$

and

(7.2) $$\limsup_{\max |T_{k-1} - T_k| \to 0} \sum_{k=1}^n \left[U_\varepsilon(-T_{k-1}) - U_\varepsilon(-T_k) - \int_{-T_k}^{-T_{k-1}} \langle h^k, u_t \rangle dt \right] \leq 0.$$

Then $V_\varepsilon \geq U_\varepsilon$.

Note that condition (7.2) basically requires that h^k belong to the superdifferential of U_ε at the point $u(-T_k)$ with a uniform control over the smooth path $u(t)$, $t \in [-T, 0]$. Equation (7.1) is thus a viscosity formulation of $\mathbb{H}(\rho, DU_\varepsilon) \leq 0$. As a matter of fact, the proof of the above theorem is quite similar to the proof of the lower bound in Theorem 2.3 in Section 5 and so we only sketch the argument.

PROOF. Let $U_\varepsilon : M \to \mathbb{R}$ be as in the statement of the theorem. In view of Lemma 5.8 and the lower semicontinuity of U_ε, it is enough to prove the following statement: Fix $\rho \in M^o$, $T > 0$, and a smooth path $u \in C([-T, 0]; M^o)$ such that $u(-T) = \bar{\rho}_\varepsilon$, $u(0) = \rho$; then $I_{[-T,0]}(u) \geq U_\varepsilon(\rho)$.

Consider a partition $[-T, 0) = \bigcup_{k=1}^n [-T_k, -T_{k-1})$ with $T_0 = 0$ and $T_n = T$. For $k = 1, \ldots, n$, let $h^k \in \mathcal{H}_0^1$ as in the statement of the theorem and choose in (5.9) the piecewise constant path $H(t) = h^k$ for $t \in [-T_k, -T_{k-1})$, $k = 1, \ldots, n$, and $H(0) = h^1$. By assumption, $\mathbb{H}(u(-T_k), h^k) \leq 0$. By the smoothness of the path u and the assumption that $|h^k|_{\mathcal{H}_0^1}$ is uniformly bounded, we have

$$\limsup_{n \to \infty} \sum_{k=1}^n \int_{-T_k}^{-T_{k-1}} \mathbb{H}(u(t), h^k) dt \leq 0$$

provided the mesh of the partition vanishes as $n \to \infty$.

Again by assumption,

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \left\{ \int_{-T_k}^{-T_{k-1}} \langle h^k, u_t \rangle dt - [U_\varepsilon(u(-T_{k-1})) - U_\varepsilon(u(-T_k))] \right\} \geq 0$$

provided the mesh of the partition vanishes as $n \to \infty$. We then deduce that

$$I_{[-T,0]}(u) \geq U(u(0)) - U(u(-T)) = U(\rho),$$

which concludes the proof. $\square$

We next discuss a formulation of the Hamilton-Jacobi equation in a strong topology and in the whole set M. Recalling (4.9), the basic idea is to look for an equation for Λ^* or, equivalently, to exploit the symplectic transformation (3.5).

Consider the space $L^1([0, 1])$ equipped with the strong topology and denote by $\widehat{M}$ its (closed) subset given by $\widehat{M} := \{\rho \in L^1([0, 1]) : 0 \leq \rho \leq 1\}$. There is nothing really peculiar about the choice of $L^1([0, 1])$; any $L^p([0, 1])$ with $p \in [1, \infty)$ would lead to the same results. $L^\infty([0, 1])$ might look more natural, but its lack of separability (with respect to the strong topology) prevents its use. Let

$$\mathcal{W} := \{w \in W^{2,1}([0, 1]) : w(0) = \varphi_0, w(1) = \varphi_0\},$$

where $W^{2,1}([0, 1])$ is the Sobolev space of the functions whose second (weak) derivative belongs to $L^1([0, 1])$. Let $\widehat{\mathbb{H}} : \widehat{M} \times \mathcal{W} \to \mathbb{R}$ be the Hamiltonian defined by

(7.3)
$$\begin{aligned} \widehat{\mathbb{H}}(\rho, w) &:= -\varepsilon \langle w_x, \sigma(\rho) w_x \rangle - \varepsilon \langle \rho, w_{xx} \rangle \\ &\quad + \langle \sigma(\rho), w_x \rangle - \rho_1[1 - \varepsilon w_x(1)] + \rho_0[1 - \varepsilon w_x(0)]. \end{aligned}$$

The relationship of $\widehat{\mathbb{H}}$ to the original Hamiltonian $\mathbb{H}$ is the following.: Consider the antisymplectic transformation $(\rho, h) \mapsto (\rho, w)$ where $w = s'(\rho) - h$. The associated Hamiltonian $\widehat{\mathbb{H}}(\rho, w) = -\mathbb{H}(\rho, s'(\rho) - w)$ is then the one defined above. We remark that while the identity $\widehat{\mathbb{H}}(\rho, w) = -\mathbb{H}(\rho, s'(\rho) - w)$ holds only for $\rho \in M^o$ (recall (4.1)), the Hamiltonian $\widehat{\mathbb{H}}$ is defined for all ρ in $\widehat{M}$.

We claim that the functional $\widehat{\Lambda}$ in (4.11) is a viscosity solution of the Hamilton-Jacobi equation $\widehat{\mathbb{H}}(\rho, DU) = 0$, $\rho \in \widehat{M}$. In order to state this result precisely, we recall the notion of Fréchet subdifferentials. Let B be a Banach space and denote by B^* its dual. The norms in B and B^* are denoted by $|\cdot|_B$ and $|\cdot|_{B*}$, respectively. A function $f : B \to \mathbb{R}$ is *Fréchet differentiable* at $x \in B$ if and only if there exists $\ell \in B^*$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle \ell, y - x \rangle}{|y - x|_B} = 0;$$

in this case we denote ℓ by $D_F f(x)$. In general the *Fréchet subdifferential* $D_F^- f(x)$ and *Fréchet superdifferential* $D_F^+ f(x)$ of f at the point x are defined as the (possibly empty) convex subsets of B^*

$$D_F^+ f(x) := \left\{ \ell \in B^* : \limsup_{y \to x} \frac{f(y) - f(x) - \langle \ell, y - x \rangle}{|y - x|_B} \leq 0 \right\},$$

$$D_F^- f(x) := \left\{ \ell \in B^* : \liminf_{y \to x} \frac{f(y) - f(x) - \langle \ell, y - x \rangle}{|y - x|_B} \geq 0 \right\}.$$

THEOREM 7.3 *The functional* $\widehat{\Lambda} : (L^1([0,1]), \text{strong}) \to \mathbb{R}$ *is convex and continuous, in particular, locally Lipschitz. Moreover, for each* $\rho \in L^1([0,1])$ *we have* $D_F^{\pm} \widehat{\Lambda}(\rho) \subset \mathcal{W}$. *Finally,* $\widehat{\Lambda}$ *is a Fréchet viscosity solution of* $\widehat{\mathbb{H}}(\rho, DU) = 0$ *in* $\widehat{M}$; *namely, the two following inequalities hold for any* $\rho \in \widehat{M}$:

$$\widehat{\mathbb{H}}(\rho, w) \leq 0 \quad \forall w \in D_F^+ \widehat{\Lambda}(\rho),$$

$$\widehat{\mathbb{H}}(\rho, w) \geq 0 \quad \forall w \in D_F^- \widehat{\Lambda}(\rho).$$

PROOF. Clearly $\widehat{\Lambda}$ is convex and, by the argument used in Lemma 4.4, continuous. Fix $\rho \in \widehat{M}$. Proposition 4.6 and the argument presented below Theorem 4.8 show that $D_F^- \widehat{\Lambda}(\rho) = \text{co} \, \mathcal{F}(\rho)$ and $D_F^+ \widehat{\Lambda}(\rho) = \{\varphi\}$ when $\mathcal{F}(\rho) = \{\varphi\}$ and $D_F^+ \widehat{\Lambda}(\rho) = \varnothing$ otherwise. In particular, by the inclusion $\mathcal{F}(\rho) \subset \mathcal{P}(\rho)$ and Proposition 4.1(ii), $D_F^{\pm} \widehat{\Lambda}(\rho) \subset \mathcal{W}$.

As in the proof of Lemma 4.3, we consider the integral operator K_ρ defined in (2.12) also for $\rho \in L^1([0,1])$ and denote by $\widehat{\mathcal{P}}(\rho) \subset \mathcal{F}$ the set of fixed points of K_ρ. As in Proposition 4.1, if $\varphi \in \widehat{\mathcal{P}}(\rho)$, then $\varphi \in C^1([0,1])$ and there exist $\delta \in (0,1)$ and $C < \infty$, depending only on $|\rho|_{L^1}$, such that $\delta \leq \varepsilon \varphi_x \leq 1 - \delta$ and $|\varphi_{xx}|_{L^1} \leq C$. In particular, the Euler-Lagrange equation (2.11) holds a.e.

We now prove the subsolution statement; that is, $\widehat{\mathbb{H}}(\rho, w) \leq 0$ for any $\rho \in \widehat{M}$ and $w \in D_F^+ \widehat{\Lambda}(\rho)$. Since $D_F^+ \widehat{\Lambda}(\rho) = \varnothing$ if $\mathcal{F}(\rho)$ is not a singleton, we need only to consider the case $\mathcal{F}(\rho) = \{\varphi\}$ for some $\varphi \in \mathcal{F}$. Hence, as shown in the proof of Lemma 4.3, $\varphi \in \widehat{\mathcal{P}}(\rho)$. It is therefore enough to show that for any $\rho \in \widehat{M}$ and $\varphi \in \widehat{\mathcal{P}}(\rho)$ we have $\widehat{\mathbb{H}}(\rho, \varphi) = 0$. This is essentially the same computation as the one used in the proof of Proposition 4.1(iii). Recall $\varphi_i = \log[\rho_i/(1 - \rho_i)]$, $i = 0, 1$, and observe that, in view of the bounds stated above, if $\varphi \in \widehat{\mathcal{P}}(\rho)$, then

$$\rho_1[1 - \varepsilon\varphi_x(1)] - \rho_0[1 - \varepsilon\varphi_x(0)]$$
$$= \left\langle \left(\frac{e^\varphi}{1 + e^\varphi} \right)_x, 1 - \varepsilon\varphi_x \right\rangle - \left\langle \frac{e^\varphi}{1 + e^\varphi}, \varepsilon\varphi_{xx} \right\rangle$$
$$= \left\langle \frac{e^\varphi}{(1 + e^\varphi)^2}, \varphi_x(1 - \varepsilon\varphi_x) \right\rangle - \left\langle \frac{e^\varphi}{1 + e^\varphi}, \varepsilon\varphi_{xx} \right\rangle$$

so that, recalling (7.3),

$$\widehat{\mathbb{H}}(\rho, \varphi) = \left\langle \sigma(\rho) - \frac{e^{\varphi}}{(1 + e^{\varphi})^2}, \varphi_x(1 - \varepsilon\varphi_x) \right\rangle + \left\langle \frac{e^{\varphi}}{1 + e^{\varphi}} - \rho, \varepsilon\varphi_{xx} \right\rangle.$$

At this point we use the special form of σ, i.e., $\sigma(\rho) = \rho(1 - \rho)$, which implies

$$\sigma(\rho) - \frac{e^{\varphi}}{(1 + e^{\varphi})^2} = \left(\frac{e^{\varphi}}{1 + e^{\varphi}} - \rho \right)\left(\rho - \frac{1}{1 + e^{\varphi}} \right).$$

We then deduce

$$\widehat{\mathbb{H}}(\rho, \varphi) = \left\langle \frac{e^{\varphi}}{1 + e^{\varphi}} - \rho, \varepsilon\varphi_{xx} + \varphi_x(1 - \varepsilon\varphi_x)\left(\rho - \frac{1}{1 + e^{\varphi}} \right) \right\rangle = 0$$

since φ satisfies (2.11) a.e.

It remains to prove the supersolution statement, that is $\widehat{\mathbb{H}}(\rho, \varphi) \geq 0$ for any $\rho \in \widehat{M}$ and $\varphi \in D_{\mathrm{F}}^{-}\widehat{\Lambda}(\rho)$. To this end, consider first the case $\varphi \in \partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$. By definition, there exists a sequence $\{\rho^n\} \subset L^1([0, 1])$ converging to $\rho \in \widehat{M}$ strongly in $L^1([0, 1])$ such that $\partial\widehat{\Lambda}(\rho^n) = \{\varphi^n\}$ and $\varphi^n \to \varphi$ weak-* in $L^{\infty}([0, 1])$. In particular, $\varphi^n \in \widehat{\mathcal{P}}(\rho^n)$ and therefore, by the bounds stated at the beginning of this proof, $\varphi \in \widehat{\mathcal{P}}(\rho)$. By the computation presented above, we then deduce $\widehat{\mathbb{H}}(\rho, \varphi) = 0$.

We now consider the general case $\varphi \in D_{\mathrm{F}}^{-}\widehat{\Lambda}(\rho) = \partial\widehat{\Lambda}(\rho)$. The concavity of $\widehat{\mathbb{H}}(\rho, \cdot)$ implies $\widehat{\mathbb{H}}(\rho, \varphi) \geq 0$ for any $\varphi \in \mathrm{co}\,\partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$. In view of Theorem 4.8, it is now enough to take the weak-* closure in $L^{\infty}([0, 1])$. This is easily accomplished by noticing that there exist $\delta \in (0, 1)$ and $C < \infty$ depending only on ρ such that any $\varphi \in \mathrm{co}\,\partial_{\mathrm{app}}\widehat{\Lambda}(\rho)$ satisfies $\delta \leq \varepsilon\varphi_x \leq 1 - \delta$ and $|\varphi_{xx}|_{L^{\infty}} \leq C$. $\qquad\square$

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