Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) 10.1007/s00205-008-0197-2

Γ-Entropy Cost for Scalar Conservation Laws

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Communicated by A. Bressan

Abstract

We are concerned with a control problem related to the vanishing viscosity approximation to scalar conservation laws. We investigate the Γ-convergence of the control cost functional, as the viscosity coefficient tends to zero. A first-order Γ-limit is established, which characterizes the measure-valued solutions to the conservation laws as the zeros of the Γ-limit. A second-order Γ-limit is then investigated, providing a characterization of entropic solutions to conservation laws as the zeros of the Γ-limit.

1. Introduction

We are concerned with the scalar one-dimensional conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

where, given $T > 0$, $u = u(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}$, subscripts denote partial derivatives, and the *flux* f is a Lipschitz function. As well known, even if the initial datum $u(0) = u(0, \cdot)$ is smooth, the flow (1.1) may develop singularities for some positive time. In general, these singularities appear as discontinuities of u and are called *shocks*. It is therefore natural to interpret (1.1) weakly; in the weak formulation uniqueness is however lost, if no further conditions are imposed. Given a smooth function η, called *entropy*, the conjugated *entropy flux* q is defined up to an additive constant as $q(u) = \int^u \mathrm{d}v\, \eta'(v)\, f'(v)$. A weak solution to (1.1) is called *entropic* iff for each entropy–entropy flux pair (η, q) with η convex, the inequality $\eta(u)_t + q(u)_x \leqq 0$ holds in the sense of distributions. Note that the entropy condition is always satisfied for smooth solutions to (1.1). The classical theory, see, for example [5,15], shows existence and uniqueness in $C\big([0, T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$ of the entropic solution to the Cauchy problem associated to (1.1). While the flow (1.1) is invariant with respect to $(t, x) \mapsto (-t, -x)$, the entropy condition breaks

Published online: 29 November 2008

such invariance and selects the "physical" direction of time. In the conservation law (1.1) the viscosity effects are neglected. This approximation is no longer valid if the gradients become large as it happens when shocks appear. A more accurate description is then given by the parabolic equation

$$u_t + f(u)_x = \frac{\varepsilon}{2}\,(D(u)u_x)_x \tag{1.2}$$

in which $(t,x) \in [0,T] \times \mathbb{R}$, D, assumed uniformly positive, is the diffusion coefficient and $\varepsilon > 0$ is the viscosity. In this context of scalar conservation laws, it is also well known that, as $\varepsilon \to 0$, equibounded solutions to (1.2) converge in $L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$ to entropic solutions to (1.1), see, for example [5,15]. This approximation result shows that the entropy condition is relevant.

Perhaps less well known, at least in the hyperbolic literature, is the fact that entropic solutions to (1.1) can be obtained as scaling limit of discrete stochastic models of lattice gases, see, for example [11, Ch. 8]. In a little more detail, consider particles living on a one-dimensional lattice and randomly jumping to their neighboring sites. It is then proven that, under hyperbolic scaling, the empirical density of particles converges in probability to entropic solutions to (1.1). A much studied example is the *totally asymmetric simple exclusion process*, where there is at most one particle in each site and only jumps heading to the right are allowed. In this case, the empirical density takes values in [0, 1] and its scaling limit is given by (1.1) with flux $f(u) = u(1-u)$. In this stochastic framework, it is also worth looking at the large deviations asymptotic associated to the aforementioned law of large numbers. Basically, this amounts to estimate the probability that the empirical density lies in a neighborhood of a given trajectory. In general this probability is exponentially small, and the corresponding decay rate is called the large deviations rate functional. For the totally asymmetric simple exclusion process, this issue has been analyzed in [9,17]. It is there shown that the large deviations rate functional is infinite off the set of weak solutions to (1.1); on such solutions the rate functional is given by the total positive mass of the entropy production $h(u)_t + g(u)_x$ where h is the Bernoulli entropy, that is $h(u) = u\log u + (1-u)\log(1-u)$ and g is its conjugated entropy flux.

A stochastic framework can also be naturally introduced in a PDE setting by adding to (1.2) a random perturbation, namely

$$u_t + f(u)_x = \frac{\varepsilon}{2}\,(D(u)u_x)_x + \sqrt{\gamma}\left(\sqrt{\sigma(u)}\alpha_\gamma\right)_x \qquad (t,x) \in (0,T) \times \mathbb{R} \tag{1.3}$$

where $\sigma(u) \geqq 0$ is a conductivity coefficient and α_γ is a Gaussian random forcing term white in time and with spatial correlations on a scale much smaller than γ. Let $u^{\varepsilon,\gamma}$ be the corresponding solution; if $\gamma \ll \varepsilon$ then $u^{\varepsilon,\gamma}$ still converges in probability to the entropic solution to (1.1) and the large deviations asymptotic becomes a relevant issue. Referring to [13] for this analysis, we formulate the problem from a purely variational point of view quantifying, in terms of the parabolic problem (1.2), the asymptotic *cost* of non-entropic solutions to (1.1). Introducing in (1.2) a *control* $E \equiv E(t,x)$ we get

$$u_t + f(u)_x = \frac{\varepsilon}{2}\,(D(u)u_x)_x - (\sigma(u)E)_x \qquad (t,x) \in (0,T) \times \mathbb{R} \tag{1.4}$$

If we think of u as a density of charge, then E can be naturally interpreted as the "controlling" external electric field and $\sigma(u) \geqq 0$ as the conductivity. The flow (1.4) conserves the *total charge* $\int \mathrm{d}x\, u(t,x)$, whenever it is well defined.

The *cost functional* I_ε associated with (1.2) can be now informally defined as the work done by the optimal controlling field E in (1.4), namely

$$I_\varepsilon(u) = \inf_E \frac{1}{2} \int_{[0,T]} \mathrm{d}t\, \mathrm{d}x\, \sigma(u) E^2 = \inf_E \frac{1}{2} \int_{[0,T]} \mathrm{d}t\ \|E\|^2_{L_2(\mathbb{R},\sigma(u)\mathrm{d}x)} \tag{1.5}$$

where the infimum is taken over the controls E such that (1.4) holds. For a suitable choice of the random perturbation α_γ, I_ε is the large deviations rate functional of the process $u^{\varepsilon,\gamma}$ solution to (1.3), when ε is fixed and $\gamma \to 0$. To avoid the technical problems connected to the possible unboundedness of the density u, we assume that the conductivity σ has compact support. In this case, if u is such that $I_\varepsilon(u) < +\infty$ then u takes values in the support of σ, see Proposition 3.4 for the precise statement. For the sake of simplicity, we assume that σ is supported by [0, 1]. The case of strictly positive σ also fits in the description below, provided, however, that the analysis is a priori restricted to equibounded densities u.

In this paper we analyze the variational convergence of I_ε as $\varepsilon \to 0$. Our first result holds for a Lipschitz flux f, and identifies the so-called Γ-limit of I_ε, which is naturally studied in a Young measures setting. The limiting cost of a Young measure $\mu \equiv \mu_{t,x}(\mathrm{d}\lambda)$ is

$$\mathcal{I}(\mu) = \frac{1}{2} \int_{[0,T]} \mathrm{d}t\ \left\| [\mu(\lambda)]_t + [\mu(f(\lambda))]_x \right\|^2_{H^{-1}(\mathbb{R},\mu(\sigma(\lambda))\mathrm{d}x)}$$

where, given $F \in C([0,1])$ we set $[\mu(F(\lambda))](t,x) = \int \mu_{t,x}(\mathrm{d}\lambda)\, F(\lambda)$ and, with a little abuse of notation, $\|\varphi\|_{H^{-1}(\mathbb{R},\mu_{t,\cdot}(\sigma(\lambda))\mathrm{d}x)}$ is the dual norm to $\left[\int \mathrm{d}x\, \mu_{t,x}(\sigma(\lambda))\, \varphi_x^2\right]^{1/2}$.

Note that $\mathcal{I}(\mu)$ vanishes iff μ is a *measure-valued* solution to (1.1). Hence we can obtain such solutions as limits of solutions to (1.4) with a suitable sequence E_ε with vanishing cost. On the other hand, if we set in (1.4) $E = 0$ we obtain, in the limit $\varepsilon \to 0$, an entropic solution to (1.1). If the flux f is non-linear, the set of measure-valued solutions to (1.1) is larger than the set of entropic solutions; it is thus natural to study the Γ-convergence of the rescaled cost functional $H_\varepsilon := \varepsilon^{-1} I_\varepsilon$, which formally corresponds to the scaling in [9,17]. Our second result concerns the Γ-convergence of H_ε which is studied under the additional hypotheses that the flux f is smooth and such that there are no intervals in which f is affine. A compensated compactness argument shows that H_ε has enough coercivity properties to force its convergence in a functions setting and not in a Young measures' one.

To informally define the candidate Γ-limit of H_ε, we first introduce some preliminary notions. We say that a weak solution u to (1.1) is *entropy-measure* iff for each *smooth* entropy η the distribution $\eta(u)_t + q(u)_x$ is a Radon measure on $(0,T) \times \mathbb{R}$. If u is an entropy-measure solution to (1.1), then there exists a measurable map ϱ_u from [0, 1] to the set of Radon measures on $(0,T) \times \mathbb{R}$, such that for each $\eta \in C^2([0,1])$ and $\varphi \in C_c^\infty((0,T) \times \mathbb{R})$, $-\int \mathrm{d}t\, \mathrm{d}x\ [\eta(u)\varphi_t + q(u)\varphi_x] =$

$\int \mathrm{d}v\, \varrho_u(v; \mathrm{d}t, \mathrm{d}x)\eta''(v)\varphi(t,x)$, see Proposition 2.3. The candidate Γ-limit of H_ε is the functional H defined as follows. If u is not an entropy-measure solution to (1.1) then $H(u) = +\infty$. Otherwise $H(u) = \int \mathrm{d}v\, \varrho_u^+(v; \mathrm{d}t, \mathrm{d}x) D(v)/\sigma(v)$, where ϱ_u^+ denotes the positive part of ϱ_u. Note that while I_ε and $\mathcal{I}$ are non-local functionals, H is local. On the other hand, while I_ε, respectively $\mathcal{I}$, quantifies in a suitable squared Hilbert norm the violation of equation (1.2), respectively (1.1), this quadratic structure is lost in H. In Proposition 2.6 we show that H is a coercive lower semicontinuous functional, this matching the necessary properties for being the Γ-limit of a sequence of equicoercive functionals. Note also that H depends on the diffusion coefficient D and the conductivity coefficient σ only through their ratio, which is an expected property of well-behaving driven diffusive systems, in hydrodynamical-like limits. We discuss this issue in Remark 2.11, where a link between the functional H and the large deviations rate functional introduced in [9,17] is also investigated. In particular, H comes as a natural generalization of the functional introduced in [9,17], whenever the flux f is neither convex nor concave.

In this paper we prove that for each sequence $u^\varepsilon \to u$ in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ we have $\underline{\lim}_\varepsilon H_\varepsilon(u^\varepsilon) \geqq H(u)$, namely Γ–$\underline{\lim}\, H_\varepsilon \geqq H$. Since the functional H vanishes only on entropic solutions to (1.1), its zero-level set coincides with the limit points of the minima of I_ε. Concerning the Γ-limsup inequality, for each weak solution u to (1.1) in a suitable set $\mathcal{S}_\sigma$, see Definition 2.4, we construct a sequence $u^\varepsilon \to u$ such that $H_\varepsilon(u^\varepsilon) \to H(u)$. The above statements imply $(\Gamma\text{–}\lim H_\varepsilon)(u) = H(u)$ for $u \in \mathcal{S}_\sigma$. To complete the proof of the Γ-convergence of H_ε to H on the whole set of entropy-measure solutions, an additional density argument is needed. This seems to be a difficult problem, as Varadhan [17] puts it: "…one does not see at the moment how to produce a 'general' non-entropic solution, partly because one does not know what it is."

The above results imply that if u^ε solves (1.4) for some control E^ε such that $\varepsilon^{-1}\int_{[0,T]}\mathrm{d}t\, \|E^\varepsilon\|^2_{L_2(\mathbb{R},\sigma(u^\varepsilon)\mathrm{d}x)}$ vanishes as $\varepsilon \to 0$ (respectively remains uniformly bounded), then any limit point of u^ε is an entropic solution to (1.1) (respectively an entropy-measure solution). This statement is sharp in the sense that there are sequences $\{E^\varepsilon\}$ with $\underline{\lim}_\varepsilon\, \varepsilon^{-1}\int_{[0,T]}\mathrm{d}t\, \|E^\varepsilon\|^2_{L_2(\mathbb{R},\sigma(u^\varepsilon)\mathrm{d}x)} > 0$ such that any limit point of the corresponding u^ε is not an entropic solutions to (1.1). More generally, the variational description of conservation laws here introduced allows the following point of view. Measure-valued solutions to (1.1) are the points in the zero-level set of the Γ-limit of I_ε, while entropic weak solutions are the points in the zero-level set of the Γ-limit of $\varepsilon^{-1}I_\varepsilon$. In Appendix B we introduce a sequence $\{J_\varepsilon\}$ of functionals related to the viscous approximation of Hamilton–Jacobi equations. In [14] a Γ-limsup inequality for a related family of functionals has been independently investigated in a BV setting. Following closely the proofs of the Γ-convergence of $\{I_\varepsilon\}$, we establish the corresponding Γ-convergence results, thus obtaining a variational characterization of measure-valued and viscosity solutions to Hamilton–Jacobi equations. Although this "variational" point of view is consistent with the standard concepts of solution in the current setting of scalar conservation laws and Hamilton–Jacobi equations, it might be helpful for less understood model equations.

2. Notation and results

Hereafter in this paper, we assume that f is a Lipschitz function on [0, 1], D and σ are continuous functions on [0, 1], with D uniformly positive and σ strictly positive on (0, 1). We understand that these assumptions are supposed to hold in every statement below.

We also let $\langle \cdot, \cdot \rangle$ denote the inner product in $L_2(\mathbb{R})$, for $T > 0$ $\langle\langle \cdot, \cdot \rangle\rangle$ stands for the inner product in $L_2([0, T] \times \mathbb{R})$, and for O an open subset of $\mathbb{R}^n$, $C_c^\infty(O)$ denotes the collection of compactly supported infinitely differentiable functions on O.

Scalar conservation law

Our analysis will be restricted to equibounded densities u that take values in [0, 1]. Let U denote the compact separable metric space of measurable functions $u : \mathbb{R} \to [0, 1]$, equipped with the following H^{-1}_{loc}-like metric d_U. For $L > 0$, set

$$\|u\|_{-1,L} := \sup \left\{ \langle u, \varphi \rangle,\ \varphi \in C_c^\infty\left((-L, L)\right),\ \langle \varphi_x, \varphi_x \rangle = 1 \right\}$$

and define the metric d_U in U by

$$d_U(u, v) := \sum_{N=1}^{\infty} 2^{-N} \frac{\|u - v\|_{-1,N}}{1 + \|u - v\|_{-1,N}} \tag{2.1}$$

Given $T > 0$, let $\mathcal{U}$ be the set $C\left([0, T]; U\right)$ endowed with the uniform metric

$$d_{\mathcal{U}}(u, v) := \sup_{t \in [0,T]} d_U\left(u(t), v(t)\right) \tag{2.2}$$

An element $u \in \mathcal{U}$ is a *weak solution* to (1.1) iff for each $\varphi \in C_c^\infty\left((0, T) \times \mathbb{R}\right)$ (in particular $\varphi(0) = \varphi(T) = 0$) it satisfies

$$\langle\langle u, \varphi_t \rangle\rangle + \langle\langle f(u), \varphi_x \rangle\rangle = 0$$

We also introduce a suitable space $\mathcal{M}$ of Young measures and recall the notion of measure-valued solution to (1.1). Consider the set $\mathcal{N}$ of measurable maps μ from $[0, T] \times \mathbb{R}$ to the set $\mathcal{P}([0, 1])$ of Borel probability measures on [0, 1]. The set $\mathcal{N}$ can be identified with the set of positive Radon measures μ on $[0, 1] \times [0, T] \times \mathbb{R}$ such that $\mu([0, 1], \mathrm{d}t, \mathrm{d}x) = \mathrm{d}t\, \mathrm{d}x$. Indeed, by existence of a regular version of conditional probabilities, for such measures μ there exists a measurable kernel $\mu_{t,x}(\mathrm{d}\lambda) \in \mathcal{P}([0, 1])$ such that $\mu(\mathrm{d}\lambda, \mathrm{d}t, \mathrm{d}x) = \mathrm{d}t\, \mathrm{d}x\, \mu_{t,x}(\mathrm{d}\lambda)$. For $\imath : [0, 1] \to [0, 1]$ the identity map, we set

$$\mathcal{M} := \left\{ \mu \in \mathcal{N} \,:\, \text{the map } [0, T] \ni t \mapsto \mu_{t,\cdot}(\imath) \text{ is in } \mathcal{U} \right\} \tag{2.3}$$

in which, for a bounded measurable function $F : [0, 1] \to \mathbb{R}$, the notation $\mu_{t,x}(F)$ stands for $\int_{[0,1]} \mu_{t,x}(\mathrm{d}\lambda) F(\lambda)$. We endow $\mathcal{M}$ with the metric

$$d_{\mathcal{M}}(\mu, \nu) := d_{\mathrm{w}}(\mu, \nu) + d_{\mathcal{U}}\left(\mu(\imath), \nu(\imath)\right) \tag{2.4}$$

where d_{w} is a distance generating the relative topology on $\mathcal{N}$ regarded as a subset of the Radon measures on $[0, 1] \times [0, T] \times \mathbb{R}$ equipped with the weak* topology. $(\mathcal{M}, d_{\mathcal{M}})$ is a complete separable metric space.

An element $\mu \in \mathcal{M}$ is a *measure-valued solution* to (1.1) iff for each $\varphi \in C_{\mathrm{c}}^{\infty}((0, T) \times \mathbb{R})$ it satisfies

$$\langle\langle \mu(\imath), \varphi_t \rangle\rangle + \langle\langle \mu(f), \varphi_x \rangle\rangle = 0$$

If $u \in \mathcal{U}$ is a weak solution to (1.1), then $\delta_{u(t,x)}(\mathrm{d}\lambda) \in \mathcal{M}$ is a measure-valued solution. On the other hand, there exist measure-valued solutions which do not have this form.

Parabolic cost functional

We next give the definition of the parabolic cost functional informally introduced in (1.5). Given $u \in \mathcal{U}$ we write $u_x \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$ iff u admits a locally square integrable weak x-derivative. For $\varepsilon > 0$, $u \in \mathcal{U}$ such that $u_x \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$, and $\varphi \in C_{\mathrm{c}}^{\infty}((0, T) \times \mathbb{R})$ we set

$$\ell_{\varepsilon}^{u}(\varphi) := -\langle\langle u, \varphi_t \rangle\rangle - \langle\langle f(u), \varphi_x \rangle\rangle + \frac{\varepsilon}{2} \langle\langle D(u) u_x, \varphi_x \rangle\rangle \tag{2.5}$$

and define $I_{\varepsilon} : \mathcal{U} \to [0, +\infty]$ as follows. If $u_x \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$ we set

$$I_{\varepsilon}(u) := \sup_{\varphi \in C_{\mathrm{c}}^{\infty}((0,T)\times\mathbb{R})} \left[\ell_{\varepsilon}^{u}(\varphi) - \frac{1}{2} \langle\langle \sigma(u) \varphi_x, \varphi_x \rangle\rangle \right] \tag{2.6}$$

letting $I_{\varepsilon}(u) := +\infty$ otherwise. Note that $I_{\varepsilon}(u)$ vanishes iff $u \in \mathcal{U}$ is a weak solution to (1.2); more generally, by Riesz representation theorem, it is not difficult to prove the connection of I_{ε} with the perturbed parabolic problem (1.4), see Lemma 3.1 below for the precise statement.

In order to discuss the behavior of I_{ε} as $\varepsilon \to 0$, we lift it to the space of Young measures $(\mathcal{M}, d_{\mathcal{M}})$. We thus define $\mathcal{I}_{\varepsilon} : \mathcal{M} \to [0, +\infty]$ by

$$\mathcal{I}_{\varepsilon}(\mu) := \begin{cases} I_{\varepsilon}(u) & \text{if} \quad \mu_{t,x} = \delta_{u(t,x)} \ \text{ for some } \ u \in \mathcal{U} \\ +\infty & \text{otherwise} \end{cases} \tag{2.7}$$

Asymptotic parabolic cost

As is well known, a most useful notion of variational convergence is the so-called Γ-convergence which, together with some compactness estimates, implies convergence of the minima. Let X be a complete separable metrizable space; recall that a sequence of functionals $F_{\varepsilon} : X \to [-\infty, +\infty]$ is *equicoercive* on X iff for each $M > 0$ there exists a compact set K_M such that for any $\varepsilon \in (0, 1]$ we have $\{x \in X \ : \ F_{\varepsilon}(x) \leqq M\} \subset K_M$. We briefly recall the basic definitions of the Γ-convergence theory, see, for example [3,6]. Given $x \in X$ we define

$$\left(\Gamma\text{–}\varliminf_{\varepsilon \to 0} F_{\varepsilon} \right)(x) := \inf \left\{ \varliminf_{\varepsilon \to 0} F_{\varepsilon}(x^{\varepsilon}), \ \{x^{\varepsilon}\} \subset X \ : \ x^{\varepsilon} \to x \right\}$$
$$\left(\Gamma\text{–}\varlimsup_{\varepsilon \to 0} F_{\varepsilon} \right)(x) := \inf \left\{ \varlimsup_{\varepsilon \to 0} F_{\varepsilon}(x^{\varepsilon}), \ \{x^{\varepsilon}\} \subset X \ : \ x^{\varepsilon} \to x \right\}$$

Whenever $\Gamma\text{–}\underline{\lim}_\varepsilon F_\varepsilon = \Gamma\text{–}\overline{\lim}_\varepsilon F_\varepsilon = F$ we say that F_ε Γ-converges to F in X. Equivalently, F_ε Γ-converges to F iff for each $x \in X$ we have:

– for any sequence $x^\varepsilon \to x$ we have $\underline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon) \geqq F(x)$ (*Γ-liminf inequality*);
– there exists a sequence $x^\varepsilon \to x$ such that $\overline{\lim}_\varepsilon F_\varepsilon(x^\varepsilon) \leqq F(x)$ (*Γ-limsup inequality*).

Equicoercivity and Γ-convergence of a sequence $\{F_\varepsilon\}$ imply an upper bound of infima over open sets, and a lower bound of infima over closed sets, see, for example [3, Prop. 1.18]; therefore it is the relevant notion of variational convergence in the control setting introduced above.

Theorem 2.1. *The sequence $\{\mathcal{I}_\varepsilon\}$ defined in (2.6), (2.7) is equicoercive on $\mathcal{M}$ and, as $\varepsilon \to 0$, Γ-converges in $\mathcal{M}$ to*

$$\mathcal{I}(\mu) := \sup_{\varphi \in C_c^\infty((0,T)\times\mathbb{R})} \left\{ -\langle\langle \mu(\imath), \varphi_t \rangle\rangle - \langle\langle \mu(f), \varphi_x \rangle\rangle - \frac{1}{2} \langle\langle \mu(\sigma)\varphi_x, \varphi_x \rangle\rangle \right\} \tag{2.8}$$

Note that $\mathcal{I}(\mu) = 0$ iff μ is a measure-valued solution to (1.1). From Theorem 2.1 we deduce the Γ-limit of I_ε, see (2.6), on $\mathcal{U}$ by projection.

Corollary 2.2. *The sequence of functionals $\{I_\varepsilon\}$ is equicoercive on $\mathcal{U}$ and, as $\varepsilon \to 0$, Γ-converges in $\mathcal{U}$ to the functional $I : \mathcal{U} \to [0, +\infty]$ defined by*

$$I(u) := \inf \left\{ \int \mathrm{d}t\, \mathrm{d}x\, R_{f,\sigma}(u(t,x), \Phi(t,x)), \right. \\ \left. \Phi \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R}) : \Phi_x = -u_t \text{ weakly} \right\}$$

where $R_{f,\sigma} : [0,1] \times \mathbb{R} \to [0, +\infty]$ is defined by

$$R_{f,\sigma}(w, c) := \inf\{(\nu(f) - c)^2/\nu(\sigma),\ \nu \in \mathcal{P}([0,1]) : \nu(\imath) = w\}$$

in which we understand $(c-c)^2/0 = 0$.

From the proof of Corollary 2.2 it follows $I(\cdot) \leqq \mathcal{I}(\delta_\cdot)$, and the equality holds iff f is linear. If we restrict to *stationary* u's, namely to the case $u_t = 0$, Corollary 2.2 can be regarded as a negative-Sobolev version of classical relaxation results for integral functionals in weak topology. More precisely, from the proofs of Theorem 4.1 and Corollary 2.2 it follows that if we define the functional $\tilde{F} : U \to [0, +\infty]$ by

$$\tilde{F}(u) := \inf_{c\in\mathbb{R}} \int \mathrm{d}x\, \frac{[f(u(x)) - c]^2}{\sigma(u(x))}$$

then its lower semicontinuous envelope with respect to the d_U-distance (2.1) is given by

$$F(u) := \inf_{c\in\mathbb{R}} \int \mathrm{d}x\, R_{f,\sigma}(u(x), c)$$

Note also that $R_{f,\sigma}$ can be explicitly calculated in some cases. Let $\underline{f}, \overline{f} : [0,1] \to \mathbb{R}$ be, respectively, the convex and concave envelope of f. Then, in the case $\sigma = 1$, we have $R_{f,1}(w,c) = \Big[\text{distance}(c, [\underline{f}(w), \overline{f}(w)])\Big]^2$. In the case $f = \sigma$ (which includes the example mentioned in the introduction $f(u) = \sigma(u) = u(1-u)$) then

$$R_{f,f}(w,c) = \begin{cases} 2(|c| - c) & \text{if } |c| \in [\underline{f}(w), \overline{f}(w)] \\ \frac{(\overline{f}(w)-c)^2}{\overline{f}(w)} & \text{if } |c| > \overline{f}(w) \\ \frac{(\underline{f}(w)-c)^2}{\underline{f}(w)} & \text{if } |c| < \underline{f}(w) \end{cases}$$

Entropy-measure solutions

Recalling (2.2), we let $\mathcal{X}$ be the same set $C([0,T];U)$ endowed with the metric

$$d_{\mathcal{X}}(u,v) := \sum_{N=1}^{\infty} \frac{1}{2^N} \|u - v\|_{L_1([0,T]\times[-N,N])} + d_{\mathcal{U}}(u,v) \tag{2.9}$$

Convergence in $\mathcal{X}$ is equivalent to convergence in $\mathcal{U}$ *and* in $L_{p,\mathrm{loc}}([0,T]\times\mathbb{R})$ for $p \in [1,+\infty)$.

Let $C^2([0,1])$ be the set of twice differentiable functions on $(0,1)$ whose derivatives are continuous up to the boundary. A function, respectively a convex function, $\eta \in C^2([0,1])$ is called an *entropy*, respectively a *convex entropy*, and its *conjugated entropy flux* $q \in C([0,1])$ is defined up to a constant by $q(u) := \int^u \mathrm{d}v\, \eta'(v) f'(v)$. For u a weak solution to (1.1), for (η, q) an entropy–entropy flux pair, the η*-entropy production* is the distribution $\wp_{\eta,u}$ acting on $C_c^{\infty}((0,T)\times\mathbb{R})$ as

$$\wp_{\eta,u}(\varphi) := -\langle\langle \eta(u), \varphi_t \rangle\rangle - \langle\langle q(u), \varphi_x \rangle\rangle \tag{2.10}$$

Let $C_c^{2,\infty}([0,1]\times(0,T)\times\mathbb{R})$ be the set of compactly supported maps $\vartheta : [0,1]\times(0,T)\times\mathbb{R} \ni (v,t,x) \mapsto \vartheta(v,t,x) \in \mathbb{R}$, that are twice differentiable in the v variable, with derivatives continuous up to the boundary of $[0,1]\times(0,T)\times\mathbb{R}$, and that are infinitely differentiable in the (t,x) variables. For $\vartheta \in C_c^{2,\infty}([0,1]\times(0,T)\times\mathbb{R})$, we denote by ϑ' and ϑ'' its partial derivatives with respect to the v variable. We say that a function $\vartheta \in C_c^{2,\infty}([0,1]\times(0,T)\times\mathbb{R})$ is an *entropy sampler*, and its *conjugated entropy flux sampler* $Q : [0,1]\times(0,T)\times\mathbb{R} \to \mathbb{R}$ is defined up to an additive function of (t,x) by $Q(u,t,x) := \int^u \mathrm{d}v\, \vartheta'(v,t,x) f'(v)$. Finally, given a weak solution u to (1.1), the ϑ*-sampled entropy production* $P_{\vartheta,u}$ is the real number

$$P_{\vartheta,u} := -\int \mathrm{d}t\, \mathrm{d}x\, \left[(\partial_t \vartheta)(u(t,x),t,x) + (\partial_x Q)(u(t,x),t,x)\right] \tag{2.11}$$

If $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some entropy η and some $\varphi \in C_c^{\infty}((0,T)\times\mathbb{R})$, then $P_{\vartheta,u} = \wp_{\eta,u}(\varphi)$.

The next proposition introduces a suitable class of solutions to (1.1), which will be needed in the following. We denote by $M((0,T)\times\mathbb{R})$ the set of Radon measures on $(0,T)\times\mathbb{R}$ that we consider equipped with the weak* topology. In the following, for $\varrho \in M((0,T)\times\mathbb{R})$ we denote by $\varrho^{\pm}$ the positive and negative part of ϱ. For u a weak solution to (1.1) and η an entropy, recalling (2.10) we set

$$\begin{aligned}
\|\wp_{\eta,u}\|_{\mathrm{TV},L} &:= \sup\left\{\wp_{\eta,u}(\varphi),\ \varphi \in C_{\mathrm{c}}^{\infty}\left((0,T)\times(-L,L)\right),\ |\varphi| \leqq 1\right\} \\
\|\wp^{+}_{\eta,u}\|_{\mathrm{TV},L} &:= \sup\left\{\wp_{\eta,u}(\varphi),\ \varphi \in C_{\mathrm{c}}^{\infty}\left((0,T)\times(-L,L)\right),\ 0 \leqq \varphi \leqq 1\right\}
\end{aligned} \tag{2.12}$$

Proposition 2.3. *Let $u \in \mathcal{X}$ be a weak solution to* (1.1). *The following statements are equivalent:*

(i) *there exists $c > 0$ such that $\|\wp^{+}_{\eta,u}\|_{\mathrm{TV},L} < +\infty$ for any $L > 0$ and $\eta \in C^2([0,1])$ with $0 \leqq \eta'' \leqq c$;*
(ii) *for each entropy η, the η-entropy production $\wp_{\eta,u}$ can be extended to a Radon measure on $(0,T)\times\mathbb{R}$, namely $\|\wp_{\eta,u}\|_{\mathrm{TV},L} < +\infty$ for each $L > 0$;*
(iii) *there exists a bounded measurable map $\varrho_u : [0,1] \ni v \to \varrho_u(v; \mathrm{d}t, \mathrm{d}x) \in M((0,T)\times\mathbb{R})$ such that for any entropy sampler ϑ*

$$P_{\vartheta,u} = \int \mathrm{d}v\, \varrho_u(v; \mathrm{d}t, \mathrm{d}x)\, \vartheta''(v,t,x) \tag{2.13}$$

A weak solution $u \in \mathcal{X}$ that satisfies any of the equivalent conditions in Proposition 2.3 is called an *entropy-measure solution* to (1.1). We denote by $\mathcal{E} \subset \mathcal{X}$ the set of entropy-measure solutions to (1.1). Proposition 2.3 establishes a so-called *kinetic formulation* for entropy-measure solutions, see also [7, Prop. 3.1] for a similar result. If $f \in C^2([0,1])$ is such that there are no intervals in which f is affine, using the results in [4] we show that entropy-measure solutions have some regularity properties, see Lemma 5.1.

A weak solution $u \in \mathcal{X}$ to (1.1) is called an *entropic solution* iff for each convex entropy η the inequality $\wp_{\eta,u} \leqq 0$ holds in distribution sense, namely $\|\wp^{+}_{\eta,u}\|_{\mathrm{TV},L} = 0$ for each $L > 0$. In particular entropic solutions are entropy-measure solutions such that $\varrho_u(v; \mathrm{d}t, \mathrm{d}x)$ is a negative Radon measure for each $v \in [0,1]$. It is well known, see, for example [5,15], that for each $u_0 \in U$ there exists a unique entropic solution $\bar{u} \in C([0,T]; L_{1,\mathrm{loc}}(\mathbb{R}))$ to (1.1) such that $\bar{u}(0) = u_0$. Such a solution $\bar{u}$ is called the *Kruzkov solution* with initial datum u_0.

Γ-entropy cost of non-entropic solutions

We next introduce a rescaled cost functional and prove in particular that entropic solutions are the only ones with vanishing rescaled asymptotic cost. Recalling that I_ε has been introduced in (2.6), the *rescaled cost functional* $H_\varepsilon : \mathcal{X} \to [0,+\infty]$ is defined by

$$H_\varepsilon(u) := \varepsilon^{-1} I_\varepsilon(u) \tag{2.14}$$

In the Γ-convergence theory, the asymptotic behavior of the rescaled functional H_ε is usually referred to as the development by Γ-convergence of I_ε, see, for example [3, §1.10]. In our case, while we lifted I_ε to the space of Young measures $\mathcal{M}$, we can consider the rescaled cost functional H_ε on $\mathcal{X}$. In fact, as shown below, H_ε has much better compactness properties than I_ε and it is equicoercive on $\mathcal{X}$. Therefore the Γ-convergence of the lift of H_ε to $\mathcal{M}$ can be immediately retrieved from the

Γ-convergence of H_ε on $\mathcal{X}$. Indeed, since $\delta_{u_\varepsilon} \to \delta_u$ in $\mathcal{M}$ implies $u_\varepsilon \to u$ in $\mathcal{X}$, the metric (2.9) generates the relative topology of $\mathcal{X}$ regarded as a subset of $\mathcal{M}$.

Recall that $\mathcal{E} \subset \mathcal{X}$ denotes the set of entropy-measure solutions to (1.1), and that for $u \in \mathcal{E}$ there exists a bounded measurable map $\varrho_u : [0,1] \to M((0,T) \times \mathbb{R})$ such that (2.13) holds. Let ϱ_u^+ be the positive part of ϱ_u, and define $H : \mathcal{X} \to [0,+\infty]$ by

$$H(u) := \begin{cases} \displaystyle\int \mathrm{d}v\, \varrho_u^+(v; \mathrm{d}t, \mathrm{d}x)\, \frac{D(v)}{\sigma(v)} & \text{if } u \in \mathcal{E} \\ +\infty & \text{otherwise} \end{cases} \tag{2.15}$$

As shown in the proof of Theorem 2.5, if u is a weak solution to (1.1) and $H(u) < +\infty$, then $H(u) = \sup_\vartheta P_{\vartheta,u}$, where the supremum is taken over the entropy samplers ϑ such that $0 \leqq \sigma(v)\vartheta''(v,t,x) \leqq D(v)$, for each $(v,t,x) \in [0,1] \times [0,T] \times \mathbb{R}$.

Definition 2.4. An entropy-measure solution $u \in \mathcal{E}$ is *entropy-splittable* iff there exist two closed sets $E^+, E^- \subset [0,T] \times \mathbb{R}$ such that

(i) For almost every $v \in [0,1]$, the support of $\varrho_u^+(v; \mathrm{d}t, \mathrm{d}x)$ is contained in E^+, and the support of $\varrho_u^-(v; \mathrm{d}t, \mathrm{d}x)$ is contained in E^-.

(ii) For each $L > 0$, the set $\big\{t \in [0,T] \,:\, (\{t\} \times [-L,L]) \cap E^+ \cap E^- \neq \emptyset\big\}$ is nowhere dense in $[0,T]$.

The set of entropy-splittable solutions to (1.1) is denoted by $\mathcal{S}$. An entropy-splittable solution $u \in \mathcal{S}$ such that $H(u) < +\infty$ and

(iii) For each $L > 0$ there exists $\delta_L > 0$ such that $\sigma(u(t,x)) \geqq \delta_L$ for almost everywhere $(t,x) \in [0,T] \times [-L,L]$.

is called *nice with respect to* σ. The set of nice (with respect to σ) solutions to (1.1) is denoted by $\mathcal{S}_\sigma$.

Note that $\mathcal{S}_\sigma \subset \mathcal{S} \subset \mathcal{E} \subset \mathcal{X}$, and that, if σ is uniformly positive on $[0,1]$, then $\mathcal{S}_\sigma = \mathcal{S}$. In Remark 2.9 we exhibit a few classes of entropy-splittable solutions to (1.1).

Theorem 2.5. *Let H_ε and H be the functionals on $\mathcal{X}$ as respectively defined in* (2.14) *and* (2.15).

1. *The sequence of functionals $\{H_\varepsilon\}$ satisfies the Γ-liminf inequality Γ-$\underline{\lim}_\varepsilon H_\varepsilon \geqq H$ on $\mathcal{X}$.*
2. *Assume that there is no interval where f is affine. Then the sequence of functionals $\{H_\varepsilon\}$ is equicoercive on $\mathcal{X}$.*
3. *Assume furthermore that $f \in C^2([0,1])$, and $D, \sigma \in C^\alpha([0,1])$ for some $\alpha > 1/2$. Define*

$$\overline{H}(u) := \inf\big\{\underline{\lim}\, H(u_n),\ \{u_n\} \subset \mathcal{S}_\sigma \,:\, u_n \to u \text{ in } \mathcal{X}\big\}$$

Then the sequence of functionals $\{H_\varepsilon\}$ satisfies the Γ-limsup inequality Γ-$\overline{\lim}_\varepsilon H_\varepsilon \leqq \overline{H}$ on $\mathcal{X}$.

From the lower semicontinuity of H on $\mathcal{X}$, see Proposition 2.6, it follows that $\overline{H} \geqq H$ on $\mathcal{X}$ and $\overline{H} = H$ on $\mathcal{S}_\sigma$, namely the Γ-convergence of H_ε to H holds on $\mathcal{S}_\sigma$. To get the full Γ-convergence on $\mathcal{X}$, the inequality $H(u) \geqq \overline{H}(u)$ is required also for $u \notin \mathcal{S}_\sigma$. This requires one to show that $\mathcal{S}_\sigma$ is H-dense in $\mathcal{X}$, namely that for $u \in \mathcal{X}$ such that $H(u) < +\infty$ there exists a sequence $\{u^n\} \subset \mathcal{S}_\sigma$ converging to u in $\mathcal{X}$ such that $H(u^n) \to H(u)$. As mentioned at the end of the introduction, this appears to be a difficult problem. A preliminary step in this direction is to obtain a chain rule formula for bounded vector fields on $[0, T] \times \mathbb{R}$ the divergence of which is a Radon measure (divergence-measure fields). This is a classical result for locally BV fields [2]. However, while entropic solutions to (1.1) are in $BV_{\mathrm{loc}}([0, T] \times \mathbb{R})$ [1, Corollary 1.3] whenever f is uniformly convex or concave, as shown in Example 2.8 below, the set $\{u \in \mathcal{X} : H(u) < +\infty\}$ is not contained in $BV_{\mathrm{loc}}([0, T] \times \mathbb{R})$ even under this assumptions on f; see [8] for similar examples, including estimates in Besov norms. Chain rule formulae out of the BV setting have been investigated in several recent papers; in particular in [7], a chain rule formula for divergence-measure fields is addressed, providing some partial results. In the remaining of this section we discuss some properties of H, and some issues related to the H-density of $\mathcal{S}_\sigma$.

In the following proposition we show that H is lower semicontinuous, and that it is coercive under the same hypotheses used for the equicoercivity of $\{H_\varepsilon\}$. Moreover, we prove that the minimizers of H are limit points of the minimizers of I_ε as $\varepsilon \to 0$, so that no further rescaling of $\{I_\varepsilon\}$ has to be investigated.

Proposition 2.6. *The functional H is lower semicontinuous on $\mathcal{X}$ and $H(u) = 0$ iff u is an entropic solution to* (1.1). *If furthermore there are no intervals where f is affine then H is also coercive on $\mathcal{X}$.*

From Proposition 5.1 and the aforementioned regularity of entropy-measure solutions, see Lemma 5.1, it follows that if $f \in C^2([0, 1])$ then the zero-level set of H coincides with the set of Kruzkov solutions to (1.1).

If $u \in \mathcal{X}$ is a weak solution with locally bounded variation, Vol'pert chain rule, see [2], gives a formula for $H(u)$ in terms of the normal traces of u on its jump set, as shown in the following remark.

Remark 2.7. Let $u \in \mathcal{X} \cap BV_{\mathrm{loc}}([0, T] \times \mathbb{R})$ be a weak solution to (1.1). Denote by $J_u \subset [0, T] \times \mathbb{R}$ its jump set, by $\mathcal{H}^1 \llcorner J_u$ the one-dimensional Hausdorff measure restricted to J_u, by $n = \left(n^t, n^x\right)$ a unit normal to J_u (which is well defined $\mathcal{H}^1 \llcorner J_u$ almost everywhere), and by $u^\pm$ the normal traces of u on J_u with respect to n. Then the Rankine–Hugoniot condition $(u^+ - u^-)n^t + \left(f(u^+) - f(u^-)\right) n^x = 0$ holds. In particular we can choose n so that n^x is uniformly positive, and thus u^+ is the right trace of u and u^- is the left trace of u. Then $u \in \mathcal{E}$ and

$$\varrho_u(v; \mathrm{d}t, \mathrm{d}x) = \frac{\mathrm{d}\mathcal{H}^1 \llcorner J_u}{\left\{(u^+ - u^-)^2 + [f(u^+) - f(u^-)]^2\right\}^{1/2}} \rho(v, u^+, u^-)$$

where, denoting by $u^- \wedge u^+$ and $u^- \vee u^+$, respectively, the minimum and maximum of $\{u^-, u^+\}$, $\rho : [0, 1]^3 \to \mathbb{R}$ is defined by

$$\rho(v, u^+, u^-) := \big[f(u^-)(u^+ - v) + f(u^+)(v - u^-) - f(v)(u^+ - u^-)\big] \times \mathbb{I}_{[u^- \wedge u^+, u^- \vee u^+]}(v)$$

Hence, denoting by ρ^+ the positive part of ρ

$$\begin{aligned} H(u) &= \int_{J_u} \frac{\mathrm{d}\mathcal{H}^1}{\{(u^+ - u^-)^2 + [f(u^+) - f(u^-)]^2\}^{1/2}} \int \mathrm{d}v\, \rho^+(v, u^+, u^-) \frac{D(v)}{\sigma(v)} \\ &= \int_{J_u} \mathrm{d}\mathcal{H}^1 |n^x| \int \mathrm{d}v\, \frac{\rho^+(v, u^+, u^-)}{|u^+ - u^-|} \frac{D(v)}{\sigma(v)} \end{aligned} \tag{2.16}$$

Note $\rho(v, u^+, u^-) \leqq 0$ iff $\frac{f(v) - f(u^-)}{v - u^-} \geqq \frac{f(u^+) - f(v)}{u^+ - v}$. This corresponds to the well known geometrical secant condition for entropic solutions, see, for example [5,15]. Therefore $H(u)$ quantifies the violation of the entropy condition along the non-entropic shocks of u.

In the following Example 2.8, we show that neither the domain of H, neither the H-closure of $\mathcal{S}_\sigma$ are contained in $BV_{\mathrm{loc}}([0, T] \times \mathbb{R})$.

Example 2.8. Let $f(u) = u(1-u)$ and pick a decreasing sequence $\{b_i\}$ of positive reals such that $b_1 < 1/2$, $\sum_i b_i = +\infty$ and $\sum_i b_i^3 < +\infty$. Let u be defined by (see Fig. 1)

$$u(t, x) := \begin{cases} 1/2 + b_i & \text{if } T(b_1 - b_i) < x + b_i\, t < T\,(b_1 - b_{i+1}) \text{ for some } i \\ 1/2 & \text{otherwise} \end{cases}$$

Then $H(u) = \frac{T}{2} \sum_i \int_{[0, b_i]} \mathrm{d}v\, \frac{D(1/2+v)}{\sigma(1/2+v)} v(b_i - v) < +\infty$. Note that, even if the initial datum is in $BV(\mathbb{R})$ and f is concave, $u \notin BV_{\mathrm{loc}}([0, T] \times \mathbb{R})$. However $H(u) = \overline{H}(u)$. Indeed the sequence $\{u^n\} \subset \mathcal{S}_\sigma$ defined by

$$u^n(t, x) := \begin{cases} u(t, x) & \text{if } x + b_n\, t < T(b_1 - b_{n+1}) \\ 1/2 & \text{otherwise} \end{cases}$$

is such that $u^n \to u$ in $\mathcal{X}$ and $\lim_n H(u^n) = H(u)$.

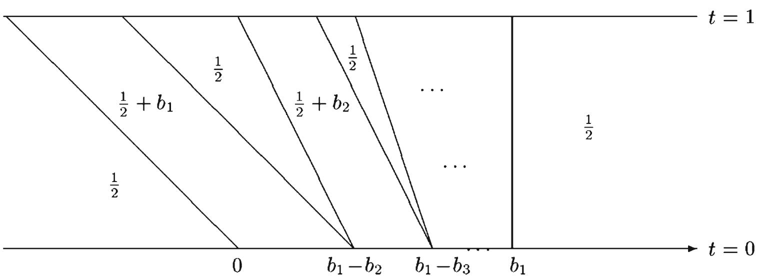

Fig. 1. The values of u in Example 2.8 for $T = 1$

In the following remarks we identify some classes of entropy-splittable solutions to (1.1), see Definition 2.4.

Remark 2.9. Weak solutions to (1.1) such that, for each convex entropy η, $\wp_{\eta,u} \leqq 0$ (entropic solutions) or $\wp_{\eta,u} \geqq 0$ (anti-entropic solutions) are entropy-splittable. Indeed they are entropy-measure solutions (see Proposition 2.6) and they fit in Definition 2.4 with the choice $E^- = [0,T]\times\mathbb{R}$ and $E^+ = \emptyset$ (for entropic solutions), and, respectively, $E^+ = [0,T]\times\mathbb{R}$ and $E^- = \emptyset$ (for anti-entropic solutions).

Let $u \in BV_{\mathrm{loc}}([0,T]\times\mathbb{R})$ be a weak solution to (1.1). In the same setting of Remark 2.7, let us define $J_u^{\pm} := \mathrm{Closure}\big(\{(t,x) \in J_u \,:\, \exists v \in [0,1] \,:\, \pm\varrho(v; u^+, u^-) > 0\}\big)$. Suppose that for each $L > 0$ the set $\{t \in [0,T] \,:\, (\{t\}\times[-L,L]) \cap J_u^+ \cap J_u^-\}$ is nowhere dense in $[0,T]$. Then u is an entropy-splittable solution. If f is convex or concave the sign of $\rho(v, u^+, u^-)$ does not depend on $v \in [u^- \wedge u^+, u^- \vee u^+]$. Therefore, under this convexity hypothesis, weak solutions to (1.1) with locally bounded variations and with a jump set J_u consisting of a locally finite number of Lipschitz curves, intersecting each other at a locally finite number of points are entropy splittable.

For a general (possibly neither convex nor concave) flux f, even piecewise constant solutions to (1.1) may fail to be entropy-splittable. However, in the following Example 2.10 we introduce a family of weak solutions u to (1.1) that are not entropy-splittable, and show that they are in the H-closure of $\mathcal{S}_\sigma$, and thus $\overline{H}(u) = H(u)$. However, while Example 2.10 can be widely generalized to prove $\overline{H}(u) = H(u)$ for u in suitable classes of piecewise smooth solutions, it does not seem that the ideas suggested by this example may work in the general setting of entropy-measure solutions $u \in \mathcal{E}$.

Example 2.10. Let $\gamma : [0,T] \to \mathbb{R}$ be a Lipschitz map, let u be a weak solution of bounded variation to (1.1), and suppose that the jump set of u coincides with γ. Let $u^- \equiv u^-(t)$ and $u^+ \equiv u^+(t)$ be the traces of u on γ, and suppose that there exists $u^0 \in (0,1)$ such that $u^-(t) < u^0 < u^+(t)$ for each t and $\frac{f(v)-f(u^-)}{v-u^-} \geqq \frac{f(u^+)-f(v)}{u^+-v}$ for $v \in [u^-, u^0]$ and $\frac{f(v)-f(u^-)}{v-u^-} \leqq \frac{f(u^+)-f(v)}{u^+-v}$ for $v \in [u^-, u^0]$. Then, if these inequalities are strict at some v and t, u is not entropy-splittable. However defining $u^n \in \mathcal{X}$ by

$$u^n(t,x) := \begin{cases} u(t, x+n^{-1}) & \text{if } x \leqq \gamma(t) - n^{-1} \\ u^0 & \text{if } \gamma(t) - n^{-1} < x < \gamma(t) + n^{-1} \\ u(t, x-n^{-1}) & \text{if } x \leqq \gamma(t) + n^{-1} \end{cases}$$

we have that $u^n \in \mathcal{S}$, $u^n \to u$ in $\mathcal{X}$ and $H(u^n) = H(u)$. In particular, if $\sigma(u)$ is uniformly positive on compact subsets of $[0,T]\times\mathbb{R}$, then $\overline{H}(u) = H(u)$. It is easy to extend this example to the case in which the jump set of u consists of a locally finite number of Lipschitz curves non-intersecting each other, provided that on each curve the quantity $\frac{f(v)-f(u^-)}{v-u^-} - \frac{f(u^+)-f(v)}{u^+-v}$ changes its sign a finite number of times for $v \in [u^+ \wedge u^-, u^+ \vee u^-]$.

We next discuss the link between this paper and [9,17]. In the introduction we informally described the connection between the problem (1.4) and stochastic particles systems under Euler scaling. It is interesting to note that such a quantitative connection can also be established for the limiting functionals. The key point is that we expect the functional H defined in (2.15) to coincide with the large deviations rate functional introduced in [9,17], provided the functions f, D and σ are chosen correspondingly. Unfortunately, we cannot establish such an identification off the set of weak solutions to (1.1) with locally bounded variation.

Remark 2.11. Let $H' : \mathcal{X} \to [0, +\infty]$ be defined as follows. If $u \in \mathcal{E}$ we set

$$H'(u) := \sup \left\{ \|\wp^{+}_{\eta,u}\|_{\mathrm{TV},L},\ L > 0,\ \eta \in C^2([0,1]) \,:\, 0 \leqq \sigma\, \eta'' \leqq D \right\}$$

letting $H'(u) := +\infty$ otherwise. Then $H \geqq H'$ and $H(u) = H'(u)$ whenever there exists a Borel set $E^+ \subset [0,T] \times \mathbb{R}$ such that for almost everywhere $v \in [0,1]$ the measure $\varrho^+_u(v; \mathrm{d}t, \mathrm{d}x)$ is concentrated on E^+ and $\varrho^-_u(v; \mathrm{d}t, \mathrm{d}x) = 0$ on E^+. In particular if f is convex or concave and $u \in BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$, then $H(u) = H'(u)$. If f is neither convex nor concave, then there exists $u \in \mathcal{X}$ such that $H(u) > H'(u)$.

A general connection between dynamical transport coefficients and thermodynamic potentials in driven diffusive systems is the so-called *Einstein relation*, see for example [16, II.2.5]. For a physical model described by (1.4), this relation states that the *Einstein entropy* $h \in C^2((0,1)) \cap C([0,1]; [0,+\infty])$ defined by

$$\sigma(v) h''(v) = D(v) \quad v \in (0,1)$$

is a physically relevant entropy in the limit $\varepsilon \to 0$. We let g be the conjugated flux to h, that is $g(u) := \int_{1/2}^{u} \mathrm{d}v\, h'(v) f'(v)$. Note that h, g may be unbounded if σ vanishes at the boundary of $[0,1]$ and that $g \leqq C_1 + C_2\, h$ for some constants $C_1, C_2 \geqq 0$. If u is a weak solution to (1.1) such that $h(u) \in L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$ and such that the distribution $h(u)_t + g(u)_x$ acts as a Radon measure on $(0,T) \times \mathbb{R}$, we let $\|\wp^{+}_{h,u}\|_{\mathrm{TV}}$ be the total variation of the positive part of such a measure. By monotone convergence $H'(u) \geqq \|\wp^{+}_{h,u}\|_{\mathrm{TV}}$ for such a u, and if f is convex or concave and u has locally bounded variation, then indeed $H'(u) = \|\wp^{+}_{h,u}\|_{\mathrm{TV}}$. If f is convex or concave, we do not know whether $H(u) = H'(u) = \|\wp^{+}_{h,u}\|_{\mathrm{TV}}$ for all $u \in \mathcal{X}$, since a chain rule formula for divergence-measure fields is missing.

The problem investigated in [9,17] formally corresponds to the case $f(u) = \sigma(u) = u(1-u)$ and $D(u) = 1$, so that the Einstein entropy h coincides with the Bernoulli entropy $h(u) = u \log u + (1-u)\log(1-u)$. The (candidate) large deviations rate functional H^{JV} introduced in [9,17] is defined as $+\infty$ off the set of weak solutions to (1.1), while $H^{JV}(u) = \|\wp^{+}_{h,u}\|_{\mathrm{TV}}$ for u a weak solution (this is well defined, since h is bounded). We thus have $H \geqq H^{JV}$, and in view of the Γ-liminf inequality, H comes as a natural generalization of H^{JV} for diffusive systems with no convexity assumptions on the flux f.

Outline of the proofs

Standard parabolic a priori estimates on u in terms of $I_\varepsilon(u)$ imply equicoercivity of $\mathcal{I}_\varepsilon$ on $\mathcal{M}$. Equicoercivity of H_ε on $\mathcal{X}$ is obtained by the same bounds and a classical compensated compactness argument.

The Γ-liminf inequality in Theorem 2.1 follows from the variational definition (2.6) of I_ε. The Γ-liminf inequality in Theorem 2.5 still follows from (2.6) by choosing test functions of the form $\varepsilon\vartheta(u^\varepsilon(t,x),t,x)$, with $\sigma\vartheta'' \leqq D$.

The Γ-limsup inequality in Theorem 2.1 is not difficult if $\mu_{t,x} = \delta_{u(t,x)}$ for some smooth u; the general result is obtained by taking the lower semicontinuous envelope. The Γ-limsup statement in Theorem 2.5 is proven by building, for each $u \in \mathcal{S}_\sigma$, a recovery sequence $\{u^\varepsilon\}$ such that a priori $H_\varepsilon(u^\varepsilon) \to H(u)$. The convergence $u^\varepsilon \to u$ is then obtained by a stability analysis of the parabolic equation (1.4) with respect to small variations of the control E.

Eventually, in Appendix B we apply our results to Hamilton–Jacobi equations.

3. Representation of I_ε and a priori bounds

Given a bounded measurable function $a \geqq 0$ on $[0,T]\times\mathbb{R}$ let $\mathcal{D}^1_a$ be the Hilbert space obtained by identifying and completing the functions $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$ with respect to the seminorm $\langle\langle\varphi_x, a\,\varphi_x\rangle\rangle^{1/2}$. Let $\mathcal{D}^{-1}_a$ be its dual space. The corresponding norms are denoted, respectively, by $\|\cdot\|_{\mathcal{D}^1_a}$ and $\|\cdot\|_{\mathcal{D}^{-1}_a}$.

We first establish the connection between the cost functional I_ε and the perturbed parabolic problem (1.4). The following lemma is a standard tool in large deviations theory, see, for example [11, Lemma 10.5.3]. We, however, detail its proof for sake of completeness.

Lemma 3.1. *Fix $\varepsilon > 0$ and let $u \in \mathcal{U}$. Then $I_\varepsilon(u) < +\infty$ iff there exists $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that u is a weak solution to (1.4) with $E = \Psi^{\varepsilon,u}_x$, namely for each $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$*

$$\begin{aligned}&\langle u(T),\varphi(T)\rangle - \langle u(0),\varphi(0)\rangle\\&\quad -\Big[\langle\langle u,\varphi_t\rangle\rangle + \Big\langle\Big\langle f(u) - \frac{\varepsilon}{2}D(u)u_x + \sigma(u)\Psi^{\varepsilon,u}_x, \varphi_x\Big\rangle\Big\rangle\Big] = 0\end{aligned} \tag{3.1}$$

In such a case $\Psi^{\varepsilon,u}$ is unique and

$$I_\varepsilon(u) = \frac{1}{2}\left\|u_t + f(u)_x - \frac{\varepsilon}{2}(D(u)u_x)_x\right\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} = \frac{1}{2}\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}} \tag{3.2}$$

Proof. Fix $\varepsilon > 0$ and $u \in \mathcal{U}$ such that $I_\varepsilon(u) < +\infty$. The functional ℓ^u_ε defined in (2.5) can be extended to a linear functional on $C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R})$ by setting

$$\begin{aligned}\ell^u_\varepsilon(\varphi) &= \langle u(T),\varphi(T)\rangle - \langle u(0),\varphi(0)\rangle - \langle\langle u,\varphi_t\rangle\rangle - \langle\langle f(u),\varphi_x\rangle\rangle\\&\quad + \frac{\varepsilon}{2}\langle\langle D(u)u_x,\varphi_x\rangle\rangle\end{aligned} \tag{3.3}$$

Since for any $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$ the map $[0,T] \ni t \mapsto \langle u(t), \varphi(t)\rangle \in \mathbb{R}$ is continuous, it is easily seen that

$$I_\varepsilon(u) = \sup_{\varphi\in C_c^\infty([0,T]\times\mathbb{R})} \left\{ \ell_\varepsilon^u(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\varphi_x, \varphi_x\rangle\rangle \right\}$$

We claim that ℓ_ε^u defines a bounded linear functional on $\mathcal{D}^1_{\sigma(u)}$. Indeed, since $I_\varepsilon(u) < +\infty$

$$\ell_\varepsilon^u(\varphi) \leqq I_\varepsilon(u) + \frac{1}{2}\langle\langle \sigma(u)\,\varphi_x, \varphi_x\rangle\rangle = I_\varepsilon(u) + \frac{1}{2}\|\varphi\|^2_{\mathcal{D}^1_{\sigma(u)}}$$

which shows that $\ell_\varepsilon^u(\varphi) = 0$ whenever $\langle\langle \sigma(u)\,\varphi_x, \varphi_x\rangle\rangle = 0$ (as $\ell_\varepsilon^u(\cdot)$ is 1-homogeneous), namely ℓ_ε^u is compatible with the identification in the definition of $D^1_{\sigma(u)}$ above. We also get that $\ell_\varepsilon^u(\varphi)$ is bounded by the $\mathcal{D}^1_{\sigma(u)}$-norm of φ (up to a multiplicative constant), and it can, therefore, be extended by compatibility and density to a continuous linear functional on $\mathcal{D}^1_{\sigma(u)}$. Still denoting by ℓ_ε^u such a functional, we get

$$I_\varepsilon(u) = \sup_{\varphi\in \mathcal{D}^1_{\sigma(u)}} \left\{ \ell_\varepsilon^u(\varphi) - \frac{1}{2}\langle\langle \sigma(u)\varphi_x, \varphi_x\rangle\rangle \right\} \tag{3.4}$$

which is equivalent to the first equality in (3.2). By Riesz representation theorem, we now get existence and uniqueness of $\Psi^{\varepsilon,u} \in \mathcal{D}^1_{\sigma(u)}$ such that $\ell_\varepsilon^u(\varphi) = (\Psi^{\varepsilon,u}, \varphi)_{\mathcal{D}^1_{\sigma(u)}}$ for any $\varphi \in \mathcal{D}^1_{\sigma(u)}$, which implies (3.1). Riesz representation also yields $I_\varepsilon(u) = \frac{1}{2}\|\Psi^{\varepsilon,u}\|^2_{\mathcal{D}^1_{\sigma(u)}}$. The converse statements are obvious. □

In the following lemma we give some regularity results for $u \in \mathcal{U}$ with finite cost, and we prove some a priori bounds.

Lemma 3.2. *Let $\varepsilon > 0$ and $u \in \mathcal{U}$ be such that $I_\varepsilon(u) < +\infty$. Then $u \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Moreover for each entropy–entropy flux pair (η, q), each $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$, and each $t \in [0,T]$*

$$\begin{aligned}
&\langle \eta(u(t)), \varphi(t)\rangle - \langle \eta(u(0)), \varphi(0)\rangle - \int_{[0,t]} \mathrm{d}s\; [\langle \eta(u), \varphi_s\rangle + \langle q(u), \varphi_x\rangle] \\
&\quad = -\frac{\varepsilon}{2}\int_{[0,t]} \mathrm{d}s\; \big[\langle \eta''(u)D(u)u_x, \varphi\, u_x\rangle + \langle \eta'(u)D(u)u_x, \varphi_x\rangle\big] \\
&\qquad + \int_{[0,t]} \mathrm{d}s\, \big[\langle \eta''(u)\sigma(u)\, u_x, \Psi_x^{\varepsilon,u}\,\varphi\rangle + \langle \eta'(u)\sigma(u)\,\Psi_x^{\varepsilon,u}, \varphi_x\rangle\big]
\end{aligned} \tag{3.5}$$

where $\Psi^{\varepsilon,u}$ is as in Lemma 3.1. Finally, there exists a constant $C > 0$ depending only on f, D and σ such that for any $\varepsilon,\ L > 0$

$$\varepsilon\int \mathrm{d}t \int_{[-L,L]} \mathrm{d}x\; u_x^2 \leqq C\left[\varepsilon^{-1} I_\varepsilon(u) + L + 1\right] \tag{3.6}$$

Proof. Recall that the linear functional ℓ_ε^u on $\mathcal{D}^1_{\sigma(u)}$ is defined as the extension of (3.3). Let $\theta := -f(u)+\frac{\varepsilon}{2}D(u)\,u_x - \sigma(u)\,\Psi_x^{\varepsilon,u} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$; by (3.1) $u_t = \theta_x$ holds weakly. Since $I_\varepsilon(u) < +\infty$ we also have $u_x \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$, so that $u \in C\big([0,T]; L_{2,\mathrm{loc}}(\mathbb{R})\big)$ by standard interpolations arguments, see, for example [12]. Since u is bounded, this is equivalent to the statement $u \in C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$.

This fact implies that integrations by parts are allowed in the first line on the right-hand side of (3.3), namely for each measurable compactly supported $\phi : [0,T]\times\mathbb{R} \to \mathbb{R}$ with $\phi_x \in L_2([0,T]\times\mathbb{R})$

$$\ell_\varepsilon^u(\phi) = \langle\langle u_t, \phi\rangle\rangle + \langle\langle f(u)_x, \phi\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u)u_x, \phi_x\rangle\rangle \tag{3.7}$$

where indeed we understand $\langle\langle u_t, \phi\rangle\rangle \equiv -\langle\langle \theta, \phi_x\rangle\rangle$. Since u_x is locally square integrable, if $\eta \in C^2([0,1])$ and $\varphi \in C_c^\infty([0,T]\times\mathbb{R})$, then $\eta'(u)\varphi$ has compact support and its weak x-derivative is square integrable. We can thus evaluate (3.7) with ϕ replaced by $\eta'(u)\varphi$; since $\ell_\varepsilon^u(\eta'(u)\varphi) = \big(\Psi^{\varepsilon,u}, \eta'(u)\varphi\big)_{\mathcal{D}^1_{\sigma(u)}}$ and $u \in C\big([0,T]; L_{2,\mathrm{loc}}(\mathbb{R})\big)$ we get (3.5).

To prove the last statement, consider an entropy–entropy flux pair (η, q). By (3.4) and (3.7)

$$\begin{aligned}
I_\varepsilon(u) &\geqq \ell_\varepsilon^u(\varepsilon\,\eta'(u)\,\varphi) - \frac{\varepsilon^2}{2}\Big\langle\Big\langle \big(\eta'(u)\varphi\big)_x, \sigma(u)\,\big(\eta'(u)\varphi\big)_x\Big\rangle\Big\rangle \\
&= \varepsilon\langle\eta(u(T)), \varphi(T)\rangle - \varepsilon\langle\eta(u(0)), \varphi(0)\rangle - \varepsilon\,[\langle\langle\eta(u), \varphi_t\rangle\rangle + \langle\langle q(u), \varphi_x\rangle\rangle] \\
&\quad + \frac{\varepsilon^2}{2}\Big[\langle\langle D(u)\eta''(u)u_x^2, \varphi\rangle\rangle + \langle\langle\eta'(u)D(u)u_x, \varphi_x\rangle\rangle \\
&\qquad - \langle\langle\sigma(u)\eta''(u)^2u_x^2, \varphi^2\rangle\rangle - \langle\langle\sigma(u)\eta'(u)^2\varphi_x, \varphi_x\rangle\rangle \\
&\qquad - 2\langle\langle\sigma(u)\eta''(u)\eta'(u)u_x, \varphi\,\varphi_x\rangle\rangle\Big]
\end{aligned}$$

We now choose $\eta \geqq 0$, uniformly convex and such that $\sigma\eta'' \leqq D$, and for such a η, we let $\alpha := \max_v\big[D(v)\eta'(v)^2/\eta''(v)\big]$, so that $\sigma\,(\eta')^2 \leqq \alpha$. By Cauchy–Schwarz inequality

$$\begin{aligned}
&2\left|\langle\langle\sigma(u)\eta''(u)\eta'(u)u_x, \varphi\,\varphi_x\rangle\rangle\right| \\
&\quad\leqq \langle\langle\sigma(u)\eta''(u)^2u_x^2, \varphi^2\rangle\rangle + \langle\langle\sigma(u)\eta'(u)^2, \varphi_x\varphi_x\rangle\rangle \\
&\quad\leqq \langle\langle D(u)\eta''(u)u_x^2, \varphi^2\rangle\rangle + \alpha\langle\langle\varphi_x, \varphi_x\rangle\rangle
\end{aligned}$$

Letting $\zeta : [0,1] \to \mathbb{R}$ be such that $\zeta' = \eta' D$, and integrating by parts we get $\langle\eta'(u)D(u)u_x, \varphi_x\rangle = -\langle\zeta(u), \varphi_{xx}\rangle$. Collecting all the bounds

$$\begin{aligned}
&\langle\eta(u(T)), \varphi(T)\rangle + \frac{\varepsilon}{2}\langle\langle D(u)\eta''(u)u_x^2, \varphi - 2\varphi^2\rangle\rangle \\
&\quad\leqq \varepsilon^{-1}I_\varepsilon(u) + \langle\eta(u(0)), \varphi(0)\rangle + \langle\langle\eta(u), \varphi_t\rangle\rangle + \langle\langle q(u), \varphi_x\rangle\rangle \\
&\qquad + \frac{\varepsilon}{2}\langle\langle\zeta(u), \varphi_{xx}\rangle\rangle + \varepsilon\,\alpha\,\langle\langle\varphi_x, \varphi_x\rangle\rangle.
\end{aligned}$$

We now choose φ independent of t and such that $\varphi(x) = 1/4$ for $|x| \leqq L$, $0 \leqq \varphi(x) \leqq 1/4$ for $L \leqq |x| \leqq L+1$, $\varphi(x) = 0$ for $|x| \geqq L+1$, and $\langle \varphi_x, \varphi_x \rangle + \langle \varphi_{xx}, \varphi_{xx} \rangle \leqq 2$. Since q, ζ are bounded and $\eta \geqq 0$, estimate (3.6) easily follows. □

Lemma 3.3. *The sequence of functionals $\{I_\varepsilon\}$ is equicoercive on $\mathcal{U}$.*

Proof. Let $u \in \mathcal{U}$ be such that $I_\varepsilon(u) < +\infty$ and $\Psi^{\varepsilon,u}$ be as in Lemma 3.1. By (3.1), (3.2) and the bound (3.6), for each $s, t \in [0, T]$, each $L > 0$, each $\varphi \in C_c^\infty(\mathbb{R})$ supported by $[-L, L]$

$$
\begin{aligned}
|\langle u(t) - u(s), \varphi \rangle| &= \left| \int_{[s,t]} \mathrm{d}r \left\langle f(u) - \frac{\varepsilon}{2} D(u) u_x + \sigma(u) \Psi_x^{\varepsilon,u}, \varphi_x \right\rangle \right| \\
&\leqq \left\{ 2 \int_{[s,t]\times[-L,L]} \mathrm{d}r\, \mathrm{d}x \left[f(u)^2 + \frac{\varepsilon^2}{4} D(u)^2 u_x^2 \right] \right\}^{1/2} [|t-s| \langle \varphi_x, \varphi_x \rangle]^{1/2} \\
&\quad + \left[\int_{[s,t]} \mathrm{d}r \langle \sigma(u) \Psi_x^{\varepsilon,u}, \Psi_x^{\varepsilon,u} \rangle \right]^{1/2} [|t-s| \langle \sigma(u) \varphi_x, \varphi_x \rangle]^{1/2} \\
&\leqq C\, [1 + L + I_\varepsilon(u)]^{1/2}\, |t-s|^{1/2} \langle \varphi_x, \varphi_x \rangle^{1/2}
\end{aligned}
$$

for a suitable constant C depending only on f, D, and σ. Since (U, d_U) is compact, see (2.1), recalling (2.2) and the Ascoli–Arzelá theorem, the equicoercivity of $\{I_\varepsilon\}$ on $\mathcal{U}$ follows. □

As mentioned in the introduction, the assumption that σ is supported by $[0, 1]$ allows us to consider only functions u that take values in $[0, 1]$. More precisely, consider a cost functional $\hat{I}_\varepsilon$ analogous to I_ε but defined on $L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$. We next prove that, if $u \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ is such that $\hat{I}_\varepsilon(u) < +\infty$ and satisfies some growth conditions, then u takes values in $[0, 1]$.

Proposition 3.4. *Let f, D, $\sigma : \mathbb{R} \to \mathbb{R}$; assume f Lipschitz, σ and D continuous and bounded, with $\sigma \geqq 0$ and D uniformly positive. Let $\hat{I}_\varepsilon : L_{1,\mathrm{loc}}([0, T] \times \mathbb{R}) \to [0, +\infty]$ be defined as follows. If $f(u) \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$, we define $\hat{I}_\varepsilon(u)$ as in (2.6), and we set $\hat{I}_\varepsilon(u) = +\infty$ otherwise. Suppose that $u \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ is such that $\hat{I}_\varepsilon(u) < +\infty$. Then $u \in C\left([0, T]; L_{1,\mathrm{loc}}(\mathbb{R})\right)$. Moreover, if σ is supported by $[0, 1]$, and u is such that $u(0) \in U$ and $\int \mathrm{d}t\, \mathrm{d}x\, |u(t, x)| e^{-r|x|} < +\infty$ for some $r > 0$, then u takes values in $[0, 1]$; hence $u \in \mathcal{U}$.*

Proof. Let $u \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ be such that $\hat{I}_\varepsilon(u) < +\infty$. By the same arguments of Lemma 3.2, since $f(u) \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$, $u_t = \theta_x$ holds weakly for some $\theta \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$. Hence, as in Lemma 3.2, $u \in C\left([0, T]; L_{1,\mathrm{loc}}(\mathbb{R})\right)$. Suppose now that σ is supported by $[0, 1]$. Pick a sequence of strictly convex, strictly positive entropies $\eta_n \in C^2(\mathbb{R})$ such that: $|\eta_n'(u)|$, $\eta_n''(u) \leqq C_n$ for some $C_n > 0$; for $u \in (0, 1)$, $\eta_n(u)$ does not depend on n and satisfies $0 < c \leqq \eta_n''(u) \leqq D(u)/\sigma(u)$; η_n is decreasing for $u < 0$ and increasing for $u > 1$; for $u \notin [0, 1]$ the sequence $\{\eta_n(u)\}$ increases pointwise to $+\infty$ as $n \to \infty$. Still following the proof of Lemma 3.2, for $t \in [0, T]$ and $\varphi \in C_c^\infty(\mathbb{R})$

$$
\begin{aligned}
&\langle \eta_n(u(t)), \varphi \rangle + \frac{\varepsilon}{2} \int_{[0,t]} \mathrm{d}s \, \langle D(u) \eta_n''(u) u_x^2, \varphi - 2\varphi^2 \rangle \leqq \varepsilon^{-1} \hat{I}_\varepsilon(u) \\
&+ \langle \eta_n(u(0)), \varphi \rangle + \int_{[0,t]} \mathrm{d}s \left[\langle q_n(u), \varphi_x \rangle + \frac{\varepsilon}{2} \langle \zeta_n(u), \varphi_{xx} \rangle + \varepsilon \, \alpha \, \langle \varphi_x, \varphi_x \rangle \right]
\end{aligned}
$$

where q_n and ζ_n are defined (up to a constant) by $q_n(v) = \int^v \mathrm{d}w \, \eta_n'(w) \, f'(w)$ and $\zeta_n' = \eta_n' \, D$, and $\alpha := \max_{u \in [0,1]} D(u) \eta_n'(u)^2 / \eta_n''(u)$ is a constant independent of n, since σ is supported by $[0, 1]$. Since f is Lipschitz and D is bounded, it is possible to choose the arbitrary constants in the definition of q_n and ζ_n such that $|q_n|$, $|\zeta_n| \leqq C \eta_n$ for some constant $C > 0$ independent of n. In particular $\zeta_n, \; q_n \in L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$; for each $\varphi \in C_c^\infty(\mathbb{R})$ such that $0 \leqq \varphi(x) \leqq 1/2$

$$
\begin{aligned}
&\langle\langle \eta_n(u), \varphi \rangle\rangle \leqq T \, \varepsilon^{-1} \hat{I}_\varepsilon(u) + T \, \langle \eta_n(u(0)), \varphi \rangle \\
&\quad + \int_{[0,T]} \mathrm{d}t \int_{[0,t]} \mathrm{d}s \, \left[\langle q_n(u), \varphi_x \rangle + \frac{\varepsilon}{2} \langle \zeta_n(u), \varphi_{xx} \rangle + \varepsilon \, \alpha \, \langle \varphi_x, \varphi_x \rangle \right]
\end{aligned}
$$

Let now r be such that $\int \mathrm{d}t \, \mathrm{d}x \, e^{-r|x|} |u(t, x)| < +\infty$. By a limiting procedure, the above bound holds for any $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leqq \varphi \leqq 1/2$ and $\sup_{x \in \mathbb{R}} e^{r|x|} [|\varphi(x)| + |\varphi_x(x)| + |\varphi_{xx}(x)|] < +\infty$. For such φ, by the choice of $q_n, \; \zeta_n$

$$
\begin{aligned}
&\frac{1}{T} \langle\langle \eta_n(u), \varphi \rangle\rangle \leqq \varepsilon^{-1} \hat{I}_\varepsilon(u) + \langle \eta_n(u(0)), \varphi \rangle \\
&\qquad\qquad + \varepsilon \frac{\alpha T}{2} \langle \varphi_x, \varphi_x \rangle + C \langle\langle \eta_n(u), |\varphi_x| + \frac{\varepsilon}{2} |\varphi_{xx}| \rangle\rangle
\end{aligned}
$$

It is easy to verify that, given $L > 0$ large enough, we can choose φ such that $\varphi(x) = 1/2$ for $|x| \leqq L$, $\varphi(x) = \frac{1}{2} e^{-r|x-L|}$ for $|x| > 2L$ and $|\varphi_{xx}(x)| \leqq r |\varphi_x(x)| \leqq r^2 \, \varphi(x) \leqq r^2/2$ for $|x| > L$. Moreover, with no loss of generality, we can assume that $\frac{1}{T} - C \, (r + \frac{\varepsilon}{2} r^2) > 0$, otherwise we can suppose T small enough and iterate this proof. Therefore

$$
\begin{aligned}
&\left[\frac{1}{T} - C \left(r + \frac{\varepsilon}{2} r^2 \right) \right] \int_{[0,T] \times [-L,L]} \mathrm{d}t \, \mathrm{d}x \, \eta_n(u) \\
&\qquad \leqq \varepsilon^{-1} \hat{I}_\varepsilon(u) + \langle \eta_n(u(0)), \varphi \rangle + \varepsilon \frac{\alpha T}{2} \langle \varphi_x, \varphi_x \rangle
\end{aligned}
$$

If $u(0) \in U$ the right-hand side of this formula is finite and independent of n, and therefore the left-hand side is bounded uniformly in n. Taking the limit $n \to \infty$, by the choice of η_n necessarily $u(t, x) \in [0, 1]$ for almost every $(t, x) \in [0, T] \times \mathbb{R}$. $\square$

The following result is not used in the sequel, but together with Lemma 3.1 and Proposition 3.4, motivates the choice of I_ε as the cost functional related to (1.2).

Proposition 3.5. *For each $\varepsilon > 0$ the functional $I_\varepsilon : \mathcal{U} \to [0, +\infty]$ is lower semicontinuous.*

Proof. Let $\{u^n\} \subset \mathcal{U}$ be a sequence converging to u in $\mathcal{U}$, and such that $I_\varepsilon(u^n)$ is bounded uniformly in n. By (3.6), for each $L > 0$ we have that $\int_{[0,T]\times[-L,L]} \mathrm{d}t\, \mathrm{d}x\, (u^n_x)^2$ is also bounded uniformly in n. Therefore, recalling definition (2.6), the lower semicontinuity of I_ε is established once we show that u^n converges to u strongly in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$. Fix $L > 0$ and pick $\chi_L \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $0 \leqq \chi_L \leqq 1$ with $\chi_L(x) = 1$ for $x \in [-L, L]$. We show that $u^{n,L} := u^n\, \chi_L$ converges to $u^L := u\, \chi_L$ in $L_2([0,T]\times\mathbb{R})$. Choose a sequence of mollifiers $\jmath_k : \mathbb{R} \to \mathbb{R}^+$ with $\int \mathrm{d}x\, \jmath_k(x) = 1$, then

$$
\begin{aligned}
\left\| u^{n,L} - u^L \right\|_{L_2([0,T]\times\mathbb{R})} &\leqq \left\| u^{n,L} - \jmath_k * u^{n,L} \right\|_{L_2([0,T]\times\mathbb{R})} \\
&+ \left\| \jmath_k * u^{n,L} - \jmath_k * u^L \right\|_{L_2([0,T]\times\mathbb{R})} + \left\| \jmath_k * u^L - u^L \right\|_{L_2([0,T]\times\mathbb{R})}
\end{aligned}
$$

where the convolution is only in the space variable. For each k the second term on the right-hand side above vanishes as $n \to \infty$ by the convergence $u^n \to u$ in $\mathcal{U}$. Since the third term vanishes as $k \to \infty$ it remains to show that the first one vanishes as $k \to \infty$ uniformly in n. Integration by parts and Young inequality for convolutions yield

$$
\begin{aligned}
&\left\| u^{n,L} - \jmath_k * u^{n,L} \right\|_{L_2([0,T]\times\mathbb{R})} \\
&\quad \leqq \left\| 1\!\!\mathrm{I}_{[0,+\infty)} - \int_{-\infty}^{\cdot} \mathrm{d}y\, \jmath_k(y) \right\|_{L_1(\mathbb{R})} \left\| u^{n,L}_x \right\|_{L_2([0,T]\times\mathbb{R})}
\end{aligned}
$$

The uniform boundedness of $\int_{[0,T]\times[-L,L]} \mathrm{d}t\, \mathrm{d}x\, (u^n_x)^2$, (3.6) and the choice of χ_L imply that the second term on the right-hand side is bounded uniformly in n, while the first term vanishes as $k \to \infty$. $\square$

4. Γ-convergence of $\mathcal{I}_\varepsilon$

In this section we prove the Γ-convergence of the parabolic cost functional $\mathcal{I}_\varepsilon$ as $\varepsilon \to 0$, see Theorem 2.1. Some technical steps are postponed in Appendix A.

Proof of Theorem 2.1: equicoercivity of $\mathcal{I}_\varepsilon$. Recall that $(\mathcal{M}, d_{\mathcal{M}})$ has been defined in (2.3), (2.4) and note that $(\mathcal{N}, d_{\mathrm{w}})$ is compact. By Lemma 3.3, for each $C > 0$ there exists a compact $K_C \subset \mathcal{U}$, such that for any ε small enough $\{\mu \in \mathcal{M} : \mathcal{I}_\varepsilon(\mu) \leqq C\} \subset \{\mu \in \mathcal{M} : \mu_{t,x} = \delta_{u(t,x)}$for some $u \in K_C\} =: \mathcal{K}_C$. In order to prove that $\mathcal{K}_C$ is compact in $(\mathcal{M}, d_{\mathcal{M}})$, consider a sequence $\{\mu^n = \delta_{u^n}\} \subset \mathcal{K}_C$. Then there exists a subsequence $\{\mu^{n_j}\}$ such that, for some $\mu \in \mathcal{N}$ and $u \in \mathcal{U}$, $\mu^{n_j} \to \mu$ in $(\mathcal{N}, d_{\mathrm{w}})$, and $\mu^{n_j}(\imath) = u^{n_j} \to u$ in $\mathcal{U}$, hence $\mu(\imath) = u$. Therefore $\mu \in \mathcal{M}$ and $\mu^{n_j} \to \mu$ in $(\mathcal{M}, d_{\mathcal{M}})$. $\square$

Proof of Theorem 2.1: Γ-liminf inequality. Let $\{\mu^\varepsilon\} \subset \mathcal{M}$ be a sequence converging to μ in $\mathcal{M}$. In order to prove $\underline{\lim}_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\mu^\varepsilon) \geqq \mathcal{I}(\mu)$, it is not restrictive to assume $\mathcal{I}_\varepsilon(\mu^\varepsilon) < +\infty$, and therefore $\mu^\varepsilon_{t,x} = \delta_{u^\varepsilon(t,x)}$ for some $u^\varepsilon \in \mathcal{U}$. For each $\varphi \in C^\infty_{\mathrm{c}}((0,T)\times\mathbb{R})$, recalling definition (2.6)

$$\begin{aligned}\mathcal{I}_\varepsilon(\mu^\varepsilon) &\geqq \ell^{u^\varepsilon}_\varepsilon(\varphi) - \frac{1}{2}\|\varphi\|^2_{\mathcal{D}^1_{\sigma(u^\varepsilon)}}\\ &= -\langle\langle\mu^\varepsilon(\imath),\varphi_t\rangle\rangle - \langle\langle\mu^\varepsilon(f),\varphi_x\rangle\rangle - \frac{1}{2}\langle\langle\mu^\varepsilon(\sigma)\varphi_x,\varphi_x\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x,\varphi_x\rangle\rangle\end{aligned}$$

Let $d \in C^1([0,1])$ be such that $d'(u) = D(u)$. Then $D(u^\varepsilon)u^\varepsilon_x = d(u^\varepsilon)_x$, and an integration by parts shows that the last term on the right-hand side of the previous formula vanishes as $\varepsilon \to 0$. Hence

$$\varliminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\mu^\varepsilon) \geqq -\langle\langle\mu(\imath),\varphi_t\rangle\rangle - \langle\langle\mu(f),\varphi_x\rangle\rangle - \frac{1}{2}\langle\langle\mu(\sigma)\varphi_x,\varphi_x\rangle\rangle$$

By optimizing over $\varphi \in C^\infty_{\mathrm{c}}((0,T)\times\mathbb{R})$ the Γ-liminf inequality follows. □

Proof of Theorem 2.1: Γ-limsup inequality. Let

$$\mathcal{M}_g := \Big\{\mu \in \mathcal{M} \,:\, \mathcal{I}(\mu) < +\infty,\ \exists r,\ L > 0,\ \exists \mu_\infty \in \mathcal{P}([0,1]) \text{ such that}$$
$$\mu(\imath),\ \mu(\sigma) \geqq r,\ \mu_{t,x} = \mu_\infty \text{ for } |x| > L\Big\} \tag{4.1}$$

$$\mathcal{M}_0 := \Big\{\mu \in \mathcal{M}_g \,:\, \mu = \delta_u \text{ for some } u \in C^1([0,T]\times\mathbb{R};[0,1])\Big\} \tag{4.2}$$

and define $\widetilde{\mathcal{I}} : \mathcal{M} \to [0,+\infty]$ by

$$\widetilde{\mathcal{I}}(\mu) := \begin{cases}\mathcal{I}(\mu) & \text{if } \mu \in \mathcal{M}_0\\ +\infty & \text{otherwise}\end{cases} \tag{4.3}$$

We claim that for $\mu \in \mathcal{M}_0$, a recovery sequence is simply given by $\mu^\varepsilon = \mu$. Indeed, if $\mu = \delta_u$ for some $u \in C^1([0,T]\times\mathbb{R};[0,1])$, we have

$$\begin{aligned}\mathcal{I}_\varepsilon(\mu^\varepsilon) = I_\varepsilon(u) &= \tfrac{1}{2}\left\|u_t + f(u)_x - \tfrac{\varepsilon}{2}(D(u)u_x)_x\right\|^2_{\mathcal{D}^{-1}_{\sigma(u)}}\\ &\leqq \tfrac{1+\varepsilon}{2}\|u_t + f(u)_x\|^2_{\mathcal{D}^{-1}_{\sigma(u)}} + \tfrac{1+\varepsilon^{-1}}{2}\left\|\tfrac{\varepsilon}{2}(D(u)u_x)_x\right\|^2_{\mathcal{D}^{-1}_{\sigma(u)}}\end{aligned}$$

As $\mu \in \mathcal{M}_g$, u is constant for $|x|$ large enough, in particular $u_x \in L_2([0,T]\times\mathbb{R})$. Since we have also $\sigma(u) \geqq r > 0$, the last term in the above formula vanishes as $\varepsilon \to 0$. Hence Γ-$\overline{\lim}_\varepsilon \mathcal{I}_\varepsilon \leqq \widetilde{\mathcal{I}}$. As well known, see, for example [3, Prop. 1.28], any Γ-limsup is lower semicontinuous; the proof is then completed by Theorem 4.1 below. □

The relaxation of the functional $\widetilde{\mathcal{I}}$ on $\mathcal{M}$ defined in (4.3) might have an independent interest; in the following result we show it coincides with $\mathcal{I}$, as defined in (2.8).

Theorem 4.1. *$\mathcal{I}$ is the lower semicontinuous envelope of $\widetilde{\mathcal{I}}$.*

The following representation of $\mathcal{I}$ is proven similarly to Lemma 3.1.

Lemma 4.2. *Let $\mu \in \mathcal{M}$. Then $\mathcal{I}(\mu) < +\infty$ iff there exists $\Psi^\mu \in \mathcal{D}^1_{\mu(\sigma)}$ such that μ is a measure-valued solution to $u_t + f(u)_x = -\left(\sigma(u)\Psi^\mu_x\right)_x$, namely*

$$\mu(\imath)_t + \mu(f)_x = -\left(\mu(\sigma)\Psi^\mu_x\right)_x \tag{4.4}$$

holds weakly. In such a case Ψ^μ is unique and

$$\mathcal{I}(\mu) = \frac{1}{2}\,\|\mu(\imath)_t + \mu(f)_x\|^2_{\mathcal{D}^{-1}_{\mu(\sigma)}} = \frac{1}{2}\|\Psi^\mu\|^2_{\mathcal{D}^1_{\mu(\sigma)}}$$

Furthermore, suppose that $\mu(\sigma) \geqq r$ for some constant $r > 0$. Then $\mathcal{I}(\mu) < +\infty$ iff there exists $G^\mu \in L_2([0,T]\times\mathbb{R})$ such that weakly

$$\mu(\imath)_t + \mu(f)_x = -G^\mu_x \tag{4.5}$$

In such a case Ψ^μ_x can be identified with a function in $L_2([0,T]\times\mathbb{R})$, and

$$G^\mu = \mu(\sigma)\Psi^\mu_x, \quad \mathcal{I}(\mu) = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}x\,\frac{(G^\mu(t,x))^2}{\mu_{t,x}(\sigma)} \tag{4.6}$$

The following remark is a consequence of Lemma 4.2.

Remark 4.3. Let $\{\mu^k\} \subset \mathcal{M}$ be such that $\mu^k \to \mu$ in $\mathcal{M}$, $\mathcal{I}(\mu^k) < +\infty$ and $\mu^k(\sigma) \geqq r$ for some $r > 0$. Let also G^{μ^k} be defined as in Lemma 4.2. If $\mu^k(\sigma) \to \mu(\sigma)$ strongly in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ and $\{G^{\mu^k}\}$ is strongly precompact in $L_2([0,T]\times\mathbb{R})$, then $\mathcal{I}(\mu^k) \to \mathcal{I}(\mu)$.

Throughout the proof of Theorem 4.1, approximation of Young measures by piecewise smooth measures is a much used procedure. In particular we will refer repeatedly to the following result, which is a simple restatement of the Rankine–Hugoniot condition for the divergence-free vector field $(\mu(\imath),\ \mu(f) + G^\mu)$ on $(0,T)\times\mathbb{R}$.

Lemma 4.4. *Let $\gamma : (0,T) \to \mathbb{R}$ be a Lipschitz map with almost everywhere derivative $\dot\gamma$, and let $O^\mp \subset (0,T)\times\mathbb{R}$ be a left, respectively a right, open neighborhood of the graph of γ; namely* $\mathrm{Graph}(\gamma) \subset \mathrm{Closure}(O^-) \cap \mathrm{Closure}(O^+)$*, and for all $(t,x) \in O^-$, respectively $(t,x) \in O^+$, the inequality $x < \gamma(t)$, respectively $x > \gamma(t)$, holds. Let also $O := O^+ \cup O^- \cup$* $\mathrm{Graph}(\gamma)$. *Suppose that a Young measure $\mu \in \mathcal{M}$ is such that, for each continuous function $F \in C([0,1])$ the map $(t,x) \mapsto \mu_{t,x}(F)$ is continuously differentiable in $O^- \cup O^+$, and such that there exist the respective traces $\mu^\mp(F)$ of $\mu(F)$ on the graph of γ. Then there exists a map $G : O \to \mathbb{R}$, defined up to an additive measurable function of the t variable, which is continuous in $O^- \cup O^+$, admits traces $G^\mp$ on the graph of γ, and is such that (4.5) holds weakly in O. Moreover the Rankine–Hugoniot condition holds for almost every $t \in [0,T]$, namely*

$$G^+ - G^- = \left[\mu(\imath)^+ - \mu(\imath)^-\right]\dot\gamma - \left[\mu(f)^+ - \mu(f)^-\right] \tag{4.7}$$

Proof of Theorem 4.1. Since $\mathcal{I}$ is lower semicontinuous, it is enough to prove that $\mathcal{M}_0$, as defined in (4.2), is $\mathcal{I}$-dense in $\mathcal{M}$, namely that for each $\mu \in \mathcal{M}$ with $\mathcal{I}(\mu) < +\infty$, there exists a sequence $\{\mu^k\} \subset \mathcal{M}_0$ such that $\mu^k \to \mu$ in $\mathcal{M}$ and $\overline{\lim}_k \mathcal{I}(\mu^k) \leqq \mathcal{I}(\mu)$ (we will also say that μ^k $\mathcal{I}$-converges to μ). We split the proof in several steps.
Step 1. Here we show that $\mathcal{M}_0$ is $\mathcal{I}$-dense in the set of Young measures which are a finite convex combination of Dirac masses for almost every (t, x). More precisely, recalling definition (4.1), we set

$$\mathcal{M}_1^n := \Big\{\mu \in \mathcal{M}_g \;:\; \mu = \textstyle\sum_{i=1}^n \alpha^i \delta_{u^i} \text{ for some } \alpha^i \in L_\infty\left([0,T]\times\mathbb{R};[0,1]\right) \\ \text{with } \textstyle\sum_{i=1}^n \alpha^i = 1 \text{ and } u^i \in L_\infty\left([0,T]\times\mathbb{R};[0,1]\right)\Big\}$$

and

$$\mathcal{M}_1 := \bigcup_{n=1}^{\infty} \mathcal{M}_1^n$$

In this step, we prove that $\mathcal{M}_0$ is $\mathcal{I}$-dense in $\mathcal{M}_1$. We proceed by induction on n; to this aim, for $n \geqq 1$, we introduce the auxiliary sets

$$\overline{\mathcal{M}}_1^n := \Big\{\mu \in \mathcal{M}_g \;:\; \exists r > 0 \text{ such that } \mu = \textstyle\sum_{i=1}^n \alpha^i \delta_{u^i}, \\ \text{for some } \alpha^i \in L_\infty\left([0,T]\times\mathbb{R};[r,1]\right) \text{ with } \textstyle\sum_{i=1}^n \alpha^i = 1 \\ \text{and } u^i \in C^0\left([0,T]\times\mathbb{R};[0,1]\right)\Big\}$$

$$\widetilde{\mathcal{M}}_1^n := \Big\{\mu \in \mathcal{M}_g \;:\; \exists r > 0 \text{ such that } \mu = \textstyle\sum_{i=1}^n \alpha^i \delta_{u^i} \\ \text{for some } \alpha^i \in C^1\left([0,T]\times\mathbb{R};[r,1]\right) \text{ with } \textstyle\sum_{i=1}^n \alpha^i = 1 \\ \text{and } u^i \in C^1\left([0,T]\times\mathbb{R};[r,1-r]\right) \text{ with } u^{i+1} \geqq u^i + r\Big\}$$

Note that $\widetilde{\mathcal{M}}_1^n \subset \overline{\mathcal{M}}_1^n \subset \mathcal{M}_1^n$, and $\widetilde{\mathcal{M}}_1^1 \subset \mathcal{M}_0$. We claim that for each $n \geqq 1$, $\widetilde{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\overline{\mathcal{M}}_1^n$, that $\overline{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\mathcal{M}_1^n$, and that $\mathcal{M}_1^n$ is $\mathcal{I}$-dense in $\widetilde{\mathcal{M}}_1^{n+1}$. The $\mathcal{I}$-density of $\mathcal{M}_0$ in $\mathcal{M}_1$ then follows by induction. The previous claims are proven in Appendix A.
Step 2. In this step we prove that $\mathcal{M}_1$ is $\mathcal{I}$-dense in $\mathcal{M}_g$, see (4.1). We use the following elementary extension of the mean value theorem.

Lemma 4.5. *Let X be a connected compact separable metric space, $F_1, \ldots, F_d \in C(X)$ be continuous functions on X, and $\mathbb{P} \in \mathcal{P}(X)$ be a Borel probability measure on X. Then there exist $\alpha^1, \ldots, \alpha^d \geqq 0$ with $\sum_i \alpha^i = 1$, $x^1, \ldots, x^d \in X$ such that $\mathbb{P}(F^i) = \sum_{j=1}^d \alpha^j F^i(x^j)$, $i = 1, \ldots, d$. Furthermore there exists a sequence $\{\mathbb{P}^n\} \subset \mathcal{P}(X)$ converging weakly* to $\mathbb{P}$, such that each $\mathbb{P}^n$ is a finite convex combination of Dirac masses, $\mathbb{P}^n(F^i) = \mathbb{P}(F^i)$ for $i = 1, \ldots, d$, and for each n the map $\mathcal{P}(X) \ni \mathbb{P} \mapsto \mathbb{P}^n \in \mathcal{P}(X)$ is Borel measurable with respect to the weak* topology.*

Proof. It is easy to see that the point $\mathbb{P}(F) := (\mathbb{P}(F_1), \dots, \mathbb{P}(F_d)) \in \mathbb{R}^d$ belongs to the closed convex hull of the set $B := \{(F_1(x), \dots, F_d(x)), x \in X\} \subset \mathbb{R}^d$. Since B is compact and connected, the Caratheodory theorem implies that $\mathbb{P}(F)$ is a convex combination of at most d points in B, namely the first statement of the lemma holds. Since X is compact, for each integer $n \geqq 1$, there exist an integer $k = k(n)$ and pairwise disjoint measurable sets $A_1^n, \dots, A_k^n \subset X$, such that $\mathbb{P}(X \setminus \cup_{l=1}^k A_l^n) = 0$, $\mathbb{P}(A_l^n) > 0$, and diameter$(A_l^n) \leqq n^{-1}$, $l = 1, \dots, k$. For $l = 1, \dots, k$, let $\mathbb{P}(\cdot|A_l^n) \in \mathcal{P}(X)$ be defined by $\mathbb{P}(B|A_l^n) := \mathbb{P}(A_l^n \cap B)/\mathbb{P}(A_l^n)$ for any Borel set $B \subset X$. By the first part of the lemma, there exists a probability measure $\mathbb{P}_l^n \in \mathcal{P}(X)$, which is a convex combination of d Dirac masses, such that $\mathbb{P}_l^n(F_i) = \mathbb{P}(F_i|A_l^n)$. The sequence $\{\mathbb{P}^n\}$ defined as $\mathbb{P}^n(\cdot) := \sum_{l=1}^k \mathbb{P}(A_l^n)\mathbb{P}_l^n(\cdot)$ satisfies the requirements of the lemma. $\square$

Let $\mu \in \mathcal{M}_g$. By Lemma 4.5, there exists a sequence $\{\mu^n\} \subset \mathcal{M}$ converging to μ in $\mathcal{M}$ such that $\mu_{t,x}$ is a convex combination of Dirac masses (t, x) for almost every (t, x), and $\mu^n(\imath) = \mu(\imath)$, $\mu^n(f) = \mu(f)$, $\mu^n(\sigma) = \mu(\sigma)$. Hence $\mathcal{I}(\mu^n) = \mathcal{I}(\mu)$ and $\mu^n \in \mathcal{M}_1$.

Step 3. Recall Lemma 4.2 and set

$$\begin{aligned}\mathcal{M}_3 := \Big\{\mu \in \mathcal{M} \,:\, \mathcal{I}(\mu) < +\infty,\ \exists r > 0 \text{ such that } \mu(\imath),\ \mu(\sigma) \geqq r,\\ G^\mu \in C^1([0, T] \times \mathbb{R}) \cap L_\infty([0, T] \times \mathbb{R}),\\ \text{for each } F \in C([0, 1])\ \ \mu(F) \in C^1([0, T] \times \mathbb{R})\Big\}\end{aligned}$$

In this step we prove that $\mathcal{M}_g$ is $\mathcal{I}$-dense in $\mathcal{M}_3$.

Let $\mu \in \mathcal{M}_3$, and choose a constant $u_\infty > 0$ such that $\mu(\imath) - u_\infty > \delta$ for some $\delta > 0$. Define the maps $\gamma_\pm^k \in C([0, T]) \cap C^1((0, T))$ as the solutions to the Cauchy problems

$$\begin{cases} \dot{\gamma}(t) = \dfrac{G^\mu(t, \gamma(t)) + \mu_{t,\gamma(t)}(f) - f(u_\infty)}{\mu_{t,\gamma(t)}(\imath) - u_\infty} \\ \gamma(0) = \pm k \end{cases}$$

$\gamma_\pm^k$ are well defined by the smoothness hypotheses on μ and G^μ. On the other hand, since we assumed G^μ to be uniformly bounded, $|\gamma_\pm^k(t) \mp k| \leqq C$, for some constant $C > 0$ not depending on k. We define, for $k > C$, μ^k by $\mu_{t,x}^k = \mu_{t,x}$ if $\gamma_-^k(t) < x < \gamma_+^k(t)$ and $\mu_{t,x}^k = \delta_{u_\infty}$ otherwise. Clearly $\mu^k \to \mu$ in $\mathcal{M}$ as $k \to \infty$. We also let $G^{\mu^k}(t, x) = G^\mu(t, x)$ if $\gamma_-^k(t) < x < \gamma_+^k(t)$, and $G^{\mu^k}(t, x) = 0$ otherwise. By (4.7) and the definition of $\gamma_\pm^k$, the equation $\mu^k(\imath)_t + \mu^k(f)_x = -G_x^{\mu^k}$ holds weakly in $(0, T) \times \mathbb{R}$. In particular, by Lemma 4.2, $\mathcal{I}(\mu^k) \leqq \mathcal{I}(\mu)$.

Step 4. Here we prove that $\mathcal{M}_3$ is $\mathcal{I}$-dense in

$$\mathcal{M}_4 := \{\mu \in \mathcal{M} \,:\, I(\mu) < +\infty,\ \exists r > 0 \text{ such that } \mu(\imath),\ \mu(\sigma) \geqq r\}$$

Let $\mu \in \mathcal{M}_4$ and $\{\jmath^k\}_{k \geq 1} \subset C_c^\infty(\mathbb{R} \times \mathbb{R})$ be a sequence of smooth mollifiers supported by $[-T/k, T/k] \times [-1, 1]$. For $k \geqq 1$, let us define the rescaled time-space variables $b^k : [0, T] \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$b^k(t, x) := \left(\frac{t + T/k}{1 + 2T/k}, \frac{x}{1 + 2T/k} \right) \tag{4.8}$$

For $k \geqq 1$ we also define the Young measure μ^k by setting for $F \in C([0, 1])$ and $(t, x) \in [0, T] \times \mathbb{R}$

$$\mu^k_{t,x}(F) := \int \mathrm{d}y \, \mathrm{d}s \; \jmath^k(t - s, x - y)\mu_{b^k(s,y)}(F)$$

It is immediate to see that $\mu^k \in \mathcal{M}_3$. Moreover, as $k \to \infty$, $\mu^k \to \mu$ in $\mathcal{M}$ and $\mu^k(F) \to \mu(F)$ strongly in $L_{1,\mathrm{loc}}([0, T] \times \mathbb{R})$ for each $F \in C([0, 1])$.

Let us also define $G^{\mu^k} \in L_2([0, T] \times \mathbb{R})$ by

$$G^{\mu^k}_{t,x} := \int \mathrm{d}y \, \mathrm{d}s \; \jmath^k(t - s, x - y) G^{\mu}\left(b^k(s, y)\right)$$

Then $\mu^k(\imath)_t + \mu^k(f)_x = -G^{\mu^k}_x$ holds weakly, and $G^{\mu^k} \to G^{\mu}$ in $L_2([0, T] \times \mathbb{R})$ as $k \to \infty$. The proof is then achieved by Remark 4.3.

Step 5. $\mathcal{M}_4$ is $\mathcal{I}$-dense in $\mathcal{M}$. For $\mu \in \mathcal{M}$ with $\mathcal{I}(\mu) < +\infty$, we define $\mu^k := (1 - k^{-1})\mu + k^{-1}\delta_{1/2}$. Clearly $\mu^k \to \mu$ in $\mathcal{M}$, and $\mu^k(\imath) \geqq k^{-1}/2$, $\mu^k(\sigma) \geqq k^{-1}\sigma(1/2)$. Therefore $\mu^k \in \mathcal{M}_4$. From (2.8) it follows that $\mathcal{I}$ is convex, and since $\mathcal{I}(\delta_{1/2}) = 0$, we have $\mathcal{I}(\mu^k) \leqq (1 - k^{-1})\mathcal{I}(\mu)$. $\square$

The following proposition is easily proven, and will be used in the proof of Corollary 2.2.

Proposition 4.6. *Let X, Y be complete separable metrizable spaces, and let $\omega : X \to Y$ be continuous. Let also $\{\mathcal{F}_\varepsilon\}$ be a family of functionals $\mathcal{F}_\varepsilon : X \to [-\infty, +\infty]$. Let us define $F_\varepsilon : Y \to [-\infty, +\infty]$ by*

$$F_\varepsilon(y) = \inf_{x \in \omega^{-1}(y)} \mathcal{F}_\varepsilon(x)$$

Then

$$\left(\Gamma\text{–}\overline{\lim_{\varepsilon \to 0}}\, F_\varepsilon \right)(y) \leqq \inf_{x \in \omega^{-1}(y)} \left(\Gamma\text{–}\overline{\lim_{\varepsilon \to 0}}\, \mathcal{F}_\varepsilon \right)(x)$$

Furthermore if $\{\mathcal{F}_\varepsilon\}$ is equicoercive on X then $\{F_\varepsilon\}$ is equicoercive on Y. In such a case

$$\left(\Gamma\text{–}\underline{\lim_{\varepsilon \to 0}}\, F_\varepsilon \right)(y) \geqq \inf_{x \in \omega^{-1}(y)} \left(\Gamma\text{–}\underline{\lim_{\varepsilon \to 0}}\, \mathcal{F}_\varepsilon \right)(x)$$

Proof of Corollary 2.2. Since the map $\mathcal{M} \ni \mu \mapsto \mu(\imath) \in \mathcal{U}$ is continuous, by Proposition 4.6 we have that I_ε is equicoercive on $\mathcal{U}$ (which we already knew from Lemma 3.3) and Γ-converges to $I : \mathcal{U} \to [0, +\infty]$ defined by

$$I(u) = \inf_{\mu \in \mathcal{M} \,:\, \mu(\imath) = u} \mathcal{I}(\mu)$$

Recall that, if $\mathcal{I}(\mu) < +\infty$, Ψ_x^μ has been defined in Lemma 4.2. Equality (4.4) yields

$$I(u) = \inf\big\{\langle\langle \mu(\sigma)\Psi_x^\mu, \Psi_x^\mu\rangle\rangle,\ \Phi \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R}),\ \mu\in\mathcal{M}:\\ \mathcal{I}(\mu) < +\infty,\ \mu(\imath) = u,\ \Phi_x = -\mu(\imath)_t \text{ weakly},\ \mu(\sigma)\Psi_x^\mu = \Phi - \mu(f)\big\}$$

The corollary then follows by direct computations. □

5. Γ-convergence of H_ε

Proof of Proposition 2.3. (i) ⇒ (ii). We first show that $\|\wp_{\eta,u}\|_{\mathrm{TV},L}$ is finite for each η such that $0 \leqq \eta'' \leqq c$. It is easily seen that for each $\varphi \in C_c^\infty((0,T)\times(-L,L);[0,1])$ there exists $\bar\varphi \in C_c^\infty((0,T)\times(-L,L);[0,1])$ such that $\bar\varphi \geqq \varphi$ and $\||\bar\varphi_t| + |\bar\varphi_x|\|_{L_1} \leqq 2(2L+T)$. Therefore

$$\begin{aligned}\wp_{\eta,u}(-\varphi) &= \wp_{\eta,u}(\bar\varphi - \varphi) - \wp_{\eta,u}(\bar\varphi)\\ &\leqq \|\wp^+_{\eta,u}\|_{\mathrm{TV},L} + \langle\langle \eta(u), \bar\varphi_t\rangle\rangle + \langle\langle q(u), \bar\varphi_x\rangle\rangle\\ &\leqq \|\wp^+_{\eta,u}\|_{\mathrm{TV},L} + 2(\|\eta\|_\infty + \|q\|_\infty)(2L+T)\end{aligned}$$

and thus $\|\wp_{\eta,u}\|_{\mathrm{TV},L} \leqq 2\|\wp^+_{\eta,u}\|_{\mathrm{TV},L} + 2(\|\eta\|_\infty + \|q\|_\infty)(2L+T)$.

Let now $\tilde\eta(v) := c\,v^2/2$, and for $\eta \in C^2([0,1])$ arbitrary, let $\alpha := c^{-1}\max_v |\eta''(v)|$. Then $\wp_{\eta,u} = -\alpha\wp_{\tilde\eta-\eta/\alpha,u} + \alpha\wp_{\tilde\eta,u}$. Since both $\tilde\eta - \eta/\alpha$ and $\tilde\eta$ are convex with second derivative bounded by c, $\wp_{\eta,u}$ is a linear combination of Radon measures, and thus a Radon measure itself.

(ii) ⇒ (iii). Throughout this proof, we say that $\eta_1,\ \eta_2 \in C^2([0,1])$ are equivalent, and we write $\eta_1 \sim \eta_2$, iff $\eta_1'' = \eta_2''$. We identify $C^2([0,1])/\sim$ with $C([0,1])$, which we equip with the topology of uniform convergence. For $u \in \mathcal{X}$ a weak solution to (1.1), for $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$, the linear mapping $C^2([0,1]) \ni \eta \mapsto \wp_{\eta,u}(\varphi) \in \mathbb{R}$ is compatible with $\sim$, and it thus defines a linear mapping $P_{\varphi,u}: C([0,1]) \to \mathbb{R}$. It is immediate to see that $P_{\varphi,u}$ is continuous, and by (ii) for each $\eta \in C^2([0,1])$ and $L > 0$

$$\sup\big\{P_{\varphi,u}(\eta''),\ \varphi\in C_c^\infty((0,T)\times(-L,L)),\ |\varphi|\leqq 1\big\} = \|\wp_{\eta,u}\|_{TV,L} < +\infty$$

By Banach–Steinhaus theorem

$$\sup\Big\{P_{\varphi,u}(e),\ \varphi\in C_c^\infty((0,T)\times(-L,L)),\ |\varphi|\leqq 1,\\ e\in C([0,1]),\ |e|\leqq 1\Big\} < +\infty$$

Therefore the linear mapping $P_u^L: C([0,1]) \times C_c^\infty((0,T)\times(-L,L)) \to \mathbb{R}$, $P_u^L(e,\varphi) := P_{\varphi,u}(e)$ can be extended to a finite Borel measure on $[0,1]\times(0,T)\times(-L,L)$. The collection $\{P_u^L\}_L$ defines a unique Radon measure P_u on $[0,1]\times(0,T)\times\mathbb{R}$, since two elements of this collection coincide on the intersection of their domains. Recalling (2.10), we thus gather for each $\eta \in C^2([0,1])$, for each $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$ and for some constant $C>0$ depending only on f

$$\left|\int P_u(\mathrm{d}v,\mathrm{d}t,\mathrm{d}x)\,\eta''(v)\varphi(t,x)\right| = \left|\wp_{\eta,u}(\varphi)\right| \leqq C\|\varphi\|_{C^1((0,T)\times\mathbb{R})}\int \mathrm{d}v\,|\eta''(v)|$$

P_u thus defines a linear continuous functional on $L_1([0,1])\times C_c^1((0,T)\times\mathbb{R})$. This implies that the Radon measure P_u can be disintegrated as $P_u = \mathrm{d}v\,\varrho_u(v;\mathrm{d}t,\mathrm{d}x)$, for some bounded measurable map $\varrho_u : [0,1]\to M((0,T)\times\mathbb{R})$. From the definition of P_u, we obtain for $\eta\in C^2([0,1])$, $\varphi\in C_c^\infty((0,T)\times\mathbb{R})$ and $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$

$$\begin{aligned} P_{\vartheta,u} = \wp_{\eta,u}(\varphi) &= \int P_u(\mathrm{d}v,\mathrm{d}t,\mathrm{d}x)\eta''(v)\varphi(t,x)\\ &= \int \mathrm{d}v\,\varrho_u(v;\mathrm{d}t,\mathrm{d}x)\,\vartheta''(v,t,x)\end{aligned}$$

By linearity and density (2.13) holds for each entropy sampler ϑ.
(iii) $\Rightarrow$ (i). It follows by choosing $\vartheta(v,t,x)=\eta(v)\varphi(t,x)$ in equation (2.13) for $\varphi\in C_c^\infty((0,T)\times\mathbb{R};[0,1])$ and $\eta\in C^2([0,1])$ with $0\leqq\eta''\leqq c$ for an arbitrary $c>0$. □

Proof of Theorem 2.5, item (ii): equicoercivity of H_ε. The equicoercivity of H_ε with respect to the topology generated by the $d_{\mathcal{U}}$-distance (2.2) follows from Lemma 3.3. It remains to show that, if u^ε is such that $H_\varepsilon(u^\varepsilon)$ is bounded uniformly in ε, then $\{u^\varepsilon\}$ is precompact in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$. By equicoercivity of $\{\mathcal{I}_\varepsilon\}$, the sequence $\{\mu^\varepsilon\}$ defined by $\mu^\varepsilon_{t,x}=\delta_{u^\varepsilon(t,x)}$ is precompact in $\mathcal{M}$. Therefore we have only to show that any limit point $\mu\in\mathcal{M}$ of $\{\mu^\varepsilon\}$ has the form $\mu_{t,x}=\delta_{u(t,x)}$ for some $u\in\mathcal{X}$, to obtain the existence of limit points for $\{u^\varepsilon\}$ in $\mathcal{X}$. This is implied by a compensated compactness argument due to Tartar, see [15, Ch. 9], provided that there is no interval where f is affine, and that, for any entropy–entropy flux pair (η,q), the sequence $\{\eta(u^\varepsilon)_t+q(u^\varepsilon)_x\}$ is precompact in $H^{-1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Let us show the latter. By (3.5), there exists $C>0$ such that for each $\varphi\in C_c^\infty((0,T)\times(-L,L))$

$$\begin{aligned}&\left|\langle\langle\eta(u^\varepsilon)_t+q(u^\varepsilon)_x,\varphi\rangle\rangle\right|\\ &\quad\leqq\frac{\varepsilon}{2}\left|\langle\langle\eta''(u^\varepsilon)D(u^\varepsilon)u^\varepsilon_x,\varphi\,u^\varepsilon_x\rangle\rangle\right|+\frac{\varepsilon}{2}\left|\langle\langle\eta'(u^\varepsilon)D(u^\varepsilon)u^\varepsilon_x,\varphi_x\rangle\rangle\right|\\ &\qquad+\left|\langle\langle\eta''(u^\varepsilon)\sigma(u^\varepsilon)\,u^\varepsilon_x,\Psi^{\varepsilon,u^\varepsilon}_x\varphi\rangle\rangle\right|+\left|\langle\langle\eta'(u^\varepsilon)\sigma(u^\varepsilon)\,\Psi^{\varepsilon,u^\varepsilon}_x,\varphi_x\rangle\rangle\right|\\ &\quad\leqq C\left[1+H_\varepsilon(u^\varepsilon)\right]\left[\varepsilon\int_{[0,T]\times[-L,L]}\mathrm{d}t\,\mathrm{d}x\,(u^\varepsilon_x)^2\right]\|\varphi\|_{L_\infty([0,T]\times\mathbb{R})}\\ &\qquad+C\left[\varepsilon H_\varepsilon(u^\varepsilon)+\varepsilon^2\int_{[0,T]\times[-L,L]}\mathrm{d}t\,\mathrm{d}x\,(u^\varepsilon_x)^2\right]^{1/2}\|\varphi_x\|_{L_2([0,T]\times\mathbb{R})}\end{aligned}$$

By the bound (3.6), $\eta(u^\varepsilon)_t+q(u^\varepsilon)_x$ is the sum of a term bounded in $L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$ and a term vanishing in $H^{-1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$ as $\varepsilon\to0$. By Sobolev compact embedding and boundedness of η, q, the sequence $\{\eta(u^\varepsilon)_t+q(u^\varepsilon)_x\}$ is compact in $H^{-1}_{\mathrm{loc}}([0,T]\times\mathbb{R})$. □

Proof of Theorem 2.5, item (i): Γ-liminf inequality. Let $\{u^\varepsilon\}$ be a sequence converging to u in $\mathcal{X}$. If u is not a weak solution to (1.1), by Theorem 2.1 we have $\underline{\lim}_{\varepsilon\to 0} I_\varepsilon(u^\varepsilon) \geqq \mathcal{I}(\delta_u) > 0$, and therefore $\underline{\lim}_{\varepsilon\to 0} H_\varepsilon(u^\varepsilon) = +\infty$. Let now u be a weak solution to (1.1). With no loss of generality we can suppose $H_\varepsilon(u^\varepsilon) \leqq C_H$. We now consider an entropy sampler–entropy sampler flux pair (ϑ, Q) such that

$$0 \leqq \sigma(v)\vartheta''(v,t,x) \leqq D(v), \quad (v,t,x) \in [0,1]\times(0,T)\times\mathbb{R} \tag{5.1}$$

We also let $\varphi^\varepsilon(t,x) = \varepsilon\vartheta'(u^\varepsilon(t,x),t,x)$, and introduce the short hand notation $\left(\vartheta'(u^\varepsilon)\right)(t,x) \equiv \vartheta'(u^\varepsilon(t,x),t,x)$, $\left(\vartheta''(u^\varepsilon)\right)(t,x) \equiv \vartheta''(u^\varepsilon(t,x),t,x)$, $((\partial_x\vartheta')(u^\varepsilon))(t,x) \equiv \left(\partial_x\vartheta'\right)(u^\varepsilon(t,x),t,x)$. As we assumed $H_\varepsilon(u^\varepsilon) < +\infty$, u^ε_x is locally square integrable, see (2.6), and since ϑ is compactly supported we have $\varphi^\varepsilon_x = \varepsilon\vartheta''(u^\varepsilon)\,u^\varepsilon_x + \varepsilon(\partial_x\vartheta')(u^\varepsilon) \in L_2([0,T]\times\mathbb{R})$. The representation (3.7) of $\ell^{u^\varepsilon}_\varepsilon(\varphi^\varepsilon)$ thus holds, and recalling (2.11) we get

$$\begin{aligned}
H_\varepsilon(u^\varepsilon) &\geqq \varepsilon^{-1}\ell^{u^\varepsilon}_\varepsilon(\varphi^\varepsilon) - \frac{\varepsilon^{-1}}{2}\|\varphi^\varepsilon\|^2_{\mathcal{D}^1_{\sigma(u^\varepsilon)}}\\
&= \langle\langle u^\varepsilon_t, \vartheta'(u^\varepsilon)\rangle\rangle + \langle\langle f(u^\varepsilon)_x, \vartheta'(u^\varepsilon)\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle\\
&\quad + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle\\
&\quad - \varepsilon\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\left(\partial_x\vartheta'\right)(u^\varepsilon), \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle\\
&= -\int \mathrm{d}t\,\mathrm{d}x\,\left[(\partial_t\vartheta)\left(u^\varepsilon(t,x),t,x\right) + (\partial_x Q)\left(u^\varepsilon(t,x),t,x\right)\right]\\
&\quad + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon), \vartheta''(u^\varepsilon)(u^\varepsilon_x)^2\rangle\rangle + \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon)u^\varepsilon_x, \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle\\
&\quad - \varepsilon\langle\langle \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)u^\varepsilon_x, \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle - \frac{\varepsilon}{2}\langle\langle \sigma(u^\varepsilon)\left(\partial_x\vartheta'\right)(u^\varepsilon), \left(\partial_x\vartheta'\right)(u^\varepsilon)\rangle\rangle
\end{aligned}$$

By the bound (3.6), the last three terms in the above formula vanish as $\varepsilon \to 0$, while $\langle\langle [D(u^\varepsilon) - \sigma(u^\varepsilon)\vartheta''(u^\varepsilon)]u^\varepsilon_x, \vartheta''(u^\varepsilon)u^\varepsilon_x\rangle\rangle \geqq 0$ for each entropy sampler ϑ satisfying (5.1). Therefore, taking the limit $\varepsilon \to 0$ and optimizing over ϑ

$$\begin{aligned}
&\underline{\lim}_{\varepsilon\to 0}\, H_\varepsilon(u^\varepsilon)\\
&\quad\geqq \sup_\vartheta \underline{\lim}_{\varepsilon\to 0} -\int \mathrm{d}t\,\mathrm{d}x\,\left[(\partial_t\vartheta)\left(u^\varepsilon(t,x),t,x\right) + (\partial_x Q)\left(u^\varepsilon(t,x),t,x\right)\right]\\
&\quad= \sup_\vartheta P_{\vartheta,u}
\end{aligned}$$

where the supremum is taken on the $\vartheta \in C^{2,\infty}_{\mathrm{c}}([0,1]\times(0,T)\times\mathbb{R})$ satisfying (5.1). Recalling that we assumed the left-hand side of this formula to be finite, we next show that this inequality implies that $u \in \mathcal{E}$, and that the right-hand side is equal to $H(u)$. By taking $\vartheta(v,t,x) = \eta(v)\varphi(t,x)$ for some $\varphi \in C^\infty_{\mathrm{c}}([0,T]\times\mathbb{R};[0,1])$ and entropy η such that $0 \leqq \sigma(v)\eta''(v) \leqq D(v)$, we get $\wp_{\eta,u}(\varphi) \leqq \underline{\lim}_\varepsilon H_\varepsilon(u^\varepsilon)$. Optimizing over φ it follows that u fulfills condition (i) in Proposition 2.3 with

$c = \min_v D(v)/\sigma(v) > 0$, and thus $u \in \mathcal{E}$. By (iii) in Proposition 2.3 and monotone convergence, we then get

$$\begin{aligned}\varliminf_{\varepsilon\to 0} H_\varepsilon(u^\varepsilon) &\geqq \sup_\vartheta P_{\vartheta,u} = \sup_\vartheta \int \mathrm{d}v\, \varrho_u(v;\mathrm{d}t,\mathrm{d}x)\, \vartheta''(v,t,x) \\ &= \int \mathrm{d}v\, \varrho_u^+(v;\mathrm{d}t,\mathrm{d}x)\, \frac{D(v)}{\sigma(v)} = H(u) \end{aligned} \tag{5.2}$$

which concludes the proof. □

Lemma 5.1. *Let $f \in C^2([0,1])$ and assume that there is no interval where f is affine. Then entropy-measure solutions to (1.1) belong to the space $C\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Let furthermore*

$$V_f^+ := \max_{v\in[0,1]} f'(v) \quad V_f^- := \min_{v\in[0,1]} f'(v) \tag{5.3}$$

Then for each $u \in \mathcal{E}$, $x \in \mathbb{R}$, $V > V_f^+$ or $V < V_f^-$

$$\lim_{\zeta\to 0} \int \mathrm{d}t\, |u\,(t, x+\zeta+V\,t) - u\,(t, x+V\,t)| = 0 \tag{5.4}$$

Proof. With the same hypotheses of this lemma, in [4, Sect. 4], it is shown that if a weak solution u to (1.1) is such that $\wp_{f,u}$ is a Radon measure, then, for each $L > 0$ and $t \in [0,T)$, $\lim_{s\downarrow t} \int_{[-L,L]} |u(s,x) - u(t,x)|\mathrm{d}x = 0$. Therefore, by item (ii) in Proposition 2.3, entropy-measure solutions enjoy this property. Since the set $\mathcal{E}$ of entropy-measure solutions is invariant under the symmetry $(t,x) \mapsto (-t,-x)$, the same holds true also for $s \uparrow t$, and thus $\mathcal{E} \subset C\,\big([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\big)$.

If u is an entropy-measure solution to the conservation law (1.1), then $u^{V,\pm}(t,x) := u(\pm t, x \pm Vt)$ is an entropy-measure solution to the conservation law with flux $f^\pm$, where $f^\pm(w) = f(w) \mp Vw$. With no loss of generality, we can thus prove (5.4) only in the case $V = 0$ with the assumption $V_f^- > 0$. In this case f is invertible on its range $[a,b]$, and we let $g \in C^2([a,b])$ be its inverse. We define $v : \mathbb{R} \times [0,T] \mapsto [a,b]$ by $v(x,t) = f(u(t,x))$. Then v satisfies

$$v_x + g(v)_t = 0 \tag{5.5}$$

Furthermore, if $l, m \in C^2([a,b])$ satisfy $m' = l'g'$, then by chain rule $l(v)_x + m(v)_t = \wp_{\eta,u}$, where $\eta(w) := \int^w \mathrm{d}z\, l'(f(z))$. Therefore v is an entropy-measure solution to (5.5), and by the first part of this lemma

$$\lim_{\zeta\to 0} \int \mathrm{d}s\ |v(x+\zeta,s) - v(x,s)| = 0$$

The result then follows by recalling $u(t,x) = g(v(x,t))$. □

Proof of Theorem 2.5, item (iii): Γ-limsup inequality. Given an nice (with respect to σ) solution $\tilde{u} \in \mathcal{S}_\sigma$, let $E^\pm$ be as in Definition 2.4. We want to construct a recovery sequence $\{u^\varepsilon\} \subset \mathcal{X}$ that converges to $\tilde{u}$ in $\mathcal{X}$ as $\varepsilon \to 0$, and such that $\varlimsup_\varepsilon H_\varepsilon(u^\varepsilon) \leqq H(\tilde{u})$. We split the proof in four steps. In *Step 1* we build a suitable family of rectangles contained in $[0, T]\times\mathbb{R}$. In *Step 2*, for $\varepsilon,\ \delta,\ L \geqq 1$, we introduce two collections $\{v^{\varepsilon,\delta,L,\pm}\}$ of auxiliary functions on $[0, T]\times\mathbb{R}$. In *Step 3*, for $N \in \mathbb{N}$ we define a collection $\{u^{\varepsilon,\delta,N,L}\} \subset \mathcal{X}$, and we prove

$$\varlimsup_{\delta\to 0}\varlimsup_{\varepsilon\to 0} H_\varepsilon(u^{\varepsilon,\delta,N,L}) \leqq H(\tilde{u}) \tag{5.6}$$

In particular $\{u^{\varepsilon,\delta,N,L}\}$ is precompact in $\mathcal{X}$. In *Step 4* we show that any limit point of $\{u^{\varepsilon,\delta,N,L}\}$ coincides with $\tilde{u}$ in $\mathcal{X}$, provided we consider the limit in $\varepsilon,\ \delta,\ N,\ L$ in a suitable order. More precisely we show

$$\lim_{L\to\infty}\lim_{N\to\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0} u^{\varepsilon,\delta,N,L} = \tilde{u} \tag{5.7}$$

By (5.6) and (5.7) it follows that there exist subsequences $\{\delta^\varepsilon\}$, $\{L^\varepsilon\} \subset (0, +\infty)$ and $\{N^\varepsilon\} \subset \mathbb{N}$ such that $u^\varepsilon := u^{\varepsilon,\delta^\varepsilon,N^\varepsilon,L^\varepsilon}$ provides the required recovery sequence for $\tilde{u}$.

Throughout this proof, we assume f' to be uniformly positive in $[0, 1]$, namely that V_f^-, as defined in (5.3), is positive. As noted in the proof of Lemma 5.1, this assumption is not restrictive. Note also that the calculations carried out below make sense also if $E^+ = \emptyset$ or $E^- = \emptyset$.

Step 1. For each t such that $(\{t\} \times [-L, L]) \cap E^+ \cap E^- = \emptyset$, the compact sets $(\{t\} \times [-L, L]) \cap E^\pm$ are disjoint, hence strictly separated. By (ii) in Definition 2.4, there exists a countable collection of pairwise disjoint time intervals $\{(s_i^L, t_i^L)\}_{i\in\mathbb{N}}$, with $(s_i^L, t_i^L) \subset (0, T)$ such that $\tau^L := \cup_i (s_i^L, t_i^L)$ is dense in $[0, T]$, and for each $i \in \mathbb{N}$ the two sets $E_i^{L,\pm} := \big((s_i^L, t_i^L) \times [-L, L]\big) \cap E^\pm$ are strictly separated. By splitting each of these intervals in a finite number of intervals, with no loss of generality, we can assume

$$t_i^L - s_i^L < \frac{1}{10 + 8\, V_f^+}\ \text{distance}\left(E_i^{L,+}, E_i^{L,-}\right) \tag{5.8}$$

where V_f^+ is defined in (5.3), and it coincides with the Lipschitz constant of f since we supposed $V_f^- > 0$.

For $i \in \mathbb{N}$ let $n_i^L \in \mathbb{N}$ be such that

$$\frac{L}{n_i^L} \leqq \frac{1}{10}\min\left\{1, \text{distance}\left(E_i^{L,+}, E_i^{L,-}\right)\right\} \tag{5.9}$$

and consider the rectangles $R_{i,j}^L := (s_i^L, t_i^L) \times (\frac{j}{n_i^L} L, \frac{j+1}{n_i^L} L)$, for $j = -n_i^L, -n_i^L + 1, \ldots, n_i^L - 1$. By the definition (5.9) of n_i^L and condition (5.8), for $j = -n_i^L + 1, -n_i^L + 2, \ldots, n_i^L - 2$

$$\text{diameter}(R_{i,j-1}^L \cup R_{i,j}^L \cup R_{i,j+1}^L \cup R_{i,j+2}^L) < \frac{1}{2}\text{distance}\left(E_i^{L,+}, E_i^{L,-}\right) \tag{5.10}$$

In particular each $R^L_{i,j}$ has non-empty intersection with at most one of the sets E^+, E^-. We define

$$R_i^{L,\pm} := \bigcup_{\substack{j=-n_i^L+1, \\ j\,:\,(R^L_{i,j-1}\cup R^L_{i,j}\cup R^L_{i,j+1}\cup R^L_{i,j+2})\cap E^{\mp}=\emptyset}}^{n_i^L-2} R^L_{i,j} \tag{5.11}$$

and for $N \in \mathbb{N}$

$$R^{N,L,\pm} := \cup_{i=1}^N R_i^{L,\pm}, \qquad R^{L,\pm} := \cup_N R^{N,L,\pm} \tag{5.12}$$

Note that by (5.8) and (5.9)

$$R^L_{i,j} \subset \Bigg\{ (r,x)\ :\ s_i^L < r < t_i^L, \\ \frac{j-1}{n_i^L}L + V_f^+(r-s_i^L) \leqq x \leqq \frac{j+2}{n_i^L}L - V_f^+(r-s_i^L) \Bigg\} \tag{5.13}$$

and by (5.10)

$$R^{L,+} \cup R^{L,-} = \bigcup_i \bigcup_{j=-n_i^L+1}^{n_i^L-2} R^L_{i,j} \tag{5.14}$$

Step 2. For $L \geqq 1$ and $\delta \in (0,1/2)$, let $\tilde{u}^{\delta,L} \in \mathcal{X}$ be defined by

$$\tilde{u}^{\delta,L}(t,x) := \begin{cases} \tilde{u}(t,x) & \text{if } |x| \leqq L \quad \text{and} \quad \tilde{u}(t,x) \in [\delta, 1-\delta] \\ \delta & \text{if } |x| \leqq L \quad \text{and} \quad \tilde{u}(t,x) \leqq \delta \\ 1-\delta & \text{if } |x| \leqq L \quad \text{and} \quad \tilde{u}(t,x) \geqq 1-\delta \\ 1/2 & \text{if } |x| > L \end{cases} \tag{5.15}$$

For $\varepsilon > 0$, $i \in \mathbb{N}$, we define $v_i^{\varepsilon,\delta,L,-} : (s_i^L, t_i^L) \times \mathbb{R} \to \mathbb{R}$ as the solution to the forward-parabolic Cauchy problem

$$\begin{cases} v_t + f(v)_x = \dfrac{\varepsilon}{2}\,(D(v)v_x)_x \\ v(s_i^L) = \tilde{u}^{\delta,L}(s_i^L) \end{cases} \tag{5.16}$$

and $v_i^{\varepsilon,\delta,L,+} : (s_i^L, t_i^L) \times \mathbb{R} \to \mathbb{R}$ as the solution to the backward-parabolic Cauchy problem

$$\begin{cases} v_t + f(v)_x = -\dfrac{\varepsilon}{2}\,(D(v)v_x)_x \\ v(t_i^L) = \tilde{u}^{\delta,L}(t_i^L) \end{cases} \tag{5.17}$$

We define $v^{\varepsilon,\delta,L,\pm} : \tau^L \times \mathbb{R} \to \mathbb{R}$ by requiring $v^{\varepsilon,\delta,L,\pm}(r,x) = v_i^{\varepsilon,\delta,L,\pm}(r,x)$ for $r \in (s_i^L, t_i^L)$. Note that $v^{\varepsilon,\delta,L,\pm} \in C\left(\tau^L; U\right)$ and $v^{\varepsilon,\delta,L,\pm}(t,x) \in [\delta, 1-\delta]$

by maximum principle. Furthermore $v_x^{\varepsilon,\delta,L,\pm} \in L_{2,\mathrm{loc}}(\tau^L \times \mathbb{R})$, and indeed by standard parabolic estimates

$$\varepsilon \int_{R^{N,L,\pm}} \mathrm{d}r\, \mathrm{d}x\, \left(v_x^{\varepsilon,\delta,L,\pm}(r,x)\right)^2 \leqq \sum_{i=1}^{N} \varepsilon \int_{[s_i^L,t_i^L]\times[-L,L]} \mathrm{d}r\, \mathrm{d}x\, \left(v_x^{\varepsilon,\delta,L,\pm}(r,x)\right)^2 \leqq C^{N,L} \tag{5.18}$$

for some constant $C^{N,L} > 0$ independent of ε and δ.

We claim

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int_{R^{N,L,\pm}} \mathrm{d}r\, \mathrm{d}x\, \left| v^{\varepsilon,\delta,L,\pm}(r,x) - \tilde{u}(r,x)\right| = 0 \tag{5.19}$$

We show (5.19) for $v^{\varepsilon,\delta,L,-}$. The analogous statement for $v^{\varepsilon,\delta,L,+}$ follows by the fact that the set $\mathcal{S}_\sigma$ is invariant with respect to the symmetry $(t,x) \mapsto (-t,-x)$, while the supports of $\varrho_u^\pm$ are exchanged under this symmetry. By the well known results of convergence of the vanishing viscosity approximations to conservation laws (and as it also follows from the Γ-liminf inequality in Theorem 2.5 item (i))

$$\lim_{\varepsilon\to 0} \int_{[s_i^L,t_i^L]\times[-L,L]} \mathrm{d}r\, \mathrm{d}x\, \left| v^{\varepsilon,\delta,L,-}(r,x) - \bar{u}_i^{\delta,L}(r - s_i^L, x)\right| = 0 \tag{5.20}$$

where $\bar{u}_i^{\delta,L}$ is the Kruzkov solution to (1.1) with initial condition $\bar{u}_i^{\delta,L}(0,\cdot) = \tilde{u}^{\delta,L}(s_i^L,\cdot)$. On the other hand, by the definition (5.11) of $R_i^{L,-}$, if j is such that $R_{i,j}^L \subset R_i^{L,-}$, then $\tilde{u}$ is entropic in the rectangle $(s_i^L, t_i^L) \times (\frac{j-1}{n_i^L}L, \frac{j+2}{n_i^L}L)$, namely $\wp_{\eta,\tilde{u}}(\varphi) \leqq 0$ for each convex entropy η and each positive test function φ compactly supported in $(s_i^L, t_i^L) \times (\frac{j-1}{n_i^L}L, \frac{j+2}{n_i^L}L)$. Therefore, by Kruzkov theorem [15]

$$\begin{aligned}
&\overline{\lim_{\delta\to 0}} \sup_{s_i^L \leqq r \leqq t_i^L} \int_{\frac{j-1}{n_i^L}L + V_f^+(r-s_i^L)}^{\frac{j+2}{n_i^L}L - V_f^+(r-s_i^L)} \mathrm{d}x\, \left|\bar{u}_i^{\delta,L}(r - s_i^L, x) - \tilde{u}(r,x)\right| \\
&\quad \leqq \overline{\lim_{\delta\to 0}} \int_{\frac{j-1}{n_i^L}L}^{\frac{j+2}{n_i^L}L} \mathrm{d}x\, \left|\bar{u}_i^{\delta,L}(0,x) - \tilde{u}(s_i^L,x)\right| \\
&\quad = \overline{\lim_{\delta\to 0}} \int_{\frac{j-1}{n_i^L}L}^{\frac{j+2}{n_i^L}L} \mathrm{d}x\, \left|\tilde{u}^{\delta,L}(s_i^L,x) - \tilde{u}(s_i^L,x)\right| = 0
\end{aligned}$$

and thus, fixed $N \in \mathbb{N}$, by (5.13) the convergence claimed in (5.19) holds on each $R_{i,j}^L$ for each $i \leqq N$ and each j such that $R_{i,j}^L \subset R_i^{L,-}$, and therefore on $R^{N,L,-}$ itself.

Next we claim that for each $L \geqq 1$, $N \in \mathbb{N}$ and $\varphi \in C_c^\infty\left(R^{N,L,+};[0,1]\right)$

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}} \frac{\varepsilon}{2} \left\langle\!\!\left\langle D(v^{\varepsilon,\delta,L,+}) v_x^{\varepsilon,\delta,L,+}, \varphi \frac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})} v_x^{\varepsilon,\delta,L,+} \right\rangle\!\!\right\rangle \leqq H(\tilde{u}) \tag{5.21}$$

Note that the left-hand side of this formula is well defined, since $\delta \leqq v^{\varepsilon,\delta,L,+} \leqq 1-\delta$ and thus $\sigma(v^{\varepsilon,\delta,L,+})$ is uniformly positive. For each $\varphi \in C_c^\infty(([0,T]\times\mathbb{R})\setminus E^-; [0,1])$ and $\eta \in C^2([0,1])$ such that $\sigma\eta'' \leqq D$ we have

$$H(\tilde{u}) \geqq \int \mathrm{d}w\, \varrho_{\tilde{u}}(w; \mathrm{d}t, \mathrm{d}x)\, \eta''(w)\varphi(t,x) = \wp_{\eta,\tilde{u}}(\varphi) \tag{5.22}$$

By (5.17) and (5.18) for each $\eta \in C^2([0,1])$, $N \in \mathbb{N}$ and $\varphi \in C_c^\infty\left(R^{N,L,+}\right)$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{\varepsilon}{2} \langle\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, \varphi\eta''(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+} \rangle\rangle = \wp_{\eta,\tilde{u}}(\varphi) \tag{5.23}$$

This implies (5.21) if σ is uniformly positive on $[0,1]$, since we can evaluate (5.23) on an entropy η such that $\eta'' = D/\sigma$ and use the trivial bound (5.22). On the other hand, if $\sigma(0) = 0$, respectively if $\sigma(1) = 0$, then by condition (iii) in Definition 2.4, we have that $\tilde{u}(t,x) \geqq \zeta_L$, respectively $\tilde{u}(t,x) \leqq 1-\zeta_L$, for almost everywhere $(t,x) \in (0,T)\times(-L,L)$ and for some $\zeta_L > 0$. By the definition of $\tilde{u}^{\delta,L}$ and maximum principle, we have also $v^{\varepsilon,\delta,L,+} \geqq \zeta_L$, respectively $v^{\varepsilon,\delta,L,+} \leqq 1-\zeta_L$, and thus (5.19) follows by evaluating (5.23) on an entropy η such that $\eta''(w) = D(w)/\sigma(w)$ for all $w \geqq \zeta_L$, respectively $w \leqq 1-\zeta_L$.
Step 3. In this step, with a little abuse of notation, we denote by f and D two bounded continuous functions on $\mathbb{R}$, such they their restrictions to $[0,1]$ coincide with f and D, and f is uniformly Lipschitz and D uniformly positive. We also let $\sigma^\delta \in C^\alpha([0,1])$ be such that $\sigma^\delta(w) = \sigma(w)$ for $w \in [\delta, 1-\delta]$, $\sigma^\delta(w) \leqq \sigma(w)$ for $w \in [0,1]$, and $\sigma^\delta(w) = 0$ for $w \leqq \delta/2$ or $w \geqq 1-\delta/2$.

For $L \geqq 1$ and $N \in \mathbb{N}$, let $\Xi^{N,L} \in C_c^\infty\left(R^{N,L,+}; [0,1]\right)$, and define

$$\begin{aligned} P^{N,L,+} &:= \text{Interior}\left(\left\{(t,x) \in R^{N,L,+} \;:\; \Xi^{N,L}(t,x) = 1\right\}\right) \\ P^{N,L,-} &:= \text{Interior}\left(\left\{(t,x) \in R^{N,L,-} \;:\; \Xi^{N,L}(t,x) = 0\right\}\right) \end{aligned} \tag{5.24}$$

For each fixed $L \geqq 1$, we require the sequence $\{\Xi^{N,L}\}$ to be increasing in N and such that

$$\bigcup_N P^{N,L,+} = R^{L,+} \tag{5.25}$$

For δ, $L \geqq 1$ and $N \in \mathbb{N}$ define $u^{\varepsilon,\delta,N,L} : [0,T]\times\mathbb{R} \to \mathbb{R}$ as the solution to the Cauchy problem

$$\begin{cases} u_t + f(u)_x = \dfrac{\varepsilon}{2}(D(u)u_x)_x \\ \qquad\qquad - \varepsilon\left[\Xi^{N,L}\dfrac{\sqrt{\sigma^\delta(u)}}{\sqrt{\sigma(v^{\varepsilon,\delta,L,+})}}D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}\right]_x \\ u(0,x) = \tilde{u}^{\delta,L}(0,x) \quad x \in \mathbb{R} \end{cases} \tag{5.26}$$

Note that the term in square brackets in (5.26) is well defined since $v^{\varepsilon,\delta,L,+}$ is well defined on the support of $\Xi^{N,L}$, and since $\delta \leqq v^{\varepsilon,\delta,L,+} \leqq 1-\delta$, $\sigma(v^{\varepsilon,\delta,L,+})$ is uniformly positive.

It is easily seen that the problem (5.26) admits at least a solution $u^{\varepsilon,\delta,N,L} \in L_\infty([0,T]\times\mathbb{R})$ with $u_x^{\varepsilon,\delta,N,L} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$. By (5.26) we also gather

$$\left\| u_t^{\varepsilon,\delta,N,L} + f(u^{\varepsilon,\delta,N,L})_x - \frac{\varepsilon}{2}\left(D(u^{\varepsilon,\delta,N,L})u_x^{\varepsilon,\delta,N,L}\right)_x \right\|^2_{\mathcal{D}^{-1}_{\sigma^\delta(u)}}$$
$$= \varepsilon^2 \left\langle\!\!\left\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2 \frac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})} v_x^{\varepsilon,\delta,L,+} \right\rangle\!\!\right\rangle < +\infty$$

Therefore, replacing σ with σ^δ in the statement of Proposition 3.4, we have $\delta \leqq u^{\varepsilon,\delta,N,L} \leqq 1-\delta$ and $u^{\varepsilon,\delta,N,L} \in \mathcal{X}$. Since $(\Xi^{N,L})^2 \in C_c^\infty(R^{N,L,+};[0,1])$, by the same estimate and (5.21)

$$\begin{aligned}
&\overline{\lim_\delta}\,\overline{\lim_\varepsilon}\, H_\varepsilon(u^{\varepsilon,\delta,N,L})\\
&\quad = \overline{\lim_\delta}\,\overline{\lim_\varepsilon}\, \frac{\varepsilon}{2}\left\langle\!\!\left\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2 \right.\right.\\
&\qquad\qquad \left.\left. \times \frac{\sigma^\delta(u^{\varepsilon,\delta,N,L})}{\sigma(u^{\varepsilon,\delta,N,L})}\frac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})} v_x^{\varepsilon,\delta,L,+}\right\rangle\!\!\right\rangle\\
&\quad \leqq \overline{\lim_\delta}\,\overline{\lim_\varepsilon}\, \frac{\varepsilon}{2}\left\langle\!\!\left\langle D(v^{\varepsilon,\delta,L,+})v_x^{\varepsilon,\delta,L,+}, (\Xi^{N,L})^2 \frac{D(v^{\varepsilon,\delta,L,+})}{\sigma(v^{\varepsilon,\delta,L,+})} v_x^{\varepsilon,\delta,L,+}\right\rangle\!\!\right\rangle\\
&\quad \leqq H(\tilde{u})
\end{aligned}$$

so that (5.6) holds.

Step 4. Since $\{H_\varepsilon\}$ is equicoercive on $\mathcal{X}$ and (5.6) holds, there exist δ_0, $\varepsilon_0 \equiv \varepsilon_0(\delta_0)$ small enough and a compact set $\mathcal{K}_0 \subset \mathcal{X}$ such that $u^{\varepsilon,\delta,N,L} \in \mathcal{K}_0$ for each $\varepsilon < \varepsilon_0$, $\delta < \delta_0$, $N \in \mathbb{N}$ and $L \geqq 1$. In this step we show that any limit point u of $\{u^{\varepsilon,\delta,N,L}\}$ coincide with $\tilde{u}$, provided the limits in ε, δ, N and L are taken in a suitable order, see (5.7). This will conclude the proof.

Let $z^{\varepsilon,\delta,N,L,\pm} : \tau^L \times \mathbb{R} \to [-1,1]$, $z^{\varepsilon,\delta,N,L,\pm} := u^{\varepsilon,\delta,N,L} - v^{\varepsilon,\delta,L,\pm}$. By (3.6), (5.6) and (5.18), for each $N \in \mathbb{N}$

$$\varepsilon \int_{R^{N,L,\pm}} \mathrm{d}t\,\mathrm{d}x\,(z_x^{\varepsilon,\delta,N,L,\pm})^2 \leqq \tilde{C}^{N,L} \tag{5.27}$$

for some constant $\tilde{C}^{N,L} > 0$ independent of ε and δ.

Since we will first perform the limit $\varepsilon \to 0$, we now fix δ, N, L as above, and we drop for a few lines these indexes, thus writing $u^\varepsilon \equiv u^{\varepsilon,\delta,N,L}$, $v^{\varepsilon,\pm} \equiv v^{\varepsilon,\delta,L,\pm}$, $z^{\varepsilon,\pm} \equiv z^{\varepsilon,\delta,N,L,\pm}$, $\Xi \equiv \Xi^{N,L}$. Recalling the definition (5.24), by (5.26) and (5.16), we have weakly on $P^{N,L,-}$

$$z_t^{\varepsilon,-} + \left(f(u^\varepsilon) - f(v^{\varepsilon,-})\right)_x = \frac{\varepsilon}{2}\left(D(u^\varepsilon)z_x^{\varepsilon,-}\right)_x + \frac{\varepsilon}{2}\left(\left[D(u^\varepsilon) - D(v^{\varepsilon,-})\right]v_x^{\varepsilon,-}\right)_x$$

Let now $l \in C^2([-1,1])$ and $\varphi \in C_c^\infty\left(P^{N,L,-}\right)$. It follows

$$
\begin{aligned}
&-\langle\langle l(z^{\varepsilon,-}), \varphi_t\rangle\rangle - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,-}), l'(z^{\varepsilon,-})\varphi_x\rangle\rangle \\
&\quad - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,-}), l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle \\
&= -\frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,-}, l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,-}, l'(z^{\varepsilon,-})\varphi_x\rangle\rangle \\
&\quad - \frac{\varepsilon}{2}\langle\langle \left[D(u^\varepsilon) - D(v^{\varepsilon,-})\right] v_x^{\varepsilon,-}, l''(z^{\varepsilon,-}) z_x^{\varepsilon,-}\varphi\rangle\rangle \\
&\quad - \frac{\varepsilon}{2}\langle\langle \left[D(u^\varepsilon) - D(v^{\varepsilon,-})\right] v_x^{\varepsilon,-}, l'(z^{\varepsilon,-})\varphi_x\rangle\rangle
\end{aligned}
\tag{5.28}
$$

In the same fashion, by (5.17), weakly on $P^{N,L,+}$

$$
\begin{aligned}
z_t^{\varepsilon,+} + \left(f(u^\varepsilon) - f(v^{\varepsilon,+})\right)_x &= \frac{\varepsilon}{2}\left(D(u^\varepsilon) z_x^{\varepsilon,+}\right)_x + \frac{\varepsilon}{2}\left(\left[D(u^\varepsilon) - D(v^{\varepsilon,+})\right] v_x^{\varepsilon,+}\right)_x \\
&\quad + \varepsilon\left(\left[\sqrt{\sigma(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\right] \frac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}\right)_x
\end{aligned}
$$

Since $v^{\varepsilon,+}$ takes values in $[\delta, 1-\delta]$, we have $\sigma^\delta(v^{\varepsilon,+}) = \sigma(v^{\varepsilon,+})$ and thus, in the same fashion as above, for each $l \in C^2([-1,1])$ and $\varphi \in C_c^\infty\left(P^{N,L,+}\right)$

$$
\begin{aligned}
&-\langle\langle l(z^{\varepsilon,+}), \varphi_t\rangle\rangle - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,+}), l'(z^{\varepsilon,+})\varphi_x\rangle\rangle \\
&\quad - \langle\langle f(u^\varepsilon) - f(v^{\varepsilon,+}), l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle \\
&= -\frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,+}, l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle D(u^\varepsilon) z_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\rangle\rangle \\
&\quad - \frac{\varepsilon}{2}\langle\langle \left[D(u^\varepsilon) - D(v^{\varepsilon,+})\right] v_x^{\varepsilon,+}, l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\varphi\rangle\rangle \\
&\quad - \frac{\varepsilon}{2}\langle\langle \left[D(u^\varepsilon) - D(v^{\varepsilon,+})\right] v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\rangle\rangle \\
&\quad - \varepsilon\left\langle\left\langle \left[\sqrt{\sigma^\delta(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\right] \frac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, \varphi l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\right\rangle\right\rangle \\
&\quad - \varepsilon\left\langle\left\langle \left[\sqrt{\sigma^\delta(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\right] \frac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x\right\rangle\right\rangle
\end{aligned}
\tag{5.29}
$$

For l convex and φ non-negative, the first term in the second lines of (5.28) and (5.29) is non-positive. With these assumptions on l and φ we thus define $B_l \equiv B_{l,\varphi}^{\varepsilon,\delta,N,L,\pm} := [\langle\langle D(u^\varepsilon) z_x^{\varepsilon,\pm}, l''(z^{\varepsilon,\pm}) z_x^{\varepsilon,\pm}\varphi\rangle\rangle]^{1/2}$ and let, for $F \in C([0,1])$,

$$
C_{F,l}^\delta := \max\left\{l''(z)|F(v+z) - F(v)|^2 \,:\, v \in [\delta, 1-\delta],\ z \in [-1,1],\ v+z \in [0,1]\right\}
$$

Since $v_x^{\varepsilon,\pm},\ z_x^{\varepsilon,\pm} \in L_{2,\mathrm{loc}}(P^{N,L,\pm})$, by (5.27), Cauchy–Schwarz inequality and the fact that D is uniformly positive, we have for each non-negative $\varphi^\pm \in C_c^\infty(P^{N,L,\pm})$, and for some constant $C \equiv C_{\varphi^\pm}^{\varepsilon,\delta,N,L}$ independent of l

$$\begin{aligned}
&\left|\langle\langle f(u^\varepsilon) - f(v^{\varepsilon,\pm}), l''(z^{\varepsilon,\pm}) z_x^{\varepsilon,\pm}\varphi^\pm\rangle\rangle\right| \\
&\quad + \left|\frac{\varepsilon}{2}\langle\langle\big[D(u^\varepsilon) - D(v^{\varepsilon,\pm})\big]\, v_x^{\varepsilon,\pm}, l''(z^{\varepsilon,\pm}) z_x^{\varepsilon,\pm}\varphi^\pm\rangle\rangle\right| \\
&\quad + \left|\varepsilon\langle\langle\Big[\sqrt{\sigma^\delta(v^{\varepsilon,+})} - \sqrt{\sigma^\delta(u^\varepsilon)}\Big]\, \frac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, \varphi^+ l''(z^{\varepsilon,+}) z_x^{\varepsilon,+}\rangle\rangle\right| \\
&\leqq C\left[\sqrt{C_{f,l}^\delta} + \sqrt{C_{D,l}^\delta} + \sqrt{C_{\sqrt{\sigma^\delta},l}^\delta}\right] B_l
\end{aligned}$$

We also let $C_l := \max_{z\in[-1,1]} |l'(z)|$ and note that, in view of (5.18) and (5.27), for any non-negative $\varphi^\pm \in C_c^\infty\left(P^{N,L,\pm}\right)$ and for some constant $\tilde{C} = \tilde{C}_{\varphi^\pm}^{\delta,N,L}$ independent of ε and l

$$\begin{aligned}
&\frac{\varepsilon}{2}\left|\langle\!\langle D(u^\varepsilon) z_x^{\varepsilon,\pm}, l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\!\rangle\right| + \frac{\varepsilon}{2}\left|\langle\!\langle\big[D(u^\varepsilon) - D(v^{\varepsilon,\pm})\big]\, v_x^{\varepsilon,\pm}, l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\!\rangle\right| \\
&\quad + \varepsilon\left|\left\langle\!\!\left\langle\big[\sqrt{\sigma(v^{\varepsilon,+})} - \sqrt{\sigma(u^\varepsilon)}\big]\frac{D(v^{\varepsilon,+})}{\sqrt{\sigma(v^{\varepsilon,+})}} v_x^{\varepsilon,+}, l'(z^{\varepsilon,+})\varphi_x^+\right\rangle\!\!\right\rangle\right| \\
&\leqq \tilde{C}\, C_l\, \sqrt{\varepsilon}
\end{aligned}$$

Patching all together, for each non-negative $\varphi^\pm \in C_c^\infty\left(P^{N,L,\pm}\right)$, we gather

$$\begin{aligned}
&-\langle\!\langle l(z^{\varepsilon,\pm}), \varphi_t^\pm\rangle\!\rangle - \langle\!\langle f(u^\varepsilon) - f(v^{\varepsilon,\pm}), l'(z^{\varepsilon,\pm})\varphi_x^\pm\rangle\!\rangle \\
&\quad\leqq -\frac{\varepsilon}{2}B_l^2 + C\left[\sqrt{C_{f,l}^\delta} + \sqrt{C_{D,l}^\delta} + \sqrt{C_{\sqrt{\sigma^\delta},l}^\delta}\right] B_l + \tilde{C} C_l\sqrt{\varepsilon} \\
&\quad\leqq \frac{3}{2\varepsilon}C^2\left[C_{f,l}^\delta + C_{D,l}^\delta + C_{\sqrt{\sigma^\delta},l}^\delta\right] + \tilde{C} C_l\sqrt{\varepsilon}
\end{aligned} \tag{5.30}$$

It is then easily seen that we can take a sequence of convex smooth functions $\{l_n\} \subset C^2([-1, 1])$ such that $|l_n'(z)| \leqq 1$, $l_n(z) \to |z|$, $z l_n'(z) \to |z|$ uniformly on $[-1, 1]$, and such that, by the Hölder continuity hypotheses on D and σ

$$\lim_{n\to\infty}\left(C_{f,l_n}^\delta + C_{D,l_n}^\delta + C_{\sqrt{\sigma^\delta},l_n}^\delta\right) = 0$$

Evaluating (5.30) for $l \equiv l_n$, taking the limit $n \to \infty$, and recalling that we assumed f' to be positive on $[0, 1]$, we gather for each non-negative $\varphi^\pm \in C_c^\infty\left(P^{N,L,\pm}\right)$

$$-\langle\langle |u^\varepsilon - v^{\varepsilon,\pm}|, \varphi_t^\pm\rangle\rangle - \langle\langle \left|f(u^\varepsilon) - f(v^{\varepsilon,\pm})\right|, \varphi_x^\pm\rangle\rangle \leqq \tilde{C}\sqrt{\varepsilon} \tag{5.31}$$

We now reintroduce the dropped indexes δ, N, L, and recall that for $\delta \leqq \delta_0$, $\varepsilon \leqq \varepsilon_0(\delta_0)$, $N \in \mathbb{N}$ and $L \geqq 1$ we have $u^{\varepsilon,\delta,N,L} \in \mathcal{K}_0$ for some compact $\mathcal{K}_0 \subset \mathcal{X}$. Let $u^{N,L} \in \mathcal{K}_0$ be a generic limit point of $\{u^{\varepsilon,\delta,N,L}\}$ in $\mathcal{X}$ as $\varepsilon \to 0$ and successively $\delta \to 0$. By (5.19) and (5.31), for each non-negative $\varphi \in C_c^\infty\left(P^{N,L,-} \cup P^{N,L,+}\right)$

$$-\left\langle\!\!\left\langle |u^{N,L} - \tilde{u}|, \varphi_t\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \left|f(u^{N,L}) - f(\tilde{u})\right|, \varphi_x\right\rangle\!\!\right\rangle \leqq 0 \tag{5.32}$$

Since $u^{N,L} \in \mathcal{K}_0$, there exist $u^L \in \mathcal{X}$ and a subsequence $\{N_k\} \subset \mathbb{N}$ such that $u^{N_k,L} \to u^L$ in $\mathcal{X}$ as $k \to +\infty$. By (5.25) and (5.32), it follows that for each non-negative $\varphi \in C_{\mathrm{c}}^{\infty}\left(R^{L,-} \cup R^{L,+}\right)$

$$-\left\langle\!\left\langle |u^L - \tilde{u}|, \varphi_t \right\rangle\!\right\rangle - \left\langle\!\left\langle \left| f(u^L) - f(\tilde{u}) \right|, \varphi_x \right\rangle\!\right\rangle \leqq 0 \tag{5.33}$$

Since τ^L is dense in $[0,T]$, by (5.14) and (5.9), we have that, for $L \geqq 1$, $R^{L,+} \cup R^{L,-}$ is dense in $[0,T] \times \left[-L + \frac{1}{4L}, L - \frac{1}{4L}\right]$. Note also that $\tilde{u} \in \mathcal{S}_\sigma \subset \mathcal{E}$ by hypotheses. Furthermore, since u^L is a limit point of a sequence with uniformly bounded H_ε-cost, we also have $u^L \in \mathcal{E}$ by item (ii) in Theorem 2.5, namely $\tilde{u}$ and u^L are entropy-measure solutions to (1.1). By Lemma 5.1, $\tilde{u},\ u^L \in C\left([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\right)$. By the same Lemma 5.1 and the assumption $V_f^- > 0$, we have that the maps $x \mapsto \tilde{u}(t,x)$ and $x \mapsto u^L(t,x)$ are continuous from $\mathbb{R}$ to $L_1([0,T])$. Therefore, since the boundaries of $R^{L,+}$ and $R^{L,-} \setminus R^{L,+}$ are countable unions of segments parallel to the x and t axes, we have that (5.33) holds for each non-negative $\varphi \in C_{\mathrm{c}}^{\infty}\left((0,T) \times (-L + \frac{1}{4L}, L - \frac{1}{4L})\right)$.

Recalling $\{u^L\} \subset \mathcal{K}_0$, let u be a limit point of $\{u^L\}$ along a subsequence $L_k \to \infty$. From (5.33) we get for each non-negative $\varphi \in C_{\mathrm{c}}^{\infty}((0,T) \times \mathbb{R})$

$$-\langle\langle |u - \tilde{u}|, \varphi_t \rangle\rangle - \langle\langle |f(u) - f(\tilde{u})|, \varphi_x \rangle\rangle \leqq 0 \tag{5.34}$$

Reasoning as above, we also have $u \in \mathcal{E}$, and thus setting $z := u - \tilde{u}$, by Lemma 5.1, $u,\ \tilde{u},\ z \in C\left([0,T]; L_{1,\mathrm{loc}}(\mathbb{R})\right)$. By (5.34), it is then easily seen that for each bounded non-negative Lipschitz function φ on $[0,T] \times \mathbb{R}$ such that $\int \mathrm{d}t\, \mathrm{d}x\, [|\varphi| + |\varphi_t| + |\varphi_x|] < +\infty$, and for each $t \in [0,T]$

$$\begin{aligned} &\langle |z(t)|, \varphi(t) \rangle - \langle |z(0)|, \varphi(0) \rangle \\ &\quad - \int_{[0,t]} \mathrm{d}r \left[\langle |z(r)|, \varphi_r(r) \rangle + \langle |f(\tilde{u}(r) + z(r)) - f(\tilde{u}(r))|, \varphi_x(r) \rangle \right] \leqq 0 \end{aligned} \tag{5.35}$$

Fixed $L \geqq 1$, we evaluate the inequality (5.35) for $\varphi(t,x) \equiv \varphi^L(x)$ defined as

$$\varphi^L(x) := \begin{cases} e^{-(L-x)} & \text{if } x < -L \\ 1 & \text{if } -L \leqq x \leqq L \\ e^{-(x-L)} & \text{if } x > L \end{cases}$$

so that setting $Z^L(t) := \langle |z(t)|, \varphi^L \rangle$ we have

$$Z^L(t) - Z^L(0) \leqq V_f^+ \int_{[0,t]} \mathrm{d}r\, \langle |z(r)|, |\varphi_x^L| \rangle \leqq V_f^+ \int_{[0,t]} \mathrm{d}r\, Z^L(r)$$

By Gronwall inequality, for each $L \geqq 1$ and each $t \in [0,T]$, we have $Z^L(r) \leqq \exp[V_f^+ t] Z^L(0)$. Note that $u(0,x) = \tilde{u}(0,x)$ by (5.15) and the definition of convergence in $\mathcal{X}$. Therefore $Z^L(0) = 0$, and thus $Z^L(t) = 0$ for each $t \in [0,T]$ and $L \geqq 1$. Hence $u = \tilde{u}$. $\square$

Proof of Proposition 2.6. In order to show that H is lower semicontinuous, first note that the set of weak solutions is closed in $\mathcal{X}$. Moreover for each entropy sampler ϑ the map $\mathcal{X} \ni u \mapsto P_{\vartheta,u} \in \mathbb{R}$ is continuous. On the other hand, if u is a weak solution to (1.1) then the equalities in (5.2) holds; thus H is a supremum of continuous maps.

Since $D(\cdot)/\sigma(\cdot)$ is uniformly positive on $[0, 1]$, $H(u) = 0$ iff $u \in \mathcal{E}$ and $\varrho_u^+ = 0$, thus u is entropic. Conversely, entropic solutions u are in $\mathcal{E}$ by item (i) in Proposition 2.3, and the entropic condition is thus equivalent to $\varrho_u^+ = 0$.

The coercivity of H follows from the Tartar's method of compensated compactness, that we already applied in the proof of Theorem 2.5 item (ii). Suppose indeed that we are given a sequence $\{u^n\} \subset \mathcal{X}$ such that $H(u^n) \leqq C_H < +\infty$ for each n. Then each u^n is an entropy-measure solution to (1.1) by the definition of H. For each entropy η, each n, $L > 0$, by the same bound in the proof of Proposition 2.3, $\|\wp_{\eta,u^n}\|_{\mathrm{TV},L} \leqq 2\|\wp^+_{\eta,u^n}\|_{\mathrm{TV},L} + 2\,(\|\eta\|_\infty + \|q\|_\infty)(2L+T)$. On the other hand, for each $\eta \in C^2([0,1])$ such that $\sigma\eta'' \leqq D$, $\|\wp^+_{\eta,u^n}\|_{\mathrm{TV},L} \leqq H(u^n)$ and therefore $\|\wp_{\eta,u^n}\|_{\mathrm{TV},L}$ is bounded uniformly in n. Since η and q are bounded, we have that $\{\eta(u^n)_t + q(u^n)_x\}$ is precompact in $H^{-1}_{\mathrm{loc}}([0,T] \times \mathbb{R})$. As we already noted in the proof of Theorem 2.5 item (ii), see [15, Ch. 9], this yields the compactness of $\{u^n\}$ in $\mathcal{X}$. □

Proof of Remark 2.7. By well known properties of functions of locally bounded variation, for each entropy η and $u \in \mathcal{X} \cap BV_{\mathrm{loc}}([0,T] \times \mathbb{R})$, we have that $\wp_{\eta,u}$ is a Radon measure on $(0,T) \times \mathbb{R}$. If u is a weak solution to (1.1), by Vol'pert chain rule [2], the absolutely continuous and Cantor parts of $\wp_{\eta,u}$ with respect to the Lebesgue measure on $(0,T) \times \mathbb{R}$ vanish, and we get

$$d\wp_{\eta,u} = \Big\{\big[\eta(u^+) - \eta(u^-)\big]\, n^t + \big[q(u^+) - q(u^-)\big]\, n^x\Big\}\ \mathrm{d}\mathcal{H}^1 \llcorner J_u$$

On the other hand the Rankine–Hugoniot condition $\big[u^+ - u^-\big]\, n^t + \big[f(u^+) - f(u^-)\big]\, n^x = 0$ holds. The statement of the remark follows by direct calculations. □

Proof of Remark 2.11. For $u \in \mathcal{E}$ we have

$$H'(u) = \sup\Big\{\wp_{\eta,u}(\varphi),\ \varphi \in C_{\mathrm{c}}^\infty\left((0,T) \times \mathbb{R}; [0,1]\right), \\ \eta \in C^2([0,1]) \,:\, 0 \leqq \sigma\eta'' \leqq D\Big\}$$

so that the inequality $H \geqq H'$ follows from the equalities in (5.2). The same inequality yields $H(u) = H'(u)$ if there exists a set E^+ as in the statement of the remark. If f is convex or concave and u has locally bounded variation, we can take $E^+ = \{(t,x) \in J_u \,:\, \exists v \in [0,1] \,:\, \rho(v, u^+, u^-) > 0\}$, where J_u, $u^\pm$ and ρ are defined as in Remark 2.7.

If f is neither convex nor concave, then there exist u^-, u^+, v', $v'' \in (0,1)$ such that $\rho(v', u^+, u^-) > 0$ and $\rho(v'', u^+, u^-) < 0$, where ρ is defined as in

Remark 2.7. Let $V := \frac{f(u^+)-f(u^-)}{u^+-u^-}$, and define $u : [0,T]\times\mathbb{R} \to [0,1]$ by

$$u(t,x) := \begin{cases} u^+ & \text{for } x < V\,t \\ u^- & \text{for } x > V\,t \end{cases}$$

Then $u \in \mathcal{E}$ and by a direct computation $H(u) > H'(u)$. □

Appendix A: $\mathcal{I}$-approximation of atomic Young measures

Here we prove the claims stated in the proof of Theorem 4.1, *Step 1*, where the sets $\mathcal{M}_n^1$, $\widetilde{\mathcal{M}}_n^1$, $\overline{\mathcal{M}}_n^1$ are defined.

Claim 1. $\widetilde{\mathcal{M}}_1^n$ is $\mathcal{I}$-dense in $\overline{\mathcal{M}}_1^n$. For $n \geqq 1$, let $\mu \in \overline{\mathcal{M}}_1^n$, let G^μ be defined as in Lemma 4.2. Let also r, α^i, u^i be as in the definition of $\overline{\mathcal{M}}_1^n$ and L, μ_∞ be as in the definition (4.1) of $\mathcal{M}_g$. With no loss of generality, we can assume that $u^{i+1} \geqq u^i$, $i = 1,\dots,n-1$, since we can reorder the $u^i(t,x)$ for all (t,x) preserving continuity of the u^i and measurability of the α^i. Analogously it is not restrictive to assume, for $|x| > L$, $u^i(t,x) = u^i_\infty$, $\alpha^i(t,x) = \alpha^i_\infty$ for some constants u^i_∞, $\alpha^i_\infty \in (0,1]$; in particular $\mu_\infty = \sum_i \alpha^i_\infty \delta_{u^i_\infty}$.

Let now $\{\jmath^k\} \subset C_c^\infty(\mathbb{R}\times\mathbb{R})$ be a sequence of smooth mollifiers supported by $[-T/k, T/k]\times[-1,1]$, and recall the definition (4.8) of b^k. For $i = 1,\dots,n$ and $h,k \geqq 1$ define $\alpha^{i;k} \in C^1([0,T]\times\mathbb{R};[r,1])$, and $u^{i;h,k} \in C^1([0,T]\times\mathbb{R};[h^{-1},1-h^{-1}])$ by

$$\begin{aligned}
\alpha^{i;k}(t,x) &:= \int \mathrm{d}y\,\mathrm{d}s\; \jmath^k(t-s,x-y)\alpha^i\left(b^k(s,y)\right)\\
u^{i;h,k}(t,x) &:= h^{-1}\left[1 + \frac{i}{n\sum_{i'} i'\alpha^{i';k}(t,x)}\right]\\
&\quad + \frac{1-3\,h^{-1}}{\alpha^{i;k}(t,x)} \int \mathrm{d}y\,\mathrm{d}s\; \jmath^k(t-s,x-y)\,\alpha^i\left(b^k(s,y)\right)\,u^i\left(b^k(s,y)\right)
\end{aligned} \tag{A.1}$$

Clearly $\alpha^{i;k}$ and $u^{i;h,k}$ are smooth, with $\alpha^{i;k} \geqq r$, $\sum_i \alpha^{i;k} = 1$, and $\alpha^{i;k}$, $u^{i;h,k}$ are constant for $|x| > L+1$. Furthermore for $i = 1,\dots,n-1$ and $(t,x) \in [0,T]\times\mathbb{R}$

$$\begin{aligned}
&\lim_{k\to\infty}\left[u^{i+1;h,k}(t,x) - u^{i;h,k}(t,x) - \frac{h^{-1}}{n^2}\right]\\
&\quad\geqq \lim_{k\to\infty}\left[u^{i+1;h,k}(t,x) - u^{i;h,k}(t,x) - \frac{h^{-1}}{n\sum_{i'} i'\alpha^{i';k}(t,x)}\right]\\
&\quad= \left[1-3h^{-1}\right]\left[u^{i+1}(t,x) - u^i(t,x)\right]
\end{aligned}$$

Since the u^i are continuous, it is not difficult to see that convergence in the last line above is uniform on compact subsets of $[0,T]\times\mathbb{R}$. On the other hand, since the u^i and $u^{i;h,k}$ are constant for $|x| > L+1$, we have that convergence is indeed uniform on $[0, T] \times \mathbb{R}$. It follows that for each $h > 1$ there exists $K^h \geqq 1$ such that $u^{i+1;h,k} \geqq u^{i;h,k} + h^{-1}n^{-2}/2$ for each $k \geqq K^h$. Therefore, defining $\mu^{h,k} \in \mathcal{M}$ by

$$\mu^{h,k}_{t,x} := \sum_{i=1}^{n} \alpha^{i;k}(t,x)\delta_{u^{i;h,k}(t,x)}$$

we get, for $k \geqq K^h$, $\mu^{h,k} \in \widetilde{\mathcal{M}}^n_1$ provided $\mathcal{I}(\mu^{h,k}) < +\infty$. Recalling Lemma 4.2, this follows by the existence of $G^{\mu^{h,k}} \in L_2([0,T] \times \mathbb{R})$ satisfying weakly on $(0,T) \times \mathbb{R}$:

$$\mu^{h,k}(\imath)_t + \mu^{h,k}(f)_x = -G^{\mu^{h,k}}_x$$

Indeed $G^{\mu^{h,k}}$ can be computed explicitly as

$$\begin{aligned} G^{\mu^{h,k}}(t,x) := & \left(1 - 3h^{-1}\right) \int \mathrm{d}s\, \mathrm{d}y\, \jmath^k(t-s, x-y) G^\mu \left(b^k(s,y)\right) \\ & + \left(1 - 3h^{-1}\right) \int \mathrm{d}s\, \mathrm{d}y\, \jmath^k(x-y, t-s)\, \mu_{b^k(s,y)}(f) \\ & - \mu^{h,k}(f) - (1 - 3h^{-1})\mu_\infty(f) + \mu^{h,k}_\infty(f) \end{aligned}$$

where

$$\mu^{h,k}_\infty(f) := \sum_i \alpha^i_\infty f \left(h^{-1} + \frac{i\, h^{-1}}{n \sum_{i'} i' \alpha^{i'}_\infty} + \left(1 - 3h^{-1}\right) u^i_\infty \right)$$

It immediately follows that $\lim_{h\to\infty} \lim_{k\to\infty} \|G^{\mu^{h,k}} - G^\mu\|_{L_2([0,T]\times\mathbb{R})} = 0$, and it is also straightforward to see that, for each $F \in C([0,1])$

$$\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k}(F) = \mu(F) \quad \text{strongly in } L_{1,\mathrm{loc}}([0,T] \times \mathbb{R})$$

By Remark 4.3, we can extract a subsequence $\{\mu^k\}$ from $\{\mu^{h,k}\}$ that $\mathcal{I}$-converges to μ.

Claim 2. $\overline{\mathcal{M}}^n_1$ is $\mathcal{I}$-dense in $\mathcal{M}^n_1$. For $n \geqq 1$, let $\mu \in \mathcal{M}^n_1$. Let also α^i, u^i and L be as in the definition of $\mathcal{M}^n_1$ and $\mathcal{M}_g$. With no loss of generality, we can assume that $\alpha^i > 0$, since we do not require the u^i to be distinct. As in *Claim 1* above, we can also assume that, for $|x| > L$, $u^i(t,x) = u^i_\infty$, $\alpha^i(t,x) = \alpha^i_\infty$ for some constants u^i_∞, $\alpha^i_\infty \in [0,1]$.

With these assumptions, for $h,\ k \geqq 1$ and $i = 1, \ldots, n$, let us define $\alpha^{i;k}$ as in (A.1), and $u^{i;k}$ by

$$u^{i;k}(t,x) := \frac{1}{\alpha^{i;k}} \int \mathrm{d}y\, \mathrm{d}s\, \jmath^k(t-s, x-y) \alpha^i \left(b^k(s,y)\right) u^i \left(b^k(s,y)\right)$$

Letting

$$\mu^k_{t,x} := \sum_{i=1}^{n} \alpha^{i;k}(t,x)\delta_{u^{i;k}(t,x)}$$

we gather $\mu^k \in \overline{\mathcal{M}}^n_1$. A computation similar to the one carried out in *Claim 1* shows that μ^k $\mathcal{I}$-converges to μ as $k \to \infty$.

Claim 3. $\mathcal{M}_1^n$ is $\mathcal{I}$-dense in $\widetilde{\mathcal{M}}_1^{n+1}$. This is the key step in the proof of Theorem 4.1. For $n \geqq 1$, let $\mu \in \widetilde{\mathcal{M}}_1^{n+1}$, and let G^μ be defined as in Lemma 4.2. Let also r, α^i, u^i be as in the definition of $\widetilde{\mathcal{M}}_1^{n+1}$, and L, μ_∞ as in the definition (4.1) of $\mathcal{M}_g$. Note that for $|x| > L$, $\alpha^i(t,x) = \alpha^i_\infty$ and $u^i(t,x) = u^i_\infty$ for some constants $\alpha^i_\infty \in [r, 1-r]$, $u^i_\infty \in [r, 1-r]$, with $u^{i+1}_\infty \geqq u^i_\infty + r$, $i = 1, \dots, n$.

Let us define the Young measures $\nu^1,\ \nu^0 \in \widetilde{\mathcal{M}}_1^n$ by

$$\nu^1_{t,x} := \delta_{u^{n+1}(t,x)}, \qquad \nu^0_{t,x} := \sum_{i=1}^n \frac{\alpha^i(t,x)}{1-\alpha^{n+1}(t,x)} \delta_{u^i(t,x)}$$

so that, letting $\beta(t,x) := \alpha^{n+1}(t,x) \leqq 1-r$

$$\mu_{t,x} = \beta(t,x)\nu^1_{t,x} + (1-\beta(t,x))\,\nu^0_{t,x}$$

The basic idea is to build up a sequence $\{\mu^k\}$ $\mathcal{I}$-converging to μ, as follows: we first slice up $[0,T] \times \mathbb{R}$ in small strips, alternating a strip of width $\beta\, k^{-1}$ with a strip of width $(1-\beta)k^{-1}$; we then set $\mu^k_{t,x} = \nu^1_{t,x}$ for (t,x) in the first family of strips, and $\mu^k_{t,x} = \nu^0_{t,x}$ for (t,x) in the second family of strips. As we let $k \to \infty$, we easily get $\mu^k \to \mu$; however, to get also $\mathcal{I}(\mu^k) \to \mathcal{I}(\mu)$, we will have to carefully define these strips.

For $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, let us consider the maps $\gamma^k_j : [0,T] \to \mathbb{R}$ solutions to

$$\begin{cases} \dot{\gamma} = \dfrac{\nu^1_{t,\gamma}(f) - \nu^0_{t,\gamma}(f)}{\nu^1_{t,\gamma}(\imath) - \nu^0_{t,\gamma}(\imath)} \\ \gamma(0) = \dfrac{j}{k} \end{cases} \tag{A.2}$$

These equations are well posed since $\nu^1(f)$, $\nu^0(f)$, $\nu^1(\imath)$, $\nu^0(\imath)$ are Lipschitz functions in the (t,x) variables, and $\nu^1(\imath) - \nu^0(\imath) \geqq r$, by the definition of $\widetilde{\mathcal{M}}_1^{n+1}$. Furthermore, by standard theory for (A.2), $\gamma^k_j \in C^0([0,T]) \cap C^1((0,T))$; $|\dot{\gamma}^k_j| \leqq 2r^{-1} \max_{v\in[0,1]} |f(v)|$; $\gamma^k_{j+1} > \gamma^k_j$; and $\gamma^k_{j+1}(t) - \gamma^k_j(t) \leqq Ck^{-1}$ for some constant C independent of k, j and t.

We next define the maps $\beta^k_j : [0,T] \to \mathbb{R}$ by

$$\int_{\gamma^k_j(t)}^{\gamma^k_j(t)+\beta^k_j(t)} \mathrm{d}x \left[\nu^1_{t,x}(\imath) - \nu^0_{t,x}(\imath)\right] = \int_{\gamma^k_j(t)}^{\gamma^k_{j+1}(t)} \mathrm{d}x\, \beta(t,x) \left[\nu^1_{t,x}(\imath) - \nu^0_{t,x}(\imath)\right] \tag{A.3}$$

Since $\nu^1_{t,x}(\imath) - \nu^0_{t,x}(\imath) \geqq r > 0$, for any fixed $t \in [0,T]$ the left-hand side of this equation is strictly increasing in $\beta^k_j(t)$. Since it vanishes for $\beta^k_j(t) = 0$ and it is larger than the right-hand side for $\beta^k_j(t) = \gamma^k_{j+1}(t) - \gamma^k_j(t)$ (recall $\beta(t,x) \in [r, 1-r]$), there exists a unique $0 < \beta^k_j(t) < \gamma^k_{j+1}(t) - \gamma^k_j(t)$ satisfying (A.3). Furthermore, since β and the γ_j^k are smooth, we have $\beta_j^k \in C^0([0,T]) \cap C^1((0,T))$. The mean value theorem then implies

$$\left|\beta_j^k(t) - \int_{\gamma_j^k(t)}^{\gamma_{j+1}^k(t)} \mathrm{d}x\, \beta(t,x)\right| \leqq C\left[\gamma_{j+1}^k(t) - \gamma_j^k(t)\right]^2 \leqq C' k^{-2} \tag{A.4}$$

for suitable constants $C,\ C'$. For h and k two positive integers, we next define the Young measure $\mu^{h,k} \in \mathcal{M}$ by

$$\mu_{t,x}^{h,k} := \begin{cases} \nu_{t,x}^0 & \text{if } \exists j \in \mathbb{Z},\ |j| \leqq h\,k \text{ such that } \gamma_j^k(t) + \beta_j^k(t) < x < \gamma_{j+1}^k(t) \\ \nu_{t,x}^1 & \text{otherwise} \end{cases}$$

Since $\nu_{t,x}^1$ is constant for $|x|$ sufficiently large, we have $\mu^{h,k} \in \mathcal{M}_g$ for h large enough. Furthermore, since convergence in $\mathcal{M}$ is *local*, (A.4) yields $\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k} = \mu$ in $\mathcal{M}$, and for each $F \in C([0,1])$

$$\lim_{h\to\infty} \lim_{k\to\infty} \mu^{h,k}(F) = \mu(F) \quad \text{strongly in } L_{1,\mathrm{loc}}([0,T]\times\mathbb{R})$$

We next prove that $\mathcal{I}(\mu^{h,k}) < +\infty$ and $\lim_h \lim_k G^{\mu^{h,k}} = G$ in $L_2([0,T]\times\mathbb{R})$; so that, reasoning as in the proof of *Claim 1*, by Remark 4.3 we get the existence of a subsequence $\{\mu^k\}$ $\mathcal{I}$-converging to μ. For each $F \in C([0,1])$, $(t,x) \mapsto \mu_{t,x}^{h,k}(F)$ is smooth outside the graph of the curves γ_j^k. Therefore, by Lemma 4.4, there exists $G^{h,k} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$ such that $\mu^{h,k}(\imath)_t + \mu^{h,k}(f)_x = -G_x^{h,k}$ holds weakly. First we show that we can choose $G^{h,k}$ to be compactly supported, so that $G^{h,k} \in L_2([0,T]\times\mathbb{R})$, and thus $\mathcal{I}(\mu^{h,k}) < +\infty$ with $G^{h,k} = G^{\mu^{h,k}}$, according to the definition given in Lemma 4.2.

Since $G^{h,k}$ is defined up to a measurable function of t, and $G_x^{h,k}(t,x) = 0$ for $x < \gamma_{-hk}^k(t)$ (we are considering h large enough as above), we can assume $G^{h,k}(t,x) = G^{\mu}(t,x) = 0$ for $x < \gamma_{-hk}^k(t)$. Furthermore, by (4.7) and (A.2), for each $j \in \mathbb{Z}$, $G^{h,k}$ is continuous in the regions $\{(t,x) : \gamma_j^k(t) + \beta_j^k(t) < x < \gamma_{j+1}^k(t) + \beta_{j+1}^k(t)\}$. Let now $j \in \mathbb{Z}$ with $|j| \leqq hk$, and $t \in [0,T]$; by (4.7) and (A.2)

$$\begin{aligned} &-\left[G^{h,k}\left(t, [\gamma_j^k(t) + \beta_j^k(t)]^-\right) - G^{h,k}\left(t, \gamma_j^k(t)\right)\right] \\ &\quad = \nu^1_{t,\gamma_j^k(t)+\beta_j^k(t)}(f) - \nu^1_{t,\gamma_j^k(t)}(f) + \int_{\gamma_j^k(t)}^{\gamma_j^k(t)+\beta_j^k(t)} \mathrm{d}x\,\left[\nu_{t,x}^1(\imath)\right]_t \end{aligned}$$

and

$$\begin{aligned} &-\left[G^{h,k}\left(t, \gamma_{j+1}^k(t)\right) - G^{h,k}\left(t, [\gamma_j^k(t) + \beta_j^k(t)]^-\right)\right] \\ &= \nu^0_{t,\gamma_{j+1}^k(t)}(f) - \nu^0_{t,\gamma_j^k(t)+\beta_j^k(t)}(f) \end{aligned}$$

$$+\int_{\gamma_j^k(t)+\beta_j^k(t)}^{\gamma_{j+1}^k(t)} \mathrm{d}x\, \left[\nu^0_{t,x}(\iota)\right]_t + \left[\nu^0_{t,\gamma_j^k(t)+\beta_j^k(t)}(f) - \nu^1_{t,\gamma_j^k(t)+\beta_j^k(t)}(f)\right]$$

$$-\left[\nu^0_{t,\gamma_j^k(t)+\beta_j^k(t)}(\iota) - \nu^1_{t,\gamma_j^k(t)+\beta_j^k(t)}(\iota)\right]\left[\dot{\gamma}_j^k(t) + \dot{\beta}_j^k(t)\right]$$

By (A.2), (A.3) and simple algebraic manipulations

$$\begin{aligned}
&G^{h,k}\left(t, \gamma_{j+1}^k(t)\right) - G^{h,k}\left(t, \gamma_j^k(t)\right) \\
&\quad = -\left[\mu_{t,\gamma_{j+1}^k(t)}(f) - \mu_{t,\gamma_j^k(t)}(f)\right] - \int_{\gamma_j^k(t)}^{\gamma_{j+1}^k(t)} \mathrm{d}x\, \left[\mu_{t,x}(\iota)\right]_t \\
&\quad = G^{\mu}\left(t, \gamma_{j+1}^k(t)\right) - G^{\mu}\left(t, \gamma_j^k(t)\right)
\end{aligned}$$

Since $G^{h,k}(t, \gamma_{-hk}^k(t)) = G^{\mu}(t, \gamma_{-hk}^k(t)) = 0$, we deduce that for each $j \in \mathbb{Z}$. we have $G^{h,k}(t, \gamma_j^k(t)) = G^{\mu}(t, \gamma_j^k(t))$. In particular, since $G^{\mu}(t, \gamma_{hk}^k(t)) = 0$ and $G_x^{h,k}(t,x) = G_x^{\mu}(t,x) = 0$ for $x > \gamma_{hk}^k(t)$, we have $G^{h,k}(t,x) = G^{\mu}(t,x) = 0$ for $x > \gamma_{hk}^k(t)$ and $x < \gamma_{-hk}^k(t)$. That is, $G^{h,k}$ and G^{μ} are compactly supported. Thus $\mathcal{I}(\mu^{h,k}) < +\infty$ and $G^{h,k} = G^{\mu^{h,k}}$.

Finally, by the definition of G^{μ} and $G^{\mu^{h,k}}$, recalling $G^{h,k}(t, \gamma_j^k(t)) = G^{\mu}(t, \gamma_j^k(t))$, we have

$$\left\| G^{\mu^{h,k}} - G^{\mu} \right\|^2_{L_2([0,T]\times\mathbb{R})} = \sum_{j=-hk}^{hk} \int_{[0,T]} \mathrm{d}t \int_{\gamma_j^k(t)}^{\gamma_{j+1}^k(t)} \mathrm{d}x\, \left(G^{\mu^{h,k}}(t,x) - G^{\mu}(t,x)\right)^2$$

$$\begin{aligned}
&= \sum_{j=-hk}^{hk} \int_{[0,T]} \mathrm{d}t \\
&\times \left\{ \int_{\gamma_j^k(t)}^{\gamma_j^k(t)+\beta_j^k(t)} \mathrm{d}x \left[\int_{\gamma_j^k(t)}^{x} \mathrm{d}y [\nu^1_{t,y}(\iota) - \mu_{t,y}(\iota)]_t + [\nu^1_{t,y}(f) - \mu_{t,y}(f)]_y\right]^2 \right. \\
&\left. + \int_{\gamma_j^k(t)+\beta_j^k(t)}^{\gamma_{j+1}^k(t)} \mathrm{d}x \left[\int_{\gamma_{j+1}^k(t)}^{x} \mathrm{d}y\, [\nu^0_{t,y}(\iota) - \mu_{t,y}(\iota)]_t + [\nu^0_{t,y}(f) - \mu_{t,y}(f)]_y\right]^2 \right\}
\end{aligned}$$

Since all the integrands in the last two lines of this formula are bounded uniformly in h and k, each term of the sum is bounded by $C\,k^{-3}$ for some constant $C > 0$. Therefore the sum itself is bounded by $2\,C\,h\,k^{-2}$, and we get $\lim_{h\to\infty} \lim_{k\to\infty} \mathcal{I}(\mu^{h,k}) = \mathcal{I}(\mu)$.

Appendix B: Γ-viscosity cost for scalar Hamilton–Jacobi equations

In this appendix we establish a Γ-convergence result for a sequence of functionals associated with the Hamilton–Jacobi equation (1.1)

$$b_t + f(b_x) = 0 \tag{B.1}$$

which is related to (1.1) via the transformation $u = b_x$. In (B.1) we understand $(t, x) \in [0, T] \times \mathbb{R}$ and $b(t, x) \in \mathbb{R}$. As usual, we assume f to be a Lipschitz function on $[0, 1]$, D and σ continuous functions on $[0, 1]$, with D uniformly positive and σ strictly positive on $(0, 1)$. We will just sketch most of the proofs, since they are similar to the proofs of the corresponding statements for (1.1).

We introduce the equivalence $\sim$ on $C([0, T]; L_{2,\mathrm{loc}}(\mathbb{R}))$ by setting $b^1 \sim b^2$ iff $b^1 - b^2$ is constant in $[0, T] \times \mathbb{R}$. We let $\mathcal{B}$ be the set of functions $b \in C([0, T]; L_{2,\mathrm{loc}}(\mathbb{R}))/\sim$ such that $b_x \in \mathcal{U}$. The requirement $b_x \in \mathcal{U}$ is clearly compatible with $\sim$, so that $\mathcal{B}$ is well defined. We equip $\mathcal{B}$ with the metric

$$d_{\mathcal{B}}(b^1, b^2) := d_{\mathcal{U}}(b^1_x, b^2_x) + \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \sum_{N=1}^{\infty} \frac{1}{2^N} \| b^1(t, \cdot) - b^2(t, \cdot) + c \|_{L_2([-N,N])} \tag{B.2}$$

Note that the second term in the right-hand side of (B.2) is the projection of the $C([0, T]; L_{2,\mathrm{loc}}(\mathbb{R}))$-distance with respect to the $\sim$ equivalence. $(\mathcal{B}, d_{\mathcal{B}})$ is a complete separable metric space.

For $b \in \mathcal{B}$ such that $b_{xx} \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$ and $\varepsilon > 0$ we next define the linear functional a^b_ε on $C^\infty_{\mathrm{c}}((0, T) \times \mathbb{R})$ by

$$a^b_\varepsilon(\varphi) := -\langle\langle b, \varphi_t \rangle\rangle + \langle\langle f(b_x), \varphi \rangle\rangle - \frac{\varepsilon}{2} \langle\langle D(b_x) b_{xx}, \varphi \rangle\rangle \tag{B.3}$$

and the functional $J_\varepsilon : \mathcal{B} \mapsto [0, +\infty]$ as follows. If $b_{xx} \in L_{2,\mathrm{loc}}([0, T] \times \mathbb{R})$ we set

$$J_\varepsilon(b) := \sup_{\varphi\in C^\infty_{\mathrm{c}}((0,T)\times\mathbb{R})} \left[a^b_\varepsilon(\varphi) - \frac{1}{2} \langle\langle \sigma(b_x)\varphi, \varphi \rangle\rangle \right] \tag{B.4}$$

letting $J_\varepsilon(b) = +\infty$ otherwise. We want to study the Γ-convergence of $\{J_\varepsilon\}$. As shown below, this problem is strictly related to the Γ-convergence of $\{I_\varepsilon\}$ defined in (2.6).

We introduce the set $\mathcal{A} := \{(b, \mu) \in \mathcal{B} \times \mathcal{M} \,:\, b_x = \mu(\imath)\}$, which we equip with the metric

$$d_{\mathcal{A}}\Big((b^1, \mu^1), (b^2, \mu^2)\Big) := d_{\mathcal{B}}(b^1, b^2) + d_{\mathcal{M}}(\mu^1, \mu^2) \tag{B.5}$$

We say that $(b, \mu) \in \mathcal{A}$ is a measure-valued solution to (B.1) iff $b_t + \mu(f) = 0$ weakly in $(0, T) \times \mathbb{R}$. We lift J_ε to a functional $\mathcal{J}_\varepsilon : \mathcal{A} \to [0, +\infty]$ by setting

$$\mathcal{J}_\varepsilon(b, \mu) := \begin{cases} J_\varepsilon(b) & \text{if} \quad \mu_{t,x} = \delta_{b_x(t,x)} \\ +\infty & \text{otherwise} \end{cases} \tag{B.6}$$

Theorem B.1. *The sequence $\{\mathcal{J}_\varepsilon\}$ is equicoercive on $\mathcal{A}$ and Γ-converges to*

$$\mathcal{J}((b,\mu)) := \sup_{\varphi \in C_c^\infty((0,T)\times\mathbb{R})} \left\{ -\langle\langle b, \varphi_t\rangle\rangle + \langle\langle \mu(f), \varphi\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\varphi, \varphi\rangle\rangle \right\}$$

Note that $\mathcal{J}((b,\mu)) = 0$ iff (b,μ) is a measure-valued solution to (B.1).

On the set $\mathcal{B}$ we next introduce the metric $d_{\mathcal{Y}}$

$$d_{\mathcal{Y}}(b^1, b^2) := d_{\mathcal{X}}(b^1_x, b^2_x) + \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \sum_{N=1}^{\infty} \frac{1}{2^N} \|b^1(t) - b^2(t) + c\|_{L_2([-N,N])}$$

and denote by $(\mathcal{Y}, d_{\mathcal{Y}})$ the complete separable metric space consisting of the same set $\mathcal{B}$ equipped with the distance $d_{\mathcal{Y}}$. We say that $b \in \mathcal{Y}$ is a weak solution to (B.1) iff $-\langle\langle b, \varphi_t\rangle\rangle + \langle\langle f(b_x), \varphi\rangle\rangle = 0$ for each $\varphi \in C_c^\infty((0,T)\times\mathbb{R})$. We denote by $\mathcal{W} \subset \mathcal{Y}$ the set of weak solutions to (B.1). We rescale the functional J_ε defining $K_\varepsilon : \mathcal{Y} \to [0, +\infty]$ as

$$K_\varepsilon := \varepsilon^{-1} J_\varepsilon \tag{B.7}$$

Theorem B.2. *Let K_ε be the functional on $\mathcal{W}$ as defined in* (B.7) *and* (B.4).

(i) *The sequence of functionals $\{K_\varepsilon\}$ satisfies the Γ-liminf inequality*

$$\left(\Gamma\text{–}\varliminf_{\varepsilon\to 0} K_\varepsilon\right)(b) \geqq \begin{cases} H(b_x) & \text{if } b \in \mathcal{W} \\ +\infty & \text{otherwise} \end{cases}$$

(ii) *Assume there is no interval where f is affine. Then the sequence $\{K_\varepsilon\}$ is equicoercive on $\mathcal{Y}$.*

(iii) *Suppose furthermore $f \in C^2([0,1])$ and $D, \sigma \in C^\alpha([0,1])$ for some $\alpha > 1/2$. Then*

$$\left(\Gamma\text{–}\varlimsup_{\varepsilon\to 0} K_\varepsilon\right)(b) \leqq \begin{cases} \overline{H}(b_x) & \text{if } b \in \mathcal{W} \\ +\infty & \text{otherwise} \end{cases}$$

Since $b(0,\cdot)$ is bounded and Lipschitz, by a well known connection between entropic solutions to (1.1) and viscosity solutions to (B.1), see, for example [10, Theorem 1.1], we gather $(\Gamma\text{–}\varliminf_\varepsilon K_\varepsilon)(b) = 0$ iff b is a viscosity solutions to (B.1). It follows that if b^ε satisfies the equation

$$b_t + f(b_x) = \frac{\varepsilon}{2} D(b_x) b_{xx} - \sigma(b_x) E^\varepsilon \tag{B.8}$$

for some E^ε such that $\lim_\varepsilon \varepsilon \|E^\varepsilon\|^2_{L_2([0,T]\times\mathbb{R},\sigma(b_x)\mathrm{d}t\,\mathrm{d}x)} = 0$, then limit points of $\{b^\varepsilon\}$ are viscosity solutions to (B.1). On the other hand if b^ε solves (B.8) for some E^ε with $\varepsilon\|E^\varepsilon\|^2_{L_2([0,T]\times\mathbb{R},\sigma(b_x)\mathrm{d}t\,\mathrm{d}x)}$ uniformly bounded, then limit points b of $\{b^\varepsilon\}$ are such that $b_x \in \mathcal{E}$.

In order to prove Theorem B.1 and Theorem B.2, we first establish some preliminary results. Given a measurable map $a : [0,T]\times\mathbb{R} \to [0,+\infty]$, we let $\mathcal{L}_a$ be the Hilbert space obtained by identifying and completing the set $\{\varphi \in C_c^\infty((0,T)\times\mathbb{R}) : \langle\langle a\varphi, \varphi\rangle\rangle < +\infty\}$ with respect to the seminorm $\langle\langle a\varphi, \varphi\rangle\rangle$.

Lemma B.3. *Let $\varepsilon > 0$ and $b \in \mathcal{B}$ be such that $J_\varepsilon(b) < +\infty$. Then there exists $E^{\varepsilon,b} \in \mathcal{L}_{\sigma(b_x)}$ such that*

$$b_t + f(b_x) = \frac{\varepsilon}{2} D(b_x)\, b_{xx} - \sigma(b_x)\, E^{\varepsilon,b} \tag{B.9}$$

holds weakly on $(0, T)\times\mathbb{R}$ and $J_\varepsilon(b) = \frac{1}{2}\|E^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}}$. Furthermore $I_\varepsilon(b_x) < +\infty$ and there exists $\gamma^{\varepsilon,b} \in \mathcal{L}_{\sigma(b_x)^{-1}}$ such that $\gamma^{\varepsilon,b}_x = 0$ and $\sigma(b_x)E^{\varepsilon,b} = \sigma(b_x)\Psi^{\varepsilon,b_x} + \gamma^{\varepsilon,b}$, where Ψ^{ε,b_x} is defined as in Lemma 3.1. In particular

$$J_\varepsilon(b) = \frac{1}{2}\|\Psi^{\varepsilon,b_x}\|^2_{\mathcal{D}^1_{\sigma(b_x)}} + \frac{1}{2}\|\gamma^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}} = I_\varepsilon(b_x) + \frac{1}{2}\langle\langle \sigma(b_x)^{-1}\gamma^{\varepsilon,b}, \gamma^{\varepsilon,b}\rangle\rangle \tag{B.10}$$

Proof. The existence of $E^{\varepsilon,b}$, (B.9) and the equality $J_\varepsilon(b) = \frac{1}{2}\|E^{\varepsilon,b}\|^2_{\mathcal{L}_{\sigma(b_x)}}$ are achieved as in Lemma 3.1. We also have

$$\begin{aligned} J_\varepsilon(b) &= \sup_{\varphi\in C^\infty_c((0,T)\times\mathbb{R})} \left\{ a^b_\varepsilon(\varphi) - \frac{1}{2}\langle\langle \sigma(b_x)\varphi, \varphi\rangle\rangle \right\} \\ &\geqq \sup_{\phi\in C^\infty_c((0,T)\times\mathbb{R})} \left\{ a^b_\varepsilon(\phi_x) - \frac{1}{2}\langle\langle \sigma(b_x)\phi_x, \phi_x\rangle\rangle \right\} \\ &= \sup_{\phi\in C^\infty_c((0,T)\times\mathbb{R})} \left\{ \ell^\varepsilon_{b_x}(\phi) - \frac{1}{2}\langle\langle \sigma(b_x)\phi_x, \phi_x\rangle\rangle \right\} = I_\varepsilon(b_x) \end{aligned}$$

By (3.1) and (B.9) there exists $\Psi^{\varepsilon,b_x} \in \mathcal{D}^1_{\sigma(b_x)}$ such that $(\sigma(b_x)E^{\varepsilon,b})_x = (\sigma(b_x)\Psi^{\varepsilon,b_x}_x)_x$, namely $\sigma(b_x)E^{\varepsilon,b} = \sigma(b_x)\Psi^{\varepsilon,b_x}_x + \gamma^{\varepsilon,b}(t)$ for some measurable map $\gamma^{\varepsilon,b} : [0, T] \to [-\infty, +\infty]$. It is then easy to check (B.10). $\square$

The following lemma is proven analogously.

Lemma B.4. *Let $(b, \mu) \in \mathcal{A}$ be such that $\mathcal{J}((b, \mu)) < +\infty$. Then there exists $E^{(b,\mu)} \in \mathcal{L}_{\mu(\sigma)}$ such that*

$$b_t + \mu(f) = -\mu(\sigma)E^{(b,\mu)}$$

and $\mathcal{J}((b, \mu)) = \frac{1}{2}\|E^{(b,\mu)}\|^2_{\mathcal{L}_{\mu(\sigma)}}$. Furthermore $\mathcal{I}(\mu) < +\infty$ and there exists $\gamma^{(b,\mu)} \in \mathcal{L}_{\mu(\sigma)^{-1}}$ such that $\gamma^{(b,\mu)}_x = 0$ and

$$\mathcal{J}((b, \mu)) = \frac{1}{2}\|\Psi^\mu\|^2_{\mathcal{D}^1_{\mu(\sigma)}} + \frac{1}{2}\|\gamma^{(b,\mu)}\|^2_{\mathcal{L}_{\mu(\sigma)}} = \mathcal{I}(\mu) + \frac{1}{2}\langle\langle \mu(\sigma)^{-1}\gamma^{(b,\mu)}, \gamma^{(b,\mu)}\rangle\rangle \tag{B.11}$$

where Ψ^μ is defined as in Lemma 4.2.

Lemma B.5. *The sequence of functional $\{J_\varepsilon\}$ is equicoercive on $(\mathcal{B}, d_{\mathcal{B}})$.*

Proof. Let $\{b^\varepsilon\} \subset \mathcal{B}$ be such that $J_\varepsilon(b^\varepsilon) \leqq C_J$ for some $C_J < +\infty$. By (B.10) $I_\varepsilon(b^\varepsilon_x) \leqq C_J$, and thus $\{b^\varepsilon_x\}$ is precompact in $\mathcal{U}$ by Lemma 3.3. We are left with the proof of the compactness of $\{b^\varepsilon\}$ with respect to the second term on the right-hand side of (B.2). By (B.10) and (3.6) we have that for any $N > 0$, $\varepsilon^2 \int_{[0,T]\times[-N,N]} \mathrm{d}t\, \mathrm{d}x\, (b^\varepsilon_{xx})^2 \leqq C(C_J + \varepsilon N + 1)$ for some constant $C > 0$ depending only on f and D. It then follows by (B.9) that for each $N > 0$, $\|b^\varepsilon_t\|_{L_2([0,T]\times[-N,N]}$ is bounded uniformly in ε. Since $b_x \in \mathcal{U}$ for each $b \in \mathcal{B}$, we also have $0 \leqq b^\varepsilon_x \leqq 1$. Recalling that elements in b are defined up to a constant, we see that the conclusion follows by these bounds on b^ε_t, b^ε_x and compact Sobolev embedding. □

The following remark follows by Proposition 3.3 and Lemma B.3 and the definition (B.2) of $d_{\mathcal{B}}$.

Remark B.6. For each $\varepsilon > 0$, J^ε is lower semicontinuous on $(\mathcal{B}, d_{\mathcal{B}})$.

Lemma B.7. *For each $u \in \mathcal{U}$ such that $I_\varepsilon(u) < +\infty$ there exists $b^{\varepsilon,u} \in \mathcal{B}$ such that $b^{\varepsilon,u}_x = u$ and $J_\varepsilon(b^{\varepsilon,u}) = I_\varepsilon(u)$. Furthermore if $b \in \mathcal{B}$ is such that $b_x = u$ and $J_\varepsilon(b) < +\infty$, then $b_t = b^{\varepsilon,u}_t + \gamma^{\varepsilon,b}$, where $\gamma^{\varepsilon,b} \in \mathcal{L}_{\sigma(u)^{-1}}$ is defined as in Lemma B.3. Conversely, given $\gamma \in \mathcal{L}_{\sigma(u)^{-1}}$ with $\gamma_x = 0$, there exists a unique $b \in \mathcal{B}$ such that $b_x = u$ and $b_t = b^{\varepsilon,u}_t + \gamma$.*

Proof. From the definitions (2.6) and (B.4), it is not difficult to gather

$$I_\varepsilon(u) = \inf_{b\in\mathcal{B}\,:\, b_x=u} J_\varepsilon(b)$$

Since J_ε is coercive and lower semicontinuous on $\mathcal{B}$, and $\{b \in \mathcal{B} : b_x = u\}$ is a closed subset of $\mathcal{B}$, there exists a $b^{\varepsilon,u}$ on which the infimum is attained.

If b is such that $b_x = u$ and $J_\varepsilon(b) < +\infty$, then by the decomposition of $E^{\varepsilon,\cdot}$ in Lemma B.3 we have $(b - b^{\varepsilon,u})_t = \sigma(u)(E^{\varepsilon,b} - E^{\varepsilon,b^{\varepsilon,u}}) = \gamma^{\varepsilon,b}$. The converse statement follows by choosing $b(t,x) = b^{\varepsilon,u}(t,x) + \int^t \mathrm{d}s\,\gamma(s)$, which identifies a unique $b \in \mathcal{B}$. □

Proof of Theorem B.1. Equicoercivity follows by (B.10), the equicoercivity statement in Theorem 2.1 and Lemma 3.3.

In order to prove the Γ-liminf inequality, let $\{(b^\varepsilon, \mu^\varepsilon)\} \subset \mathcal{A}$ converge to some $(b,\mu) \in \mathcal{A}$. It is not restrictive to assume $J_\varepsilon(b^\varepsilon) < +\infty$, and thus $b^\varepsilon_{xx} \in L_{2,\mathrm{loc}}([0,T]\times\mathbb{R})$ and $\mu^\varepsilon = \delta_{b^\varepsilon_x}$. Then for each $\varphi \in C^\infty_{\mathrm{c}}((0,T)\times\mathbb{R})$

$$\begin{aligned}
&\mathcal{J}_\varepsilon\left((b^\varepsilon, \mu^\varepsilon)\right) = J_\varepsilon(b^\varepsilon)\\
&\geqq -\langle\langle b^\varepsilon, \varphi_t\rangle\rangle + \langle\langle f(b^\varepsilon_x), \varphi\rangle\rangle - \frac{\varepsilon}{2}\langle\langle D(b^\varepsilon_x) b^\varepsilon_{xx}, \varphi\rangle\rangle - \frac{1}{2}\langle\langle \varphi, \sigma(b^\varepsilon_x)\varphi\rangle\rangle
\end{aligned}$$

As in the proof of the Γ-liminf inequality in Theorem 2.1, an integration by parts shows that the third term in the left-hand side vanishes as $\varepsilon \to 0$. Hence

$$\varliminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon\left((b^\varepsilon, \mu^\varepsilon)\right) \geqq -\langle\langle b, \varphi_t\rangle\rangle + \langle\langle \mu(f), \varphi\rangle\rangle - \frac{1}{2}\langle\langle \mu(\sigma)\varphi, \varphi\rangle\rangle$$

and the Γ-liminf inequality is achieved by optimizing over φ.

Let $(b, \mu) \in \mathcal{A}$ be such that $\mathcal{J}((b, \mu)) < +\infty$. By Lemma B.4 $\mathcal{I}(\mu) < +\infty$ and by the Γ-limsup inequality in Theorem 2.1 there exists a sequence $\{u^\varepsilon\} \subset \mathcal{U}$ such that $\delta_{u^\varepsilon} \to \mu$ in $\mathcal{M}$ and $\overline{\lim}_\varepsilon I_\varepsilon(u^\varepsilon) = \overline{\lim}_\varepsilon \mathcal{I}_\varepsilon(\delta_{u^\varepsilon}) \leqq \mathcal{I}(\mu)$. By Corollary B.7 there exists $b^{\varepsilon,u^\varepsilon} \in \mathcal{B}$ such that $b_x^{\varepsilon,u^\varepsilon} = u^\varepsilon$ and $J_\varepsilon(b^{\varepsilon,u^\varepsilon}) = I_\varepsilon(u^\varepsilon)$. Letting $\gamma^{(b,\mu)}$ be defined as in Lemma B.4, it is also easily seen that there exists a sequence $\gamma^\varepsilon \in \mathcal{L}_{\sigma(u^\varepsilon)^{-1}}$ such that $\gamma_x^\varepsilon = 0$, $\gamma^\varepsilon \to \gamma^{(b,\mu)}$ weakly in $L_2([0,T])$, and $\|\gamma^\varepsilon\|_{\mathcal{L}_{\sigma(u^\varepsilon)^{-1}}} \to \|\gamma^{(b,\mu)}\|_{\mathcal{L}_{\mu(\sigma)^{-1}}}$. Recalling Corollary B.7, we define the sequence b^ε by the requirements $b_x^\varepsilon = u^\varepsilon$ and $b_t^\varepsilon = b_t^{\varepsilon,u^\varepsilon} + \gamma^\varepsilon$. We have

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\, \mathcal{J}_\varepsilon\left(b^\varepsilon, \delta_{b_x^\varepsilon}\right) &= \overline{\lim_{\varepsilon\to 0}}\, J_\varepsilon(b^{\varepsilon,u^\varepsilon}) + \frac{1}{2}\langle\langle \sigma(u^\varepsilon)\gamma^\varepsilon, \gamma^\varepsilon\rangle\rangle \\ &\leqq \mathcal{I}(\mu) + \frac{1}{2}\langle\langle \mu(\sigma)\gamma^{(b,\mu)}, \gamma^{(b,\mu)}\rangle\rangle = \mathcal{J}\left((b,\mu)\right)\end{aligned}$$

On the other hand $\delta_{b_x^\varepsilon} \to \mu$ in $\mathcal{M}$, and it is not difficult to check $b_t^\varepsilon \to b_t$ weakly. Therefore any limit point in $\mathcal{A}$ of $\{(b^\varepsilon, \delta_{b_x^\varepsilon})\}$ coincides with (b, μ). $\square$

Proof of Theorem B.2. If $b \in \mathcal{Y}$ is such that (b, δ_{b_x}) is a measure-valued solution to (B.1), then $b \in \mathcal{W}$. By the Γ-liminf inequality in Theorem B.1 we thus obtain $(\Gamma\text{–}\underline{\lim}_\varepsilon K_\varepsilon)(b) = +\infty$ if $b \notin \mathcal{W}$. The Γ-liminf inequality on $\mathcal{W}$ follows immediately by (i) in Theorem 2.5 and (B.10).

Equicoercivity is a consequence of (ii) in Theorem 2.5 and Lemma B.5.

In order to prove the Γ-limsup inequality, let $b \in \mathcal{W}$ be such that $\overline{H}(b_x) < +\infty$. By (iii) in Theorem 2.5 there exists a sequence $\{u^\varepsilon\} \subset \mathcal{X}$ converging to $u := b_x$ in $\mathcal{X}$ and such that $\overline{\lim}_\varepsilon H_\varepsilon(u^\varepsilon) \leqq \overline{H}(u)$. Let $b^\varepsilon := b^{\varepsilon,u^\varepsilon}$; by Corollary B.7 $\overline{\lim}_\varepsilon K_\varepsilon(b^{\varepsilon,u^\varepsilon}) \leqq K(b)$. Furthermore, by (i) and (ii) proven above, $\{b^\varepsilon\}$ is precompact in $\mathcal{Y}$ and its limit points are in $\mathcal{W}$. Let $\tilde{b} \in \mathcal{W}$ be a limit point of $\{b^\varepsilon\}$. Then $\tilde{b}_x = b_x$, since $b_x^\varepsilon = u^\varepsilon \to u = b_x$ in $\mathcal{X}$; on the other hand $b_t + f(b_x) = 0 = \tilde{b}_t + f(\tilde{b}_x)$, so that we also gather $b_t = \tilde{b}_t$. It follows $\tilde{b} = b$. $\square$

Acknowledgments. We thank Stefano Bianchini and Corrado Mascia for fruitful discussions. The authors gratefully acknowledge the hospitality and the support of the Centro De Giorgi of the Scuola Normale Superiore di Pisa, where this paper was partially written.

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(*Received December 17, 2007 / Accepted October 18, 2008*)