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Strong Asymmetric Limit of the Quasi-Potential of the Boundary Driven Weakly Asymmetric Exclusion Process

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Abstract: We consider the weakly asymmetric exclusion process on a bounded interval with particles reservoirs at the endpoints. The hydrodynamic limit for the empirical density, obtained in the diffusive scaling, is given by the viscous Burgers equation with Dirichlet boundary conditions. In the case in which the bulk asymmetry is in the same direction as the drift due to the boundary reservoirs, we prove that the quasi-potential can be expressed in terms of the solution to a one-dimensional boundary value problem which has been introduced by Enaud and Derrida [16]. We consider the strong asymmetric limit of the quasi-potential and recover the functional derived by Derrida, Lebowitz, and Speer [15] for the asymmetric exclusion process.

1. Introduction

The study of steady states of non-equilibrium systems has motivated a lot of work over the last decades. It is now well established that the steady states of non-equilibrium systems exhibit in general long-range correlations and that the thermodynamic functionals, such as the free energy, are neither local nor additive.

The analysis of the large deviations asymptotics of stochastic lattice gases with particle reservoirs at the boundary has proven itself to be an important step in the physical description of *nonequilibrium stationary states* and a rich source of mathematical problems. We refer to [6,14] for two recent reviews on this topic.

We consider a boundary driven one-dimensional lattice gas whose dynamics can be informally described as follows. Fix an integer $N \geq 1$, an external force E in $\mathbb{R}$ and boundary densities $0 < \rho_- < \rho_+ < 1$. At any given time each site of the interval $\{-N+1, \dots, N-1\}$ is either empty or occupied by one particle. In the bulk, each particle attempts to jump to the right at rate $1 + E/2N$ and to the left at rate $1 - E/2N$. To respect the exclusion rule, the particle jumps only if the target site is empty, otherwise nothing happens. At the boundary sites $\pm(N-1)$ particles are created and removed for

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the local density to be $\rho_\pm$: at rate $\rho_\pm$ a particle is created at $\pm(N-1)$ if the site is empty and at rate $1-\rho_\pm$ the particle at $\pm(N-1)$ is removed if the site is occupied.

The dynamics just described defines an irreducible Markov process on a finite state space which has a unique stationary state denoted by μ_E^N. Let $\varphi_\pm := \log[\rho_\pm/(1-\rho_\pm)]$ be the chemical potential of the boundary reservoirs and set $E_0 := (\varphi_+ - \varphi_-)/2$. When $E = E_0$, the drift caused by the external field E matches the drift due to the boundary reservoirs, and the process becomes reversible.

In the limit $N \uparrow \infty$, the typical density profile $\bar\rho_E$ under the stationary state μ_E^N can be described as follows. For each $E \le E_0$ there exists a unique $J_E \le 0$ such that

$$\frac{1}{2}\int_{\rho_-}^{\rho_+} dr \, \frac{1}{E\chi(r) - J_E} = 1,$$

where χ is the mobility of the system: $\chi(a) = a(1-a)$. The profile $\bar\rho_E$ is then obtained by solving

$$\bar\rho_E' - E\,\chi(\bar\rho_E) = -J_E$$

with the boundary condition $\bar\rho_E(-1) = \rho_-$.

In the same limit $N \uparrow \infty$, the probability of observing a density profile γ different from $\bar\rho_E$ can be expressed as

$$\mu_E^N\{\gamma\} \sim \exp\{-N V_E(\gamma)\}. \tag{1.1}$$

The large deviations functional V_E, which also depends on ρ_-, ρ_+, is an extension of the notion of free energy in the context of non-equilibrium systems.

The free energy of a boundary driven lattice gas has first been derived for the symmetric simple exclusion process by Derrida, Lebowitz and Speer [15] based on the so-called matrix method, introduced by Derrida, which permits to express the stationary state μ_E^N as a product of matrices. Bertini et al. [4] derived the same result through a dynamical approach which we extend here to the weakly asymmetric case.

We consider only the situation $E < E_0$ for the bulk asymmetry to be in the same direction as the drift due to the boundary. The reversible case $E = E_0$ lacks interest because the stationary state is product and does not exhibit long range correlations. In contrast, the analysis of the quasi-potential V_E for $E > E_0$, not treated here, appears a most interesting problem. For instance, a representation of V_E as a supremum of trial functionals analogous to (2.14) below seems to be ruled out.

In the boundary driven weakly asymmetric exclusion process, for $E < E_0$, the quasi-potential takes the following form:

$$\begin{aligned} V_E(\gamma) := \int_{-1}^{1} du \, \Big\{ &\gamma\log\gamma + (1-\gamma)\log(1-\gamma) + (1-\gamma)\varphi - \log\big(1+e^{\varphi}\big) \\ &+ \frac{1}{E}\big[\varphi'\log\varphi' - (\varphi'-E)\log(\varphi'-E)\big] - A_E \Big\}, \end{aligned} \tag{1.2}$$

where A_E is the constant given by

$$A_E := \log(-J_E) + \frac{1}{2}\int_{\gamma_-}^{\gamma_+} dr \, \frac{1}{E\chi(r)} \log\left[1 - \frac{E\chi(r)}{J_E}\right];$$

and where φ is the unique solution of the Euler-Lagrange equation

$$\frac{\varphi''}{\varphi'(\varphi' - E)} + \frac{1}{1 + e^{\varphi}} = \gamma$$

satisfying $\varphi(\pm 1) = \varphi_\pm$, $\varphi' > \max\{0, E\}$.

This result, stated in a different form, has been proved by Enaud and Derrida [16] based on the matrix method. We prove this result in Sect. 4 below by the dynamical approach introduced in [4,5]. We also show that the quasi-potential is convex and lower semi-continuous.

In Sect. 5, we show that V_E Γ-converges, as $E \downarrow -\infty$, to the free energy of the boundary driven asymmetric exclusion process, first derived by Derrida, Lebowitz and Speer [15]. This asymptotic behavior is somewhat surprising since the hydrodynamic time scales at which the weakly asymmetric exclusion process and the asymmetric exclusion process evolve are different. We also prove convergence of the solutions of the Euler-Lagrange equations as the external force E diverges.

The dynamical approach followed here permits to compute the fluctuation probabilities (1.1) in great generality, in any dimension and for a large class of processes. However, it is only in dimension one and for very few interacting particle systems that an explicit expression of type (1.2) is available for the non-equilibrium free energy V_E.

2. Notation and Results

The boundary driven weakly asymmetric exclusion process. Fix an integer $N \geq 1$, $E \in \mathbb{R}$, $0 < \rho_- \leq \rho_+ < 1$ and let $\Lambda_N := \{-N+1, \dots, N-1\}$. The configuration space is $\Sigma_N := \{0,1\}^{\Lambda_N}$; elements of Σ_N are denoted by η so that $\eta(x) = 1$, resp. 0, if site x is occupied, resp. empty, for the configuration η. We denote by $\sigma^{x,y}\eta$ the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$, i.e.

$$(\sigma^{x,y}\eta)(z) := \begin{cases} \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \\ \eta(z) & \text{if } z \neq x, y, \end{cases}$$

and by $\sigma^x\eta$ the configuration obtained from η by flipping the configuration at x, i.e.

$$(\sigma^x\eta)(z) := \begin{cases} 1 - \eta(x) & \text{if } z = x \\ \eta(z) & \text{if } z \neq x. \end{cases}$$

The one-dimensional boundary driven weakly asymmetric exclusion process is the Markov process on Σ_N whose generator L_N can be decomposed as

$$L_N = L_{0,N} + L_{-,N} + L_{+,N}, \tag{2.1}$$

where the generators $L_{0,N}$, $L_{-,N}$, $L_{+,N}$ act on functions $f : \Sigma_N \to \mathbb{R}$ as

$$(L_{0,N} f)(\eta) = \frac{N^2}{2} \sum_{x=-N+1}^{N-2} e^{-E/(2N)\,[\eta(x+1)-\eta(x)]} \Big[f(\sigma^{x,x+1}\eta) - f(\eta) \Big],$$

$$(L_{-,N} f)(\eta) = \frac{N^2}{2} c_-\,(\eta(-N+1)) \Big[f(\sigma^{-N+1}\eta) - f(\eta) \Big],$$

$$(L_{+,N} f)(\eta) = \frac{N^2}{2} c_+\,(\eta(N-1)) \Big[f(\sigma^{N-1}\eta) - f(\eta) \Big],$$

where $c_\pm : \{0,1\} \to \mathbb{R}$ are given by

$$c_\pm(\zeta) \;:=\; \rho_\pm e^{\mp E/(2N)}(1-\zeta) + (1-\rho_\pm)e^{\pm E/(2N)}\zeta.$$

Notice that the (weak) external field is $E/(2N)$ and, in view of the diffusive scaling limit, the generator has been speeded up by N^2. We denote by η_t the Markov process on Σ_N with generator L_N and by $\mathbb{P}^N_\eta$ its distribution if the initial configuration is η. Note that $\mathbb{P}^N_\eta$ is a probability measure on the path space $D(\mathbb{R}_+, \Sigma_N)$, which we consider endowed with the Skorohod topology and the corresponding Borel σ-algebra. Expectation with respect to $\mathbb{P}^N_\eta$ is denoted by $\mathbb{E}^N_\eta$.

Since the Markov process η_t is irreducible, for each $N \ge 1$, $E \in \mathbb{R}$, and $0 < \rho_- \le \rho_+ < 1$ there exists a unique invariant measure μ^N_E in which we drop the dependence on $\rho_\pm$ from the notation. Let $\varphi_\pm := \log[\rho_\pm/(1-\rho_\pm)]$ be the chemical potential of the boundary reservoirs and set $E_0 := (\varphi_+ - \varphi_-)/2$. A simple computation shows that if $E = E_0$ then the process η_t is reversible with respect to the product measure

$$\mu^N_{E_0}(\eta) = \prod_{x=-N+1}^{N-1} \frac{e^{\bar\varphi^N_{E_0}(x)\,\eta(x)}}{1+e^{\bar\varphi^N_{E_0}(x)}}, \tag{2.2}$$

where

$$\bar\varphi^N_{E_0}(x) := \varphi_- \frac{N-x}{2N} + \varphi_+ \frac{N+x}{2N}.$$

On the other hand, for $E \ne E_0$ the invariant measure μ^N_E cannot be written in a simple form.

The dynamical large deviation principle. We denote by $u \in [-1,1]$ the macroscopic space coordinate and by $\langle\cdot,\cdot\rangle$ the inner product in $L_2([-1,1], du)$. We set

$$\mathcal{M} := \{\rho \in L_\infty([-1,1], du) \;:\; 0 \le \rho \le 1\}, \tag{2.3}$$

which we equip with the topology induced by the weak convergence of measures, namely a sequence $\{\rho^n\} \subset \mathcal{M}$ converges to ρ in $\mathcal{M}$ if and only if $\langle \rho^n, G\rangle \to \langle \rho, G\rangle$ for any continuous function $G : [-1,1] \to \mathbb{R}$. Note that $\mathcal{M}$ is a compact Polish space that we consider endowed with the corresponding Borel σ-algebra. The empirical density of the configuration $\eta \in \Sigma_N$ is defined as $\pi^N(\eta)$, where the map $\pi^N : \Sigma_N \to \mathcal{M}$ is given by

$$\pi^N(\eta)\,(u) \;:=\; \sum_{x=-N+1}^{N-1} \eta(x)\, \mathbf{1}\left\{\left[\frac{x}{N} - \frac{1}{2N}, \frac{x}{N} + \frac{1}{2N}\right)\right\}(u), \tag{2.4}$$

in which $\mathbf{1}\{A\}$ stands for the indicator function of the set A. Let $\{\eta^N\}$ be a sequence of configurations with $\eta^N \in \Sigma_N$. If the sequence $\{\pi^N(\eta^N)\} \subset \mathcal{M}$ converges to ρ in $\mathcal{M}$ as $N \to \infty$, we say that $\{\eta^N\}$ is *associated* to the macroscopic density profile $\rho \in \mathcal{M}$.

Given $T > 0$, we denote by $D([0,T];\mathcal{M})$ the Skorohod space of paths from $[0,T]$ to $\mathcal{M}$ equipped with its Borel σ-algebra. Elements of $D([0,T],\mathcal{M})$ will be denoted by $\pi \equiv \pi_t(u)$ and sometimes by $\pi(t,u)$. Note that the evaluation map $D([0,T];\mathcal{M}) \ni \pi \mapsto \pi_t \in \mathcal{M}$ is not continuous for $t \in (0,T)$ but is continuous for $t = 0, T$. We denote by π^N also the map from $D([0,T];\Sigma_N)$ to $D([0,T];\mathcal{M})$ defined by $\pi^N(\eta_\cdot)_t := \pi^N(\eta_t)$. The notation $\pi^N(t,u)$ is also used.

Fix a profile $\gamma \in \mathcal{M}$ and consider a sequence $\{\eta^N : N \ge 1\}$ associated to γ. Let η_t^N be the boundary driven weakly asymmetric exclusion process starting from η^N. In [13,21,23] it is proven that as $N \to \infty$ the sequence of random variables $\{\pi^N(\eta_\cdot^N)\}$, which take values in $\mathcal{D}([0,T];\mathcal{M})$, converges in probability to the path $\rho \equiv \rho_t(u)$, $(t,u) \in [0,T] \times [-1,1]$ which solves the viscous Burgers equation with Dirichlet boundary conditions at ± 1, i.e.

$$\begin{cases} \partial_t \rho + \dfrac{E}{2} \nabla \chi(\rho) = \dfrac{1}{2} \Delta \rho \\ \rho_t(\pm 1) = \rho_\pm \\ \rho_0(u) = \gamma(u) \end{cases}, \tag{2.5}$$

where $\chi : [0,1] \to \mathbb{R}_+$ is the mobility of the system, $\chi(a) = a(1-a)$, and ∇, resp. Δ, denotes the derivative, resp. the second derivative, with respect to u. In fact the proof presented in [13,21] is in real line, while the one in [23] is on the torus. The arguments however can be adapted to the boundary driven case, see [18,19,22] for the hydrodynamic limit of different boundary driven models.

A large deviation principle for the empirical density can also be proven following [23,25,26], adapted to the open boundary context in [5]. In order to state this result some more notation is required. Fix $T > 0$ and let $\Omega_T = (0,T) \times (-1,1)$, $\overline{\Omega_T} = [0,T] \times [-1,1]$. For positive integers m, n, we denote by $C^{m,n}(\overline{\Omega_T})$ the space of functions $G \equiv G_t(u) \colon \overline{\Omega_T} \to \mathbb{R}$ with m derivatives in time, n derivatives in space which are continuous up the the boundary. We improperly denote by $C_0^{m,n}(\overline{\Omega_T})$ the subset of $C^{m,n}(\overline{\Omega_T})$ of the functions which vanish at the endpoints of $[-1,1]$, i.e. $G \in C^{m,n}(\overline{\Omega_T})$ belongs to $C_0^{m,n}(\overline{\Omega_T})$ if and only if $G_t(\pm 1) = 0$, $t \in [0,T]$.

Let the energy $\mathcal{Q} : D([0,T],\mathcal{M}) \to [0,\infty]$ be given by

$$\mathcal{Q}(\pi) = \sup_H \left\{ \int_0^T dt \int_{-1}^1 du\, \pi(t,u)\, (\nabla H)(t,u) - \frac{1}{2} \int_0^T dt \int_{-1}^1 du\, H(t,u)^2\, \chi(\pi(t,u)) \right\},$$

where the supremum is carried over all smooth functions $H : \Omega_T \to \mathbb{R}$ with compact support. If $\mathcal{Q}(\pi)$ is finite, π has a generalized space derivative, $\nabla \pi$, and

$$\mathcal{Q}(\pi) = \frac{1}{2} \int_0^T dt \int_{-1}^1 du\, \frac{(\nabla \pi_t)^2}{\chi(\pi_t)} \,.$$

Fix a function $\gamma \in \mathcal{M}$ which corresponds to the initial profile. For each H in $C_0^{1,2}(\overline{\Omega_T})$, let $\hat{J}_H = \hat{J}_{T,H,\gamma} \colon D([0,T],\mathcal{M}) \longrightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} \hat{J}_H(\pi) := \langle \pi_T, H_T \rangle - \langle \gamma, H_0 \rangle - \int_0^T dt\, \langle \pi_t, \partial_t H_t \rangle \\ - \frac{1}{2} \int_0^T dt\, \langle \pi_t, \Delta H_t \rangle + \frac{\rho_+}{2} \int_0^T dt\, \nabla H_t(1) - \frac{\rho_-}{2} \int_0^T dt\, \nabla H_t(-1) \\ - \frac{E}{2} \int_0^T dt\, \langle \chi(\pi_t), \nabla H_t \rangle - \frac{1}{2} \int_0^T dt\, \left\langle \chi(\pi_t), (\nabla H_t)^2 \right\rangle. \end{aligned}$$

Let $\hat{I}_T(\cdot\,|\gamma) \colon D([0,T],\mathcal{M}) \longrightarrow [0,+\infty]$ be the functional defined by

$$\hat{I}_T(\pi|\gamma) := \sup_{H \in C_0^{1,2}(\overline{\Omega_T})} \hat{J}_H(\pi). \tag{2.6}$$

The rate functional $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,\infty]$ is given by

$$I_T(\pi|\gamma) \;=\; \begin{cases} \hat{I}_T(\pi|\gamma) & \text{if } \mathcal{Q}(\pi) < \infty\,, \\ \infty & \text{otherwise.} \end{cases} \tag{2.7}$$

It is proven in [8], for any E in $\mathbb{R}$, that the functional $I_T(\cdot|\gamma)$ is lower semicontinuous, has compact level sets and that a dynamical large deviations principle for the empirical measure holds.

Theorem 2.1. *Fix $T > 0$ and an initial profile γ in $\mathcal{M}$. Consider a sequence $\{\eta^N : N \geq 1\}$ of configurations associated to γ. Then, the sequence of probability measures $\{\mathbb{P}^N_{\eta^N} \circ (\pi^N)^{-1} : N \geq 1\}$ on $D([0,T];\mathcal{M})$ satisfies a large deviation principle with speed N and good rate function $I_T(\cdot|\gamma)$. Namely, $I_T(\cdot|\gamma) : D([0,T];\mathcal{M}) \to [0,\infty]$ has compact level sets and for each closed set $\mathcal{C} \subset D([0,T];\mathcal{M})$ and each open set $\mathcal{O} \subset D([0,T];\mathcal{M})$,*

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}^N_{\eta^N}\left(\pi^N \in \mathcal{C}\right) \;\leq\; -\inf_{\pi\in\mathcal{C}} I_T(\pi|\gamma),$$
$$\varliminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}^N_{\eta^N}\left(\pi^N \in \mathcal{O}\right) \;\geq\; -\inf_{\pi\in\mathcal{O}} I_T(\pi|\gamma).$$

The quasi-potential. From now on we consider only the case $E \leq E_0 = (\varphi_+ - \varphi_-)/2$, where $\varphi_\pm = \log[\rho_\pm/(1-\rho_\pm)]$. Simple computations, which are omitted, show that the unique stationary solution $\bar{\rho}_E \in \mathcal{M}$ of the hydrodynamic equation (2.5) can be described as follows. For each $E \leq E_0$ there exists a unique $J_E \leq 0$ such that

$$\frac{1}{2}\int_{\rho_-}^{\rho_+} dr\, \frac{1}{E\chi(r) - J_E} = 1. \tag{2.8}$$

The profile $\bar{\rho}_E$ is then obtained by solving

$$\bar{\rho}_E' - E\,\chi(\bar{\rho}_E) = -J_E \tag{2.9}$$

with the boundary condition $\bar{\rho}_E(-1) = \rho_-$. Note that $J_E/2$ is the current maintained by the stationary profile $\bar{\rho}_E$. The solution to (2.9) can easily be written in an explicit form, see [16]. We shall however only use, as can be easily checked, that $\bar{\rho}_E$ is strictly increasing and that the inequality $J_E/E > \max_{r\in[\rho_-,\rho_+]} \chi(r)$ holds for $E < 0$.

Given $E \leq E_0$, the *quasi-potential* for the rate function I_T is the functional $V_E : \mathcal{M} \to [0,+\infty]$ defined by

$$V_E(\rho) \;:=\; \inf_{T>0} \inf\left\{I_T(\pi|\bar{\rho}_E)\,,\ \pi \in D([0,T];\mathcal{M}) \,:\, \pi_T = \rho\right\} \tag{2.10}$$

so that $V_E(\rho)$ measures the minimal cost to produce the profile ρ starting from $\bar{\rho}_E$.

Recall that μ^N_E is the unique invariant measure of the boundary driven weakly asymmetric exclusion process. The following result, which states that the quasi-potential gives the rate function of the empirical density when particles are distributed according to μ^N_E is proven in [10,20] in the case $E = 0$. Thanks to Theorem 2.1, the proof applies also to the weakly asymmetric case, see [20] for more details on this topic.

Theorem 2.2. *For each E in $\mathbb{R}$, the sequence of probability measures on $\mathcal{M}$ given by $\{\mu_E^N \circ (\pi^N)^{-1}\}$ satisfies a large deviation principle with speed N and rate function V_E. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mu_E^N \Big(\pi^N \in \mathcal{C}\Big) \;\leq\; -\inf_{\rho\in\mathcal{C}} V_E(\rho),$$
$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \mu_E^N \Big(\pi^N \in \mathcal{O}\Big) \;\geq\; -\inf_{\rho\in\mathcal{O}} V_E(\rho).$$

In this paper we prove that the quasi-potential V_E can be expressed in terms of the solution to a one-dimensional boundary value problem. This result has been obtained in [16] by analyzing directly the invariant measure μ_E^N through combinatorial techniques; while we here follow instead the dynamic approach [4,5] by characterizing the optimal path, as also described in [17], for the variational problem (2.10).

For $E < E_0$, let $C^{1+1}([-1,1])$ be the set of continuously differentiable functions on $[-1,1]$ with Lipshitz derivative and set

$$\mathcal{F}_E := \Big\{\varphi \in C^{1+1}([-1,1]) \,:\, \varphi(\pm 1) = \varphi_\pm\,,\ \varphi' > 0 \vee E\Big\}, \tag{2.11}$$

where, given $a, b \in \mathbb{R}$, the notation $a \vee b$, resp. $a \wedge b$, stands for $\max\{a,b\}$, resp. $\min\{a,b\}$. Note that $\mathcal{F}_E = \mathcal{F}_{E'}$ for $E, E' < 0$.

For $E < E_0$, $E \neq 0$, let $\mathcal{G}_E : \mathcal{M} \times \mathcal{F}_E \to \mathbb{R}$ be given by

$$\begin{aligned}\mathcal{G}_E(\rho,\varphi) := \int_{-1}^{1} du\, \Big\{&\rho\log\rho + (1-\rho)\log(1-\rho) + (1-\rho)\varphi - \log\big(1+e^{\varphi}\big)\\ &+ \frac{1}{E}\big[\varphi'\log\varphi' - (\varphi'-E)\log(\varphi'-E)\big] - A_E\Big\},\end{aligned} \tag{2.12}$$

where, by convention, $0\log 0 = 0$ and A_E is the constant given by

$$A_E \;:=\; \log(-J_E) \;+\; \frac{1}{2}\int_{\rho_-}^{\rho_+} dr\, \frac{1}{E\chi(r)} \log\Big[1 - \frac{E\chi(r)}{J_E}\Big]. \tag{2.13}$$

The right-hand side is well defined as $J_E < 0$ and $J_E/E > \max_{r\in[\rho_-,\rho_+]} \chi(r)$ for $E < 0$.

For $E = 0$, $\mathcal{G}_0 : \mathcal{M} \times \mathcal{F}_E \to \mathbb{R}$ is defined by continuity as

$$\begin{aligned}&\mathcal{G}_0(\rho,\varphi)\\ &= \int_{-1}^{1} du\, \big\{\rho\log\rho + (1-\rho)\log(1-\rho) + (1-\rho)\varphi - \log\big(1+e^{\varphi}\big) + \log\varphi' + 1 - A_0\big\},\end{aligned}$$

where $A_0 = \log[(\rho_+ - \rho_-)/2] + 1$.

For $E < E_0$, define the functional $S_E : \mathcal{M} \to \mathbb{R}$ by

$$S_E(\rho) \;:=\; \sup_{\varphi\in\mathcal{F}_E} \mathcal{G}_E(\rho,\varphi). \tag{2.14}$$

Note that S_E is a positive functional because a simple computation relying on (2.9) shows that

$$S_E(\rho) \;\geq\; \mathcal{G}_E(\rho,\bar\varphi_E) \;=\; \int_{-1}^{1} du\, \Big\{\rho\log\frac{\rho}{\bar\rho_E} + (1-\rho)\log\frac{1-\rho}{1-\bar\rho_E}\Big\} \tag{2.15}$$

if $\bar\varphi_E := \log[\bar\rho_E/(1-\bar\rho_E)]$.

In the special case $E = E_0$, as already observed, the weakly asymmetric exclusion process is reversible and the stationary state $\mu^N_{E_0}$ is a product measure. In particular, the rate functional S_{E_0} of the static large deviations principle for the empirical density can be explicitly computed. It is given by

$$S_{E_0}(\rho) \;=\; \int_{-1}^{1} du\, \left\{ \rho \log \frac{\rho}{\bar\rho_{E_0}} + (1-\rho) \log \frac{1-\rho}{1-\bar\rho_{E_0}} \right\}. \tag{2.16}$$

The Euler-Lagrange equation associated to the variational problem (2.14) is

$$\frac{\varphi''}{\varphi'(\varphi' - E)} + \frac{1}{1+e^{\varphi}} \;=\; \rho. \tag{2.17}$$

A function $\varphi \in \mathcal{F}_E$ solves the above equation when it is satisfied Lebesgue a.e. Recalling that the stationary profile $\bar\rho_E$ satisfies (2.9) and (2.8), it is easy to check that if $\rho = \bar\rho_E$ then $\bar\varphi_E$ solves (2.17) and $\mathcal{G}_E(\bar\rho_E, \bar\varphi_E) = 0$.

The analysis of the quasi-potential for the boundary driven symmetric exclusion process, i.e. the case $E = 0$ of the current setting, has been considered in [5]. In particular it is there shown that V_0 coincides with S_0. We prove in this article an analogous statement for any $E \le E_0$.

Theorem 2.3. *Let $E \le E_0$ and $V_E, S_E : \mathcal{M} \to [0, +\infty]$ be the functionals defined in* (2.10), (2.14) *and* (2.16).

(i) *The functional S_E is bounded, convex, and lower semicontinuous on $\mathcal{M}$.*
(ii) *Fix $E < E_0$. For each $\rho \in \mathcal{M}$ there exists in $\mathcal{F}_E$ a unique solution to* (2.17) *denoted by $\Phi(\rho)$. Moreover*

$$S_E(\rho) = \max_{\varphi \in \mathcal{F}_E} \mathcal{G}_E(\rho, \varphi) = \mathcal{G}_E(\rho, \Phi(\rho)). \tag{2.18}$$

(iii) *The equality $V_E = S_E$ holds on $\mathcal{M}$.*

The proof of the last item of the previous theorem is achieved by characterizing the optimal path, as also described in [17], for the variational problem (2.10) defining the quasi-potential. For $E < E_0$ it is obtained by the following algorithm. Given $\rho \in \mathcal{M}$ let $\Phi(\rho) \in \mathcal{F}_E$ be the solution to (2.17) and define $G = e^{\Phi(\rho)}/[1 + e^{\Phi(\rho)}]$. Let $F \equiv F_t(u)$ be the solution to the viscous Burgers equation (2.5) with initial condition G and set $\psi = \log[F/(1-F)]$, note that $\psi_0 = \Phi(\rho)$ and $\psi_t \to \bar\varphi_E$ as $t \to \infty$. Let $\rho^*_t = \Phi^{-1}(\psi_t)$, i.e. ρ^*_t is given by the l.h.s. of (2.17) with φ replaced by ψ_t. Observe that $\rho^*_0 = \rho$ and $\rho^*_t \to \bar\rho_E$ as $t \to \infty$. The optimal path for (2.10) is then $\pi^*_t = \rho^*_{-t}$; the fact that it is defined on the time interval $(-\infty, 0]$ instead of $[0, \infty)$ makes no real difference. As discussed in [4,6], this description of the optimal path π^* is related to the possibility of expressing the hydrodynamic limit for the process on Σ_N whose generator is the adjoint of L_N in $L_2(d\mu^N_E)$ in terms of (2.5) via the nonlocal map Φ.

The asymmetric limit. Consider the boundary driven asymmetric exclusion process, that is the process on Σ_N with generator given by (2.1), where the external field E is replaced by $N\alpha$ and the generator is speeded up by N instead of N^2. We consider only the case $\alpha < 0$. According to the previous notation, denote by $\mu^N_{N\alpha}$ the unique invariant measure of the boundary driven asymmetric exclusion process with external field αN. In the hydrodynamic scaling limit, it is proved in [1] that the empirical density converges to the unique entropy solution to the inviscid Burgers equation with Bardos-le Roux-Nédélec boundary conditions [2], namely (2.5) with $E/2$ replaced by $\sinh(\alpha/2)$ and no viscosity.

Let $\bar\rho_{\rm a} \in \{\rho_-, \rho_+, 1/2\}$ be such that $\max_{r\in[\rho_-,\rho_+]} \chi(r) = \chi(\bar\rho_{\rm a})$. It is not difficult to check that the stationary profile $\bar\rho_E$ converges, as $E \to -\infty$, to the constant density profile equal to $\bar\rho_{\rm a}$, which is the unique stationary solution to the inviscid Burgers equation with the prescribed boundary conditions.

By using combinatorial techniques, it is shown in [15] that the sequence of probability measures $\{\mu^N_{N\alpha} \circ (\pi^N)^{-1}\}$ on $\mathcal{M}$ satisfies a large deviation principle with speed N and rate function $S_{\rm a}$ defined as follows. Let

$$\mathcal{F}_{\rm a} := \Big\{\varphi \in C^1\left([-1,1]\right) \,:\, \varphi(\pm 1) = \varphi_\pm\,,\ \varphi' > 0\Big\}. \tag{2.19}$$

Note that $\mathcal{F}_E \subset \mathcal{F}_{\rm a}$.

Given $\rho \in \mathcal{M}$ and $\varphi \in \mathcal{F}_{\rm a}$ set

$$\mathcal{G}_{\rm a}(\rho,\varphi) := \int_{-1}^{1} du\ \left\{\rho \log\rho + (1-\rho)\log(1-\rho) + (1-\rho)\varphi - \log\left(1+e^{\varphi}\right) - A_{\rm a}\right\}, \tag{2.20}$$

in which the constant $A_{\rm a}$ is

$$A_{\rm a} := \max_{r\in[\rho_-,\rho_+]} \log\chi(r) \ = \ \log\chi(\bar\rho_{\rm a}). \tag{2.21}$$

Let

$$S_{\rm a}(\rho) \ := \ \sup_{\varphi\in\mathcal{F}_{\rm a}} \ \mathcal{G}_{\rm a}(\rho,\varphi). \tag{2.22}$$

The functional $S_{\rm a}$ is written in a somewhat different form in [15]. The above expression is however simply obtained by replacing the trial function F in [15] by $e^{\varphi}/(1+e^{\varphi})$. The advantage of the above formulation is that for each $\rho \in \mathcal{M}$ the functional $\mathcal{G}_{\rm a}(\rho,\cdot)$ is concave on $\mathcal{F}_{\rm a}$. By choosing $\varphi = \log[\bar\rho_{\rm a}/(1-\bar\rho_{\rm a})]$ as trial function in (2.22) we get a lower bound analogous to (2.15):

$$S_{\rm a}(\rho) \ge \int_{-1}^{1} du\ \left\{\rho\log\frac{\rho}{\bar\rho_{\rm a}} + (1-\rho)\log\frac{1-\rho}{1-\bar\rho_{\rm a}}\right\}.$$

Note finally that $S_{\rm a}$ does not depend on $\alpha < 0$.

We prove in Sect. 5 that the functional S_E converges, as $E \downarrow -\infty$, to $S_{\rm a}$. As discussed in [7, Lemma 4.3], the appropriate notion of variational convergence for rate functionals is the so-called Γ-convergence. Referring e.g. to [11] for more details, we just recall its definition. Let X be a metric space. A sequence of functionals $F_n : X \to [0,+\infty]$ is said to Γ-*converge* to a functional $F : X \to [0,+\infty]$ if the following two conditions hold for each $x \in X$. There exists a sequence $x_n \to x$ such that $\overline{\lim}_n F_n(x_n) \le F(x)$ (Γ-*limsup inequality*) and for any sequence $x_n \to x$ we have $\underline{\lim}_n F_n(x_n) \ge F(x)$ (Γ-*liminf inequality*).

Theorem 2.4. *Let $S_E : \mathcal{M} \to [0,+\infty]$ be as defined in* (2.14). *As $E \downarrow -\infty$, the sequence of functionals $\{S_E\}$ Γ-converges in $\mathcal{M}$ to $S_{\rm a}$ defined in* (2.22).

While the above result deals only with the variational convergence of the quasi-potential, it is reasonable to expect also the convergence of the dynamical rate functional. More precisely, the dynamic rate functional (2.7) of the weakly asymmetric exclusion process should converge, in the appropriate scaling, to the one for the asymmetric exclusion process. We refer to [9] for a discussion of this topic and we mention that the above result has been proven in [3] for general scalar conservation laws on the real line.

Γ-convergence implies an upper bound for the infimum over open sets and a lower bound for the infimum over compact sets: For each compact set $\mathcal{K} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$,

$$\varliminf_{E\to-\infty} \inf_{\rho\in\mathcal{K}} S_E(\rho) \ \geq\ \inf_{\rho\in\mathcal{K}} S_{\mathrm{a}}(\rho),$$
$$\varlimsup_{E\to-\infty} \inf_{\rho\in\mathcal{O}} S_E(\rho) \ \leq\ \inf_{\rho\in\mathcal{O}} S_{\mathrm{a}}(\rho).$$

The proof of this statement is straightforward and can be found in [11]. Since $\mathcal{M}$ is compact, the previous fact and Theorems 2.2, 2.3 (iii), 2.4 provide the following asymptotics for the invariant measure μ_E^N.

Corollary 2.5. *For each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$,*

$$\varlimsup_{E\to-\infty}\ \varlimsup_{N\to\infty}\ \frac{1}{N}\log \mu_E^N\Big(\pi^N \in \mathcal{C}\Big) \ \leq\ -\inf_{\rho\in\mathcal{C}} S_{\mathrm{a}}(\rho),$$
$$\varliminf_{E\to-\infty}\ \varliminf_{N\to\infty}\ \frac{1}{N}\log \mu_E^N\Big(\pi^N \in \mathcal{O}\Big) \ \geq\ -\inf_{\rho\in\mathcal{O}} S_{\mathrm{a}}(\rho).$$

The last topic we discuss is the asymptotic behavior as, $E \to -\infty$, of the solution to the Euler-Lagrange equation (2.17). More precisely, we show that it converges to the unique maximizer for (2.22).

Consider the set $\mathcal{F}_{\mathrm{a}}$ equipped with the topology inherited from the weak convergence of measures on $[-1, 1)$: $\varphi^n \to \varphi$ in $\mathcal{F}_{\mathrm{a}}$ if and only if $\int_{-1}^{1} d\varphi^n\, G \to \int_{-1}^{1} d\varphi\, G$ for any function G in $C_0([-1, 1))$, the set of continuous functions $G : [-1, 1) \to \mathbb{R}$ such that $\lim_{u\uparrow 1} G(u) = 0$. The closure of $\mathcal{F}_{\mathrm{a}}$, denoted by $\overline{\mathcal{F}}_{\mathrm{a}}$, consists of all nondecreasing, càdlàg functions $\varphi : [-1, 1) \to [\varphi_-, \varphi_+]$ such that $\varphi(-1) = \varphi_-$, $\lim_{u\uparrow 1} \varphi(u) \leq \varphi_+$. By the Helly theorem $\overline{\mathcal{F}}_{\mathrm{a}}$ is a compact Polish space. Moreover, if $\varphi^n \to \varphi$ in $\overline{\mathcal{F}}_{\mathrm{a}}$ then $\varphi^n(u) \to \varphi(u)$ Lebesgue a.e.

Theorem 2.6. *Fix $\rho \in \mathcal{M}$. There exists a unique $\phi \in \overline{\mathcal{F}}_{\mathrm{a}}$ such that $S_{\mathrm{a}}(\rho) = \max_{\varphi\in\mathcal{F}_{\mathrm{a}}} \mathcal{G}_{\mathrm{a}}(\rho, \varphi) = \mathcal{G}_{\mathrm{a}}(\rho, \phi)$. Let $\phi_E := \Phi(\rho) \in \mathcal{F}_E$ be the optimal profile for (2.14). As $E \to -\infty$ the sequence $\{\phi_E\}$ converges to ϕ in $\mathcal{F}_{\mathrm{a}}$.*

3. The Nonequilibrium Free Energy

In this section we analyze the variational problem (2.14) and prove items (i) and (ii) in Theorem 2.3. We start by proving an existence and uniqueness result for the Euler-Lagrange equation (2.17) together with a C^1 dependence of the solution with respect to ρ. We consider the space $C^1([-1, 1])$ endowed with the norm $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$, where $\|g\|_\infty := \sup_{u\in[-1,1]} |g(u)|$. For each $E < E_0$ the set $\mathcal{F}_E$ defined in (2.11) is a convex subset of $C^1([-1, 1])$; we denote by $\overline{\mathcal{F}}_E = \big\{\varphi \in C^1([-1, 1]) \,:\, \varphi(\pm 1) = \varphi_\pm\,,\ \varphi' \geq 0 \vee E\big\}$ its closure in $C^1([-1, 1])$.

Theorem 3.1. *Let $E < E_0$. For each $\rho \in \mathcal{M}$ there exists in $\mathcal{F}_E$ a unique solution to* (2.17)*, denoted by $\Phi(\rho)$. Furthermore,*

(i) *If $\rho \in C([-1,1];[0,1])$ then $\Phi(\rho) \in C^2([-1,1])$.*
(ii) *Let $\{\rho^n\} \subset \mathcal{M}$ be a sequence converging to ρ in $\mathcal{M}$. Then $\{\Phi(\rho^n)\} \subset \mathcal{F}_E$ converges to $\Phi(\rho)$ in $C^1([-1,1])$.*

Proof. The proof is divided into several steps.
Existence of solutions. For $E \le 0$, resp. $E \in (0, E_0)$, we formulate (2.17) as an integral-differential equation informally obtained multiplying (2.17) by $\varphi' - E$, resp. by φ', and integrating the resulting equation. Existence of solutions will be deduced from the Schauder fixed point theorem.

Given $E < E_0$, $\rho \in \mathcal{M}$, and $\varphi \in \overline{\mathcal{F}}_E$, let

$$\mathcal{R}^{(1)}(\rho,\varphi;u) := \Big[\rho - \frac{1}{1+e^{\varphi(u)}}\Big]\big[\varphi'(u) - E\big],$$
$$\mathcal{R}^{(2)}(\rho,\varphi;u) := \Big[\rho - \frac{1}{1+e^{\varphi(u)}}\Big]\varphi'(u).$$

For a fixed $\rho \in \mathcal{M}$ and $i = 1,2$ we define the integral-differential operators $\mathcal{K}_\rho^{(i)} : \overline{\mathcal{F}}_E \to C^1([-1,1])$ by

$$\mathcal{K}_\rho^{(1)}(\varphi)(u) := \varphi_- + (\varphi_+ - \varphi_-)\,\frac{\displaystyle\int_{-1}^{u} dv\,\exp\Big\{\int_{-1}^{v} dw\,\mathcal{R}^{(1)}(\rho,\varphi;w)\Big\}}{\displaystyle\int_{-1}^{1} dv\,\exp\Big\{\int_{-1}^{v} dw\,\mathcal{R}^{(1)}(\rho,\varphi;w)\Big\}},$$

$$\mathcal{K}_\rho^{(2)}(\varphi)(u) := \varphi_- + E(u+1) + (\varphi_+ - \varphi_- - 2E)\,\frac{\displaystyle\int_{-1}^{u} dv\,\exp\Big\{\int_{-1}^{v} dw\,\mathcal{R}^{(2)}(\rho,\varphi;w)\Big\}}{\displaystyle\int_{-1}^{1} dv\,\exp\Big\{\int_{-1}^{v} dw\,\mathcal{R}^{(2)}(\rho,\varphi;w)\Big\}}.$$

For $E \le 0$, resp. $E \in (0, E_0)$, we formulate the boundary problem (2.17) as a fixed point on $\overline{\mathcal{F}}_E$ for the operator $\mathcal{K}_\rho^{(1)}$, resp. $\mathcal{K}_\rho^{(2)}$.

Consider first the case $E \le 0$ corresponding to $i = 1$. Simple computations show that for each $\rho \in \mathcal{M}$ the map $\mathcal{K}_\rho^{(1)}$ is a continuous on $\overline{\mathcal{F}}_E$ and $\mathcal{K}_\rho^{(1)}\big(\overline{\mathcal{F}}_E\big) \subset \overline{\mathcal{F}}_E$. It is also straightforward to check that there exists a constant $C_1 = C_1(\varphi_-, \varphi_+, E) \in (0,\infty)$ such that for any $\rho \in \mathcal{M}$, $\varphi \in \mathcal{F}_E$, and $u, v \in [-1,1]$,

$$\frac{1}{C_1} \le \frac{d}{du}\mathcal{K}_\rho^{(1)}(\varphi)(u) \le C_1, \quad \Big|\frac{d}{dv}\mathcal{K}_\rho^{(1)}(\varphi)(v) - \frac{d}{du}\mathcal{K}_\rho^{(1)}(\varphi)(u)\Big| \le C_1|u-v|. \tag{3.1}$$

In particular $\mathcal{K}_\rho^{(1)}\big(\overline{\mathcal{F}}_E\big) \subset \mathcal{F}_E$. Notice that $\overline{\mathcal{F}}_E$ is a closed convex subset of $C^1([-1,1])$ and, by the previous bounds and the Ascoli-Arzelà theorem, $\mathcal{K}_\rho^{(1)}\big(\overline{\mathcal{F}}_E\big)$ has compact closure in $C^1([-1,1])$. By the Schauder fixed point theorem we get that for each $\rho \in \mathcal{M}$

there exists $\varphi^* \in \overline{\mathcal{F}}_E$ such that $\mathcal{K}_\rho^{(1)}(\varphi^*) = \varphi^*$. From (3.1) it follows that $\varphi^* \in \mathcal{F}_E$ and standard manipulations show that φ^* satisfies (2.17) Lebesgue a.e.

The case $E \in (0, E_0)$, corresponding to a fixed point for $\mathcal{K}_\rho^{(2)}$, is analyzed in the same way. In this case, it is indeed straightforward to check that there exists a constant $C_2 = C_2(\varphi_-, \varphi_+, E) \in (0, \infty)$ such that for any $\rho \in \mathcal{M}$, $\varphi \in \mathcal{F}_E$, and $u \in [-1, 1]$,

$$\frac{1}{C_2} \le \frac{d}{du}\mathcal{K}_\rho^{(2)}(\varphi)(u) - E \le C_2,$$
$$\left| \frac{d}{dv}\mathcal{K}_\rho^{(2)}(\varphi)(v) - \frac{d}{du}\mathcal{K}_\rho^{(2)}(\varphi)(u) \right| \le C_2 \, |u - v|. \tag{3.2}$$

Uniqueness of solutions. Let $\phi \in \mathcal{F}_E$, $E \neq 0$, be a solution to (2.17); by chain rule the equation

$$\left[\frac{1}{E} \log \frac{\phi' - E}{\phi'} \right]' \equiv \frac{\phi''}{\phi'(\phi' - E)} = \rho - \frac{1}{1 + e^\phi}$$

holds Lebesgue a.e. Hence, for each $u \in [-1, 1]$,

$$\frac{1}{E} \log \frac{\phi'(u) - E}{\phi'(u)} = \frac{1}{E} \log \frac{\phi'(-1) - E}{\phi'(-1)} + \int_{-1}^{u} dv \left[\rho(v) - \frac{1}{1 + e^{\phi(v)}} \right]. \tag{3.3}$$

Let $\phi_1, \phi_2 \in \mathcal{F}_E$ be two solutions to (2.17). If $\phi_1'(-1) = \phi_2'(-1)$ an application of the Gronwall inequality in (3.3) yields $\phi_1 = \phi_2$. We next assume $\phi_1'(-1) < \phi_2'(-1)$ and deduce a contradiction. Recall that $\phi_i' > 0 \vee E$ and let $\bar{u} := \inf\{v \in (-1, 1] \,:\, \phi_1(v) = \phi_2(v)\}$, which belongs to $(-1, 1]$ because $\phi_1(\pm 1) = \phi_2(\pm 1)$ and $\phi_1'(-1) < \phi_2'(-1)$. By definition of $\bar{u}$, $\phi_1(u) < \phi_2(u)$ for any $u \in (-1, \bar{u})$, $\phi_1(\bar{u}) = \phi_2(\bar{u})$ and $\phi_1'(\bar{u}) \ge \phi_2'(\bar{u})$. Note that the real function $(0 \vee E, \infty) \ni z \mapsto E^{-1} \log[(z - E)/z]$ is strictly increasing. Therefore from (3.3) we obtain $\phi_1'(\bar{u}) < \phi_2'(\bar{u})$, which is a contradiction and concludes the proof of the uniqueness.

The case $E = 0$, that was examined in [5], can be treated similarly with $-(1/\phi')'$ in place of $\{(1/E) \log[(\phi' - E)/\phi']\}'$.

Claims (i) and (ii). Claim (i) follows straightforwardly from the previous analysis. To prove (ii), let $\phi^n := \Phi(\rho^n) \in \mathcal{F}_E$. By (3.1), (3.2) and the Ascoli-Arzelà theorem, the sequence $\{\phi^n\} \subset \mathcal{F}_E$ is precompact in $C^1([-1, 1])$. It remains to show uniqueness of its limit points. Consider a subsequence n_j and assume that $\{\phi^{n_j}\}$ converges to ψ in $C^1([-1, 1])$. Since $\{\rho^{n_j}\}$ converges to ρ in $\mathcal{M}$ and $\{\phi^{n_j}\}$ converges to ψ in $C^1([-1, 1])$, for $E \le 0$, resp. for $E \in (0, E_0)$, we have that $\mathcal{K}_{\rho^{n_j}}^{(1)}(\phi^{n_j})$ converges to $\mathcal{K}_\rho^{(1)}(\psi)$, resp. $\mathcal{K}_{\rho^{n_j}}^{(2)}(\phi^{n_j})$ converges to $\mathcal{K}_\rho^{(2)}(\psi)$. In particular, $\psi = \lim_j \phi^{n_j} = \lim_j \mathcal{K}_{\rho^{n_j}}^{(i)}(\phi^{n_j}) = \mathcal{K}_\rho^{(i)}(\psi)$ for $i = 1, 2$. By the uniqueness result, $\psi = \Phi(\rho)$. This shows that $\Phi(\rho)$ is the unique possible limit point of the sequence $\{\phi^n\}$, and concludes the proof of Claim (ii). □

Fix a path $\rho \equiv \rho_t(u) \in C^{1,0}([0, T] \times [-1, 1]; [0, 1])$ and let $\phi \equiv \Phi(\rho_t)(u)$ be the solution to (2.17). We prove below that ϕ belongs to $C^{1,2}([0, T] \times [-1, 1])$. Note that, by (3.1) and (3.2), for each $E < E_0$ there exists a constant $C \in (0, \infty)$ such that for any $(t, u) \in [0, T] \times [-1, 1]$,

$$\begin{cases} C^{-1} \le \nabla\phi_t(u) \le C & \text{if } E \le 0, \\ C^{-1} \le \nabla\phi_t(u) - E \le C & \text{if } 0 < E < E_0. \end{cases} \tag{3.4}$$

Lemma 3.2. *Let $E < E_0$, $T > 0$, $\rho \in C^{1,0}([0,T] \times [-1,1]; [0,1])$, and $\phi := \Phi(\rho_t)$ be the solution to (2.17). Then $\phi \in C^{1,2}([0,T] \times [-1,1])$ and $\psi := \partial_t \phi$ is the unique classical solution to the linear boundary value problem*

$$\begin{cases} \nabla\left[\dfrac{\nabla\psi_t}{\nabla\phi_t(\nabla\phi_t - E)}\right] - \dfrac{e^{\phi_t}}{\left(1+e^{\phi_t}\right)^2}\,\psi_t = \partial_t\rho_t & (t,u) \in [0,T] \times (-1,1) \\ \psi_t(\pm 1) = 0 & t \in [0,T]. \end{cases} \tag{3.5}$$

Proof. Fix $t \in [0,T]$. For $h \neq 0$ such that $t+h \in [0,T]$ define $\psi_t^h(\cdot)$ by $\psi_t^h(u) := [\phi_{t+h}(u) - \phi_t(u)]/h$. By Theorem 3.1 (i), $\psi_t^h(\cdot)$ belongs to $C^2([-1,1])$. Set $R_t^h := [\rho_{t+h} - \rho_t]/h$; from (2.17) it follows that ψ^h solves

$$\begin{aligned} &\frac{\Delta\psi_t^h}{\nabla\phi_t(\nabla\phi_t - E)} - \frac{\Delta\phi_{t+h}\,(\nabla\phi_t + \nabla\phi_{t+h} - E)}{\nabla\phi_t(\nabla\phi_t - E)\nabla\phi_{t+h}(\nabla\phi_{t+h} - E)}\,\nabla\psi_t^h \\ &- \frac{e^{\phi_t}}{\left(1+e^{\phi_t}\right)\left(1+e^{\phi_{t+h}}\right)}\,\frac{e^{h\,\psi_t^h} - 1}{h} \;=\; R_t^h \end{aligned} \tag{3.6}$$

for $(t,u) \in [0,T] \times (-1,1)$ with the boundary conditions $\psi_t^h(\pm 1) = 0$, $t \in [0,T]$.

Multiplying the above equation by ψ_t^h and integrating in du, using the inequality $x(e^x - 1) \geq 0$ and an integration by parts we get that

$$\left\langle \nabla\psi_t^h, \frac{\nabla\psi_t^h}{\nabla\phi_t(\nabla\phi_t - E)} \right\rangle \;\leq\; -\,\langle \psi_t^h, R_t^h \rangle \;+\; \langle \psi_t^h, F(\phi_t, \phi_{t+h})\nabla\psi_t^h \rangle, \tag{3.7}$$

where

$$\begin{aligned} F(\phi_t, \phi_{t+h}) := &\frac{1}{(\nabla\phi_t)^2(\nabla\phi_t - E)^2\nabla\phi_{t+h}(\nabla\phi_{t+h} - E)} \\ &\times \{\Delta\phi_t\nabla\phi_{t+h}(\nabla\phi_{t+h} - E)(2\nabla\phi_t - E) \\ &-\; \Delta\phi_{t+h}\nabla\phi_t(\nabla\phi_t - E)(\nabla\phi_{t+h} + \nabla\phi_t - E)\}\,. \end{aligned}$$

For each $t \in [0,T]$,

$$\lim_{h\to 0} \|F(\phi_t, \phi_{t+h})\|_\infty \;=\; 0. \tag{3.8}$$

Indeed, since $\rho \in C^{1,0}([0,T] \times [-1,1])$, as $h \to 0$, $\rho_{t+h}(\cdot) \to \rho_t(\cdot)$ in $C([-1,1])$. By Theorem 3.1 (ii), $\phi_{t+h}(\cdot) \to \phi_t(\cdot)$ in $C^1([-1,1])$. By the differential equation (2.17), $\phi_{t+h}(\cdot) \to \phi_t(\cdot)$ in $C^2([-1,1])$. Together with (3.4) this concludes the proof of (3.8).

By (3.4), Cauchy-Schwarz, and the Poincaré inequality for the Dirichlet Laplacian in $[-1,1]$, we obtain from (3.7) that

$$\begin{aligned} \frac{1}{C^2}\,\langle \nabla\psi_t^h, \nabla\psi_t^h \rangle \;&\leq\; \left\langle \nabla\psi_t^h, \frac{\nabla\psi_t^h}{\nabla\phi_t(\nabla\phi_t - E)} \right\rangle \\ &\leq\; \langle \psi_t^h, \psi_t^h \rangle^{1/2} \left[\langle R_t^h, R_t^h \rangle^{1/2} \;+\; \|F(\phi_t, \phi_{t+h})\|_\infty \langle \nabla\psi_t^h, \nabla\psi_t^h \rangle^{1/2} \right] \\ &\leq\; C'\,\langle \nabla\psi_t^h, \nabla\psi_t^h \rangle^{1/2} \left[\langle R_t^h, R_t^h \rangle^{1/2} + \|F(\phi_t, \phi_{t+h})\|_\infty \langle \nabla\psi_t^h, \nabla\psi_t^h \rangle^{1/2} \right] \end{aligned} \tag{3.9}$$

for some constant $C' > 0$.

From (3.9) and (3.8) it follows that there exists a constant $C'' > 0$ such that

$$\varlimsup_{h\to 0} \langle \nabla\psi_t^h, \nabla\psi_t^h\rangle \;\le\; C'' \,\langle \partial_t\rho_t, \partial_t\rho_t\rangle\,, \quad t\in[0,T]. \tag{3.10}$$

Therefore for each $t \in [0, T]$ the sequence $\{\psi_t^h(\cdot)\}$ is precompact in $C([-1, 1])$. By taking the limit $h \to 0$ in (3.6) and using (3.8), it is now easy to show that any limit point of $\{\psi_t^h(\cdot)\}$ is a weak solution to (3.5). By the classical theory on one-dimensional elliptic problems, see e.g. [24, IV, §2.1], there exists a unique weak solution to (3.5) which is in fact the classical solution because $\partial_t\rho_t(\cdot)$ belongs to $C([-1, 1])$. This implies that there exists a unique limit point $\psi_t(\cdot) \in C^2([-1, 1])$. Finally $\psi \in C^{0,2}([0, T] \times [-1, 1])$ by the continuous dependence in the $C^2([-1, 1])$ topology of the solution to (3.5) w.r.t. $\partial_t\rho_t(\cdot)$ in the $C([-1, 1])$ topology. □

We are now in a position to prove two statements of the first main result of this article.

Proof of Theorem 2.3 (i) and (ii). We start with Claim (i). The case $E = E_0$ follows from the definition (2.16) of the functional S_{E_0}. Assume $E < E_0$. By the convexity of the map $\Xi(\rho) = \rho\log\rho + (1-\rho)\log(1-\rho)$, for each $\varphi \in \mathcal{F}_E$ the functional $\mathcal{G}_E(\cdot, \varphi)$ is convex and lower semicontinuous on $\mathcal{M}$. Hence, by (2.14), the functional S_E, being the supremum of convex lower semicontinuous functionals, is a convex lower semicontinuous functional on $\mathcal{M}$. On the other hand, since the real function $(0\vee E,\infty) \ni x \mapsto \big[x\log x - (x-E)\log(x-E)\big]/E$ is strictly concave, the Jensen inequality and $\varphi(\pm1) = \varphi_\pm$ imply that $\mathcal{G}_E(\rho, \varphi)$ is bounded by some constant depending only on $\varphi_\pm$ and E. This proves (i).

Fix $\rho \in \mathcal{M}$. The strict concavity mentioned above and the strict concavity of the real function $\mathbb{R} \ni x \mapsto -\log(1 + e^x)$ yield that the functional $\mathcal{G}_E(\rho, \cdot)$ is strictly concave on $\mathcal{F}_E$. Thanks to Theorem 3.1, it easily follows that the supremum on the r.h.s. of (2.14) is uniquely attained when $\varphi = \Phi(\rho)$. □

In the proof of the equality between the quasi-potential V_E and the functional S_E, we shall need the following simple observation.

Lemma 3.3. *For each $\rho \in \mathcal{M}$ there exists a sequence $\{\rho^n\} \subset \mathcal{M}$ converging to ρ in $\mathcal{M}$ and such that: $\rho^n \in C^2([-1, 1])$, $\rho^n(\pm1) = \rho_\pm$, $0 < \rho^n < 1$, $S_E(\rho^n) \to S_E(\rho)$.*

Proof. For $E = E_0$, this is obvious from the definition of the functional S_{E_0}. For $E < E_0$, given $\rho \in \mathcal{M}$, it is enough to consider a sequence $\{\rho^n\} \subset C^2([-1, 1])$ with $\rho^n(\pm1) = \rho_\pm$ and $0 < \rho^n < 1$, which converges to ρdu a.e. By Theorem 2.3 (ii), Theorem 3.1 (ii), and dominated convergence, $S_E(\rho^n) = \mathcal{G}_E(\rho^n, \Phi(\rho^n)) \longrightarrow \mathcal{G}_E(\rho, \Phi(\rho)) = S_E(\rho)$. □

4. The Quasi-Potential

In this section we characterize the optimal path for the variational problem (2.10) defining the quasi-potential V_E and conclude the proof of Theorem 2.3 by showing the equality $V_E = S_E$. The heuristic argument is quite simple. To the variational problem (2.10) is associated the following Hamilton-Jacobi equation [4,6]. The quasi-potential V_E is the maximal solution to

$$\frac{1}{2}\left\langle \nabla\frac{\delta V_E}{\delta\rho}\,,\,\chi(\rho)\,\nabla\frac{\delta V_E}{\delta\rho}\right\rangle + \left\langle \frac{\delta V_E}{\delta\rho}\,,\,\frac{1}{2}\Delta\rho \;-\; \frac{E}{2}\,\nabla\chi(\rho)\right\rangle \;=\; 0 \tag{4.1}$$

with the boundary condition that $\delta V_E/\delta\rho$ vanishes at the endpoints of $[-1, 1]$. Few formal computations show that S_E solves (4.1). To check that S_E is the maximal solution one constructs a suitable path for the variational problem (2.10), [4,6]. Since it is not clear how to analyze (4.1) directly, we first approximate, as in [5], paths $\pi \in D([0, T]; \mathcal{M})$ with $I_T(\pi|\bar\rho_E) < \infty$ by smooth paths bounded away from 0 and 1 which satisfy the boundary conditions $\rho_\pm$ at the endpoints of $[-1, 1]$. For such smooth paths we can make sense of (4.1) and complete the proof.

In the case $E = E_0$, the process is reversible and the picture is well known. The path which minimizes the variational formula defining the quasi-potential is the solution of the hydrodynamic equation reversed in time. The identity between S_{E_0} and V_{E_0} follows easily from this principle. The proof presented below for $E < E_0$ can be adapted with several simplifications. It is enough to set $\Phi(\rho) = \log\{\bar\rho_{E_0}/1 - \bar\rho_{E_0}\}$ everywhere.

Assume from now on that $E < E_0$. We first need to recall some notation introduced in [8]. Fix a density profile $\gamma : [-1, 1] \to [0, 1]$ and a time $T > 0$. Denote by $\mathcal{F}_2 = \mathcal{F}_2(T, \gamma, \rho_\pm)$ the set of trajectories π in $C([0, T], \mathcal{M})$ bounded away from 0 and 1 in the sense that for each $t > 0$, there exists $\varepsilon > 0$ such that $\varepsilon \le \pi \le 1 - \varepsilon$ on $[t, T]$, which satisfy the boundary conditions, $\pi_0 = \gamma$, $\pi_t(\pm 1) = \rho_\pm$, $0 \le t \le T$, and for which there exists $\delta_1, \delta_2 > 0$ such that π_t follows the hydrodynamic equation (2.5) in the time interval $[0, \delta_1]$, π_t is constant in the time interval $[\delta_1, \delta_1 + \delta_2]$ and π_t is smooth in time in the time interval $(\delta_1, T]$.

If the density profile γ is the stationary profile $\bar\rho_E$, the trajectories π in $\mathcal{F}_2$ are in fact constant in the time interval $[0, \delta_1 + \delta_2]$. Since they are also smooth in time in $(\delta_1, T]$, we deduce that they are smooth in time in the interval $[0, T]$. Moreover, since $\bar\rho_E$ is bounded away from 0 and 1, there exists $\varepsilon > 0$ such that $\varepsilon \le \pi \le 1 - \varepsilon$ on $[0, T]$.

Assume that $\gamma = \bar\rho_E$ and recall from the proof of [8, Theorem 4.6] the definition of the sequence of trajectories $\{\pi_\varepsilon : \varepsilon > 0\}$. Since a path π in $\mathcal{F}_2$ is in fact constant in the time interval $[0, b]$, each π_ε is smooth in space and time. In particular, let

$$\mathcal{D}_0 \;:=\; C^{\infty,\infty}([0, T] \times [-1, 1]) \cap \mathcal{F}_2. \tag{4.2}$$

Theorem 4.6 in [8] can be rephrased in the present context as

Theorem 4.1. *For each π in $D([0, T], \mathcal{M})$ such that $I_T(\pi|\bar\rho_E) < \infty$, there exists a sequence $\{\pi^n\} \subset \mathcal{D}_0$ converging to π in $D([0, T]; \mathcal{M})$ such that $I_T(\pi^n|\bar\rho_E)$ converges to $I_T(\pi|\bar\rho_E)$.*

The first two lemmata of this section state that, for smooth paths, the functional S_E satisfies (4.1). Recall that for $\rho \in \mathcal{M}$ we denote by $\Phi(\rho) \in \mathcal{F}_E$ the unique solution to (2.17).

Lemma 4.2. *Let $E < E_0$, $T > 0$, $\pi \in \mathcal{D}_0$, and $\Gamma : [0, T] \times [-1, 1] \to \mathbb{R}$ be defined by*

$$\Gamma_t \;:=\; \log\frac{\pi_t}{1 - \pi_t} \;-\; \Phi(\pi_t). \tag{4.3}$$

Then

$$S_E(\pi_T) - S_E(\pi_0) \;=\; \int_0^T dt\, \langle \Gamma_t, \partial_t \pi_t \rangle. \tag{4.4}$$

Proof. Let $\phi \equiv \phi_t(u) := \Phi(\pi_t)(u)$, $(t,u) \in [0,T]\times[-1,1]$. By Lemma 3.2, ϕ belongs to $C^{1,2}([0,T]\times[-1,1])$. Since $\phi_t(\pm 1) = \varphi_\pm$, then $\partial_t \phi_t(\pm 1) = 0$, $t \in [0,T]$. By Theorem 2.3 (ii), dominated convergence, an explicit computation, and an integration by parts,

$$\begin{aligned}\frac{d}{dt} S_E(\pi_t) &= \frac{d}{dt}\,\mathcal{G}_E\,(\pi_t\,,\ \Phi(\pi_t)) \\ &= \langle \Gamma_t\,,\ \partial_t \pi_t\rangle \ +\ \left\langle \partial_t\phi_t\,,\ \frac{\Delta\phi_t}{\nabla\phi_t(\nabla\phi_t - E)}\ +\ \frac{1}{1+e^{\phi_t}}\ -\ \pi_t \right\rangle.\end{aligned}$$

The lemma follows, noticing that the last term vanishes by (2.17). □

Let

$$\mathcal{M}_0 \ :=\ \left\{\rho \in C^2([-1,1])\ :\ \rho(\pm 1) = \rho_\pm\,,\ 0 < \rho < 1\right\}. \tag{4.5}$$

Lemma 4.3. *Let $E < E_0$, $\rho \in \mathcal{M}_0$, and $\Gamma : [-1,1] \to \mathbb{R}$ be defined by*

$$\Gamma \ :=\ \log\frac{\rho}{1-\rho}\ -\ \Phi(\rho). \tag{4.6}$$

Then,

$$\langle \nabla\Gamma\,,\ \chi(\rho)\,\nabla\Gamma\rangle\ -\ \langle \nabla\rho\ -\ E\chi(\rho)\,,\ \nabla\Gamma\rangle\ =\ 0. \tag{4.7}$$

Proof. As before we let $\phi \equiv \phi(u) := \Phi(\rho)(u)$, $u \in [-1,1]$. By Theorem 3.1 (i), ϕ belongs to $C^2([-1,1])$. By the definition of Γ in (4.6), statement (4.7) is equivalent to

$$\langle \nabla\rho\,,\ -\nabla\phi + E\rangle\ +\ \langle -\nabla\phi\,,\ \chi(\rho)\,(-\nabla\phi + E)\rangle\ =\ 0.$$

The above equation holds if and only if

$$\begin{aligned}&\left\langle \nabla\left(\rho - \frac{e^\phi}{1+e^\phi}\right)\,,\ \nabla\phi - E\right\rangle + \left\langle \nabla\left(\frac{e^\phi}{1+e^\phi}\right)\,,\ \nabla\phi - E\right\rangle \\ &\quad -\ \langle \nabla\phi\,,\ \chi(\rho)(\nabla\phi - E)\rangle\ =\ 0.\end{aligned}$$

Since $e^{\phi(\pm1)}/[1 + e^{\phi(\pm1)}] = e^{\varphi_\pm}/[1+e^{\varphi_\pm}] = \rho_\pm = \rho(\pm1)$, integrating by parts the previous equation, it becomes

$$\left\langle \rho - \frac{e^\phi}{1+e^\phi}\,,\ \Delta\phi\right\rangle - \left\langle\left(\frac{e^\phi}{\left(1+e^\phi\right)^2}\ -\ \chi(\rho)\right)\nabla\phi\,,\ \nabla\phi - E\right\rangle\ =\ 0.$$

At this point the explicit expression for χ given by $\chi(\rho) = \rho(1-\rho)$ plays a crucial role. Indeed, for such χ,

$$\frac{e^\phi}{(1+e^\phi)^2} - \chi(\rho)\ =\ -\left(\frac{e^\phi}{1+e^\phi} - \rho\right)\left(\frac{e^\phi}{1+e^\phi} - (1-\rho)\right),$$

so that (4.7) is equivalent to

$$\left\langle \rho - \frac{e^\phi}{1+e^\phi}\,,\ \Delta\phi\ +\ \nabla\phi\,(\nabla\phi - E)\left(1 - \rho - \frac{e^\phi}{1+e^\phi}\right)\right\rangle\ =\ 0,$$

which holds true because $\phi = \Phi(\rho)$ solves (2.17). □

We next prove the first half of the equality $V_E = S_E$. In fact the argument basically shows that any solution to the Hamilton-Jacobi equation (4.1) gives a lower bound on the quasi-potential.

Proof of Theorem 2.3: the inequality $V_E \geq S_E$. In view of the variational definition of V_E in (2.10), to prove the lemma we need to show that for each $\rho \in \mathcal{M}$ we have $S_E(\rho) \leq I_T(\pi|\bar{\rho}_E)$ for any $T > 0$ and any path $\pi \in D([0, T]; \mathcal{M})$ such that $\pi_T = \rho$.

Assume firstly that $\rho \in \mathcal{M}_0$ and consider only paths $\pi \in \mathcal{D}_0$. Of course the energy $\mathcal{Q}(\pi)$ of such a path π is finite. In view of the variational definition of $I_T(\pi|\bar{\rho}_E)$ given in (2.6), (2.7), to prove that $S_E(\rho) \leq I_T(\pi|\bar{\rho}_E)$, it is enough to exhibit some function $H \in C_0^{1,2}([0, T] \times [-1, 1])$ for which $S_E(\rho) \leq \hat{J}_{T,H,\bar{\rho}_E}(\pi)$. We claim that Γ given in (4.3) fulfills these conditions. Let $\phi \equiv \phi_t(u) := \Phi(\pi_t)(u)$. Since $\pi \in \mathcal{D}_0$, by Lemma 3.2 $\Gamma \in C^{1,2}([0, T] \times [-1, 1])$. On the other hand, since $\pi_t(\pm 1) = \rho_\pm$ and $\phi_t(\pm 1) = \varphi_\pm$, $\Gamma_t(\pm 1) = 0$, $t \in [0, T]$; whence $\Gamma \in C_0^{1,2}([0, T] \times [-1, 1])$. Recalling the definition of the functional $\hat{J}_{T,\Gamma,\bar{\rho}_E}$, after an integration by parts, we obtain that

$$\hat{J}_{T,\Gamma,\bar{\rho}_E}(\pi) = \int_0^T dt \left[\langle \Gamma_t, \partial_t \pi_t \rangle + \frac{1}{2} \langle \nabla \Gamma_t, \nabla \pi_t - E\chi(\pi_t) \rangle - \frac{1}{2} \left\langle \chi(\pi_t), (\nabla \Gamma_t)^2 \right\rangle \right].$$

By using Lemmata 4.2 and 4.3, since $\pi_0 = \bar{\rho}_E$, $S_E(\bar{\rho}_E) = 0$, it follows that $\hat{J}_{T,\Gamma,\bar{\rho}_E}(\pi) = S_E(\rho)$, which proves the statement for $\rho \in \mathcal{M}_0$ and paths $\pi \in \mathcal{D}_0$.

Let now $\rho \in \mathcal{M}$ and consider an arbitrary path $\pi \in D([0, T]; \mathcal{M})$ such that $\pi_T = \rho$. With no loss of generality we can assume $I_T(\pi|\bar{\rho}_E) < \infty$. Let $\{\pi^n\} \subset \mathcal{D}_0$ be the sequence given by Theorem 4.1. The result for $\rho \in \mathcal{M}_0$ and paths in $\mathcal{D}_0$, together with the lower semicontinuity of S_E, yield

$$I_T(\pi|\bar{\rho}_E) = \lim_{n\to\infty} I_T(\pi^n|\bar{\rho}_E) \geq \varliminf_{n\to\infty} S_E(\pi_T^n) \geq S_E(\pi_T) = S_E(\rho),$$

which concludes the proof. □

To prove the converse inequality $V_E \leq S_E$ on $\mathcal{M}$, we need to characterize the optimal path for the variational problem (2.10). The following lemma explains which is the right candidate.

Denote by $C_K^\infty(\Omega_T)$ the smooth functions $H : \Omega_T \to \mathbb{R}$ with compact support. For a trajectory π in $D([0, T], \mathcal{M})$, let $\mathcal{H}_0^1(\chi(\pi))$ be the Hilbert space induced by $C_K^\infty(\Omega_T)$ endowed with the scalar product defined by

$$\langle\langle G, H \rangle\rangle_{1,\chi(\pi)} = \int_0^T dt \int_{-1}^1 du \, (\nabla G)(t, u) \, (\nabla H)(t, u) \, \chi(\pi(t, u)).$$

Induced means that we first declare two functions F, G in $C_K^\infty(\Omega_T)$ to be equivalent if $\langle\langle F - G, F - G \rangle\rangle_{1,\chi(\pi)} = 0$ and then we complete the quotient space with respect to the scalar product. Denote by $\| \cdot \|_{1,\chi(\pi)}$ the norm associated to the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{1,\chi(\pi)}$.

Repeating the arguments of the proof of Lemma 4.7 in [8], we obtain an explicit expression of the rate function $I_T(\pi|\gamma)$ in terms of a solution to an elliptic equation.

Lemma 4.4. *Fix a trajectory π in $\mathcal{D}_0$. For each $0 \leq t \leq T$, let H_t be the unique solution to the elliptic equation*

$$\begin{cases} \partial_t \pi_t = (1/2)\Delta \pi_t - \nabla \{\chi(\pi_t) [(E/2) + \nabla H_t]\}, \\ H_t(\pm 1) = 0. \end{cases} \tag{4.8}$$

Then, H is smooth on $[0, T] \times [-1, 1]$ *and*

$$I_T(\pi|\pi_0) \;=\; \frac{1}{2}\|H\|^2_{1,\chi(\pi)}. \tag{4.9}$$

We could have used the next lemma to prove the inequality $V_E \ge S_E$; we presented the separate argument before for its simplicity. On the other hand, (4.11) clearly suggests that the optimal path for the variational problem (2.10) is obtained by taking a path which satisfies (4.10) with $K = 0$. Recall that $\Phi(\rho)$ denotes the solution to (2.17).

Lemma 4.5. *Let $E < E_0$, $T > 0$, $\gamma \in \mathcal{M}_0$, and $\pi \in \mathcal{D}_0$ be such that $I_T(\pi|\gamma) < \infty$. Then, there exists K in $C_0^{1,2}$ $([0, T] \times [-1, 1])$ such that π is a classical solution to*

$$\begin{cases} \partial_t \pi_t + \dfrac{E}{2}\,\nabla\chi(\pi_t) = -\dfrac{1}{2}\,\Delta\pi_t + \nabla\left[\chi(\pi_t)\nabla\left(\Phi(\pi_t) + K_t\right)\right] \\ \pi_t(\pm 1) = \rho_\pm \\ \pi_0 = \gamma. \end{cases} \tag{4.10}$$

Furthermore,

$$I_T(\pi|\gamma) = S_E(\pi_T) - S_E(\gamma) + \frac{1}{2}\;\|K\|^2_{1,\chi(\pi)}\;. \tag{4.11}$$

Proof. Note that $\gamma = \pi_0$ because we assume the rate function to be finite. Denote by H the smooth function introduced in Lemma 4.4 and let Γ be as defined in (4.3). We claim that $K := \Gamma - H$ meets the requirements in the lemma. As before we have that Γ belongs to $C_0^{1,2}$ $([0, T] \times [-1, 1])$. Hence, K also belongs to this space because H is smooth and vanishes at the boundary of $[-1, 1]$. The equation (4.10) follows easily from (4.8) replacing H by $\Gamma - K$. To prove identity (4.11), consider (4.4) and express $\partial_t \pi_t$ in terms of the differential equation in (4.8). Since $H = \Gamma - K$, after an integration by parts we get that $S_E(\pi_T) - S_E(\gamma)$ is equal to

$$-\frac{1}{2}\int_0^T dt\;\langle\nabla\Gamma_t, \nabla\pi_t - E\,\chi(\pi_t)\rangle \;+\; \int_0^T dt\;\langle\nabla\Gamma_t\,,\;\chi(\pi_t)\nabla\left(\Gamma_t - K_t\right)\rangle\,.$$

By Lemma 4.3, the previous expression is equal to

$$\frac{1}{2}\int_0^T dt\;\langle\nabla\Gamma_t\,,\;\chi(\pi_t)\nabla\Gamma_t\rangle \;-\; \int_0^T dt\;\langle\nabla\Gamma_t\,,\;\chi(\pi_t)\nabla K_t\rangle\,.$$

Since $K = \Gamma - H$, we finally get that

$$S_E(\pi_T) \;-\; S_E(\gamma) \;+\; \frac{1}{2}\;\|K\|^2_{1,\chi(\pi)} \;=\; \frac{1}{2}\;\|H\|^2_{1,\chi(\pi)}\,,$$

which, in view of (4.9), concludes the proof. □

We next show how a solution to the (nonlocal) Eq. (4.10) with $K = 0$ can be obtained by the algorithm presented below the statement of Theorem 2.3. Recall that such algorithm requires to solve (2.17) only for the initial datum and then to solve the (local) hydrodynamic equation (2.5). Note indeed that by setting $\pi_t^* := \rho_{-t}^*$, where ρ^* is defined in the next lemma, then π^* solves the differential equation in (4.10) with $K = 0$.

Fix $E < E_0$, $\gamma \in \mathcal{M}_0$ and set $G := e^{\Phi(\gamma)}/[1 + e^{\Phi(\gamma)}]$. By Theorem 3.1 the profile G belongs to $C^4([-1, 1])$, it is strictly increasing and satisfies $G(\pm 1) = \rho_\pm$. Denote by $F \equiv F_t(u) \in C^{1,4}$ $([0, \infty) \times [-1, 1])$ the solution to the hydrodynamic equation (2.5) with γ replaced by G. By the maximum principle, $\rho_- \le F \le \rho_+$.

Lemma 4.6. *Let $\psi := \log[F/(1-F)]$. Then, ψ belongs to $C^{1,4}([0,\infty)\times[-1,1])$ and satisfies $\nabla\psi > 0 \vee E$. Let $\rho^* \equiv \rho_t^*(u)$ be defined by*

$$\rho^* := \frac{1}{1+e^{\psi}} + \frac{\Delta\psi}{\nabla\psi(\nabla\psi - E)}\,. \tag{4.12}$$

Then, ρ^* belongs to $C^{1,2}([0,\infty)\times[-1,1])$, satisfies $\rho_t^*(\pm1) = \rho_\pm$, $0 < \rho^* < 1$, and solves

$$\begin{cases} \partial_t\rho_t^* - \dfrac{E}{2}\nabla\chi(\rho_t^*) = \dfrac{1}{2}\Delta\rho_t^* - \nabla\left[\chi(\rho_t^*)\nabla\Phi(\rho_t^*)\right] \\ \rho_t^*(\pm1) = \rho_\pm \\ \rho_0^* = \gamma. \end{cases} \tag{4.13}$$

Proof. Let $\psi : [0,\infty)\times[-1,1] \to \mathbb{R}$ be given by $\psi = \log\{F/1-F\}$ and set

$$\tau := \sup\{t \geq 0 \,:\, \nabla\psi_s(u) > 0 \vee E \text{ for all } (s,u) \in [0,t]\times[-1,1]\}\,.$$

Since $\nabla\psi_0 = \nabla\Phi(\gamma) > 0 \vee E$, $\tau > 0$ by continuity. We show at the end of the proof that $\tau = \infty$.

A straightforward computation shows that ψ solves

$$\begin{cases} \partial_t\psi = \dfrac{1}{2}\Delta\psi + \dfrac{1}{2}\dfrac{1-e^{\psi}}{1+e^{\psi}}\nabla\psi\,(\nabla\psi - E) \\ \psi_t(\pm1) = \varphi_\pm \\ \psi_0 = \Phi(\gamma). \end{cases} \tag{4.14}$$

Since $\psi \in C^{1,4}([0,\infty)\times[-1,1])$, definition (4.12) yields $\rho^* \in C^{1,2}([0,\tau)\times[-1,1])$ and $\rho_0^* = \gamma$. On the other hand, from (4.14) we deduce that for any $t \in [0,\tau)$,

$$\Delta\psi_t(\pm1) + \frac{1-e^{\varphi_\pm}}{1+e^{\varphi_\pm}}\nabla\psi_t(\pm1)\,[\nabla\psi_t(\pm1) - E] = 0.$$

Whence, again by (4.12),

$$\rho_t^*(\pm1) = \frac{1}{1+e^{\varphi_\pm}} - \frac{1-e^{\varphi_\pm}}{1+e^{\varphi_\pm}} = \rho_\pm.$$

By using (4.14), a long and tedious computation that we omit shows that ρ_t^*, $t \in [0,\tau)$, solves the differential equation in (4.13).

We next show that $0 < \rho^* < 1$. Since $\gamma \in \mathcal{M}_0$, there exists $\delta \in (0,1)$ such that $\delta \leq \gamma \leq 1-\delta$. We claim that $\min\{\rho_-, 1-\rho_+, \delta\} \leq \rho^* \leq \max\{\rho_+, 1-\rho_-, 1-\delta\}$. Fix $t \in (0,\tau)$ and assume that $\rho_t^*(\cdot)$ has a local maximum at $u_0 \in (-1,1)$. Since ρ^* solves (4.13), since $\Phi(\rho^*)$ solves (2.17) and since $\nabla\rho_t^*(u_0) = 0$, $\Delta\rho_t^*(u_0) \leq 0$,

$$\begin{aligned} \partial_t\rho_t^*(u_0) &= \frac{1}{2}\Delta\rho_t^*(u_0) \;-\; \chi(\rho_t^*(u_0))\,\Delta\Phi(\rho_t^*)(u_0) \\ &\leq -\chi(\rho_t^*(u_0))\,\nabla\Phi(\rho_t^*)(u_0)\,\left[\nabla\Phi(\rho_t^*)(u_0) - E\right]\left[\rho_t^*(u_0) - \frac{1}{1+e^{\Phi(\rho_t^*)(u_0)}}\right]. \end{aligned}$$

Assume now that $\rho_t^*(u_0) > 1-\rho_-$. Since $\Phi(\rho_t^*) \geq \varphi_-$ we deduce $\rho_t^*(u_0) - [1 + e^{\Phi(\rho_t^*)(u_0)}]^{-1} > 1-\rho_- - [1+e^{\varphi_-}]^{-1} = 0$. As $\nabla\Phi(\rho_t^*)(u_0) > 0\vee E$, we get $\partial_t\rho_t^*(u_0) < 0$.

In particular, by a standard argument, $\rho^* \le \max\{\rho_+, 1-\rho_-, 1-\delta\}$. The proof of the lower bound is analogous.

We conclude the proof showing that $\tau = \infty$. Assume that $\tau < \infty$. Since for each $t \in [0,\tau)$, ρ^*_t belongs to $\mathcal{M}_0$, it follows from (4.12) that $\Phi(\rho^*_t) = \psi_t$, $t \in [0,\tau)$. By Theorem 3.1 (ii), $\Phi(\rho^*_\tau) = \psi_\tau$ so that $\nabla\psi_\tau = \nabla\Phi(\rho^*_\tau) > E \vee 0$ because $\Phi(\rho^*_\tau)$ belongs to $\mathcal{F}_E$. By continuity, there exists $\delta > 0$ such that $\nabla\psi_t > E \vee 0$ for $\tau \le t < \tau + \delta$. This contradicts the definition of τ. □

Fix a density profile $\gamma : [-1,1] \to [0,1]$, a time $T > 0$ and consider the solution ρ^* to (4.13). Let $\lambda_t(\cdot) = \rho^*_{T-t}(\cdot)$. Clearly, λ is the solution to (4.10) in the time interval $[0,T]$ with $K = 0$ and initial condition $\lambda_0 = \rho^*_T$. In particular, by (4.11),

$$I_T(\lambda|\rho^*_T) \;=\; S_E(\gamma) \;-\; S_E(\rho^*_T).$$

In the next lemma we prove that ρ^*_T converges to $\bar\rho_E$ as $T \to \infty$. Letting $T \uparrow \infty$ in the previous formula, we see that the time reversed trajectory of (4.13) is the natural candidate to solve the variational formula defining the quasi-potential. This argument is made rigorous in the next paragraphs.

By standard properties of parabolic equations on a bounded interval, see e.g. [12], as $t \to \infty$, the solution to (2.5) converges, in a strong topology, to the unique stationary solution $\bar\rho_E$. Such convergence implies that the path ρ^*, as defined in Lemma 4.6, also converges to $\bar\rho_E$ as $t \to \infty$. This is the content of the next lemma. This result will permit to use the time reversal of ρ^* as a trial path in the variational problem (2.10).

Lemma 4.7. *Let $E < E_0$, $\gamma \in \mathcal{M}_0$, and ρ^* be defined as in Lemma 4.6. As $t \to \infty$, the profile $\rho^*_t \in \mathcal{M}_0$ converges to $\bar\rho_E$ in the $C^1([-1,1])$ topology, uniformly for $\gamma \in \mathcal{M}_0$.*

Proof. Recall the notation introduced just before Theorem 4.6. Let ρ be the solution to (2.5). In [12, Theorem 4.9] it is shown that, as $t \to \infty$, the profile ρ_t converges to $\bar\rho_E$ in the $C^1([-1,1])$ topology, uniformly for $\gamma \in \mathcal{M}_0$. By the methods there developed, it is however straightforward to prove this statement in the $C^3([-1,1])$ topology. In particular, F_t converges to $\bar\rho_E$ in the $C^3([-1,1])$ topology to $\bar\rho_E$ so that ψ_t converges to $\log[\bar\rho_E/(1+\bar\rho_E)] = \bar\varphi_E$ in the $C^3([-1,1])$ topology uniformly in $\gamma \in \mathcal{M}_0$. Since $\Phi(\bar\rho_E) = \bar\varphi_E$, the statement now follows from (4.12). □

We next show that profiles close to $\bar\rho_E$ in a strong topology can be reached with a small cost.

Lemma 4.8. *Let $E < E_0$ and $\delta \in (0,1)$. Then, there exist $T > 0$ and constant $C = C(E,\rho_\pm,\delta) > 0$ such that the following hold. For each $\rho \in C^1([-1,1])$ satisfying $\rho(\pm 1) = \rho_\pm$ and $\delta \le \rho \le 1-\delta$, there exists a path $\hat\pi \in D([0,T];\mathcal{M})$ such that $\hat\pi_T = \rho$ and*

$$I_T(\hat\pi|\bar\rho_E) \le C\,\left\|\rho - \bar\rho_E\right\|^2_{C^1}.$$

Proof. Simple computations show that $T = 1$ and the straight path $\hat\pi_t = \bar\rho_E + t(\rho - \bar\rho_E)$ meet the requirements. For $E = 0$, in [5, Lemma 5.7] a more clever path is chosen which yields a bound in terms of the L_2 norm of $\rho - \bar\rho_E$. □

We can now conclude the proof of Theorem 2.3.

Proof of Theorem 2.3: the inequality $V_E \le S_E$. Given $\rho \in \mathcal{M}$ and $\delta > 0$ we need to find $T > 0$ and a path $\pi^* \in D([0,T];\mathcal{M})$ such that $\pi^*_T = \rho$ and $I_T(\pi^*|\bar\rho_E) \le S_E(\rho) + \delta$. By Lemma 3.3, there exists a sequence $\{\rho^n\} \subset \mathcal{M}_0$ converging to ρ in $\mathcal{M}$ and such that $S_E(\rho^n) \to S_E(\rho)$. Let $\rho^{*,n}$ be the path constructed in Lemma 4.6 with γ replaced by ρ^n and pick $\varepsilon > 0$ to be chosen later. By Lemma 4.7, there exists a time $T_1 = T_1(\varepsilon) > 0$ independent of n such that $\|\rho^{*,n}_{T_1} - \bar\rho_E\|_{C^1} \le \varepsilon$. Whence, by Lemma 4.8, there exists a time $T_2 > 0$, still independent of n, and a path $\hat\pi^n_t$, $t \in [0,T_2]$ such that $\hat\pi^n_0 = \bar\rho_E$, $\hat\pi^n_{T_2} = \rho^{*,n}_{T_1}$ and $I_{T_2}(\hat\pi^n|\bar\rho_E) \le \beta_\varepsilon$, where β_ε vanishes as $\varepsilon \to 0$ and is independent of n. We now set $T := T_1 + T_2$ and let $\pi^{*,n}_t$, $t \in [0,T]$ be the path defined by

$$\pi^{*,n}_t := \begin{cases} \hat\pi^n_t & t \in [0,T_2] \\ \rho^{*,n}_{T-t} & t \in (T_2,T] \end{cases},$$

which satisfies $\pi^{*,n}_0 = \bar\rho_E$ and $\pi^{*,n}_T = \rho^n$. The covariance of I w.r.t. time shifts, Lemmata 4.5 and 4.6 yield

$$\begin{aligned} I_T(\pi^{*,n}|\bar\rho_E) &= I_{T_2}(\hat\pi^n|\bar\rho_E) + I_{T_1}(\rho^{*,n}_{T_1-\cdot}|\rho^{*,n}_{T_1}) \\ &\le \beta_\varepsilon + S_E(\rho^n) - S_E(\rho^{*,n}_{T_1}) \;\le\; \beta_\varepsilon + S_E(\rho^n). \end{aligned} \tag{4.15}$$

Since $S_E(\rho^n) \to S_E(\rho) < \infty$ and $I_T(\cdot|\bar\rho_E)$ has compact level sets, see Theorem 2.1, the bound (4.15) implies precompactness of the sequence $\{\pi^{*,n}\} \subset D([0,T];\mathcal{M})$. Therefore a path π^* and a subsequence n_j exist such that $\pi^{*,n_j} \to \pi^*$ in $D([0,T];\mathcal{M})$. In particular $\pi^*_T = \lim_j \pi^{*,n_j}_T = \lim_j \rho^{n_j} = \rho$. The lower semicontinuity of $I_T(\cdot|\bar\rho_E)$ and (4.15) now yield

$$I_T(\pi^*|\bar\rho_E) \;\le\; \varliminf_{j\to\infty} I_T(\pi^{*,n_j}|\bar\rho_E) \;\le\; \beta_\varepsilon + \lim_{j\to\infty} S_E(\rho^{n_j}) \;=\; \beta_\varepsilon + S_E(\rho),$$

which, by choosing ε so that $\beta_\varepsilon \le \delta$, concludes the proof. □

5. The Asymmetric Limit

In this section we discuss the asymmetric limit $E \to -\infty$ and prove Theorems 2.4 and 2.6.

Proof of Theorem 2.4: Γ-liminf inequality. Fix $\rho \in \mathcal{M}$ and a sequence $\{\rho_E\} \subset \mathcal{M}$ converging to ρ in $\mathcal{M}$ as $E \to -\infty$. We show that $\varliminf_E S_E(\rho_E) \ge S_{\mathrm{a}}(\rho)$.

Let J_E be such that (2.8) holds; it is straightforward to check that

$$\lim_{E\to-\infty} \frac{J_E}{E} = \max_{r\in[\rho_-,\rho_+]} \chi(r),$$

whence, recalling that A_E has been defined in (2.13) and A_{a} in (2.21),

$$\lim_{E\to-\infty} \big[A_E - \log(-E)\big] = \max_{r\in[\rho_-,\rho_+]} \log\chi(r) = A_{\mathrm{a}}. \tag{5.1}$$

Fix $\varphi \in C^{1+1}([-1,1])$ such that $\varphi(\pm1) = \varphi_\pm$ and $\varphi' > 0$. From (5.1) it easily follows that

$$\lim_{E\to-\infty} \int_{-1}^{1} du \left\{ \frac{1}{E}\big[\varphi'\log\varphi' - (\varphi' - E)\log(\varphi' - E)\big] - (A_E - A_{\mathrm{a}}) \right\} = 0. \tag{5.2}$$

Recalling (2.14), (2.12) and (2.20), from the convexity of the real function $F : [0, 1] \to \mathbb{R}$, $F(\rho) = \rho \log \rho + (1 - \rho) \log(1 - \rho)$ and (5.2) we get

$$\varliminf_{E \to -\infty} S_E(\rho_E) \geq \varliminf_{E \to -\infty} \mathcal{G}_E(\rho_E, \varphi) \geq \mathcal{G}_a(\rho, \varphi).$$

The proof of the Γ-liminf inequality is now completed by optimizing on φ. Note indeed that the supremum in (2.22) can be restricted to strictly increasing $\varphi \in C^{1+1}([-1, 1])$ such that $\varphi(\pm 1) = \varphi_\pm$. □

Proof of Theorem 2.4: Γ-limsup inequality. Fix $\rho \in \mathcal{M}$; we need to exhibit a sequence $\{\rho_E\} \subset \mathcal{M}$ converging to ρ in $\mathcal{M}$ as $E \to -\infty$ such that $\varlimsup_E S_E(\rho_E) \leq S_a(\rho)$. We claim that the constant sequence $\rho_E = \rho$ meets this condition.

Recalling item (ii) in Theorem 2.3, let $\phi_E := \Phi(\rho) \in \mathcal{F}_E$ be the solution to (2.17) in which we indicated explicitly its dependence on E. From the concavity of the real function $F : [0, \infty) \to \mathbb{R}$, $F(x) = E^{-1} \big[x \log x - (x - E) \log(x - E) \big]$, $E < 0$, the Jensen inequality, and (5.1) we deduce

$$\varlimsup_{E \to -\infty} \int_{-1}^{1} du \, \left\{ \frac{1}{E} \big[\phi'_E \log \phi'_E - (\phi'_E - E) \log(\phi'_E - E) \big] - (A_E - A_a) \right\} \leq 0. \quad (5.3)$$

Since $\mathcal{F}_E \subset \mathcal{F}_a$ and $\overline{\mathcal{F}}_a$ is compact, the sequence $\{\phi_E\}$ is precompact in $\overline{\mathcal{F}}_a$. Let now $\phi^* \in \overline{\mathcal{F}}_a$ be any limit point of $\{\phi_E\}$ and pick a subsequence $E' \to -\infty$ such that $\phi_{E'} \to \phi^*$ in $\mathcal{F}_a$. In particular $\phi_{E'}(u) \to \phi^*(u)$ Lebesgue a.e. Recalling Theorem 2.3 (ii), (2.12), (2.20), and using (5.3) we get that

$$\varlimsup_{E' \to -\infty} S_{E'}(\rho) = \varlimsup_{E' \to -\infty} \mathcal{G}_{E'}(\rho, \phi_{E'}) \leq \mathcal{G}_a(\rho, \phi^*) \leq S_a(\rho),$$

which concludes the proof. □

Proof of Theorem 2.6. Existence of a maximizer for (2.22) follows from the compactness of $\overline{\mathcal{F}}_a$ and from the continuity of $\mathcal{G}_a(\rho, \cdot)$ for the topology of $\overline{\mathcal{F}}_a$. On the other hand, the strict concavity of the function $F : [\varphi_-, \varphi_+] \to \mathbb{R}_+$, $F(\varphi) = -\log(1 + e^\varphi)$, gives the uniqueness of the maximizer.

The proof of the convergence of the maximizers follows a variational approach. Given $\rho \in \mathcal{M}$ and $E < 0$ we define $\bar{\mathcal{G}}_E(\rho, \cdot) : \overline{\mathcal{F}}_a \to [-\infty, +\infty)$ by

$$\bar{\mathcal{G}}_E(\rho, \varphi) := \begin{cases} \mathcal{G}_E(\rho, \varphi) & \text{if } \varphi \in \mathcal{F}_E \\ -\infty & \text{otherwise.} \end{cases}$$

By [11, Theorem 1.21], with all inequalities reversed since we focus on maximizers instead of minimizers, the convergence of the sequence $\{\phi_E\}$ to ϕ in $\overline{\mathcal{F}}_a$ follows from the next three conditions. Fix $\rho \in \mathcal{M}$ and $\varphi \in \overline{\mathcal{F}}_a$ then:

(i) for any sequence $\varphi_E \to \varphi$ in $\overline{\mathcal{F}_a}$, $\varlimsup_E \bar{\mathcal{G}}_E(\rho, \varphi_E) \leq \mathcal{G}_a(\rho, \varphi)$;
(ii) there exists a sequence $\varphi_E \to \varphi$ in $\overline{\mathcal{F}}_a$ such that $\varliminf_E \bar{\mathcal{G}}_E(\rho, \varphi_E) \geq \mathcal{G}_a(\rho, \varphi)$;
(iii) ϕ is the unique maximizer for the functional $\mathcal{G}_a(\rho, \cdot)$ on $\overline{\mathcal{F}}_a$.

Proof of (i). We may assume that $\varphi_E \in \mathcal{F}_E$; the proof of (i) is then achieved by noticing that (5.3) holds also if ϕ_E is replaced by φ_E. □

Proof of (ii). Assume firstly that φ belongs to $C^1([-1,1])$ and satisfies $\varphi(\pm 1) = \varphi_\pm$, $\varphi' > 0$. Since (5.2) holds for such φ, it is enough to take the constant sequence $\varphi_E = \varphi$. The proof of (ii) is completed by a density argument, see e.g. [11, Rem. 1.29]. More precisely, it is enough to show that for each $\varphi \in \overline{\mathcal{F}}_a$ there exists a sequence $\varphi^n \in C^1([-1,1])$ satisfying $\varphi^n(\pm 1) = \varphi_\pm$, $(\varphi^n)' > 0$, and such that $\varphi^n \to \varphi$ in $\overline{\mathcal{F}}_a$, $\mathcal{G}_a(\rho, \varphi^n) \to \mathcal{G}_a(\rho, \varphi)$. This is implied by classical results on the approximation of BV functions by smooth ones. □

As we have already shown (iii), the proof is completed. □

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