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Dobrushin States in the ϕ_1^4 Model

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Abstract

We consider the van der Waals free energy functional in a bounded interval with inhomogeneous Dirichlet boundary conditions imposing the two stable phases at the endpoints. We compute the asymptotic free energy cost, as the length of the interval diverges, of shifting the interface from the midpoint. We then discuss the effect of thermal fluctuations by analyzing the ϕ_1^4-measure with Dobrushin boundary conditions. In particular, we obtain a non-trivial limit in a suitable scaling in which the length of the interval diverges and the temperature vanishes. The limiting state is not translation invariant and describes a localized interface. This result can be seen as the probabilistic counterpart of the variational convergence of the associated excess free energy.

1. Introduction

The van der Waals theory of phase transition is based on the functional

$$\mathcal{F}(m) = \int \mathrm{d}x \left[\frac{1}{2} m'(x)^2 + 2V(m(x)) \right], \tag{1.1}$$

where the scalar field $m(x)$ represents the local order parameter and $V(m)$ is a smooth, symmetric, double well potential whose minimum value, chosen to be zero, is attained at $m_\pm$; we also assume $V''(m_\pm) > 0$. We restrict the discussion to the one dimensional case $x \in \mathbb{R}$. If (1.1) is considered in the whole line $\mathbb{R}$, there are infinitely many critical points. The most relevant ones are the constant profiles $m_\pm$, where $\mathcal{F}$ attains its minimum, and $\pm\bar{m}(x)$, where $\bar{m}(x)$ is the solution to

$$\bar{m}''(x) - 2V'(\bar{m}(x)) = 0, \quad \lim_{x\to\pm\infty} \bar{m}(x) = m_\pm, \quad \bar{m}(0) = 0, \tag{1.2}$$

together with its translates $\pm\bar{m}_z(x) = \pm\bar{m}(x - z)$, $z \in \mathbb{R}$. Note that $\bar{m}_z$ minimizes $\mathcal{F}$ under the constraint that $\lim_{x\to\pm\infty} m(x) = m_\pm$. Therefore $\bar{m}_z$ is the stationary profile with the two pure phases $m_\pm$ coexisting to the right and to the left of z. Accordingly, the van der Waals surface tension is $\sigma_w = \mathcal{F}(\bar{m})$. We set $\mathcal{M} = \{\bar{m}_z : z \in \mathbb{R}\}$. We emphasize that we do not consider the sharp interface limit which is obtained by introducing a scaling parameter in (1.1). In particular, even if the convergence of $\bar{m}_z$ to its asymptotic values is exponentially fast, the profile $\bar{m}_z$ describing the interface is not sharp but diffuse, we refer to it as a *mesoscopic* interface.

Our first purpose is to analyze the finite size effects in the free energy $\mathcal{F}$. More precisely, we consider (1.1) in the bounded interval $[-\ell, \ell]$ with the inhomogeneous Dirichlet boundary conditions $m(\pm\ell) = m_\pm$. If we think of m as the local magnetization, this condition models the effect of opposite magnetic fields applied at the endpoints. We denote by $\mathcal{F}_\ell$ the functional (1.1) with these stipulated boundary conditions.

It is not difficult to show that the functional $\mathcal{F}_\ell$ has a unique minimizer m^*_ℓ, which by symmetry converges to $\bar{m}_0$ as $\ell \to \infty$. On the other hand, the limiting functional $\mathcal{F}$ is minimized, under the constraint $m(\pm\infty) = m_\pm$, by any shifted interface $\bar{m}_z \in \mathcal{M}$. It is therefore natural to introduce the *excess* free energy

$$\mathcal{G}_\ell(m) = \mathrm{e}^{\alpha\ell}\left[\mathcal{F}_\ell(m) - \mathcal{F}_\ell(m^*_\ell)\right], \tag{1.3}$$

in which the exponential rescaling $\mathrm{e}^{\alpha\ell}$ is chosen to get a non-trivial limit as $\ell \to \infty$. Indeed, in this paper we show there exists $\alpha = \alpha(V)$ for which $\mathcal{G}_\ell$ converges to a limiting functional $\mathcal{G}$ which is finite only on the set $\mathcal{M}$, where it is given by

$$\mathcal{G}(\bar{m}_z) = A\,[\mathrm{ch}(\alpha z) - 1]\,,$$

for a suitable constant $A = A(V) > 0$. The quantity $\mathrm{e}^{-\alpha\ell}\mathcal{G}(\bar{m}_z)$ gives therefore the asymptotic free energy cost needed to shift the interface by z and encodes the leading finite size correction to the free energy $\mathcal{F}$.

Actually, the above variational formulation of phase transitions neglects completely the microscopic fluctuations, which play an important role in various phenomena. At the mesoscopic level, the effect of fluctuations can be modeled by considering the probability measure, on the space of order parameter profiles, informally given by

$$\mathrm{d}\mu_\varepsilon(m) = Z^{-1}\,\exp\left\{-\varepsilon^{-1}\mathcal{F}(m)\right\}\prod_x \mathrm{d}m(x). \tag{1.4}$$

In the case $V(m) = \frac{1}{4}(m^2 - 1)^2$, the above measure corresponds to the Euclidean version of the quantum anharmonic oscillator and it is usually referred to as the ϕ^4_1-measure, here the subscript one stands for one dimension. This model has been extensively analyzed because exhibits an interesting behavior in a simple setting, see [17] and references therein.

In the van der Waals theory, the local order parameter $m(x)$ represents the empirical average, on a mesoscopic scale, of the microscopic observable. Accordingly, the parameter ε is to be interpreted as the ratio between the microscopic scale (say of the order of Angstroms) to the mesoscopic one (say of the order of tens of microns). In this Gibbsian setting, the chosen inhomogeneous Dirichlet boundary conditions are usually referred to as Dobrushin boundary conditions, their effect is to force an interface in the system. We denote by $\mu_{\varepsilon,\ell}$ the probability measure defined as in (1.4) with $\mathcal{F}$ replaced by $\mathcal{F}_\ell$. For ℓ fixed and ε small, since the measure $\mu_{\varepsilon,\ell}$ concentrates on the minimizer of $\mathcal{F}_\ell$, a typical configuration is close to m_ℓ^*. On the other hand, since the model is one dimensional, for ε fixed and $\ell \to \infty$ the measure $\mu_{\varepsilon,\ell}$ forgets the prescribed boundary conditions and converges to the unique infinite volume Gibbs state. The precise statement would be that the measure $\mu_{\varepsilon,\ell}$, considered on $C(\mathbb{R})$ with the topology of uniform convergence in compacts, converges weakly as $\ell \to \infty$ to an infinite volume Gibbs measure, defined as a solution to the DLR equations. In the context of the ϕ_1^4 model, uniqueness of solution to the DLR equations follows from the analysis in [17, II.6], but we did not find in the literature a detailed proof (see however the discussion in [10, Section II.5, VII.2]) of the weak convergence of $\mu_{\varepsilon,\ell}$ to the unique infinite volume state. However this is not really relevant in the present paper, in which we investigate a diagonal limit $\varepsilon \to 0$ and $\ell \to \infty$. In particular, the aforementioned convergence of the excess free energy $\mathcal{G}_\ell$ suggests that a non-trivial limiting behavior could be obtained by choosing $\varepsilon = \mathrm{e}^{-\alpha\ell}$. We show this is indeed the case: with this choice, the measure $\mu_{\varepsilon,\ell}$ weakly converges to a measure μ with support $\mathcal{M}$ and there given by

$$\mathrm{d}\mu(\bar{m}_z) = N^{-1} \exp\{-\mathcal{G}(\bar{m}_z)\}\, \mathrm{d}z. \tag{1.5}$$

We call this limiting measure a Dobrushin state because it is not translation invariant and describes a fluctuating interface. We emphasize, however, that the order parameter profile is fixed and, with probability super-exponentially close to one as $L \to \infty$, the interface is localized in the bounded interval $(-L, L)$.

In the case of short range, ferromagnetic, lattice models of statistical mechanics (Ising models), phase transitions may occur only in dimension $d \geqq 2$. The behavior of interface fluctuations when the system is considered in a box of side ℓ and Dobrushin boundary conditions are imposed has been analyzed in detail. In $d = 2$ the interface behaves as a random walk having fluctuations of the order of $\sqrt{\ell}$; in particular, in the thermodynamic limit $\ell \to \infty$, the corresponding Gibbs measure converges to a translation invariant state which is a mixture of the pure phases, that is there are no Dobrushin states [7]. In $d \geqq 3$, for low temperature, the interface fluctuations remain bounded and a not translation invariant state is obtained in the thermodynamic limit [6]. With respect to the above context, the diagonal limit $\ell \to \infty, \varepsilon \to 0$, corresponds to a joint limit in which the size of the system diverges and the temperature vanishes. This peculiar limiting procedure allows one to get non-trivial Dobrushin states for $d < 3$. We also mention that a localized interface can be obtained for long-range (power law decay) one-dimensional Ising models [3].

2. Notation and results

It will be convenient to denote by t the space variable and by $x = x(t)$ a continuous function of t. Let

$$\mathcal{X} := \left\{ x \in C(\mathbb{R}) \; : \; \lim_{t\to\pm\infty} x(t) = \pm 1 \right\}, \tag{2.1}$$

endowed with the metric $d(x, y) := \|x - y\|_\infty := \sup_{t\in\mathbb{R}} |x(t) - y(t)|$ and the associated Borel σ-algebra. We emphasize that we need to use this topology and not the one of uniform convergence on compacts because we need to distinguish the behavior of x as $|t| \to \infty$. Given $\ell > 0$, we also let

$$\mathcal{X}_\ell := \left\{x \in C(\mathbb{R}) \;:\; x(t) = \mathrm{sgn}(t) \quad \text{for } |t| \geqq \ell\right\},$$

which is a closed subset of $\mathcal{X}$.

For the sake of concreteness, in this paper we restrict the analysis to the paradigmatic case of the symmetric double well potential, that is we choose

$$V(x) = \frac{1}{4}\left(x^2 - 1\right)^2, \tag{2.2}$$

which attains its minimum at $x = \pm 1$. In this case the solution to (1.2) is given by $\bar{m}(t) = \mathrm{th}(t)$; for $z \in \mathbb{R}$ we set $\bar{m}_z(t) := \mathrm{th}(t - z)$, and define

$$\mathcal{M} := \{\bar{m}_z : z \in \mathbb{R}\}, \tag{2.3}$$

which is a closed subset of $\mathcal{X}$. Given $\ell > 0$, we denote by $W_\ell^{1,2}$ the Sobolev space $W^{1,2}([-\ell, \ell])$ and define the finite volume free energy as the functional $\mathcal{F}_\ell : \mathcal{X} \to [0, +\infty]$ given by

$$\mathcal{F}_\ell(x) := \int_{-\ell}^{+\ell} \mathrm{d}t \left[\frac{1}{2} x'(t)^2 + 2V(x(t))\right] \tag{2.4}$$

if $x \in \mathcal{X}_\ell$ and $x\restriction_\ell \in W_\ell^{1,2}$, while $\mathcal{F}_\ell(x) := +\infty$ otherwise. Here $x\restriction_\ell$ denotes the restriction of x to $(-\ell, \ell)$.

Our first statement concerns the limiting behavior of the sequence $\mathcal{F}_\ell$. This result can be seen as a *diffuse* version of the classical Modica–Mortola result, see, for example [2, Theorem 6.4]. More precisely, the latter result deals with the sharp interface limit, and states that the limiting free energy is concentrated on profiles taking values in $\{-1; 1\}$ and counts the number of jumps. Here we instead show that any minimizer of the limiting functional $\mathcal{F}$ is a profile in $\mathcal{M}$.

Referring, for example to [2, Chapter 1] for more details, we next outline the basic definitions and results of the Γ-convergence theory. Let X be a metric space. A sequence of functionals $F_n : X \to [0, +\infty]$ is *equi-coercive* iff from any sequence x_n such that $\overline{\lim}_n F_n(x_n) < +\infty$ it is possible to extract a converging subsequence. The sequence F_n is *equi-mildly-coercive* iff there exists an non-empty compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any $n \in \mathbb{N}$. The sequence F_n Γ-converges to a functional $F : X \to [0, +\infty]$ iff the following conditions hold for each $x \in X$. There exists a sequence $x_n \to x$ such that $\overline{\lim}_n F_n(x_n) \leqq F(x)$ (*Γ-limsup inequality*) and for any sequence $x_n \to x$ we have $\underline{\lim}_n F_n(x_n) \geqq F(x)$ (*Γ-liminf inequality*). If the sequence F_n is equi-mildly-coercive and Γ-converges to F then $\inf_X F = \min_X F = \lim_n \inf_X F_n$. Moreover, if x_n is a precompact sequence such that $\lim_n F_n(x_n) = \lim_n \inf_X F_n$ then every converging subsequence of x_n converges to a minimizer of F. Finally, if the sequence F_n is equi-coercive and Γ-converges to F then, for each open set A and each closed set C, we have

$$\overline{\lim_n} \inf_A F_n \leqq \inf_A F, \quad \underline{\lim_n} \inf_C F_n \geqq \inf_C F,$$

which are the relevant estimates in the asymptotic analysis of the free energy.

Theorem 2.1. *The sequence $\mathcal{F}_\ell : \mathcal{X} \to [0, +\infty]$ is equi-mildly-coercive and as $\ell \to \infty$ Γ-converges to*

$$\mathcal{F}(x) := \begin{cases} \int_{-\infty}^{+\infty} \mathrm{d}t \left[\frac{1}{2} x'(t)^2 + 2V(x(t))\right] & \text{if } x', 1 - x^2 \in L_2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, the set of minimizers of $\mathcal{F}$ is $\mathcal{M}$, as defined in (2.3). *In particular, the* (*van der Waals*) *surface tension is*

$$\sigma_w = \inf_{\mathcal{X}} \mathcal{F} = \mathcal{F}(\bar{m}) = \frac{4}{3}. \tag{2.5}$$

We remark that $\mathcal{F}_\ell$ is not equi-coercive. Indeed, we can construct a real sequence $z_\ell \to +\infty$ and a sequence x_ℓ such that $\|x_\ell - \bar{m}_{z_\ell}\|_\infty \to 0$ and $\mathcal{F}_\ell(x_\ell) \to \frac{4}{3}$. As stated in the previous theorem, the limiting free energy $\mathcal{F}$ does not remember that m_ℓ^*, the unique minimizer of $\mathcal{F}_\ell$, converges to $\bar{m}_0$. The underlying reason is that the finite volume free energy cost of profiles close to $\bar{m}_z$, $z \in \mathbb{R}$, is infinitesimal as $\ell \to \infty$. We then introduce the *excess free energy* $\mathcal{G}_\ell : \mathcal{X} \to [0, +\infty]$ as

$$\mathcal{G}_\ell(x) := \mathrm{e}^{4\ell} \left[\mathcal{F}_\ell(x) - \mathcal{F}_\ell(m_\ell^*)\right], \tag{2.6}$$

in which the rescaling $\mathrm{e}^{4\ell}$ has been chosen to get a non-trivial limit as $\ell \to \infty$. In fact, as shown in Proposition 3.1 below, the finite volume corrections to the surface tension are $O(\mathrm{e}^{-4\ell})$, in particular $\lim_\ell \mathrm{e}^{4\ell} \left[\mathcal{F}_\ell(m_\ell^*) - \sigma_w\right] = 16$. In this setting the limiting functional $\mathcal{G}$ will be finite only on $\mathcal{M}$ and describes the asymptotic cost of shifting an interface from the origin. Indeed, in the next theorem we identify the Γ-limit of $\mathcal{G}_\ell$. This is usually referred to as *development by Γ-convergence*.

Theorem 2.2. *The sequence $\mathcal{G}_\ell : \mathcal{X} \to [0, +\infty]$ is equi-coercive and as $\ell \to \infty$ Γ-converges to*

$$\mathcal{G}(x) := \begin{cases} 16\,[\mathrm{ch}(4z) - 1] & \text{if } x = \bar{m}_z \text{ for some } z \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

We now discuss the asymptotic behavior of the ϕ_1^4-measure with Dobrushin boundary conditions. We first recall the precise definition of the measure informally introduced in (1.4). Given $\varepsilon > 0$ we denote by $\varrho_{\varepsilon,\ell}$ the probability measure on $\mathcal{X}$, whose support is $\mathcal{X}_\ell$ and having there the law of the Brownian bridge with diffusion coefficient ε, starting at time $-\ell$ from -1 and arriving at time ℓ to $+1$. In other words, $\varrho_{\varepsilon,\ell}$ is the Gaussian measure on $\mathcal{X}$ with mean

$$\bar{x}_\ell(t) := \varrho_{\varepsilon,\ell}\,(x(t)) = \begin{cases} \frac{t}{\ell} & \text{if } |t| \leqq \ell, \\ \mathrm{sgn}(t) & \text{if } |t| > \ell, \end{cases}$$

and covariance

$$\varrho_{\varepsilon,\ell}\left([x(t)-\bar{x}_\ell(t)]\,[x(s)-\bar{x}_\ell(s)]\right) = \begin{cases} \frac{\varepsilon}{2\ell}\,(\ell+s\wedge t)(\ell-s\vee t) & \text{if } s,t\in[-\ell,\ell], \\ 0 & \text{otherwise,} \end{cases}$$

where hereafter $\mu(f)$ denotes the expectation of the measurable function f with respect to the measure μ and, for $a,b\in\mathbb{R}$, $a\wedge b$ (resp. $a\vee b$) denotes the minimum (resp. maximum) between a and b.

The ϕ_1^4 model at temperature ε with Dobrushin type boundary condition is the probability measure $\mu_{\varepsilon,\ell}$ on $\mathcal{X}$ with support $\mathcal{X}_\ell$, whose density with respect to $\varrho_{\varepsilon,\ell}$ is given by

$$\frac{d\mu_{\varepsilon,\ell}}{d\varrho_{\varepsilon,\ell}}(x) = \frac{1}{Z_{\varepsilon,\ell}}\exp\left\{-\varepsilon^{-1}\int_{-\ell}^{\ell} dt\, 2V(x(t))\right\}, \tag{2.8}$$

where

$$Z_{\varepsilon,\ell} := \varrho_{\varepsilon,\ell}\left(\exp\left\{-\varepsilon^{-1}\int_{-\ell}^{\ell} dt\, 2V(x(t))\right\}\right). \tag{2.9}$$

From the Laplace–Varadhan theorem, it follows, see, for example [5, Example 4.3.11], that for ℓ fixed the probability $\mu_{\varepsilon,\ell}$ satisfies a large deviation principle with speed ε^{-1} and rate function $\mathcal{F}_\ell(x)-\mathcal{F}_\ell(m_\ell^*)$. On the other hand, by Theorem 2.2, the functional $\mathcal{F}_\ell(x)-\mathcal{F}_\ell(m_\ell^*)$ behaves like $e^{-4\ell}\mathcal{G}(x)$. Therefore we expect that, in the diagonal limit $\ell=\frac{1}{4}\log\varepsilon^{-1}$ and $\varepsilon\to 0$, the measure $\mu_{\varepsilon,\ell}$ converges to a non-degenerate limit μ, which should look like $d\mu(x)\approx e^{-\mathcal{G}(x)}\,dx$. Our main result shows that this is indeed the case.

Theorem 2.3. *Let $\ell=\frac{1}{4}\log\varepsilon^{-1}$, then the measure $\mu_{\varepsilon,\ell}$ converges weakly in $\mathcal{X}$ as $\varepsilon\to 0$ to the measure μ defined on the Borel sets $A\subset\mathcal{X}$ by*

$$\mu(A)=\widehat{\mu}\left(\{z\in\mathbb{R}\,:\,\bar{m}_z\in A\}\right), \tag{2.10}$$

where $\widehat{\mu}$ is the probability measure on $\mathbb{R}$ given by

$$\widehat{\mu}(dz)=\frac{e^{-16[\mathrm{ch}(4z)-1]}}{\int dz'\, e^{-16[\mathrm{ch}(4z')-1]}}\,dz. \tag{2.11}$$

We emphasize that this non-trivial limiting behavior is due to the particular choice $\ell=\frac{1}{4}\log\varepsilon^{-1}$, the coefficient $\frac{1}{4}$ coming from the specific form (2.2) of the double well potential V. From the analysis carried out in this paper it follows that if we had considered $\ell=\left(\frac{1}{4}-\delta\right)\log\varepsilon^{-1}$ for some $\delta>0$, the measure $\mu_{\varepsilon,\ell}$ would have converged weakly to the probability concentrated on the single configuration $\bar{m}_0$. Moreover, it should be also possible to show that if we had considered $\ell=\left(\frac{1}{4}+\delta\right)\log\varepsilon^{-1}$ for some $\delta>0$, then the family $\mu_{\varepsilon,\ell}$ would not have been tight on $\mathcal{X}$. On the other hand, the family $\mu_{\varepsilon,\ell}$, still for $\ell=\left(\frac{1}{4}+\delta\right)\log\varepsilon^{-1}$ and considered in $C(\mathbb{R})$ endowed with the topology of uniform convergence on compacts, would converge weakly to $\frac{1}{2}[\delta_{-1} + \delta_1]$, where $\delta_{\pm 1}$ denotes the Dirac measure concentrated on the configuration identically equal to ± 1: on this scale the interface "went to infinity". As it appears clear from the above discussion, the compactness property of the family $\mu_{\varepsilon,\ell}$ is a key point; in particular tightness of $\mu_{\varepsilon,\ell}$ implies that the interface remains localized in a compact set of $\mathbb{R}$ with probability close to one.

Strategy of the proof. As it is well known, see, for example [17], the measure describing the ϕ_1^4 model in the whole line can be realized as the law of the stationary process associated to the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = a_\varepsilon(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}w_t, \tag{2.12}$$

where w_t is a standard Brownian motion and the drift a_ε is the logarithmic derivative of the ground state of the (quantum) anharmonic oscillator. More precisely, let us denote by λ_ε the smallest eigenvalue of the Schroedinger operator $H_\varepsilon := -\frac{1}{2}\varepsilon^2\Delta + 2V$ on $L_2(\mathbb{R}, \mathrm{d}x)$, the corresponding eigenfunction, chosen strictly positive, is denoted by ϕ_ε. Then $a_\varepsilon = \varepsilon\nabla\log\phi_\varepsilon$; in particular $\phi_\varepsilon(x)^2\,\mathrm{d}x$ is the invariant measure of the process X_t. We mention that this representation of the infinite volume ϕ_1^4-measure allows, by means of Friedlin–Wentzell large deviations estimates [8], a detailed study of the typical configurations as $\varepsilon \to 0$. From the analysis in [12], whose main motivation lies on semiclassical limits, the following picture emerges. With probability exponentially close to one as $\varepsilon \to 0$, we see $x(t) \approx \pm 1$ for t in intervals of the order $\mathrm{e}^{\varepsilon^{-1}\sigma_w}$; the transition (tunneling) between the pure phases taking place in a small neighborhood of $\bar{m}_z$ for suitable z's. Moreover, if the lengths of the above intervals are properly normalized, they converge weakly to an independent jump process with exponential distribution, as in the case of Ising spin systems, either nearest-neighbors [16] or with long range interaction of Kac type [4].

A representation in terms of a diffusion process can be obtained also in the present setting of the ϕ_1^4 model with Dobrushin boundary condition. From a statistical mechanics viewpoint, this representation corresponds to transfer matrix arguments. The probability $\mu_{\varepsilon,\ell}$ can be realized as the law of the diffusion process (2.12) with initial condition $X_{-\ell} = -1$ *conditioned* to reach 1 at the time $t = \ell$. According to the results in [11], this conditioned process can be also realized as the solution to a stochastic differential equation with a time dependent drift. Let us denote by X_t^x the solution to (2.12) with initial condition $X_0 = x$ and introduce the transition probability density $p_t^\varepsilon(x, y)$ by requiring that for each $t > 0$ and each Borel set $B \subset \mathbb{R}$,

$$P\left(X_t^x \in B\right) = \int_B \mathrm{d}y\; p_t^\varepsilon(x, y). \tag{2.13}$$

For $(t, x) \in (-\ell, \ell) \times \mathbb{R}$, we define

$$\widehat{a}_{\varepsilon,\ell}(t, x) := -a_\varepsilon(x) + \varepsilon\partial_x \log p_{\ell-t}^\varepsilon(1, x). \tag{2.14}$$

Then, as follows from [11], the measure $\mu_{\varepsilon,\ell}$ is the law of the process Y defined as follows. For $|t| \geqq \ell$ we set $Y_t = \mathrm{sgn}(t)$ while for $|t| < \ell$ we define Y_t as the solution to the stochastic differential equation

$$\begin{cases} \mathrm{d}Y_t = \widehat{a}_{\varepsilon,\ell}(t, Y_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}w_t, \\ Y_{-\ell} = -1, \end{cases} \tag{2.15}$$

here w_t, $t \in [-\ell, \ell]$, is a standard Brownian with $w_{-\ell} = 0$.

Theorem 2.3 can, therefore, equivalently, be rephrased in terms of the limiting behavior of the solution to (2.15). We emphasize that we obtain a non-degenerate limiting behavior as $\varepsilon \to 0$ even if the noise term vanishes. This is due both to the simultaneous divergence of the time interval and to the peculiar behavior of the drift $\widehat{a}_{\varepsilon,\ell}$. We discuss the latter issue in some more detail. By the well-known ground state transformation, see, for example [17], we can rewrite the transition probability density in (2.13) in terms of the kernel of the semigroup generated by H_ε,

$$p_t^\varepsilon(x, y) = \frac{1}{\phi_\varepsilon(x)}\,\mathrm{e}^{-\varepsilon^{-1}(H_\varepsilon - \lambda_\varepsilon)t}(x, y)\,\phi_\varepsilon(y),$$

so that, recalling (2.14),

$$\widehat{a}_{\varepsilon,\ell}(t, x) = \varepsilon \partial_x \log \mathrm{e}^{-\varepsilon^{-1}(H_\varepsilon - \lambda_\varepsilon)(\ell - t)}(1, x). \tag{2.16}$$

It is also not difficult to check that, by writing $\widehat{a}_{\varepsilon,\ell}(t, x) = -\partial_x S_\varepsilon(\ell - t, x)$, the function S_ε solves the viscous Hamilton–Jacobi equation

$$\partial_t S_\varepsilon + \frac{1}{2}(\partial_x S_\varepsilon)^2 - 2V = \frac{\varepsilon}{2}\left[\partial_{xx} S_\varepsilon - \frac{1}{t}\right], \tag{2.17}$$

of course S_ε is singular as $t \downarrow 0$. To analyze the solution to (2.15), as $\varepsilon \to 0$, we therefore, need sharp estimates on the semiclassical limit of the Schrodinger operator H_ε. More precisely, we need good control on the kernel of the corresponding semigroup up to times of order $\ell = O(\log \varepsilon^{-1})$. In the context of semiclassical limits, see, for example [14], this scale of time is known as *Erhenfest time* and it is the one in which the semiclassical approximation is not—in general—anymore valid.

As it appears quite intricate to get good control on $\widehat{a}_{\varepsilon,\ell}$ by direct semiclassical methods or perturbation theory in Hamilton–Jacobi, we follow a different approach, which we might call Euclidean semiclassical approximation. If ℓ were fixed, by the Feynmann–Kac formula and Laplace–Varadhan asymptotic in (2.16), we would get, as $\varepsilon \to 0$,

$$\widehat{a}_{\varepsilon,\ell}(t, x) \approx -\partial_x S(\ell - t, x), \tag{2.18}$$

where

$$S(t, x) := \inf\left\{\int_0^t \mathrm{d}s\,\left[\frac{1}{2}\dot{\psi}(s)^2 + 2V(\psi(s))\right] \,:\, \psi(0) = 1,\ \psi(t) = x\right\} \tag{2.19}$$

is the action for a Newtonian particle of mass one in the potential $-2V$ starting at time zero from 1 and arriving at time t to x; the change of sign in the potential is due to the fact that we are looking at the Schroedinger semigroup. Note that S solves (2.17) with $\varepsilon = 0$. As the right-hand side of (2.18) makes sense, we use it as the drift term of an auxiliary diffusion process. Namely, we introduce the process ξ as the solution to

$$\mathrm{d}\xi_t = -\partial_x S(\ell - t, \xi_t)\, \mathrm{d}t + \sqrt{\varepsilon}\, \mathrm{d}w_t,$$

where w is a standard Brownian motion. Since (2.18) is not an identity, the law of ξ is not $\mu_{\varepsilon,\ell}$. On the other hand it is a good approximation of it in the sense that, as shown in Proposition 4.1 below, their Radon–Nykodim derivative is "only" of the order $\mathrm{e}^{O(\ell)}$. Moreover, even if the drift term above is not really given explicitly, standard methods for one-dimensional mechanical systems allow one to get sharp estimates on it.

By exploiting the above strategy, we get enough control on the measure $\mu_{\varepsilon,\ell}$ to show that it concentrates in a small neighborhood of $\mathcal{M}$ and that it is tight in $\mathcal{X}$. The identification of its limit points with the measure μ defined in Theorem 2.3 will be accomplished by a dynamical argument. We refer to [9] for a recent review on the dynamics of stochastic interfaces. The probability $\mu_{\varepsilon,\ell}$ can be in fact characterized, see [8, Theorem 5.1], as the unique invariant measure of the Markov process $X \equiv X_\sigma(t)$, $(\sigma, t) \in \mathbb{R}_+ \times \mathbb{R}$, in $C(\mathbb{R}_+; \mathcal{X}_\ell)$ which solves the stochastic partial differential equation

$$\begin{cases} \mathrm{d}X_\sigma = \left[\frac{1}{2}\partial_{tt} X_\sigma - V'(X_\sigma)\right] \mathrm{d}\sigma + \sqrt{\varepsilon}\, \mathrm{d}W_\sigma & \sigma > 0,\ |t| < \ell, \\ X_\sigma(t) = \mathrm{sgn}(t) & \sigma \geqq 0,\ |t| \geqq \ell, \end{cases} \tag{2.20}$$

where W is the cylindrical Wiener process on $L_2([-\ell, \ell],\ \mathrm{d}t)$. As shown in [1], in the scaling limit $\ell = \frac{1}{4}\log\varepsilon^{-1}$ and $\varepsilon \to 0$, $X_{\varepsilon^{-1}\sigma}$ converges in law to $\bar{m}_{\zeta_\sigma}$ where ζ solves

$$\mathrm{d}\zeta_\sigma = -\,24\ \mathrm{sh}\,(4\zeta_\sigma)\, \mathrm{d}\sigma + \sqrt{\frac{3}{4}}\, \mathrm{d}B_\sigma, \tag{2.21}$$

with B a standard Brownian motion. As the unique invariant measure of this one-dimensional diffusion process is $\widehat{\mu}$, see (2.11), we conclude the identification.

A final remark on the relationship between the equilibrium asymptotic stated in Theorem 2.3 and the above dynamical result is due. A basic paradigm in non-equilibrium statistical mechanics is the Einstein relation which connects dynamical transport coefficients and thermodynamic potentials, see, for example [15, I.8.8]. The general structure of this relation is [drift] $= \frac{1}{2}\cdot$ [diffusion] $\cdot$ [thermodynamic force], where the thermodynamic force is minus the derivative of the free energy. It is worth noticing that such a relationship is verified also in the present setting of a drift induced by the boundary conditions, namely

$$-\,24\ \mathrm{sh}(4z) = -\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\mathrm{d}}{\mathrm{d}z}16\,[\mathrm{ch}(4z) - 1]\,.$$

3. Asymptotic analysis of the free energy

In this section we analyze the asymptotic behavior of the free energy $\mathcal{F}_\ell$ and prove Theorems 2.1 and 2.2. We start by showing that for each $\ell > 0$ there exists a unique minimizer m^*_ℓ of $\mathcal{F}_\ell$ and discuss its behavior as $\ell \to \infty$. Recall that for the choice (2.2) of V we have $\mathcal{F}(\bar{m}) = \frac{4}{3}$.

Proposition 3.1. *The functional $\mathcal{F}_\ell$ has a unique minimizer m^*_ℓ in $\mathcal{X}_\ell$. Moreover $m^*_\ell \restriction_\ell \in C^2((-\ell, \ell))$ and for $|t| \leqq \ell$ it is the unique solution to the boundary value problem*

$$\begin{cases} \frac{1}{2}x''(t) - V'(x(t)) = 0, & t \in (-\ell, \ell), \\ x(\pm\ell) = \pm 1. \end{cases} \tag{3.1}$$

Finally,

$$\overline{\lim_{\ell\to\infty}}\ \mathrm{e}^{2\ell}\ \|m^*_\ell - \bar{m}_0\|_\infty < \infty, \tag{3.2}$$

$$\lim_{\ell\to\infty} \mathrm{e}^{4\ell}\left[\mathcal{F}_\ell(m^*_\ell) - \frac{4}{3}\right] = 16. \tag{3.3}$$

Proof. The boundary value problem (3.1) is the Euler–Lagrange equation for the stated variational problem. Equation (3.1) can be regarded as that of the motion of a Newtonian particle of mass one in the potential $-2V$. By standard Weierstrass analysis of one-dimensional mechanical systems, it is then straightforward to prove that there exists a unique twice differentiable solution m^*_ℓ to (3.1). Explicit estimates yield the bounds (3.2) and (3.3), see Lemma A.1. We here prove uniqueness of the minimizer with the given boundary conditions. The argument is rather standard, and it is reported for completeness. Let us denote by $S_\sigma(x)$, $\sigma \geqq 0$, the gradient flow associated to $\mathcal{F}_\ell$, that is $u(\sigma, t) = S_\sigma(x)(t)$ solves

$$\begin{cases} \partial_\sigma u(\sigma, t) = \dfrac{1}{2}\partial_{tt}u(\sigma, t) - V'(u(\sigma, t)), \\ u(\sigma, t) = \pm 1, \quad |t| \geqq \ell, \\ u(0, t) = x(t). \end{cases}$$

By standard theory, for each $x \in \mathcal{X}_\ell$, we have that $S_\sigma(x) \restriction_\ell \in C^2((-\ell, \ell))$ for $\sigma > 0$ and $\|S_\sigma(x) - x\|_\infty \to 0$ as $\sigma \to 0$. This implies in particular that $\mathcal{F}_\ell(S_\sigma(x)) \leqq \mathcal{F}_\ell(x)$ for any $\sigma \geqq 0$. By the compactness of the level sets of $\mathcal{F}_\ell(x)$ there exists at least one minimizer, say $\tilde{x}$. For what stated above we then have $\mathcal{F}_\ell(S_\sigma(\tilde{x})) = \mathcal{F}_\ell(\tilde{x})$ for any $\sigma \geqq 0$. By taking the derivative we conclude that, for $\sigma > 0$, $S_\sigma(\tilde{x})$ is a twice differentiable solution of (3.1); hence $S_\sigma(\tilde{x}) = m^*_\ell$. By continuity $\tilde{x} = m^*_\ell$. □

Proof of Theorem 2.1. The argument is rather standard, and it is detailed below for completeness. The equi-mildly-coerciveness of $\mathcal{F}_\ell$ follows immediately from

Proposition 3.1. We next prove the Γ-limsup inequality. Given $x \in \mathcal{X}$ we define

$$x_\ell(t) := \begin{cases} -1 & \text{if } t \in (-\infty, -\ell), \\ t + \ell - 1 + x(-\ell + 1)(t + \ell) & \text{if } t \in [-\ell, -\ell + 1], \\ x(t) & \text{if } t \in (-\ell + 1, \ell - 1), \\ t - \ell + 1 + x(\ell - 1)(\ell - t) & \text{if } t \in [\ell - 1, \ell], \\ 1 & \text{if } t \in (\ell, +\infty). \end{cases}$$

Clearly, $x_\ell \to x$ in $\mathcal{X}$. Moreover it is straightforward to check that $\mathcal{F}_\ell(x_\ell) \to \mathcal{F}(x)$ since the contribution in the set $[-\ell, -\ell+1] \cup [\ell-1, \ell]$ vanishes as $\ell \to \infty$ because $x(\pm(\ell-1)) \to \pm 1$ as $\ell \to \infty$. We finally prove the Γ-liminf inequality. Pick $x \in \mathcal{X}$ and a sequence $x_\ell \to x$; if $\mathcal{F}_\ell(x_\ell) < +\infty$ we have $x_\ell(t) = \mathrm{sgn}(t)$ for $|t| \geqq \ell$. Hence $\mathcal{F}_\ell(x_\ell) = \mathcal{F}(x_\ell)$, and we conclude by the lower semicontinuity of $\mathcal{F}$, which is established by noticing that $\mathcal{F}(x) = \sup_\ell \int_{-\ell}^{\ell} \mathrm{d}t \, \left[\frac{1}{2} x'(t)^2 + 2V(x(t))\right]$.

To prove the last statement, we first show that $\mathcal{F}(x) \geqq \mathcal{F}(\bar{m})$ for any $x \in \mathcal{X}$. Indeed, using the inequality $a^2 + b^2 \geqq 2|ab|$ we have

$$\begin{aligned} \mathcal{F}(x) &\geqq 2 \int_{-\infty}^{+\infty} \mathrm{d}t \, |x'(t)| \sqrt{V(x(t))} \geqq 2 \left| \int_{-\infty}^{+\infty} \mathrm{d}t \, x'(t) \sqrt{V(x(t))} \right| \\ &= 2 \int_{-1}^{+1} \mathrm{d}y \, \sqrt{V(y)} = \int_{-1}^{+1} \mathrm{d}y \, (1 - y^2) = \frac{4}{3} = \mathcal{F}(\bar{m}). \end{aligned}$$

On the other hand, in the above computation, we get an equality if and only if $|x'(t)| = 2\sqrt{V(x(t))} = |1 - x^2(t)|$. Since $x(\pm\infty) = \pm 1$, this implies $x = \bar{m}_z$ for some $z \in \mathbb{R}$. $\square$

Proof of Theorem 2.2. It is convenient to introduce the notation

$$\mathcal{F}_{[a,b]}(x) := \int_a^b \mathrm{d}t \, \left[\frac{1}{2} x'(t)^2 + 2V(x(t))\right]. \tag{3.4}$$

The equi-coercivity of $\mathcal{G}_\ell$ is proven in Lemma A.2. We next prove the Γ-limsup inequality. By (2.7), it is enough to consider $x \in \mathcal{M}$. Recall that m_ℓ^* is the minimizer of $\mathcal{F}_\ell$ and note that, by the symmetry of V, $m_\ell^*(0) = 0$. Given $z \in \mathbb{R}$, for $\ell \geqq |z|$, we define

$$m_z^{(\ell)}(t) := \begin{cases} -1 & \text{if } t \in (-\infty, -\ell], \\ m_{\ell+z}^*(t - z) & \text{if } t \in (-\ell, z], \\ m_{\ell-z}^*(t - z) & \text{if } t \in (z, \ell], \\ 1 & \text{if } t \in (\ell + \infty). \end{cases} \tag{3.5}$$

From (3.2) we get $m_z^{(\ell)} \to \bar{m}_z$ in $\mathcal{X}$. We claim that $m_z^{(\ell)}$ is a recovery sequence, that is

$$\lim_{\ell \to \infty} \mathcal{G}_\ell \left(m_z^{(\ell)}\right) = 16\,[\mathrm{ch}(4z) - 1]\,. \tag{3.6}$$

Indeed we have

$$\begin{aligned}\mathcal{F}_\ell\left(m_z^{(\ell)}\right) &= \mathcal{F}_{[-\ell,z]}\left(m_z^{(\ell)}\right) + \mathcal{F}_{[z,\ell]}\left(m_z^{(\ell)}\right) \\ &= \mathcal{F}_{[-\ell-z,0]}\left(m^*_{\ell+z}\right) + \mathcal{F}_{[0,\ell-z]}\left(m^*_{\ell-z}\right) \\ &= \frac{1}{2}\left[\mathcal{F}_{\ell+z}\left(m^*_{\ell+z}\right) + \mathcal{F}_{\ell-z}\left(m^*_{\ell-z}\right)\right],\end{aligned}$$

where in the second step we used the translation covariance of $\mathcal{F}_{[a,b]}$ while, in the third one, that $t \mapsto m^*_\ell(t)$ is an odd function and $x \mapsto V(x)$ is an even function. Therefore

$$\begin{aligned}\mathcal{G}_\ell\left(m_z^{(\ell)}\right) = \frac{e^{4\ell}}{2}\left[\mathcal{F}_{\ell+z}\left(m^*_{\ell+z}\right) - \frac{4}{3}\right] + \frac{e^{4\ell}}{2}\left[\mathcal{F}_{\ell-z}\left(m^*_{\ell-z}\right) - \frac{4}{3}\right] \\ + e^{4\ell}\left[\frac{4}{3} - \mathcal{F}_\ell\left(m^*_\ell\right)\right],\end{aligned}$$

and (3.6) follows from (3.3).

We finally prove the Γ-liminf inequality. Let $x \in \mathcal{X}\setminus\mathcal{M}$, from Theorem 2.1 and (3.3) it follows that, for any sequence $x_\ell \to x$, we have

$$\varliminf_{\ell\to\infty} \mathcal{F}_\ell(x_\ell) \geqq \mathcal{F}(x) > \frac{4}{3} = \lim_{\ell\to\infty} \mathcal{F}_\ell(m^*_\ell),$$

whence $\mathcal{G}_\ell(x_\ell) \to +\infty$ as $\ell \to \infty$. It remains to show that for any $z \in \mathbb{R}$ and any sequence $x_\ell \to \bar{m}_z$ we have

$$\varliminf_{\ell\to\infty} \mathcal{G}_\ell(x_\ell) \geqq 16\left[\mathrm{ch}(4z) - 1\right]. \tag{3.7}$$

It suffices to consider sequences $x_\ell \to \bar{m}_z$ such that $x_\ell(t) = \mathrm{sgn}(t)$ for $|t| \geqq \ell$. We next remark that, by symmetry, the function $m^*_\ell(t)$, $t \in [0,\ell]$, is the unique minimizer of $\mathcal{F}_{[0,\ell]}(x)$ with the boundary conditions $x(0) = 0$, $x(\ell) = 1$. Analogously, $m^*_\ell(t)$, $t \in [-\ell, 0]$, is the unique minimizer of $\mathcal{F}_{[-\ell,0]}(x)$ with the boundary conditions $x(-\ell) = -1$, $x(0) = 0$. Since $x_\ell \to \bar{m}_z$, we can find a sequence $z_\ell \to z$ such that $x_\ell(z_\ell) = 0$. Recalling (3.5), by the translation covariance of $\mathcal{F}_{[a,b]}$ we get

$$\begin{aligned}\mathcal{F}_\ell(x_\ell) &= \mathcal{F}_{[-\ell,z_\ell]}(x_\ell) + \mathcal{F}_{[z_\ell,\ell]}(x_\ell) \\ &\geqq \mathcal{F}_{[-\ell-z_\ell,0]}\left(m^*_{\ell+z_\ell}\right) + \mathcal{F}_{[0,\ell-z_\ell]}\left(m^*_{\ell-z_\ell}\right) \ = \ \mathcal{F}_\ell(m^{(\ell)}_{z_\ell}).\end{aligned}$$

The proof of (3.7) is completed by observing that (3.6) holds also if the sequence $m_z^{(\ell)}$ is replaced by $m_{z_\ell}^{(\ell)}$ with $z_\ell \to z$. □

4. Euclidean semiclassical approximation

From now on we set $\ell = \frac{1}{4}\log\varepsilon^{-1}$ and drop the subscript ℓ from the notation. We suppose given a filtered probability space $(\Omega, \mathcal{S}, \mathcal{S}_t, P)$ equipped with a standard Brownian motion w_t, $t \in [-\ell, \ell]$, with $w_{-\ell} = 0$. By, for example [13, Section 5.6.B], the Brownian bridge with diffusion coefficient ε, starting at time $-\ell$ from -1 and arriving at time ℓ to $+1$, can be realized as the solution to the stochastic differential equation

$$\begin{cases} \mathrm{d}\eta_t = \dfrac{1-\eta_t}{\ell - t}\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}w_t, \\ \eta_{-\ell} = -1, \end{cases} \tag{4.1}$$

for $t \in [-\ell, \ell)$, and $\eta_t = \mathrm{sgn}(t)$ for $|t| \geqq \ell$. Note in fact that the solution to the above equation satisfies $\lim_{t\uparrow\ell}\eta_t = 1$ almost surely.

Recalling the definition (2.19), given $\ell > 0$ and $(t, x) \in [-\ell, \ell) \times \mathbb{R}$ we set

$$b(t,x) := -\partial_x S(\ell - t, x) = \begin{cases} +\sqrt{4V(x) + E_{t,x}} & \text{if } x < 1, \\ 0 & \text{if } x = 1, \\ -\sqrt{4V(x) + E_{t,x}} & \text{if } x > 1, \end{cases} \tag{4.2}$$

where $E_{t,x}$ is such that

$$\ell - t = \left| \int_x^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + E_{t,x}}} \right|. \tag{4.3}$$

The last equality in (4.2) can be seen to follow for instance from (4.3) and (A.5).

We then define the process ξ as the solution to the one -dimensional stochastic differential equation

$$\begin{cases} \mathrm{d}\xi_t = b(t, \xi_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}w_t, \\ \xi_{-\ell} = -1, \end{cases} \tag{4.4}$$

for $t \in [-\ell, \ell)$ and $\xi_t = \mathrm{sgn}(t)$ for $|t| \geqq \ell$. We shall denote by ν_ε the law of ξ. Note that $b(t, x) > 0$ for $x < 1$ while $b(t, x) < 0$ for $x > 1$; moreover $b(t, x)$ diverges as $t \uparrow \ell$ (unless $x = 1$). Therefore the drift in (4.4) drives the process ξ from -1 at time $-\ell$ to 1 at time ℓ. Finally, for ℓ large and (t, x) in compacts, $E_{t,x} \to 0$, so that we expect the solution to (4.4) to converge, in the diagonal limit $\varepsilon \to 0$ and $\ell \to +\infty$, to some $\bar{m}_z \in \mathcal{M}$ which solves $\dot{x} = \sqrt{4V(x)}$. We emphasize that in this limit some randomness will remain, as small deviation of the random force affects the choice of z. Note indeed that $b(-\ell, -1) = \sqrt{E_{-\ell,-1}} = O(\mathrm{e}^{-2\ell})$, which is of the same order of the noise. The above picture will be substantiated in the following. The required analysis is not completely standard as it involves the joint limit $\varepsilon \to 0$ and $\ell \to \infty$, and depends crucially on the precise scaling $\ell = \frac{1}{4}\log\varepsilon^{-1}$.

Before analyzing the process ξ in itself, we show how it can be used in the study of the ϕ_1^4-measure with Dobrushin boundary conditions. Recalling (2.19), we let

$$S_0(t,x) := S(t,x) - \frac{(1-x)^2}{2t}. \tag{4.5}$$

Note that $\frac{1}{2t}(1-x)^2$ is the action of a free particle, that is the infimum in (2.19) in the case $V=0$. Since $S(t,x)$ satisfies the Hamilton–Jacobi equation with Hamiltonian $H(x,p)=\frac{1}{2}p^2-2V(x)$, we get that S_0 satisfies the equation

$$\partial_t S_0(t,x) + \frac{1}{2}\left[\partial_x S_0(t,x)\right]^2 - \frac{1-x}{t}\partial_x S_0(t,x) - 2V(x) = 0. \tag{4.6}$$

We also define $b_0(t,x) := -\partial_x S_0(\ell-t,x)$, so that $b(t,x) = b_0(t,x) + \frac{1-x}{\ell-t}$. It is shown in Theorem A.3 that $S_0(t,x)$ is regular as $t \downarrow 0$, therefore the drifts in (4.1) and (4.4) have the same singular part. Indeed, we next show that ξ is absolutely continuous with respect to the Brownian bridge η, and we obtain an explicit expression for the density of μ_ε with respect to ν_ε. Recall the definition (2.4) of $\mathcal{F}_\ell$, and its minimizer m_ℓ^* considered in Proposition 3.1.

Proposition 4.1. *We have*

$$\frac{d\mu_\varepsilon}{d\nu_\varepsilon}(x) = A_\varepsilon^{-1}\,\exp\left\{-\frac{1}{2}\int_{-\ell}^{\ell} dt\,\partial_{xx}S_0(\ell-t,x(t))\right\}, \tag{4.7}$$

where

$$A_\varepsilon := \nu_\varepsilon\left(e^{-\frac{1}{2}\int_{-\ell}^{\ell} dt\,\partial_{xx}S_0(\ell-t,x(t))}\right) = Z_\varepsilon\, e^{\varepsilon^{-1}[\mathcal{F}_\ell(m_\ell^*)-\ell^{-1}]}.$$

In Section 5, we shall analyze the measure μ_ε as a perturbation of ν_ε. Notice indeed that, while $\frac{d\mu_\varepsilon}{d\rho_\varepsilon} = e^{O(\varepsilon^{-1}\ell)}$, we have $\frac{d\mu_\varepsilon}{d\nu_\varepsilon} = e^{O(\ell)}$.

Proof. Let $\psi_\ell : \mathcal{X}_\ell \to \mathcal{X}_\ell$ be the map defined as follows. The function $y = \psi_\ell(x) \restriction_\ell$ is the unique solution to

$$y(t) = x(t) + \int_{-\ell}^{t} ds\,\frac{\ell-t}{\ell-s}\,b_0(s,y(s)), \quad t\in[-\ell,\ell). \tag{4.8}$$

By writing the integral form of (4.4) and using Duhamel formula with respect to (4.1), we find that $\nu_\varepsilon = \varrho_\varepsilon \circ \psi_\ell^{-1}$. In particular the process ξ is well defined and satisfies $\lim_{t\uparrow\ell}\xi_t = 1$ almost surely. This representation of ν_ε, together with the regularity of $b_0(t,x)$ proven in Theorem A.3, allows, by a standard truncation of which we omit the details, to use Girsanov theorem to obtain an explicit expression for the Radon-Nykodim derivative $\frac{d\nu_\varepsilon}{d\varrho_\varepsilon}$. We get

$$\frac{d\nu_\varepsilon}{d\varrho_\varepsilon}(\eta) = \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_{-\ell}^{\ell} b_0(t,\eta_t)\,dw_t - \frac{1}{2\varepsilon}\int_{-\ell}^{\ell} dt\,b_0(t,\eta_t)^2\right\}. \tag{4.9}$$

On the other hand, by Ito's formula,

$$\begin{aligned} &S_0(0,\eta_\ell) - S_0(2\ell,\eta_{-\ell}) \\ &\quad = \int_{-\ell}^{\ell} \mathrm{d}t \left[-\partial_t S_0(\ell-t,\eta_t) + \partial_x S_0(\ell-t,\eta_t)\,\frac{1-\eta_t}{\ell-t} + \frac{\varepsilon}{2}\partial_{xx} S_0(\ell-t,\eta_t) \right] \\ &\qquad + \sqrt{\varepsilon}\int_{-\ell}^{\ell} \partial_x S_0(\ell-t,\eta_t)\,\mathrm{d}w_t. \end{aligned} \tag{4.10}$$

We note that $S(2\ell,-1) = \mathcal{F}_\ell(m_\ell^*)$, whence $S_0(2\ell,-1) = \mathcal{F}_\ell(m_\ell^*) - \frac{1}{\ell}$ and $S_0(0,1) = 0$. Recalling $b_0(t,x) = -\partial_x S_0(\ell-t,x)$, by plugging (4.10) into (4.9) we obtain

$$\begin{aligned} \frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\varrho_\varepsilon}(\eta) &= \exp\left\{ \varepsilon^{-1}\left[\mathcal{F}_\ell(m_\ell^*) - \ell^{-1}\right] + \frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\,\partial_{xx} S_0(\ell-t,\eta_t) \right. \\ &\qquad -\varepsilon^{-1}\int_{-\ell}^{\ell} \mathrm{d}t \left[\partial_t S_0(\ell-t,\eta_t) - \partial_x S_0(\ell-t,\eta_t)\,\frac{1-\eta_t}{\ell-t} \right. \\ &\qquad\quad \left.\left. + \frac{1}{2}\left[\partial_x S_0(\ell-t,\eta_t)\right]^2 \right]\right\} \\ &= \exp\left\{ \varepsilon^{-1}\left[\mathcal{F}_\ell(m_\ell^*) - \ell^{-1}\right] + \frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\,\partial_{xx} S_0(\ell-t,\eta_t) \right. \\ &\qquad \left. -\ \varepsilon^{-1}\int_{-\ell}^{\ell} \mathrm{d}t\, 2V(\eta_t) \right\}, \end{aligned}$$

where in the last equality we used (4.6). Recalling (2.8), the identity (4.7) is thus proven. □

We now turn to the analysis of (4.4). Given $x \in \mathcal{X}_\ell$, we let $\mathcal{Z}(x)$ be the leftmost zero of x, that is

$$\mathcal{Z}(x) := \inf\{t \in [-\ell,\ell] \,:\, x(t) = 0\}. \tag{4.11}$$

In the next theorem, whose proof is the main content of the present section, we estimate the probability that the process ξ lies in a small neighborhood of $\mathcal{M}$ and $\mathcal{Z}(\xi)$ stays in a compact.

Theorem 4.2. *There exists $\eta_0 > 0$ such that, for any $\eta \in (0,\eta_0)$ the following holds. There exist positive reals a_0 and ε_0 such that, for any $\varepsilon \in (0,\varepsilon_0]$ and $a \in [a_0,\ell]$, we have*

$$\nu_\varepsilon\left(\left\{x \,:\, d(x,\mathcal{M}) > \varepsilon^{\frac{1}{2}-\eta}\right\}\right) \leqq \exp\left\{-\varepsilon^{-\frac{1}{2}\eta}\right\}, \tag{4.12}$$

$$\nu_\varepsilon\left(\{x \,:\, |\mathcal{Z}(x)| > a\}\right) \leqq \exp\left\{-\varepsilon^{-\frac{1}{2}\eta}\right\} + 2\exp\left\{-\mathrm{e}^{\frac{a}{4}}\right\}, \tag{4.13}$$

$$\nu_\varepsilon\left(\{x \,:\, \exists\, t \in [-\ell+a,a] \text{ such that } x(t) < \bar{m}_a(t)\}\right) \leqq \exp\left\{-\mathrm{e}^{\frac{a}{4}}\right\}. \tag{4.14}$$

By the aforementioned behavior of $b(t,x)$, since the noise is of order $\sqrt{\varepsilon}$ and the time interval is of order $\log \varepsilon^{-1}$, the process ξ will essentially move inside the interval $[-1,1]$. The precise statement is the following.

Lemma 4.3. *For each* $\delta > 0$ *we have*

$$\nu_\varepsilon\left(\sup_{t\in\mathbb{R}} |x(t)| > 1+\delta\right) \leqq \frac{8}{\sqrt{\pi}} \frac{\sqrt{\varepsilon\ell}}{\delta} \exp\left\{-\frac{\delta^2}{16\varepsilon\ell}\right\}.$$

Proof. Let ξ be the solution of (4.4), and introduce the event

$$\mathcal{B} := \left\{ w \,:\, \sup_{t\in[-\ell,\ell]} |w_t| < \frac{\delta}{2\sqrt{\varepsilon}} \right\}.$$

By the reflection principle,

$$P(\mathcal{B}^\complement) \leqq 4P\left(w_\ell \geqq \frac{\delta}{2\sqrt{\varepsilon}}\right) \leqq \frac{8}{\sqrt{\pi}} \frac{\sqrt{\varepsilon\ell}}{\delta} \exp\left\{-\frac{\delta^2}{16\varepsilon\ell}\right\}. \tag{4.15}$$

We claim that $\sup_{t\in[-\ell,\ell]} |\xi_t| < 1+\delta$ on the event $\mathcal{B}$. We shall only prove that $\inf_{t\in[-\ell,\ell]} \xi_t > -1-\delta$, a symmetric argument shows that we also have $\sup_{t\in[-\ell,\ell]} \xi_t < 1+\delta$. Let τ be the first time ξ_t hits $-1-\delta$. If there is no such τ in the interval $(-\ell,\ell)$ we are done. Otherwise let $\sigma < \tau$ be the last passage by -1 before τ. By integrating (4.4) in the time interval $[\sigma,\tau]$ and using that $b(t,x) \geqq 0$ for $(t,x) \in (-\ell,\ell)\times(-\infty,-1]$, see (4.2), we get

$$-\delta = \xi_\tau - \xi_\sigma = \int_\sigma^\tau \mathrm{d}s\, b(s,\xi_s) + \sqrt{\varepsilon}\,(w_\tau - w_\sigma) \geqq -2\sqrt{\varepsilon} \sup_{t\in[-\ell,\ell]} |w_t|,$$

which gives a contradiction. □

In the following lemma we show that, for t away from the boundary, the solution of (4.4) is in a small neighborhood of some profile $\bar{m}_z \in \mathcal{M}$ with probability close to one. We also identify z as a zero (it does not matter which one) of $t \mapsto \xi_t$.

Lemma 4.4. *For each* $\eta \in \left(0,\frac{1}{2}\right)$ *and* $\sigma \in (0, 1-2\eta)$ *there exists* ε_0 *such that for any* $\varepsilon \in (0,\varepsilon_0]$ *we have*

$$\nu_\varepsilon\left(|\mathcal{Z}(x)| < \sqrt{\ell}\,;\; \sup_{|t-\mathcal{Z}(x)|\leqq\sigma\ell} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\eta} \right) \leqq \exp\left\{-\varepsilon^{-\eta}\right\}.$$

Proof. We shorthand $\mathcal{Z}(x)$ by z and define

$$\tau_- := \sup\left\{ t \leqq z \,:\, |x(t) - \bar{m}_z(t)| > \varepsilon^{\frac{1}{2}-\eta} \right\} \vee (z - \sigma\ell),$$
$$\tau_+ := \inf\left\{ t \geqq z \,:\, |x(t) - \bar{m}_z(t)| > \varepsilon^{\frac{1}{2}-\eta} \right\} \wedge (z + \sigma\ell).$$

Let also

$$\mathcal{B}_1 := \left\{ w \,:\, \sup_{t\in[-\ell,\ell]} |w_t| < \varepsilon^{-\frac{2}{3}\eta} \right\}. \tag{4.16}$$

By the bound (4.15) we have $P\left(\mathcal{B}_1^{\complement}\right) \leqq \exp\left\{-\varepsilon^{-\eta}\right\}$ for any ε small enough. The proof will be completed by showing that on the event $\mathcal{B}_1 \cap \{|z| < \sqrt{\ell}\}$ we have $\tau_\pm = z \pm \sigma\ell$ for any ε small enough.

Let $v_t := \xi_t - \bar{m}_z(t)$. Integrating (4.4) and using $\bar{m}_z' = 1 - \bar{m}_z^2$, we obtain that for $t \in [\tau_-, \tau_+]$,

$$v_t = \int_z^t \mathrm{d}s\, \left[-2\bar{m}_z(s)v_s + R(s, v_s)\right] + \sqrt{\varepsilon}\,[w_t - w_z], \tag{4.17}$$

with

$$\begin{aligned} R(s, y) &:= b(s, \bar{m}_z(s) + y) - [1 - \bar{m}_z^2(s)] + 2\bar{m}_z(s)\, y \\ &= \sqrt{\left[1 - (\bar{m}_z(s) + y)^2\right]^2 + E_{s,\bar{m}_z(s)+y}} - [1 - (\bar{m}_z(s) + y)^2] - y^2, \end{aligned} \tag{4.18}$$

where we used (4.2), with $\bar{m}_z(s) + v_s \in (-1, 1)$, which holds for any ε small enough since, for $\eta \in \left[0, \frac{1}{2}\right)$, $\sigma \in (0, 1 - 2\eta)$, and $s \in [\tau_-, \tau_+]$,

$$|\bar{m}_z(s) + v_s| \leqq 1 - \mathrm{e}^{-2|s-z|} + \varepsilon^{\frac{1}{2}-\eta} \leqq 1 - \mathrm{e}^{-2\sigma\ell} + \mathrm{e}^{-2\ell+4\eta\ell} < 1.$$

Integration of (4.17), using that $-2\bar{m}_z = (\log \bar{m}_z')'$ and $\bar{m}'(t) = \mathrm{ch}(t)^{-2}$ yields that, on the event $\mathcal{B}_1$,

$$\begin{aligned} |v_t| &= \left|\int_z^t \mathrm{d}s\, \frac{\mathrm{ch}^2(s-z)}{\mathrm{ch}^2(t-z)} \left[R(s, v_s) - 2\sqrt{\varepsilon}\,\bar{m}_z(s)\,(w_s - w_z)\right] + \sqrt{\varepsilon}\,(w_t - w_z)\right| \\ &\leqq \left|\int_z^t \mathrm{d}s\, \frac{\mathrm{ch}^2(s-z)}{\mathrm{ch}^2(t-z)} R(s, v_s)\right| + 6\,\varepsilon^{\frac{1}{2}-\frac{2}{3}\eta}, \end{aligned} \tag{4.19}$$

where we used $\left|\int_z^t \mathrm{d}s\, \frac{\mathrm{ch}^2(s-z)}{\mathrm{ch}^2(t-z)}\right| \leqq 1$.

We claim that, for each $\eta \in \left[0, \frac{1}{2}\right)$ and $\sigma \in (0, 1 - 2\eta)$ there exists a constant $C > 0$ such that, for any ε small enough,

$$\sup_{t\in[\tau_-,\tau_+]} E_{t,\bar{m}_z(t)+v_t} \leqq C\ell^4 \mathrm{e}^{-4(\ell-z)}, \tag{4.20}$$

whose proof is given in Appendix A. By using (4.20) in (4.18) we get, for $s \in [\tau_-, \tau_+]$,

$$\begin{aligned} |R(s, v_s)| &\leqq v_s^2 + \frac{E_{s,\bar{m}_z(s)+v_s}}{2\left[1 - (\bar{m}_z(s) + v_s)^2\right]} \leqq v_s^2 + \frac{C\ell^4 \mathrm{e}^{-4(\ell-z)}}{1 - \left(|\bar{m}_z(s)| + \varepsilon^{\frac{1}{2}-\eta}\right)^2} \\ &\leqq v_s^2 + 2C\ell^4 \exp\{-4(\ell - z) + 2|s - z|\}, \end{aligned}$$

where in the last inequality we used $1 - |\bar{m}_z(s)| \geqq \mathrm{e}^{-2|s-z|}$ and $\mathrm{e}^{-2|s-z|} > \varepsilon^{\frac{1}{2}-\eta}$ for $|s - z| \leqq \sigma\ell$. Plugging this bound into (4.19) and using the estimate

$$\frac{\mathrm{ch}^2(s-z)}{\mathrm{ch}^2(t-z)} \leqq 4\,\mathrm{e}^{-2|t-s|} \quad \text{for } s \in [t \wedge z\,,\ t \vee z],$$

we get that, on the event $\mathcal{B}_1 \cap \{|z| < \sqrt{\ell}\}$, for any $t \in [\tau_-, \tau_+]$ and ε small enough,

$$\begin{aligned} |v_t| &\leqq \left| \int_z^t \mathrm{d}s \, 4 \, \mathrm{e}^{-2|t-s|} \, v_s^2 \right| + 2C\ell^4 \, \exp\{-4\ell + 4z + 2|t-z|\} + 6 \, \varepsilon^{\frac{1}{2} - \frac{2}{3}\eta} \\ &\leqq \left| \int_z^t \mathrm{d}s \, 4 \, \mathrm{e}^{-2|t-s|} \, v_s^2 \right| + 2C\ell^4 \sqrt{\varepsilon} + 6 \, \varepsilon^{\frac{1}{2} - \frac{2}{3}\eta} \\ &\leqq \left| \int_z^t \mathrm{d}s \, 4 \, \mathrm{e}^{-2|t-s|} \, v_s^2 \right| + 7 \, \varepsilon^{\frac{1}{2} - \frac{2}{3}\eta}, \end{aligned} \tag{4.21}$$

where in the second inequality we used that $-4\ell + 4z + 2|t-z| < -4\ell + 4\sqrt{\ell} + 2\sigma\ell < -2\ell$ for $\eta \in \left[0, \frac{1}{2}\right)$, $\sigma \in (0, 1-2\eta)$, and ℓ large enough. By (4.21) and a standard bootstrap argument it follows that $\tau_\pm = z \pm \sigma\ell$ for any ε small enough. □

In order to complete the proof of Theorem 4.2 we need to analyze the behavior of ξ_t for t close to $\pm\ell$. We remark that while both the measures ϱ_ε and μ_ε are invariant with respect to the map $x(t) \mapsto -x(-t)$, this symmetry property does not hold for ν_ε. We need therefore two separate arguments. We start with $t < 0$ and, in the next lemma, we give an upper bound for the probability that $\xi(t)$ gets above $\bar{m}_{-a}(t)$ for $t \leqq -a$ and a large.

Lemma 4.5. *There exist reals ε_0 and $a_0 > 0$ such that, for any $a \in [a_0, \ell]$ and $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\nu_\varepsilon\left(\{x \, : \, \exists\, t \in [-\ell, -a] \text{ such that } x(t) > \bar{m}_{-a}(t)\}\right) \leqq \exp\{-\mathrm{e}^{\frac{a}{4}}\}.$$

Proof. We introduce the event

$$\mathcal{B}_{1,a} := \left\{ w \, : \, |w_t| < \mathrm{e}^{\frac{a}{4}} + (t + \ell) \quad \forall\, t \in [-\ell, \ell] \right\}. \tag{4.22}$$

The probability of $\mathcal{B}_{1,a}$ can be computed explicitly, see, for example [13, Section 4.3.C]. We give however a short proof of the bound

$$P(\mathcal{B}_{1,a}^{\complement}) \leqq 2 \exp\left\{-2\, \mathrm{e}^{\frac{a}{4}}\right\}. \tag{4.23}$$

Indeed, let $M_t := \exp\{2w_t - 2(t+\ell)\}$, $t \in [-\ell, \ell]$. Since M_t is a mean one continuous martingale, by Doob inequality we have

$$\begin{aligned} &P\left(\left\{\exists\, t \in [-\ell, \ell] \, : \, w_t \geqq \mathrm{e}^{\frac{a}{4}} + t + \ell\right\}\right) \\ &\quad = P\left(\sup_{t \in [-\ell, \ell]} M_t \geqq \exp\left\{2\, \mathrm{e}^{\frac{a}{4}}\right\}\right) \leqq \exp\left\{-2\, \mathrm{e}^{\frac{a}{4}}\right\} \end{aligned}$$

and the bound (4.23) follows.

We next show that there exist $\varepsilon_0, a_0 > 0$ such that, for any $a \in [a_0, \ell]$ and $\varepsilon \in (0, \varepsilon_0]$, on the event $\mathcal{B}_{1,a}$ we have $\xi_t < \bar{m}_{-a}(t)$ for any $t \in [-\ell, -a]$. Let $\tau := \inf\{t \geqq -\ell \, : \, \xi_t = \bar{m}_{-a}(t)\} \wedge (-a)$. Note that, by continuity $\tau > -\ell$; we show that $\tau = -a$ on the event $\mathcal{B}_{1,a}$ arguing by contradiction. Indeed, let

$\sigma \in [-\ell, \tau)$ be the last time for which $\xi_t = m^*_\ell(t)$, the minimizer defined in Proposition 3.1. We integrate the Equation (4.4) in the interval $[\sigma, t]$ with $t \in [\sigma, \tau]$, getting

$$\xi_t = \xi_\sigma + \int_\sigma^t \mathrm{d}s\, b(s, \xi_s) + \sqrt{\varepsilon}\,(w_t - w_\sigma).$$

Since for $s \in [\sigma, \tau]$, we have $-1 \leqq m^*_\ell(s) \leqq \xi_s \leqq \bar{m}_{-a}(s) \leqq 0$, from (4.2), the inequality $\sqrt{\alpha+\beta} \leqq \sqrt{\alpha} + \sqrt{\beta}$, $\alpha, \beta \geqq 0$, and (A.28) we have

$$\xi_t \leqq m^*_\ell(\sigma) + \int_\sigma^t \mathrm{d}s\, \left[1 - \xi_s^2 + \sqrt{E_{s, m^*_\ell(s)}}\right] + \sqrt{\varepsilon}\,(w_t - w_\sigma). \tag{4.24}$$

Set $v_t := \xi_t - \bar{m}_{-a}(t)$ and note that $E_{s,m^*_\ell(s)} = E_{-\ell,-1} = E_\ell$. Since $\bar{m}'_{-a}(s) = 1 - \bar{m}^2_{-a}(s)$ and

$$1 - \xi_s^2 - [1 - \bar{m}_{-a}(s)^2] = -2\bar{m}_{-a}(s)v_s - v_s^2 \leqq -2\bar{m}_{-a}(s)v_s.$$

From (4.24) and (A.6), we get that, for ε small enough,

$$\begin{aligned} v_t \leqq& -\left[\bar{m}_{-a}(\sigma) - m^*_\ell(\sigma)\right] + \int_\sigma^t \mathrm{d}s\, \left[-2\bar{m}_{-a}(s)v_s + 9\sqrt{\varepsilon}\right] \\ &+ \sqrt{\varepsilon}\,(w_t - w_\sigma). \end{aligned} \tag{4.25}$$

Next, we show that, on the event $\mathcal{B}_{1,a}$, we have $v_\tau < 0$ provided a is large enough and ε is small enough, what contradicts the assumption $\tau \in [-\ell, -a)$. To this end we integrate the inequality (4.25), proceeding as explained when getting (4.19) from (4.17), obtaining

$$\begin{aligned} v_\tau \leqq& -\frac{\mathrm{ch}^2(\sigma + a)}{\mathrm{ch}^2(\tau + a)}\left[\bar{m}_{-a}(\sigma) - m^*_\ell(\sigma)\right] + \sqrt{\varepsilon}(w_\tau - w_\sigma) \\ &+ \sqrt{\varepsilon}\int_\sigma^\tau \mathrm{d}t\, \frac{\mathrm{ch}^2(t+a)}{\mathrm{ch}^2(\tau+a)}\left(9 - 2\bar{m}_{-a}(t)[w_t - w_\sigma]\right). \end{aligned} \tag{4.26}$$

We now observe that, by (A.8),

$$\bar{m}_{-a}(\sigma) - m^*_\ell(\sigma) \geqq \bar{m}_{-a}(\sigma) - \bar{m}_0(\sigma) - A_1\sqrt{\varepsilon} \geqq \mathrm{e}^{2\sigma}\left(\mathrm{e}^{2a} - 2\right) - A_1\sqrt{\varepsilon},$$

where we used $\mathrm{e}^{2a} \leqq 1 + \mathrm{th}(a) \leqq 2\mathrm{e}^{2a}$, $a \leqq 0$. On the other hand, since

$$\frac{1}{2}\,\mathrm{e}^{\beta-\alpha} \leqq \frac{\mathrm{ch}(\alpha)}{\mathrm{ch}(\beta)} \leqq \mathrm{e}^{\beta-\alpha}, \quad \alpha < \beta \leqq 0, \tag{4.27}$$

the inequality (4.26) yields

$$\begin{aligned} v_\tau <& -\frac{1}{4}\,\mathrm{e}^{2(\tau-\sigma)}\left[\mathrm{e}^{2\sigma}\left(\mathrm{e}^{2a} - 2\right) - A_1\sqrt{\varepsilon}\right] + 2\sqrt{\varepsilon}\left(\tau + \ell + \mathrm{e}^{\frac{a}{4}}\right) \\ &+ \sqrt{\varepsilon}\int_\sigma^\tau \mathrm{d}t\, \mathrm{e}^{2(\tau-t)}\left[9 + 4\left(t + \ell + \mathrm{e}^{\frac{a}{4}}\right)\right] \end{aligned}$$

$$\leqq -\frac{1}{4}\sqrt{\varepsilon}\,\mathrm{e}^{2(\tau+\ell)}\left\{\mathrm{e}^{2a} - 2 - A_1 - 8\,\mathrm{e}^{-2(\tau+\ell)}(\tau+\ell+\mathrm{e}^{\frac{a}{4}}) - 4\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-2s}\left[9+4\,(s+\mathrm{e}^{\frac{a}{4}})\right]\right\}.$$

By choosing a_0 large enough the term inside the curly brackets above is strictly positive for any $a \geqq a_0$. This yields $v_\tau < 0$ which is the contradiction announced and, together with (4.23), concludes the proof of the lemma. □

The analysis for $t > 0$ is somewhat more delicate. As a first step, which is the content of the next lemma, we study the process ξ_t for $t \in [-\ell + a, a]$ and show that for a large it does not get below $\bar{m}_a(\cdot)$. In particular $\xi_a \geqq 0$ with probability close to one. In Lemma 4.7 below we then show that this property yields an upper bound on the probability that ξ_t gets below $\bar{m}_a(t)$ for some $t \in [a, \ell]$.

Lemma 4.6. *There exist a_0, $\varepsilon_0 > 0$ such that, for any $a \in [a_0, \ell]$ and $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\nu_\varepsilon\left(\{x\ :\ \exists\, t \in [-\ell+a, a] \text{ such that } x(t) < \bar{m}_a(t)\}\right) \leqq \exp\left\{-\mathrm{e}^{\frac{a}{4}}\right\}.$$

Proof. The proof will be completed in three steps, each one taking place with probability close to one for a large and ε small. We first show that in the time interval $[-\ell, -\ell + a]$ the process ξ_t reaches the level $-1 + \sqrt{\varepsilon}\,\mathrm{e}^{\frac{a}{3}}$. We then show that $\xi_\cdot$ hits the level $\varepsilon^{\frac{3}{8}}$ before hitting $\bar{m}_a(\cdot)$. Finally, once the process is above $\varepsilon^{\frac{3}{8}}$ it does not go below zero.

Step 1. We introduce the event

$$\mathcal{B}_{2,a} := \left\{ w\ :\ \sup_{t\in[-\ell,-\ell+a]} |w_t| < \mathrm{e}^{\frac{a}{3}} \right\}. \tag{4.28}$$

Note that, for a large enough, we have $P(\mathcal{B}_{2,a}^{\complement}) \leqq \exp\left\{-\mathrm{e}^{\frac{a}{2}}\right\}$. We claim that on the event $\mathcal{B}_{2,a}$ there exists a time $\tau_1 \in [-\ell, -\ell + a]$ such that $\xi_{\tau_1} \geqq -1 + \sqrt{\varepsilon}\,\mathrm{e}^{\frac{a}{3}}$. We argue by contradiction. If there is no such τ_1 we have $\xi_t < -1 + \sqrt{\varepsilon}\,\mathrm{e}^{\frac{a}{3}} \leqq 0$ for any $t \in [-\ell, -\ell + a]$ (we choose ε_0 so small that $\sqrt{\varepsilon}\,\mathrm{e}^{\frac{\ell}{3}} \leqq 1$). From the inequality $(1 + \xi_t)\mathrm{e}^{\ell - t} \leqq \mathrm{e}^{\frac{a}{3}}$, (4.2), and (A.29) we get

$$\xi_t \geqq -1 + \sqrt{\varepsilon}\int_{-\ell}^{t} \mathrm{d}s\, 2\mathrm{e}^{s+\ell-\frac{a}{3}} - \sqrt{\varepsilon}\mathrm{e}^{\frac{a}{3}} \geqq -1 + 2\sqrt{\varepsilon}\,\mathrm{e}^{-\frac{a}{3}}\left(\mathrm{e}^{t+\ell} - 1\right) - \sqrt{\varepsilon}\,\mathrm{e}^{\frac{a}{3}}.$$

In particular, for a large enough, $\xi_{-\ell+a} > -1 + \sqrt{\varepsilon}\,\mathrm{e}^{\frac{a}{3}}$ which contradicts the definition of τ_1.

Step 2. Let $\tau_1 \in [-\ell, -\ell + a]$ be as in Step 1. We define $\tau_2 := \inf\{t > \tau_1\ :\ \xi_t \leqq \bar{m}_a(t)\} \wedge a$ and $\tau_3 := \inf\{t > \tau_1\ :\ \xi_t \geqq \varepsilon^{\frac{3}{8}}\} \wedge a$. On the event $\mathcal{B}_{2,a}$, for a large enough we have $\xi_{\tau_1} > \bar{m}_a(\tau_1)$; hence $\tau_2, \tau_3 > \tau_1$. Consider the event $\mathcal{B}_{1,a}$ that has been defined in (4.22). We claim that, by taking ε small enough and a large enough, in the event $\mathcal{B}_{2,a} \cap \mathcal{B}_{1,a}$ we have $\tau_3 < \tau_2$. We argue by contradiction, that is we assume that $\xi_t < \varepsilon^{\frac{3}{8}}$ for any $t \in (\tau_1, \tau_2]$. Let T such that $\bar{m}_a(T) = -\varepsilon^{\frac{3}{8}}$ and set $v_t := \xi_t - \bar{m}_a(t)$. Integrating (4.4) in the time interval $[\tau_1, t]$, with $t \in [\tau_1, \tau_2 \wedge T]$, using (4.2) and $\bar{m}_a' = 1 - \bar{m}_a^2$, we get

$$
\begin{aligned}
v_t &\geqq \xi_{\tau_1} - \bar{m}_a(\tau_1) + \int_{\tau_1}^t \mathrm{d}s\, [-\bar{m}_a(s) - \xi_s]\, v_s + \sqrt{\varepsilon}\,[w_t - w_{\tau_1}] \\
&\geqq -1 + \sqrt{\varepsilon}\, \mathrm{e}^{\frac{a}{3}} - \bar{m}_a(-\ell + a) + \int_{\tau_1}^t \mathrm{d}s \left[-\bar{m}_a(s) - \varepsilon^{\frac{3}{8}}\right] v_s \\
&\quad + \sqrt{\varepsilon}\,[w_t - w_{\tau_1}].
\end{aligned}
$$

Note that $\bar{m}_a(-\ell + a) \leqq -1 + 2\sqrt{\varepsilon}$. By integrating the above inequality, proceeding as in (4.26), we thus find

$$
\begin{aligned}
v_t \geqq\ & \frac{\mathrm{ch}(\tau_1 - a)}{\mathrm{ch}(t - a)} \mathrm{e}^{-\varepsilon^{\frac{3}{8}}(t-\tau_1)} \sqrt{\varepsilon} \left(\mathrm{e}^{\frac{a}{3}} - 2\right) + \sqrt{\varepsilon}[w_t - w_{\tau_1}] \\
& + \sqrt{\varepsilon} \int_{\tau_1}^t \mathrm{d}s\, \frac{\mathrm{ch}(s - a)}{\mathrm{ch}(t - a)} \mathrm{e}^{-\varepsilon^{\frac{3}{8}}(t-s)} \left[-\bar{m}_a(s) - \varepsilon^{\frac{3}{8}}\right] [w_s - w_{\tau_1}],
\end{aligned}
$$

whence, by (4.27) and the definition of the event $\mathcal{B}_{1,a}$ (we suppose ε so small that $\mathrm{e}^{-\varepsilon^{\frac{3}{8}}}(t - \tau_1) > \frac{1}{2}$),

$$
\begin{aligned}
v_t &> \frac{1}{4} \sqrt{\varepsilon}\, \mathrm{e}^{(t-\tau_1)} \left\{ \mathrm{e}^{\frac{a}{3}} - 2 - 8\mathrm{e}^{-(t-\tau_1)} \left[\mathrm{e}^{\frac{a}{4}} + (t + \ell) \right] \right. \\
&\quad \left. -9 \int_{\tau_1}^t \mathrm{d}s\, \mathrm{e}^{-(s-\tau_1)} \left[\mathrm{e}^{\frac{a}{4}} + s + \ell) \right] \right\} \\
&\geqq \frac{1}{4} \sqrt{\varepsilon}\, \mathrm{e}^{(t-\tau_1)} \left\{ \mathrm{e}^{\frac{a}{3}} - 2 - 17 \left[\mathrm{e}^{\frac{a}{4}} + a + 1 \right] \right\},
\end{aligned}
$$

where we used that $\tau_1 + \ell \leqq a$. By choosing a_0 large enough, we get that the term inside the curly brackets above is strictly positive for any $a \geqq a_0$, so $\tau_2 \wedge T = T$. Finally, by evaluating the above inequality for $t = T$, we conclude that $\xi_T > 4\varepsilon^{\frac{3}{8}}$ which gives the desired contradiction.

Step 3. Let τ_3 be as in Step 2. We claim that on the event $\mathcal{B}_{2,a} \cap \mathcal{B}_{1,a}$ we have $\xi_t \geqq 0$ for any $t \in [\tau_3, a]$. Assume this is not the case, let $\sigma_+ > \tau_3$ be the hitting time of the level zero, and let $\sigma_- := \sup\{t \in [\tau_3, \sigma_+) \,:\, \xi_t = \varepsilon^{\frac{3}{8}}\}$. By using (4.2) for $t \in [\sigma_-, \sigma_+]$ we have

$$
\xi_t \geqq \varepsilon^{\frac{3}{8}} + \sqrt{\varepsilon}\left(w_t - w_{\sigma_-}\right).
$$

Recalling (4.22) this gives the contradiction $\xi_{\sigma_+} > 0$. □

Lemma 4.7. *For each* $\eta \in (0, \frac{1}{2})$ *there exist* $a_0, \varepsilon_0 > 0$ *such that, for any* $a \in [a_0, \ell]$ *and* $\varepsilon \in (0, \varepsilon_0]$*, we have*

$$\nu_\varepsilon\left(\left\{x \,:\, \exists\, t \in [a,\ell] \text{ such that } x(t) < \bar m_a(t) - \varepsilon^{\frac12-\eta}\right\}\right) \leqq \exp\left\{-\varepsilon^{-\eta}\right\} + \exp\left\{-e^{\frac{a}{4}}\right\}.$$

Proof. By Lemma 4.6, for $a \geqq a_0$, $\nu_\varepsilon(\{x : x(a) < 0\}) \leqq \exp\left\{-e^{\frac{a}{4}}\right\}$. Let $\tau := \inf\{t \in [a,\ell] : \xi_t \geqq 1\} \wedge \ell$. Recalling $\mathcal{B}_1$ is defined in (4.16), we claim that on the event $\mathcal{B}_1 \cap \{\xi_a \geqq 0\}$ we have $\xi_t \geqq \bar m_a(t) - \varepsilon^{\frac12-\eta}$ for any $t \in [a,\tau]$. If $\tau = a$ there is nothing to prove, otherwise let $\varkappa := \xi_a \in [0,1]$ and define γ_t, $t \in [a,\tau]$, as the solution to

$$\gamma_t = \varkappa + \int_a^t \mathrm{d}s(1-\gamma_s^2) + \sqrt{\varepsilon}\,(w_t - w_a)\,.$$

By (4.2) we have $\xi_t \geqq \gamma_t$. Letting $a_\varkappa := a - \bar m_0^{-1}(\varkappa)$ we note $\bar m_{a_\varkappa}(t)$, $t \in [a,\tau]$, solves

$$\bar m_{a_\varkappa}(t) = \varkappa + \int_a^t \mathrm{d}s\,\left[1 - \bar m_{a_\varkappa}(s)^2\right].$$

By setting $v_t = \gamma_t - \bar m_{a_\varkappa}(t)$ and proceeding as in (4.19), on the event $\mathcal{B}_1 \cap \mathcal{B}_{2,a} \cap \mathcal{B}_{1,a}$, see (4.22) and (4.28), we get

$$|v_t| \leqq \int_a^t \mathrm{d}s\,\frac{\mathrm{ch}^2(s-a_\varkappa)}{\mathrm{ch}^2(t-a_\varkappa)}\,v_s^2 + 10\,\varepsilon^{\frac12-\frac23\eta}.$$

By a standard bootstrap argument, we deduce that $\sup_{t\in[a,\tau]}|v_t| \leqq \varepsilon^{\frac12-\eta}$ for ε small enough. This concludes the proof of the claim.

Finally, since $b(t,x) \geqq 0$ for any $(t,x) \in [-\ell,\ell]\times(-\infty,1]$, by the same argument given in the proof of Lemma 4.3 we get that, for each $\eta \in \left(0,\frac12\right)$ and ε small enough,

$$P\left(\left\{\exists\, t \in [\tau,\ell] \,:\, \xi_t < 1 - \varepsilon^{\frac12-\eta}\right\}\right) \leqq \exp\left\{-\varepsilon^{-\eta}\right\},$$

which concludes the proof. □

We have now collected all the ingredients needed to conclude the proof of the main result of this section.

Proof of Theorem 4.2. Recall that $\mathcal{Z}(x)$ denotes the leftmost zero of $x \in \mathcal{X}_\ell$. The bound (4.13) follows directly from Lemmata 4.5 and 4.7. The bound (4.14) is the content of Lemma 4.6. In order to prove (4.12), by (4.13), it is enough to prove the following. For each η small enough there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, we have

$$\nu_\varepsilon\left(\sup_{t\in[-\ell,\ell]} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\eta},\ |\mathcal{Z}(x)| \leqq a_\ell\right) \leqq \frac{1}{2}\exp\left\{\varepsilon^{-\frac{1}{2}\eta}\right\}, \tag{4.29}$$

where $a_\ell := \log^2 \ell$.

We consider separately the cases $t \in [-\ell, -\sigma\ell]$, $t \in [\sigma\ell, \ell]$, and $t \in [-\sigma\ell, \sigma\ell]$, with $\sigma < 1$ suitably chosen. For the first case, we observe that by Lemma 4.3 with $\delta = \frac{1}{2}\varepsilon^{\frac{1}{2}-\eta}$ and Lemma 4.5, for any ε small enough,

$$\begin{aligned}&\nu_\varepsilon\left(\exists\, t \in [-\ell, -\sigma\ell] \,:\, x(t) \notin \left(-1 - \tfrac{1}{2}\varepsilon^{\frac{1}{2}-\eta}, \bar{m}_{-a_\ell}(t)\right)\right)\\ &\leqq \exp\left\{-\varepsilon^{-\eta}\right\} + \exp\left\{-\mathrm{e}^{\frac{1}{4}a_\ell}\right\}.\end{aligned}$$

We observe that $\bar{m}_{-a_\ell}(t) \leqq -1 + 2\varepsilon^{\frac{\sigma}{2}}\mathrm{e}^{2a_\ell}$ for any $t \in [-\ell, -\sigma\ell]$. By choosing $\sigma \in [\sigma_0, 1)$ with $\sigma_0 := 1 - \frac{5}{3}\eta$, the previous estimate implies

$$\begin{aligned}&\nu_\varepsilon\left(\sup_{t\in[-\ell,-\sigma\ell]} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\eta}, |\mathcal{Z}(x)| \leqq a_\ell\right)\\ &\qquad\leqq \exp\left\{-\varepsilon^{-\eta}\right\} + \exp\left\{-\mathrm{e}^{\frac{1}{4}a_\ell}\right\}.\end{aligned}$$

Analogously, for the second case, by Lemma 4.3 with $\delta = \frac{1}{2}\varepsilon^{\frac{1}{2}-\eta}$ and Lemma 4.7 with η replaced by $\frac{2}{3}\eta$, for any ε small enough we have

$$\begin{aligned}&\nu_\varepsilon\left(\exists\, t \in [\sigma\ell, \ell] \,:\, x(t) \notin \left(\bar{m}_{a_\ell}(t) - \varepsilon^{\frac{1}{2}-\frac{2}{3}\eta}, 1 + \tfrac{1}{2}\varepsilon^{\frac{1}{2}-\eta}\right)\right)\\ &\leqq \exp\left\{-\varepsilon^{-\frac{2}{3}\eta}\right\} + \exp\left\{-\varepsilon^{-\eta}\right\} + \exp\left\{-\mathrm{e}^{\frac{1}{4}a_\ell}\right\}.\end{aligned}$$

By choosing $\sigma \in [\sigma_0, 1)$ with σ_0 as before, the previous estimate implies

$$\begin{aligned}&\nu_\varepsilon\left(\sup_{t\in[\sigma\ell,\ell]} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\eta}, |\mathcal{Z}(x)| \leqq a_\ell\right)\\ &\qquad\leqq \exp\left\{-\varepsilon^{-\eta}\right\} + \exp\left\{-\varepsilon^{-\frac{2}{3}\eta}\right\} + \exp\left\{-\mathrm{e}^{\frac{1}{4}a_\ell}\right\}.\end{aligned}$$

Finally, by applying Lemma 4.4 with η replaced by $\frac{2}{3}\eta$, we have that, for any $\sigma' \in \left(0, 1 - \frac{4}{3}\eta\right)$ and ε small enough (which implies $a_\ell \leqq \sqrt{\ell}$),

$$\begin{aligned}&\nu_\varepsilon\left(\sup_{t\in[-\sigma'\ell+a_\ell,\sigma'\ell-a_\ell]} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\eta}, |\mathcal{Z}(x)| \leqq a_\ell\right)\\ &\quad\leqq \nu_\varepsilon\left(\sup_{t\in[-\sigma'\ell+a_\ell,\sigma'\ell-a_\ell]} \left|x(t) - \bar{m}_{\mathcal{Z}(x)}(t)\right| > \varepsilon^{\frac{1}{2}-\frac{2}{3}\eta}, |\mathcal{Z}(x)| \leqq a_\ell\right)\\ &\quad\leqq \exp\left\{-\varepsilon^{-\frac{2}{3}\eta}\right\}.\end{aligned}$$

Since $\sigma_0 < 1 - \frac{4}{3}\eta$, we can choose $\sigma \in (\sigma_0, 1)$ and $\sigma' \in (\sigma, 1)$; the bound (4.29) follows. □

5. Weak convergence of the measure

We first show, by using the representation of the measure μ_ε given in Proposition 4.1 and the sharp estimates of the previous section, that μ_ε concentrates in a $\sqrt{\varepsilon}$-neighborhood of the manifold $\mathcal{M}$. We also show that the interface remains in a compact set of $\mathbb{R}$ with probability close to one. Recall that $\mathcal{Z}(x)$ is the leftmost zero of $x \in \mathcal{X}_\ell$.

Theorem 5.1. *For each $\eta > 0$, we have*

$$\lim_{\varepsilon\to 0} \mu_\varepsilon \left(\left\{ x \,:\, d(x, \mathcal{M}) > \varepsilon^{\frac{1}{2}-\eta} \right\} \right) = 0, \tag{5.1}$$

$$\lim_{L\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \mu_\varepsilon \left(\{ x \,:\, |\mathcal{Z}(x)| > L \} \right) = 0. \tag{5.2}$$

We first prove a rougher bound showing that, uniformly in ε, the measure μ_ε of bounded sets is close to one.

Lemma 5.2. *We have*

$$\lim_{K\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \mu_\varepsilon \left(\{ x \,:\, \|x\|_\infty > K \} \right) = 0.$$

Proof. We assume that $K \in \mathbb{N}$ ad use the representation of μ_ε in Proposition 4.1 together with the bound $|\partial_{xx} S_0(t, x)| \leqq A_3(1 + |x|)$ proven in Theorem A.3. We have

$$\begin{aligned}
\mu_\varepsilon \left(\{ x \,:\, \|x\|_\infty > K \} \right) &= \sum_{h \geqq K} \frac{\nu_\varepsilon \left(\mathrm{e}^{-\frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t))}\, 1\!\!1_{\{\|x\|_\infty \in [h, h+1)\}} \right)}{\nu_\varepsilon \left(\mathrm{e}^{-\frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t))} \right)} \\
&\leqq \sum_{h \geqq K} \frac{\mathrm{e}^{A_3 \ell (2+h)}\, \nu_\varepsilon \left(\|x\|_\infty \geqq h \right)}{\mathrm{e}^{-3A_3\ell}\, \nu_\varepsilon \left(\|x\|_\infty \leqq 2 \right)} \\
&\leqq \sum_{h \geqq K} 2\, \mathrm{e}^{A_3 \ell (5+h)}\, \mathrm{e}^{-c_0 (\varepsilon \ell)^{-1} h^2},
\end{aligned}$$

where we used that, by Lemma 4.3, if ε is small enough then $\nu_\varepsilon \left(\|x\|_\infty \leqq 2 \right) \geqq \frac{1}{2}$ and $\nu_\varepsilon \left(\|x\|_\infty \geqq h \right) \leqq \mathrm{e}^{-c_0(\varepsilon\ell)^{-1}h^2}$ for some $c_0 > 0$. □

Proof of Theorem 5.1. We first prove (5.1). By Lemma 5.2 it is enough to show that for each $K \geqq 2$ and $\eta > 0$ we have

$$\lim_{\varepsilon\to 0} \mu_\varepsilon (\mathcal{B}) = 0, \quad \mathcal{B} := \left\{ x \,:\, d(x, \mathcal{M}) > \varepsilon^{\frac{1}{2}-\eta} \right\} \cap \left\{ x \,:\, \|x\|_\infty \leqq K \right\}, \tag{5.3}$$

By the representation given in Proposition 4.1 and Theorem A.3,

$$\mu_\varepsilon(\mathcal{B}) = \frac{\nu_\varepsilon\left(\exp\left\{-\frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t))\right\} 1\!\!1_{\mathcal{B}}\right)}{\nu_\varepsilon\left(\exp\left\{-\frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t))\right\}\right)}$$
$$\leqq \mathrm{e}^{A_3(4+K)\ell} \frac{\nu_\varepsilon\left(d(x,\mathcal{M}) > \varepsilon^{\frac{1}{2}-\eta}\right)}{\nu_\varepsilon\left(\|x\|_\infty \leqq 2\right)},$$

which, by (4.12), concludes the proof of (5.1).

We next prove (5.2). By (5.1) it is enough to show that, for some $\eta > 0$,

$$\lim_{L\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \mu_\varepsilon\left(\mathcal{B}_{\varepsilon,L}\right) = 0, \quad \mathcal{B}_{\varepsilon,L} := \left\{x : |\mathcal{Z}(x)| > L,\ d(x,\mathcal{M}) \leqq \varepsilon^{\frac{1}{2}-\eta}\right\} \tag{5.4}$$

Let

$$\mathcal{I}_\ell(x) := -\frac{1}{2}\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t)) - \frac{1}{2}\log \ell.$$

By the representation given in Proposition 4.1,

$$\mu_\varepsilon\left(\mathcal{B}_{\varepsilon,L}\right) = \frac{\nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\, 1\!\!1_{\mathcal{B}_{\varepsilon,L}}\right)}{\nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\right)}.$$

We first observe that, by setting

$$\mathcal{A} := \left\{d(x,\mathcal{M}) \leqq \varepsilon^{\frac{1}{2}-\eta}\right\} \cap \{|\mathcal{Z}(x)| \leqq a\} \cap \left\{x(t) \geqq \bar{m}_a(t)\ \forall\, t \in [-\ell + a, a]\right\},$$

we have

$$\nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\right) \geqq \nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\, 1\!\!1_{\mathcal{A}}\right) \geqq \mathrm{e}^{-A_5 a}\left(1 - \nu_\varepsilon(\mathcal{A}^{\complement})\right),$$

where we used Proposition A.5. By choosing a large enough and applying Theorem 4.2 we get $\underline{\lim}_{\varepsilon\to 0}\, \nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\right) > 0$.

We next observe that, by Theorem A.3, we have $|\partial_{xx} S_0(\ell - t, x(t))| \leqq 3A_3$ on the event $d(x,\mathcal{M}) \leqq 1$, so that, for ℓ large enough,

$$\mathcal{B}_{\varepsilon,L} = \left(\mathcal{B}_{\varepsilon,L} \cap \left\{|\mathcal{I}_\ell| \leqq 2A_5 L\right\}\right)$$
$$\cup \bigcup_{h=L}^{\left[7\frac{A_3}{A_5}\ell\right]} \left(\mathcal{B}_{\varepsilon,L} \cap \left\{2hA_5 < |\mathcal{I}_\ell| \leqq 2A_5(h+1)\right\}\right).$$

Accordingly,

$$\nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\, 1\!\!1_{\mathcal{B}_{\varepsilon,L}}\right) \leqq \mathrm{e}^{2A_5 L}\nu_\varepsilon\left(|\mathcal{Z}| > L\right)$$
$$+ \sum_{h=L}^{\left[7\frac{A_3}{A_5}\ell\right]} \mathrm{e}^{2A_5(h+1)}\, \nu_\varepsilon\left(\mathcal{B}_{\varepsilon,L} \cap \{|\mathcal{I}_\ell| > 2hA_5\}\right).$$

Choosing η small enough and applying Proposition A.5, we get

$$\begin{aligned}\nu_\varepsilon\left(\mathcal{B}_{\varepsilon,L}\cap\{|\mathcal{I}_\ell|>2hA_5\}\right) &\leqq \nu_\varepsilon\left(d(x,\mathcal{M})\leqq\varepsilon^{\frac{1}{2}-\eta},\ |\mathcal{I}_\ell|>2A_5\,h\right)\\ &\leqq \nu_\varepsilon\left(|\mathcal{Z}(x)|>h\right)+\nu_\varepsilon\left(\{\exists\, t\in[-\ell+h,h]\,:\,x(t)<\bar m_h(t)\}\right).\end{aligned}$$

By choosing L large enough and applying Theorem 4.2, we thus obtain

$$\varlimsup_{\varepsilon\to0}\nu_\varepsilon\left(\mathrm{e}^{\mathcal{I}_\ell}\,\mathbb{1}_{\mathcal{B}_{\varepsilon,L}}\right)\leqq 2\exp\left\{2A_5L-\mathrm{e}^{\frac14 L}\right\}+\sum_{h=L}^{\infty}3\exp\left\{2A_5(h+1)-\mathrm{e}^{\frac14 h}\right\},$$

which concludes the proof. □

We next conclude the proof of Theorem 2.3 by characterizing the limit points of μ_ε as the invariant measure of (2.21). By [8, Theorem 5.1] μ_ε is the unique invariant measure of the process $X=X_\sigma$ in $C(\mathbb{R}_+;\mathcal{X}_\ell)$ which solves (2.20) with $\ell=\frac14\log\varepsilon^{-1}$. For $T>0$, we denote by $\mathbb{P}^\varepsilon_{x_0}$ the law of the process $X_{\varepsilon^{-1}\sigma},\sigma\in[0,T]$, where X is the solution to (2.20) with initial datum $x_0\in\mathcal{X}_\ell$. We regard $\mathbb{P}^\varepsilon_{x_0}$ as a probability on $C([0,T];\mathcal{X})$, endowed with the topology of uniform convergence. Let also P_{z_0} be the law of the one-dimensional diffusion solution to (2.21) with initial datum $z_0\in\mathbb{R}$. We finally define $\mathbb{P}_{z_0}$ as the probability measure on $C([0,T];\mathcal{X})$ with support $C([0,T];\mathcal{M})$ such that $\mathbb{P}_{z_0}(A)=P_{z_0}\left(\bar m_{\zeta_\cdot}\in A\right)$. The analysis in [1], see in particular Theorem 2.2, yields the weak convergence of $\mathbb{P}^\varepsilon_{x_0}$ to $\mathbb{P}_{\mathcal{Z}(x_0)}$, recall $\mathcal{Z}(x)$ is the leftmost zero of x. Moreover, for η small enough, the above convergence is uniform for $\mathcal{Z}(x_0)$ in compacts and x_0 such that $d(x_0,\mathcal{M})\leqq\varepsilon^{\frac12-\eta}$.

Theorem 5.3. *Let $T>0$. There exists $\eta_1>0$ such that for any $\eta\in[0,\eta_1]$ the following holds. For each $L>0$ and each uniformly continuous and bounded function F on $C([0,T];\mathcal{X})$, we have*

$$\lim_{\varepsilon\to0}\ \sup_{z_0\in[-L,L]}\ \sup_{x_0\in\mathcal{N}^\varepsilon_\eta(z_0)}\left|\mathbb{P}^\varepsilon_{x_0}(F)-\mathbb{P}_{z_0}(F)\right|=0,\tag{5.5}$$

where $\mathcal{N}^\varepsilon_\eta(z_0):=\left\{x\in\mathcal{X}_\ell\,:\,d(x,\bar m_{z_0})\leqq\varepsilon^{\frac12-\eta}\right\}$.

Proof of Theorem 2.3. We set $p_\varepsilon:=\mu_\varepsilon\circ\mathcal{Z}^{-1}$, namely p_ε is the distribution of the real random variable $\mathcal{Z}(x)$ when x is distributed according to μ_ε. Note that p_ε is tight by (5.2). Let also $Q_z(\cdot)$ be a regular version of the conditional probability $\mu_\varepsilon(\,\cdot\,|\,\mathcal{Z}=z)$.

Denote by $\pi_\sigma\,:\,C(\mathbb{R}_+;\mathcal{X})\to\mathcal{X}$ the evaluation map at σ. Since μ_ε is the invariant measure of (2.20), for each $\sigma\in\mathbb{R}_+$ and each uniformly continuous and bounded function F on $\mathcal{X}$, we have

$$\begin{aligned}\int \mathrm{d}\mu_\varepsilon(x)\,F(x)&=\int \mathrm{d}\mu_\varepsilon(x_0)\,\mathbb{P}^\varepsilon_{x_0}\left(F\circ\pi_\sigma\right)\\ &=\int_{-L}^{L}\mathrm{d}p_\varepsilon(z_0)\int_{\mathcal{N}^\varepsilon_\eta(z_0)}\mathrm{d}Q_{z_0}(x_0)\,\mathbb{P}^\varepsilon_{x_0}\left(F\circ\pi_\sigma\right)+R_{L,\varepsilon}(F),\end{aligned}\tag{5.6}$$

where, by the compactness and tube estimate of Theorem 5.1, for each $\eta > 0$ we have

$$\lim_{L\to\infty} \overline{\lim_{\varepsilon\to 0}} \left| R_{L,\varepsilon}(F) \right| \\ \leqq \|F\|_\infty \lim_{L\to\infty} \overline{\lim_{\varepsilon\to 0}} \mu_\varepsilon \left(\{|\mathcal{Z}(x)| > L\} \cup \left\{ d(x, \bar{m}_{\mathcal{Z}(x)}) > \varepsilon^{\frac{1}{2}-\eta} \right\} \right) = 0.$$

By the tightness of p_ε, there exists a probability measure p on $\mathbb{R}$ and a subsequence, still denoted by p_ε, weakly convergent to p. By Theorem 5.3, for any $\eta \in [0, \eta_1]$, the real function

$$z_0 \;\mapsto\; \int_{N_\eta^\varepsilon(z_0)} \mathrm{d}Q_{z_0}(x_0)\, \mathbb{P}_{x_0}^\varepsilon\, (F \circ \pi_\sigma)$$

converges to $\mathbb{P}_{z_0}(F \circ \pi_\sigma) = P_{z_0}(F(\bar{m}_{\zeta_\sigma}))$ uniformly for z_0 in compacts. By taking in (5.6) the limit $\varepsilon \to 0$ along the converging subsequence and then $L \to \infty$, we get

$$\lim_{\varepsilon\to 0} \int \mathrm{d}\mu_\varepsilon(x)\, F(x) = \int \mathrm{d}p(z_0)\, P_{z_0}\left(F\left(\bar{m}_{\zeta_\sigma}\right)\right). \tag{5.7}$$

As σ and F were arbitrary, (5.7) shows that p is an invariant measure for the one-dimensional diffusion process (2.21). Since the latter has a unique invariant measure given by $\widehat{\mu}$ as in Equation (2.11), we conclude that $p = \widehat{\mu}$, and, by (5.7), the proof of the theorem. □

Remark. From the above proof it follows that the stationary process associated to (2.20), as a random element in $C(\mathbb{R}; \mathcal{X})$, converges in law to $\bar{m}_{\zeta_\cdot}$, where $\zeta_\cdot$ is the stationary process associated to (5.5).

Appendix A: Weierstrass analysis of the mechanical problem

Recall that S has been defined in (2.19). Since the potential $-2V(x)$ attains its global maximum at $x = 1$, for each $(t, x) \in (0, \infty) \times \mathbb{R}$ there is a unique solution $\psi_{t,x}(\cdot)$ to the Newton equation

$$\begin{cases} \ddot{\psi}_{t,x} = 2V'(\psi_{t,x}), \\ \psi_{t,x}(0) = 1, \\ \psi_{t,x}(t) = x. \end{cases} \tag{A.1}$$

As discussed in the proof of Proposition 3.1, $\psi_{t,x}(\cdot)$ is the minimizer for $S(t, x)$, that is,

$$S(t, x) = \int_0^t \mathrm{d}s \left[\frac{1}{2} \dot{\psi}_{t,x}(s)^2 + 2V(\psi_{t,x}(s)) \right]. \tag{A.2}$$

Integration of (A.1) yields that, for $s \in (0, t)$,

$$\dot{\psi}_{t,x}(s)^2 - 4V(\psi_{t,x}(s)) = e_{t,x}, \tag{A.3}$$

for some non-negative constant $e_{t,x}$. Clearly $e_{t,1} = 0$; otherwise, integrating (A.3) by separation of variables, we get that $e_{t,x}$ solves

$$t = \left| \int_x^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + e_{t,x}}} \right|. \tag{A.4}$$

Also, substitution of (A.3) into (A.2) gives

$$S(t,x) = \left| \int_x^1 \mathrm{d}u \sqrt{4V(u) + e_{t,x}} \right| - \frac{1}{2}\, t\, e_{t,x}. \tag{A.5}$$

We finally notice that, by the symmetry of V, $\inf_{\mathcal{X}_\ell} \mathcal{F}_\ell = S(2\ell, -1)$.

In the first two lemmata we prove the estimates used in Section 3 to prove the variational convergence of $\mathcal{G}_\ell$.

Lemma A.1. *Let* $E_\ell := e_{2\ell,-1}$. *Then*

$$\lim_{\ell \to \infty} \mathrm{e}^{4\ell} E_\ell = 64, \tag{A.6}$$

$$\lim_{\ell \to \infty} \mathrm{e}^{4\ell} \left[S(2\ell,-1) - \frac{4}{3} \right] = 16. \tag{A.7}$$

Moreover, there exists a constant $A_1 > 0$ *such that for any* $\ell \geqq 1$*,*

$$\sup_{t \in [-\ell,\ell]} \left| m_\ell^*(t) - \bar{m}_0(t) \right| \leqq A_1\, \mathrm{e}^{-2\ell}. \tag{A.8}$$

Proof. Direct integration yields

$$\left| \int_x^1 \frac{\mathrm{d}u}{\sqrt{\gamma^2 (1-u)^2 + \beta}} \right| = \frac{1}{\gamma} \operatorname{arcsh} \frac{\gamma |x-1|}{\sqrt{\beta}}, \quad \beta, \gamma > 0, \tag{A.9}$$

which will be repeatedly used in the sequel. By (A.4) and the symmetry of V we thus have

$$\ell = \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + E_\ell}} = \frac{1}{2} \operatorname{arcsh} \frac{2}{\sqrt{E_\ell}} + R_1(E_\ell),$$

where

$$R_1(E) := \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + E}} - \int_0^1 \frac{\mathrm{d}u}{\sqrt{4(1-u)^2 + E}}. \tag{A.10}$$

A straightforward computation yields

$$\lim_{E \downarrow 0} R_1(E) = \frac{1}{2} \int_0^1 \frac{\mathrm{d}u}{1+u} = \frac{1}{2} \log 2. \tag{A.11}$$

Since $E_\ell \downarrow 0$ as $\ell \to +\infty$, (A.6) follows.

To prove (A.7) we first observe that $\frac{4}{3} = \int_{-1}^1 \mathrm{d}u \sqrt{4V(u)}$. We next show that

$$\lim_{\ell \to \infty} \frac{1}{E_\ell} \left[S(2\ell,-1) - \int_{-1}^1 \mathrm{d}u \sqrt{4V(u)} \right] = \frac{1}{4}, \tag{A.12}$$

which, together with (A.6), yields (A.7). The identities (A.4), (A.5), and simple computations give

$$\frac{1}{E_\ell}\left[S(2\ell,-1)-\int_{-1}^{1}\mathrm{d}u\,\sqrt{4V(u)}\right]=R_2(E_\ell),$$

where

$$R_2(E):=\int_0^1\mathrm{d}u\,\frac{E}{\sqrt{4V(u)+E}\,\left(\sqrt{4V(u)}+\sqrt{4V(u)+E}\right)^2}.$$

By the change of variable $1-u=\frac{1}{2}\sqrt{E}\,y$, we get

$$R_2(E)=\frac{1}{2}\int_0^{\frac{2}{\sqrt{E}}}\mathrm{d}y\,\frac{1}{\sqrt{1+y^2(1-\frac{1}{4}\sqrt{E}y)^2}}\\ \times\frac{1}{\left(y(1-\frac{1}{4}\sqrt{E}y)+\sqrt{1+y^2(1-\frac{1}{4}\sqrt{E}y)^2}\right)^2},$$

hence

$$\lim_{E\downarrow 0}R_2(E)=\frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}y}{\sqrt{1+y^2}\,\left(y+\sqrt{1+y^2}\right)^2}=\frac{1}{4},$$

which gives (A.12).

To prove (A.8) we first note that it is enough to consider the case $t\geqq 0$ as both m_ℓ^* and $\bar{m}_0$ are odd functions. Since m_ℓ^* is the solution to (3.1), for $t\in[0,\ell]$, we have $e_{\ell-t,m_\ell^*(t)}=e_{2\ell,-1}=E_\ell$, namely

$$t\;=\int_0^{m_\ell^*(t)}\frac{\mathrm{d}u}{\sqrt{4V(u)+E_\ell}}.$$

On the other hand, since $\bar{m}_0'=\sqrt{4V(\bar{m}_0)}$, for $t\geqq 0$,

$$t\;=\int_0^{\bar{m}_0(t)}\frac{\mathrm{d}u}{\sqrt{4V(u)}}.$$

Then,

$$\int_{\bar{m}_0(t)}^{m_\ell^*(t)}\frac{\mathrm{d}u}{\sqrt{4V(u)+E_\ell}}\;=\int_0^{\bar{m}_0(t)}\mathrm{d}u\left[\frac{1}{\sqrt{4V(u)}}-\frac{1}{\sqrt{4V(u)+E_\ell}}\right].\tag{A.13}$$

We now have

$$\frac{1}{\sqrt{4V(u)}}-\frac{1}{\sqrt{4V(u)+E_\ell}}\leqq\frac{E_\ell}{2\,[4V(u)]^{\frac{3}{2}}}$$

and

$$\int_{\bar{m}_0(t)}^{m_\ell^*(t)} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_\ell}} \geqq \frac{\big(m_\ell^*(t)-\bar{m}_0(t)\big)}{\sqrt{4V(\bar{m}_0(t))+E_\ell}} = \frac{\big(m_\ell^*(t)-\bar{m}_0(t)\big)}{\sqrt{\bar{m}_0'(t)^2+E_\ell}}.$$

After substituting in (A.13), we obtain

$$\begin{aligned} m_\ell^*(t)-\bar{m}_0(t) &\leqq E_\ell \sqrt{\bar{m}_0'(t)^2+E_\ell} \int_0^{\bar{m}_0(t)} \frac{\mathrm{d}u}{2\,[4V(u)]^{\frac{3}{2}}} \\ &= E_\ell \sqrt{\bar{m}_0'(t)^2+E_\ell} \int_0^t \frac{\mathrm{d}s}{2\,\bar{m}_0'(s)^2}, \end{aligned}$$

that, recalling (A.6), yields (A.8). □

Lemma A.2. *Let x_ℓ be a sequence in $\mathcal{X}$ such that $\overline{\lim}_\ell\, \mathcal{G}_\ell(x_\ell) < +\infty$. Then $\{x_\ell\}$ is precompact.*

Proof. Precompactness of a sequence x_ℓ in $\mathcal{X}$ is equivalent to its equi-continuity together with

$$\lim_{K\to\infty}\ \overline{\lim_{\ell\to\infty}}\ \sup_{t\in\pm[K,\infty)} |x_\ell(t)\mp 1| = 0. \tag{A.14}$$

Since $\overline{\lim}_\ell\, \mathcal{G}_\ell(x_\ell) < +\infty$, by (3.3) there exists $C>0$ such that $\mathcal{F}_\ell(x_\ell) \leqq \frac{4}{3} + C\,\mathrm{e}^{-4\ell}$. This estimate implies immediately the equi-continuity of the sequence x_ℓ.

We next prove that

$$\underline{\lim}_{K\to\infty}\ \underline{\lim}_{\ell\to\infty}\ \inf_{t\in(-\infty,-K]} x_\ell(t) \geqq -1. \tag{A.15}$$

Given $\delta>0$, let $\tau_\ell^\delta := \inf\{t\in[-\ell,\ell]\,:\,x_\ell(t) = -1-\delta\}\wedge \ell$ be the time of the first passage by $-1-\delta$. The estimate (A.15) is then equivalent to $\underline{\lim}_\ell\, \tau_\ell^\delta > -\infty$ for any $\delta>0$. Since $x_\ell\in\mathcal{X}_\ell$, we have $\tau_\ell^\delta\in(-\ell,\ell]$. If $\tau_\ell^\delta=\ell$ (A.15) holds trivially, otherwise we define $\sigma_\ell^\delta := \sup\big\{t\in[-\ell,\tau_\ell^\delta]\,:\,x_\ell(t) = -1-\frac{\delta}{2}\big\}$. Recalling the notation (3.4), the equi-boundedness of the excess free energy $\mathcal{G}_\ell(x_\ell)$ yields

$$\begin{aligned} C\,\mathrm{e}^{-4\ell} &\geqq \mathcal{F}_\ell(x_\ell) - \mathcal{F}_\ell(m_\ell^*) \\ &= \mathcal{F}_{[-\ell,\tau_\ell^\delta]}(x_\ell) - \mathcal{F}_{[-\ell,\tau_\ell^\delta]}(m_\ell^*) + \mathcal{F}_{[\tau_\ell^\delta,\ell]}(x_\ell) - \mathcal{F}_{[\tau_\ell^\delta,\ell]}(m_\ell^*) \\ &\geqq \int_{\sigma_\ell^\delta}^{\tau_\ell^\delta} \mathrm{d}t\, 2V(x_\ell(t)) - \mathcal{F}_{[-\ell,\tau_\ell^\delta]}(m_\ell^*) \\ &\quad + \inf_{\substack{x\in\mathcal{X}_\ell \\ x(\tau_\ell^\delta)=-1-\delta}} \mathcal{F}_{[\tau_\ell^\delta,\ell]}(x) - \mathcal{F}_{[\tau_\ell^\delta,\ell]}(m_\ell^*). \end{aligned}$$

Recalling (A.2), the second difference on the right-hand side above equals $S(\ell-\tau_\ell^\delta, -1-\delta) - S(\ell-\tau_\ell^\delta, m_\ell^*(\tau_\ell^\delta))$. Since $x\mapsto S(t,x)$ is increasing and $m_\ell^*(\tau_\ell^\delta) > -1$ we conclude that

$$C\,\mathrm{e}^{-4\ell} \geqq 2V\left(-1-\frac{\delta}{2}\right)\left(\tau_\ell^\delta - \sigma_\ell^\delta\right) - \mathcal{F}_{[-\ell,\tau_\ell^\delta]}(m_\ell^*),$$

whence

$$\underline{\lim}_{\ell\to\infty} \mathcal{F}_{[-\ell,\tau_\ell^\delta]}(m_\ell^*) \geqq \underline{\lim}_{\ell\to\infty} 2V\left(-1-\frac{\delta}{2}\right)\left(\tau_\ell^\delta - \sigma_\ell^\delta\right) > 0,$$

where we used the equi-continuity of x_ℓ. We then conclude that $\underline{\lim}_\ell \tau_\ell^\delta > -\infty$ and (A.15) follows since $\delta > 0$ was arbitrary. By symmetry we also have

$$\overline{\lim}_{K\to\infty} \overline{\lim}_{\ell\to\infty} \sup_{t\in[K,\infty)} x_\ell(t) \leqq 1. \tag{A.16}$$

We next prove

$$\overline{\lim}_{K\to\infty} \overline{\lim}_{\ell\to\infty} \sup_{t\in(-\infty,-K]} x_\ell(t) \leqq -1. \tag{A.17}$$

Given $\delta \in (0,1)$ let $T \equiv T_\ell^\delta := \inf\{t \in (-\ell,\ell) : x_\ell(t) = -1+\delta\}$; by continuity of x_ℓ it follows that $T \in (-\ell,\ell)$. We have

$$\begin{aligned}
C\,\mathrm{e}^{-4\ell} &\geqq \mathcal{F}_\ell(x_\ell) - \frac{4}{3} \geqq \inf_{\substack{x\in\mathcal{X}_\ell\\ x(T)=-1+\delta}} \mathcal{F}_{[-\ell,T]}(x) + \inf_{\substack{x\in\mathcal{X}_\ell\\ x(T)=-1+\delta}} \mathcal{F}_{[T,\ell]}(x) - \frac{4}{3}\\
&= S(\ell+T, 1-\delta) + S(\ell-T,-1+\delta) - \frac{4}{3}\\
&= \int_{1-\delta}^{1} \mathrm{d}u \left[\sqrt{4V(u)+E_+} - \sqrt{4V(u)}\right] - \frac{1}{2}(\ell+T)\,E_+\\
&\quad + \int_{-1+\delta}^{1} \mathrm{d}u \left[\sqrt{4V(u)+E_-} - \sqrt{4V(u)}\right] - \frac{1}{2}(\ell-T)\,E_-,
\end{aligned}$$

where $E_\pm > 0$ is the solution to

$$\ell \pm T = \int_{\pm(1-\delta)}^{1} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_\pm}} \tag{A.18}$$

and we used the symmetry of V, the identity (A.5), and $\frac{4}{3} = \int_{-1}^{1} \mathrm{d}u\,\sqrt{4V(u)}$.

By computations similar to those used in proving (A.7), we get

$$\frac{E_\pm^2}{2}\int_{\pm(1-\delta)}^{1} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_\pm}\,\left(\sqrt{4V(u)}+\sqrt{4V(u)+E_\pm}\right)^2} \leqq C\,\mathrm{e}^{-4\ell},$$

which gives that for each $\delta \in (0,1)$ we have

$$\overline{\lim}_{\ell\to\infty} \mathrm{e}^{4\ell}\,E_\pm < \infty. \tag{A.19}$$

We rewrite (A.18) for E_+ as

$$\begin{aligned}
\ell + T &= \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_+}} - \int_0^{1-\delta} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_+}}\\
&= \frac{1}{2}\,\mathrm{arcsh}\,\frac{2}{\sqrt{E_+}} + R_1(E_+) - \int_0^{1-\delta} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_+}},
\end{aligned}$$

where we recall that R_1 is defined in (A.10). By taking the limit $\ell \to \infty$ in this identity and using the estimate (A.19) for E_+ together with (A.11), we get that, for each $\delta \in (0, 1)$,

$$\varliminf_{\ell\to\infty} T_\ell^\delta \geqq \varliminf_{\ell\to\infty} \left[\frac{1}{2} \operatorname{arcsh} \frac{2}{\sqrt{E_+}} - \ell\right] + \frac{1}{2}\log 2 - \operatorname{arcth}(1-\delta) > -\infty,$$

which yields (A.17). By symmetry

$$\varliminf_{K\to\infty} \varliminf_{\ell\to\infty} \inf_{t\in[K,\infty)} x_\ell(t) \geqq 1. \tag{A.20}$$

The estimate (A.14) follows from (A.15), (A.16), (A.17), and (A.20). □

Recall that S_0 has been defined in (4.5). The regularity of S, whence of S_0, is standard for $t > 0$ and $x \in \mathbb{R}$. In the next theorem, we show that S_0 is actually regular for $t \downarrow 0$ and estimate its second derivative.

Theorem A.3. *We have*

$$\lim_{t\downarrow 0} S_0(t,x) = \lim_{t\downarrow 0} \partial_x S_0(t,x) = \lim_{t\downarrow 0} \partial_{xx} S_0(t,x) = 0,$$

uniformly for x in compacts. Moreover

$$A_3 := \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \frac{1}{1+|x|} \, |\partial_{xx} S_0(t,x)| < +\infty. \tag{A.21}$$

Proof. Set $c(x) := 2 + |x|$. Then $0 \leqq 4V(u) \leqq c(x)^2(1-u)^2$ whenever $x < 1$ and $u \in [x, 1]$ or $x > 1$ and $u \in [1, x]$. Recalling (A.9), from (A.4), we get

$$\frac{1}{c(x)} \operatorname{arcsh} \frac{c(x)\,|1-x|}{\sqrt{e_{t,x}}} \leqq t \leqq \frac{|1-x|}{\sqrt{e_{t,x}}},$$

whence

$$\sqrt{e_{t,x}} = \frac{|1-x|}{t}\,[1+g(t,x)] \quad \text{with} \quad \frac{c(x)\,t}{\operatorname{sh}[c(x)\,t]} - 1 \leqq g(t,x) \leqq 0. \tag{A.22}$$

Then, setting $A(V,e) := \sqrt{4V+e} - \sqrt{e}$ and recalling (4.5), (A.5),

$$\begin{aligned} S_0(t,x) &= |1-x|\,\sqrt{e_{t,x}} + \left|\int_x^1 \mathrm{d}u\, A(V(u), e_{t,x})\right| - \frac{1}{2}\,t\,e_{t,x} - \frac{(1-x)^2}{2t} \\ &= \left|\int_x^1 \mathrm{d}u\, A(V(u), e_{t,x})\right| - \frac{(1-x)^2}{2t} g(t,x)^2. \end{aligned}$$

Analogously, recalling (4.2),

$$\begin{aligned} \partial_x S_0(t,x) &= \operatorname{sgn}(x-1)\sqrt{4V(x)+e_{t,x}} - \frac{x-1}{t} \\ &= \operatorname{sgn}(x-1) A(V(x), e_{t,x}) - \frac{1-x}{t} g(t,x). \end{aligned}$$

Since $A(V,e) \leqq \frac{2V}{\sqrt{e}}$, we now have

$$|S_0(t,x)| \leqq \frac{t}{1+g(t,x)}\,\frac{2}{1-x}\int_x^1 \mathrm{d}u\, V(u) \;+\; \frac{(1-x)^2}{2t}g(t,x)^2$$

and

$$|\partial_x S_0(t,x)| \leqq \frac{t}{1+g(t,x)}\,\frac{2V(x)}{|1-x|} \;+\; \frac{|1-x|}{t}\,|g(t,x)|.$$

From the bound (A.22) on $g(t,x)$ we then conclude that both $S_0(t,x)$ and $\partial_x S_0(t,x)$ vanish as $t \downarrow 0$ (uniformly for x in compact sets).

Let us now consider the second derivative of $S_0(t,x)$. By differentiating the identity (A.4), we have

$$\partial_x e_{t,x} = -\frac{2}{\sqrt{4V(x)+e_{t,x}}}\left[\int_x^1 \frac{\mathrm{d}u}{[4V(u)+e_{t,x}]^{\frac{3}{2}}}\right]^{-1}. \tag{A.23}$$

Plugging (A.23) in the explicit expression of $\partial_{xx} S_0(t,x)$, we obtain

$$\begin{aligned}\partial_{xx} S_0(t,x) &= \operatorname{sgn}(x-1)\frac{2V'(x)}{\sqrt{4V(x)+e_{t,x}}}\\ &\quad + \frac{1}{4V(x)+e_{t,x}}\left|\int_x^1 \frac{\mathrm{d}u}{[4V(u)+e_{t,x}]^{\frac{3}{2}}}\right|^{-1} - \frac{1}{t}.\end{aligned} \tag{A.24}$$

We now write

$$\left|\int_x^1 \frac{\mathrm{d}u}{[4V(u)+e_{t,x}]^{\frac{3}{2}}}\right| = \frac{|1-x|}{e_{t,x}^{\frac{3}{2}}}\;(1-D(t,x))$$

with

$$\begin{aligned} 0 \;\leqq\; D(t,x) &:= 1 - \frac{e_{t,x}^{\frac{3}{2}}}{1-x}\int_x^1 \mathrm{d}u\,\frac{1}{[4V(u)+e_{t,x}]^{\frac{3}{2}}}\\ &\leqq 1 - \sqrt{\frac{(1+g(t,x))^2}{(1+g(t,x))^2+t^2c(x)^2}},\end{aligned} \tag{A.25}$$

where we used $V(u) \leqq c(x)^2(1-u)^2$, (A.22), and the identity

$$\left|\int_x^1 \frac{\mathrm{d}u}{[\gamma^2(1-u)^2+\beta]^{\frac{3}{2}}}\right| = \frac{1}{\beta}\,\frac{|x-1|}{\sqrt{\gamma^2(1-x)^2+\beta}}, \quad \beta,\gamma > 0. \tag{A.26}$$

Then (A.24) reads

$$\begin{aligned}\partial_{xx} S_0(t,x) \;&=\; \operatorname{sgn}(x-1)\frac{2V'(x)\,t}{\sqrt{4V(x)t^2+(1-x)^2[1+g(t,x)]^2}}\\ &+\frac{1}{t}\left\{\frac{(1-x)^2}{4V(x)t^2+(1-x)^2[1+g(t,x)]^2}\,\frac{[1+g(t,x)]^3}{1-D(t,x)} - 1\right\}.\end{aligned} \tag{A.27}$$

From the above expression and the bounds (A.22) and (A.25) we have $\partial_{xx} S_0(t,x) \to 0$ as $t \downarrow 0$ (uniformly for x in compacts).

To prove the bound (A.21) we notice that the first term on the right-hand side of (A.27) is bounded by $2|x|$. Simple algebraic manipulations yield that the second term can be rewritten as

$$\frac{1}{t}\frac{[D(t,x)+g(t,x)]\,[1+g(t,x)]^2}{(1-D(t,x))\,[(1+x)^2t^2+[1+g(t,x)]^2]} - \frac{(1+x)^2 t}{(1+x)^2t^2+[1+g(t,x)]^2}.$$

We analyze separately the two terms above. For the second one, by using the bound (A.22) it is easy to show that

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \frac{1}{c(x)}\frac{(1+x)^2 t}{(1+x)^2t^2+[1+g(t,x)]^2} < +\infty.$$

For the first one we first notice that, by (A.22), (A.25), and simple computations, we have

$$0 \leqq \frac{g(t,x)+D(t,x)}{c(x)t} \leqq \frac{1}{c(x)t}\left\{\frac{c(x)t}{\operatorname{sh}[c(x)t]} - 1 + \operatorname{th}^2[c(x)t]\right\},$$

which is bounded for $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. To conclude it remains to show that

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \frac{[1+g(t,x)]^2}{(1-D(t,x))\,[(1+x)^2t^2+[1+g(t,x)]^2]} < +\infty,$$

which can be easily checked using again (A.25) and (A.22). □

Lemma A.4. *Let $e_{t,x}$ be the solution to (A.4) and set $E_{t,x} := e_{\ell-t,x}$, $(t,x)\in[-\ell,\ell)\times\mathbb{R}$. Then*

$$-\ell \leqq t \leqq \ell, \quad -\infty < x \leqq y \leqq 1 \implies E_{t,x} \geqq E_{t,y}. \tag{A.28}$$

Moreover,

$$(t,x)\in[-\ell,\ell]\times(-\infty,0] \implies \sqrt{E_{t,x}} \geqq \frac{4\,\mathrm{e}^{-(\ell-t)}}{1+\mathrm{e}^{\ell-t}\,[1+x]_+}. \tag{A.29}$$

Finally,

$$\frac{4V(x)}{\operatorname{sh}^2[(1+x)(\ell-t)]} \leqq E_{t,x} \leqq \frac{4(x-1)^2}{\operatorname{sh}^2[2(\ell-t)]}, \quad (t,x)\in(-\ell,\ell)\times[1,\infty), \tag{A.30}$$

$$\frac{4(x-1)^2}{\operatorname{sh}^2[2(\ell-t)]} \leqq E_{t,x} \leqq \frac{4V(x)}{\operatorname{sh}^2[(1+x)(\ell-t)]}, \quad (t,x)\in(-\ell,\ell)\times(-1,1). \tag{A.31}$$

Proof. The inequality (A.28) follows directly from the definition of $E_{t,x}$. By (A.28), to prove (A.29) it is enough to consider $x \in [-1, 0]$. In this case, from (A.4) we get

$$\begin{aligned} \ell - t &= 2\int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + E_{t,x}}} - \int_{-x}^1 \frac{\mathrm{d}u}{\sqrt{4V(u) + E_{t,x}}} \\ &\geqq \operatorname{arcsh} \frac{2}{\sqrt{E_{t,x}}} - \frac{1}{2} \operatorname{arcsh} \frac{2\,(1+x)}{\sqrt{E_{t,x}}} \\ &\geqq \frac{1}{2} \log \frac{16}{E_{t,x} + 4\,(1+x)\sqrt{E_{t,x}}}, \end{aligned}$$

where we used $4V(u) \leqq 4(1-u)^2$ for $u \in [0, 1]$ and (A.9) in the second inequality, and that $\log(2y) \leqq \operatorname{arcsh} y \leqq \log(1+2y)$ for $y \geqq 0$ in the last inequality. We thus get

$$\sqrt{E_{t,x}} \;\geqq\; \frac{8\mathrm{e}^{-2(\ell-t)}}{1 + x + \sqrt{(1+x)^2 + 4\mathrm{e}^{-2(\ell-t)}}},$$

from which the estimate (A.29) follows. Finally, to get the estimates (A.30) and (A.31), it is enough to insert the bounds

$$\begin{cases} 4(1-u)^2 \;\leqq\; 4V(u) \;\leqq\; (1+x)^2(1-u)^2 & \text{if } 1 \leqq u \leqq x, \\ (1+x)^2(1-u)^2 \;\leqq\; 4V(u) \;\leqq\; 4(1-u)^2 & \text{if } x \leqq u \leqq 1 \end{cases} \tag{A.32}$$

in (4.3) and use (A.9). $\square$

Proposition A.5. *Let*

$$\mathcal{G}_{\varepsilon,L,a} := \Big\{x \,:\, |\mathcal{Z}(x)| \leqq L,\;\; d(x, \mathcal{M}) \leqq \varepsilon^{\frac{1}{2}-\eta},\;\; x(t) \geqq \bar{m}_a(t)\;\; \forall\, t \in [-\ell + a, a]\Big\}.$$

Then, for all η small enough there exists a real $A_5 > 0$ such that for any $L, a > 0$, and $x \in \mathcal{G}_{\varepsilon,L,a}$ we have

$$\varlimsup_{\varepsilon \to 0} \left| \int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell - t, x(t)) + \log \ell \right| \leqq A_5\,(L + a). \tag{A.33}$$

Proof. In the sequel we shall assume that ε is so small that $\varepsilon^{\frac{1}{2}-\eta} \leqq \frac{1}{2}$. We shall denote by C a generic positive constant independent on ε, L, a whose numerical value may change from line to line. Fix $x \in \mathcal{G}_{\varepsilon,L,a}$ and let z, $|z| \leqq L$, be such that $|x(t) - \bar{m}_z(t)| \leqq \varepsilon^{\frac{1}{2}-\eta}$ for any $t \in [-\ell, \ell]$. Setting $z^* = z + \operatorname{th}(1/2)$, by the assumptions on ε we have

$$\begin{cases} \bar{m}_a(t) \leqq x(t) \leqq \frac{1}{2} & \text{if } t \in [-\ell + a, a \wedge z], \\ -\frac{1}{2} \leqq x(t) \leqq \frac{3}{2} & \text{if } t \in [a \wedge z, \ell], \\ 0 \leqq x(t) \leqq \frac{3}{2} & \text{if } t \in [z^*, \ell]. \end{cases} \tag{A.34}$$

By (A.24), noticing $E_{t,x} = e_{\ell-t,x}$ and $\int_{-\ell}^{\ell-1} \mathrm{d}t\, \frac{1}{\ell-t} = \log(2\ell)$, we decompose

$$\int_{-\ell}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell-t, x(t)) + \log \ell = \sum_{i=1}^{5} I_i - \log 2, \tag{A.35}$$

where

$$\begin{aligned}
I_1 &= \int_{-\ell}^{-\ell+2a} \mathrm{d}t \left[\partial_{xx} S_0(\ell-t, x(t)) + \frac{1}{\ell-t}\right] + \int_{\ell-1}^{\ell} \mathrm{d}t\, \partial_{xx} S_0(\ell-t, x(t)),\\
I_2 &= \int_{-\ell+2a}^{\ell-1} \mathrm{d}t\ \operatorname{sgn}(x(t)-1) \frac{2V'(x(t))}{\sqrt{4V(x(t)) + E_{t,x(t)}}},\\
I_3 &= \int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, G(\ell-t, x(t)),\\
I_4 &= \int_{a\wedge z}^{z^*} \mathrm{d}t\, G(\ell-t, x(t)),\\
I_5 &= \int_{z^*}^{\ell-1} \mathrm{d}t\, G(\ell-t, x(t)),
\end{aligned}$$

with

$$G(t,x) := \frac{1}{4V(x) + e_{t,x}} \left| \int_x^1 \frac{\mathrm{d}u}{[4V(u) + e_{t,x}]^{\frac{3}{2}}} \right|^{-1}. \tag{A.36}$$

Since $|x(t)| \leqq \frac{3}{2}$, by Theorem A.3 we get $|I_1| + |I_4| \leqq C\, a$. We next estimate the other integrals separately.
Bound on I_2. Since

$$\operatorname{sgn}(x-1) \frac{2V'(x)}{\sqrt{4V(x)}} = 2x, \quad \forall\, x \geqq -1$$

and recalling that $x(t) > -1$ for any $t \in [-\ell+2a, \ell]$ we have

$$\begin{aligned}
|I_2| &\leqq 2 \int_{-\ell+2a}^{\ell-1} \mathrm{d}t\, |V'(x(t))| \left[\frac{1}{\sqrt{4V(x(t))}} - \frac{1}{\sqrt{4V(x(t)) + E_{t,x(t)}}}\right]\\
&\qquad + 2 \int_{-\ell+2a}^{\ell-1} \mathrm{d}t\ |x(t) - \bar{m}_z(t)| \,+\, 2 \left| \int_{-\ell+2a}^{\ell} \mathrm{d}t\, \bar{m}_z(t) \right|\\
&\leqq \int_{-\ell+2a}^{\ell-1} \mathrm{d}t\, \frac{|V'(x(t))|}{[4V(x(t))]^{\frac{3}{2}}}\, E_{t,x(t)} \,+\, 4\,\varepsilon^{\frac{1}{2}-\eta}\,(\ell-a) + 4\,|a-z|\\
&\leqq \frac{3}{2} \int_{-\ell+2a}^{\ell-1} \mathrm{d}t\, \frac{E_{t,x(t)}}{4V(x(t))} \,+\, C\,(L+a),
\end{aligned}$$

where we used $|x(t)| \leqq \frac{3}{2}$ in the last inequality. Now, by (A.28), (A.30), and (A.31) we have

$$\begin{aligned}\int_{-\ell+2a}^{\ell-1} \mathrm{d}t\, \frac{E_{t,x(t)}}{4V(x(t))} &\leqq \int_{-\ell+2a}^{\ell-1} \frac{\mathrm{d}t}{\mathrm{sh}^2[2(\ell-t)]}\, \mathbb{I}_{x(t)\geqq 1} \\ &+ \int_{a\wedge z}^{\ell-1} \frac{\mathrm{d}t}{\mathrm{sh}^2[(1+x(t))(\ell-t)]}\, \mathbb{I}_{x(t)<1} + \int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, \frac{E_{t,\bar{m}_a(t)}}{4V(x(t))}\, \mathbb{I}_{x(t)<1} \\ &\leqq \int_{-\ell+2a}^{\ell-1} \frac{\mathrm{d}t}{\mathrm{sh}^2[2(\ell-t)]} + \int_{a\wedge z}^{\ell-1} \frac{\mathrm{d}t}{\mathrm{sh}^2\left[\frac{1}{2}(\ell-t)\right]} + 4\int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, \frac{E_{t,\bar{m}_a(t)}}{[1+\bar{m}_a(t)]^2},\end{aligned}$$

where we used (A.34). The first two integrals on the right-hand side above are readily seen to be uniformly bounded in ℓ. For the last one we need an upper bound for $E_{t,\bar{m}_a(t)}$. To this end we observe that, for $t \leqq a$,

$$\begin{aligned}\ell - t &= \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_{t,\bar{m}_a(t)}}} + \int_0^{|\bar{m}_a(t)|} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_{t,\bar{m}_a(t)}}} \\ &\leqq \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_{t,\bar{m}_a(t)}}} + \operatorname{arcth}|\bar{m}_a(t)| \\ &= \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_{t,\bar{m}_a(t)}}} + a - t,\end{aligned}$$

from which, by (A.6), we get $E_{t,\bar{m}_a(t)} \leqq C\mathrm{e}^{-4(\ell-a)}$ for any $t \in [-\ell, a]$. Then,

$$\int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, \frac{E_{t,\bar{m}_a(t)}}{[1+\bar{m}_a(t)]^2} \leqq C\int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, \mathrm{e}^{-4(\ell-a)}\mathrm{e}^{-4(t-a)} \leqq C, \qquad \text{(A.37)}$$

so that $|I_2| \leqq C(L+a)$.

Bounds on I_3 and I_5. From (A.32) and (A.26), we have

$$G(\ell-t,x) \leqq \frac{E_{t,x}}{x-1}\, \frac{1}{\sqrt{4V(x)+E_{t,x}}}, \qquad \text{if } x \in (1,+\infty), \qquad \text{(A.38)}$$

$$G(\ell-t,x) \leqq \frac{E_{t,x}}{1-x}\, \frac{\sqrt{4(1-x)^2+E_{t,x}}}{4V(x)+E_{t,x}} \qquad \text{if } x \in (-1,1). \qquad \text{(A.39)}$$

By (A.28), (A.34), and (A.39),

$$I_3 \leqq C\int_{-\ell+2a}^{a\wedge z} \mathrm{d}t\, \frac{E_{t,\bar{m}_a(t)}}{[1+\bar{m}_a(t)]^2},$$

and the integral on the right-hand side has been bounded in (A.37). For I_5, we observe that, by (A.30), (A.31), (A.38), and (A.39),

$$G(\ell-t,x) \leqq \frac{4}{\mathrm{sh}^2[2(\ell-t)]}\, \frac{\mathrm{th}[(1+x)(\ell-t)]}{(1+x)} \qquad \text{if } x \in (1,+\infty),$$

$$G(\ell-t,x) \leqq \frac{1}{\mathrm{sh}^2[(1+x)(\ell-t)]}\sqrt{4+\frac{(1+x)^2}{\mathrm{sh}^2[(1+x)(\ell-t)]}} \qquad \text{if } x \in (-1,1).$$

Then, recalling also (A.34),

$$I_5 \leqq \int_{z^*}^{\ell-1} \mathrm{d}t \, \frac{4(\ell-t)}{\mathrm{sh}^2[2(\ell-t)]} \, \mathbb{I}_{x(t)\geqq 1} + \int_{z^*}^{\ell-1} \mathrm{d}t \, \frac{1}{\mathrm{sh}^2[(\ell-t)]} \sqrt{4+\frac{1}{(\ell-t)^2}} \, \mathbb{I}_{x(t)<1},$$

which is uniformly bounded. □

Proof of (4.20). We assume ε so small that

$$\varepsilon^{\frac{1}{2}-\eta} < \frac{1}{2}\mathrm{e}^{-2\sigma\ell}, \quad \ell > \frac{5}{2}, \quad \log\ell < \sigma\ell. \tag{A.40}$$

In particular, since $\bar{m}_0(s) > 1-2\mathrm{e}^{-2s}$ for any $s \geqq 0$, setting $T = \frac{1}{2}\log\ell$, for $t \in [\tau_-, \tau_+]$ we have

$$\xi_t = \bar{m}_z(t) + v_t \begin{cases} < 0 & \text{if } t-z < -T, \\ > 0 & \text{if } t-z > T. \end{cases}$$

We shorthand $E_t = E_{t,\bar{m}_z(t)+v_t}$ and analyze separately three cases:
(i) Assume $t \in [z-T, z+T] \cap [\tau_-, \tau_+]$. By (4.3) with $x = \xi_t = \bar{m}_z(t) + v_t$, we have:

$$\ell \leqq z + T + \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}} + \int_0^{|\xi_t|} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}}.$$

By (A.40) we have $|\xi_t| \leqq \bar{m}_0(T) + \varepsilon^{\frac{1}{2}-\eta} \leqq 1 - \frac{1}{2}\mathrm{e}^{-2T}$, so that

$$\int_0^{|\xi_t|} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}} \leqq \operatorname{arcth}\left[1-\frac{1}{2}\mathrm{e}^{-2T}\right] \leqq T + \frac{1}{2}\log 4,$$

whence

$$\ell - z - \frac{1}{2}\log 4 - \log\ell \leqq \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}}. \tag{A.41}$$

(ii) Let $t \in [z-\sigma\ell, z-T] \cap [\tau_-, \tau_+]$. Since $\xi_t < 0$, by (4.3), we have:

$$\ell \leqq \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}} + t + \int_0^{|\xi_t|} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}}.$$

By (A.40) $|\xi_t| \leqq \bar{m}_0(|t-z|) + \varepsilon^{\frac{1}{2}-\eta} \leqq 1 - \frac{1}{2}\mathrm{e}^{-|t-z|}$, so that

$$t + \int_0^{|\xi_t|} \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}} \leqq t + \operatorname{arcth}\left[1-\frac{1}{2}\mathrm{e}^{-2|t-z|}\right] \leqq z + \frac{1}{2}\log 4,$$

whence

$$\ell - z - \frac{1}{2}\log 4 \leqq \int_0^1 \frac{\mathrm{d}u}{\sqrt{4V(u)+E_t}}. \tag{A.42}$$

(iii) Finally let $t \in [z + T, \sigma\ell + z] \cap [\tau_-, \tau_+]$. Since $\xi_t \geqq \bar{m}_z(t) - \varepsilon^{\frac{1}{2}-\eta} \geqq 1 - 3e^{-2(t-z)}$, by (4.3), we have

$$\begin{aligned}\ell - t &\leqq \int_{1-3e^{-2(t-z)}}^{1} \frac{du}{\sqrt{4V(u) + E_t}} \\ &\leqq \frac{1}{2 - 3e^{-2(t-z)}} \operatorname{arcsh} \frac{3e^{-2(t-z)}\left(2 - 3e^{-2(t-z)}\right)}{\sqrt{E_t}}.\end{aligned}$$

Recalling $t - z > \frac{1}{2}\log \ell$, for a suitable constant $C > 0$ and any ε small enough, we get

$$\begin{aligned}E_t &\leqq C \exp\{-4(t-z) - 4(\ell - t) + 6(\ell - t)e^{-2(t-z)}\} \\ &\leqq C \exp\{-4(\ell - z) + 6\}.\end{aligned} \tag{A.43}$$

By comparing (A.41) and (A.42) with (A.6), we conclude that

$$\overline{\lim_{\varepsilon\to 0}} \sup_{t\in[z-T,z+T]\cap[\tau_-,\tau_+]} e^{4(\ell-z)}\ell^{-4}E_t < \infty. \tag{A.44}$$

The claim (4.20) now follows from (A.43) and (A.44). □

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