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Soft and Hard Wall in a Stochastic Reaction Diffusion Equation

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Abstract

We consider a stochastically perturbed reaction diffusion equation in a bounded interval, with boundary conditions imposing the two stable phases at the endpoints. We investigate the asymptotic behavior of the front separating the two stable phases, as the intensity of the noise vanishes and the size of the interval diverges. In particular, we prove that, in a suitable scaling limit, the front evolves according to a one-dimensional diffusion process with a non-linear drift accounting for a "soft" repulsion from the boundary. We finally show how a "hard" repulsion can be obtained by an extra diffusive scaling.

1. Introduction

Let $V(m)$ be a smooth, symmetric, double well potential whose minimum is attained at $m = m_\pm$, $V''(m_\pm) > 0$. After the pioneering paper [1], the semi-linear parabolic equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\Delta m - V'(m) \tag{1.1}$$

and its stochastic perturbations have became a basic model in the kinetics of phase separation and interface dynamics for systems with a non-conserved order parameter.

Before introducing our results, let us review the main features of (1.1) in the one-dimensional case. The corresponding evolution is the L_2 gradient flow of the functional

$$\mathcal{F}(m) = \int \mathrm{d}x \left[\frac{1}{2}m'(x)^2 + 2V(m(x))\right], \tag{1.2}$$

that is (1.1) can be rewritten as $\frac{\partial m}{\partial t} = -\frac{1}{2}\frac{\delta \mathcal{F}}{\delta m}$. In the case that (1.1) is considered in the whole line $\mathbb{R}$, there are infinitely many stationary solutions, which are the critical points of $\mathcal{F}$. The most relevant ones are the constant profiles $m_\pm$, where $\mathcal{F}$ attains its minimum, and $\pm\bar{m}$, where $\bar{m}$ is the unique increasing solution to

$$\frac{1}{2}\bar{m}'' - V'(\bar{m}) = 0, \qquad \lim_{x\to\pm\infty} \bar{m}(x) = m_\pm, \quad \bar{m}(0) = 0, \tag{1.3}$$

together with its translates $\pm\bar{m}_\zeta(x) = \pm\bar{m}(x-\zeta)$, $\zeta \in \mathbb{R}$. The profile $\bar{m}_\zeta$ is a standing wave of (1.1) that connects the two pure phases $m_\pm$. Note that $\bar{m}_\zeta$ minimizes $\mathcal{F}$ under the constraint that $\lim_{x\to\pm\infty} m(x) = m_\pm$. Therefore $\bar{m}_\zeta$ is the equilibrium profile which has the two pure phases $m_\pm$ coexisting to the right and to the left of ζ. It represents a mesoscopic interface located at ζ. We use the word "mesoscopic" because the interface is diffuse and the transition from one phase to the other, even though exponentially fast, is not sharp. In [11] it is proven that the one parameter invariant manifold $\mathcal{M} = \{\bar{m}_\zeta : \zeta \in \mathbb{R}\}$ is asymptotically stable for the evolution (1.1).

Referring to [15] for a review on stochastic interface models, we outline some results on the stochastic perturbation of (1.1). When a random forcing term of intensity $\sqrt{\varepsilon}$ is added to (1.1) and the initial datum is $\bar{m}_0$, in [7,8,14], it is shown that the solution at times $\varepsilon^{-1}t$ stays close to $\bar{m}_{\zeta_\varepsilon(t)}$ for some random $\zeta_\varepsilon(t)$, which converges to a Brownian motion as $\varepsilon \to 0$. To explain heuristically this result, let us regard the random forcing term as a source of independent small kicks, which we decompose along the directions parallel and orthogonal to $\mathcal{M}$. The orthogonal component is exponentially damped by the deterministic drift, while the parallel component, associated to the zero eigenvalue of the linearization of (1.1) around $\bar{m}_\zeta$, is not contrasted and, by independence, sums up to a Brownian motion.

We next discuss the behavior of (1.1) on the bounded interval $[-a, b]$. The case of Neumann boundary conditions is considered in [9,16], where it is shown that there exists a stationary solution $m^*_{a,b}$, close to $\bar{m}_{(b-a)/2}$ as a, b diverge. The profiles $\pm m^*_{a,b}$ are saddle points of $\mathcal{F}$, each one having a one-dimensional unstable manifold connecting it to the stable points $m_\pm$. For a, b large, solutions to (1.1) are first attracted by these manifolds, and they then move along them towards one of the stable phases, with a velocity exponentially small in the distance from the endpoints. From the analysis in [9,16], we have that there exists a constant $c_0 > 0$ (depending on the potential V) such that, if we take $a = c_0 \log \varepsilon^{-1}$, $b = \varepsilon^{-\beta}$ for some $\beta > 0$, and the initial condition is close to $\bar{m}_0$, the following holds. As $\varepsilon \to 0$, the solution of (1.1) at times $\varepsilon^{-1}t$, for t small enough, is close to $\bar{m}_{\zeta(t)}$, where $\zeta(t)$ solves the equation $\dot{\zeta} = -A\,\varepsilon^{-1}\mathrm{e}^{-(\zeta+a)/c_0} = -A\,\mathrm{e}^{-\zeta/c_0}$ for some $A > 0$. When a random forcing term of order $\sqrt{\varepsilon}$ is added to (1.1), by the analysis in [7], it follows that, by taking $a = c\log\varepsilon^{-1}$ with $c \gg c_0$, and looking at the time scale ε^{-1}, the random fluctuations are dominant so that the limiting motion of the interface is still described by a Brownian motion. On the other hand, for $c < c_0$ the deterministic drift should become dominant, the minority phase shrinking deterministically up to extinction. In the critical case $c = c_0$, at the initial state of the evolution, we should see the effect both of the drift and of the stochastic fluctuations.

In this paper, we consider a stochastic perturbation of (1.1) in a bounded interval with inhomogeneous Dirichlet boundary conditions imposing the two stable phases $m_\pm$, and analyze the competition between the stochastic fluctuations and the given boundary conditions on the motion of the interface. Let us first consider the deterministic case, that is, (1.1) in the interval $[-a, b]$ with boundary conditions $m(t, -a) = m_-$, $m(t, b) = m_+$. The meaning of these conditions is to force the m_- phase, respectively, the m_+ phase, to the left of $-a$, respectively, to the right of b. If we think of m as the local magnetization, this choice models the effect of opposite magnetic fields applied at the endpoints. To our knowledge, an analysis along the same lines of [9,16] has not been carried out in detail. However, in this case, it is straightforward to check that there exists a unique, globally attractive, stationary solution $m^*_{a,b}$, close to $\bar{m}_{(b-a)/2}$ as a, b diverge. Moreover, as it follows from the analysis of the present paper, there is a slow motion as in the case of Neumann boundary conditions. More precisely, there exists an approximately invariant manifold $\mathcal{M}_{a,b}$, close to $\mathcal{M}$ as a, b diverge. In this limit, the motion near $\mathcal{M}_{a,b}$ can be described in terms of coordinates along and transversal to $\mathcal{M}_{a,b}$. The transversal component of the flow is exponentially damped uniformly in a, b, while the motion along $\mathcal{M}_{a,b}$, parametrized by the interface location $\zeta(t)$, evolves according to $\dot{\zeta} = A\left[e^{-(\zeta+a)/c_0} - e^{-(b-\zeta)/c_0}\right]$ for A and c_0 positive constants. We emphasize that, since the boundary conditions force the presence of an interface, the drift pushes the solution toward $m^*_{a,b}$, where the two pure phases coexist.

We consider a stochastic perturbation of (1.1), given by a space–time white noise of intensity $\sqrt{\varepsilon}$. To get a non-trivial scaling limit and to see the competition between the random fluctuations and the repulsion from one endpoint $(-a)$, we choose $a = c_0 \log \varepsilon^{-1}$, $b = \varepsilon^{-\beta}$, for some $\beta > 0$, the initial condition close to $\bar{m}_0$, and look at the evolution at times $\varepsilon^{-1}t$. We prove that, as $\varepsilon \to 0$, the solution stays close to $\bar{m}_{\zeta(t)}$, where $\zeta(t)$ solves the stochastic equation $\dot{\zeta} = A\, e^{-\zeta/c_0} + \eta$, here η is a white noise. We interpret this result as a "soft wall", since the repulsion is not sharp. Actually, the solution remains close to $\mathcal{M}_{a,b}$ also on a slightly longer time scale and performing a further diffusive rescaling of the interface location, we also prove that the soft wall converges to a "hard" one: the interface dynamics behaves as a reflected Brownian motion.

2. Notation and results

Let $a, b \in \mathbb{R}_+$, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a standard filtered probability space, and $W = \{W(t), t \in \mathbb{R}_+\}$ be the cylindrical Wiener process on $L_2([-a, b], dx)$. This means that W is the $\mathcal{F}_t$-adapted mean zero Gaussian process such that, for each $\varphi, \varphi' \in C^\infty([-a, b])$ and $t, t' \in \mathbb{R}_+$,

$$\mathbb{E}\left(\langle W(t), \varphi\rangle\langle W(t'), \varphi'\rangle\right) = t \wedge t' \,\langle \varphi, \varphi'\rangle, \tag{2.1}$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$, $t \wedge t' := \min\{t, t'\}$, and $\langle \cdot, \cdot \rangle$ is the inner product in $L_2([-a, b], dx)$.

In this paper, we consider the prototypical case of the symmetric double well potential, that is we choose

$$V(m) = \frac{1}{4}\left(m^2 - 1\right)^2, \tag{2.2}$$

which attains its minimum at $m = \pm 1$. Given $\varepsilon > 0$, we consider a stochastic perturbation of the one-dimensional reaction diffusion equation (1.1) with inhomogeneous Dirichlet boundary conditions at the endpoints. More precisely, we let $m(t) \equiv m(t,x)$, $(t,x) \in \mathbb{R}_+ \times [-a,b]$ be the solution to

$$\begin{cases} \mathrm{d}m(t) = \left[\frac{1}{2}\Delta m(t) - V'(m(t))\right] \mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W(t), \\ m(t,-a) = -1, \\ m(t,b) = 1, \\ m(0,x) = m_0(x). \end{cases} \tag{2.3}$$

To give a precise meaning to the above equation for $m_0 \in C([-a,b])$ such that $m_0(-a) = -1$ and $m_0(b) = 1$, let $\nu(x) = \frac{2x}{a+b} + \frac{a-b}{a+b}$ be the solution of $\nu''(x) = 0$, $x \in (-a,b)$ with the above boundary conditions and denote by p_t^0 the heat semigroup on $(-a,b)$ with zero boundary conditions at the endpoints. Then a mild solution to (2.3) is defined as the solution to the integral equation

$$m(t) = \nu + p_t^0(m_0 - \nu) - \int_0^t \mathrm{d}s\, p_{t-s}^0 V'(m(s)) + \sqrt{\varepsilon}\int_0^t p_{t-s}^0\,\mathrm{d}W(s). \tag{2.4}$$

By, for example [12], there exists a unique $\mathcal{F}_t$-adapted process $m \in C(\mathbb{R}_+; C([-a,b])$ which solves (2.4).

As explained in the Introduction, let $\bar{m}_\zeta(x)$ be the standing wave with "center" $\zeta \in \mathbb{R}$, that is the solution to (1.3). For the specific choice (2.2) of the potential, we have $\bar{m}_\zeta(x) = \operatorname{th}(x-\zeta)$. Note that, if $a = b = \infty$ and $\varepsilon = 0$, then $\mathcal{M} = \{\bar{m}_\zeta, \zeta \in \mathbb{R}\}$ is a one parameter family of stationary solutions of (2.3). Given $p \in [1,\infty]$ we denote by $\|\cdot\|_p$ the norm in $L_p([-a,b],\mathrm{d}x)$. We consider $C(\mathbb{R}_+)$ equipped with the (metrizable) topology of uniform convergence in compacts. Our main results are stated as follows.

Theorem 2.1. *Given $\beta > 0$, set*

$$a := \frac{1}{4}\log\varepsilon^{-1}, \quad b := \varepsilon^{-\beta}, \quad \lambda := \log\varepsilon^{-1}, \tag{2.5}$$

and let $m^{(\varepsilon)}(t)$ be the solution to (2.4) with initial datum $m_0^{(\varepsilon)} \in C([-a,b])$, $m_0^{(\varepsilon)}(-a) = -1$, $m_0^{(\varepsilon)}(b) = 1$, such that for each $\eta > 0$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}+\eta}\left\|m_0^{(\varepsilon)} - \bar{m}_0\right\|_\infty = 0. \tag{2.6}$$

Then:

(i) *there exists an $\mathcal{F}_t$-adapted real process X_ε such that, for each $\theta, \eta > 0$,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,\lambda\varepsilon^{-1}\theta]} \left\|m^{(\varepsilon)}(t) - \bar{m}_{X_\varepsilon(t)}\right\|_\infty > \varepsilon^{\frac{1}{2}-\eta}\right) = 0; \tag{2.7}$$

(ii) *the real process* $Y_\varepsilon(\tau) := X_\varepsilon(\varepsilon^{-1}\tau)$, $\tau \in \mathbb{R}_+$, *converges weakly in* $C(\mathbb{R}_+)$ *to the unique strong solution* Y *to the stochastic equation*

$$\begin{cases} \mathrm{d}Y(\tau) = 12\exp\{-4Y(\tau)\}\,\mathrm{d}\tau + \mathrm{d}B(\tau), \\ Y(0) = 0, \end{cases} \tag{2.8}$$

where B *is a Brownian motion with diffusion coefficient* $\frac{3}{4}$;

(iii) *the real process* $Z_\varepsilon(\theta) := \lambda^{-1/2} X_\varepsilon(\lambda\varepsilon^{-1}\theta)$, $\theta \in \mathbb{R}_+$, *converges weakly in* $C(\mathbb{R}_+)$ *to a Brownian motion with diffusion coefficient* $\frac{3}{4}$ *reflected at zero.*

Item (i) states that, up to times $\varepsilon^{-1}\log\varepsilon^{-1}$, the solution of (2.3) with initial condition close to the one-dimensional manifold $\{\bar{m}_\zeta;\, \zeta \in (-a, b)\}$ remains close to that manifold. Items (ii) and (iii) then identify the limiting evolution of the interface $X_\varepsilon(t)$. On the time scales ε^{-1} the interface is at distance $\frac{1}{4}\log\varepsilon^{-1} + Y_\varepsilon$ from the endpoint $-a$; moreover Y_ε behaves as a Brownian motion with a strong drift toward the right for $Y_\varepsilon < 0$ and essentially no drift for $Y_\varepsilon > 0$. We interpret this as a "soft wall". On the longer time scale $\varepsilon^{-1}\log\varepsilon^{-1}$ the interface is at a distance $\frac{1}{4}\log\varepsilon^{-1} + \sqrt{\log\varepsilon^{-1}}\,Z_\varepsilon$ from the endpoint $-a$; on this time scale the repulsion is sharp: Z_ε behaves as a Brownian motion reflected at zero. We interpret this as a "hard wall". We finally remark that the choice of λ in (2.5) has been made for the sake of concreteness: it would have been enough to take λ such that $\lambda \to \infty$ and $\sqrt{\lambda}/\log\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$.

We emphasize that this non-trivial behavior is due to the choice $a = \frac{1}{4}\log\varepsilon^{-1}$ for which there is a competition between the stochastic fluctuations and the drift due to the Dirichlet boundary condition at the endpoint $-a$. Here the coefficient $\frac{1}{4}$, as well as the diffusion coefficient $\frac{3}{4}$ of the Brownian motion, depend on the special choice of the double well potential V in (2.2). Since $b = \varepsilon^{-\beta} \gg a$ the right endpoint b has no effect on the limiting motion of the interface, apart from (minor) technical details, the case $b = +\infty$ behaves as the one here considered. It follows from our analysis that if we had chosen $a = (\frac{1}{4} + \delta)\log\varepsilon^{-1}$ for some $\delta > 0$, in the limiting motion of the interface we would have seen only the effect of the stochastic force, namely Y_ε would behave as a Brownian motion.

In [2] we analyze the invariant measure μ_ε of (2.3) with $a = b = \frac{1}{4}\log\varepsilon^{-1}$ and show it has a non-trivial limit as $\varepsilon \to 0$. In fact in [2] the main effort is in proving the compactness of μ_ε, relying on the following dynamical scaling limit to identify its limit points. Fix $\tau_0 > 0$ and let $\mathbb{Q}^\varepsilon_{m_0}$ be the law of $m(\varepsilon^{-1}\tau)$, $\tau \in [0, \tau_0]$, with $m(t)$ the solution to (2.3) with $a = b = \frac{1}{4}\log\varepsilon^{-1}$. By setting $m(t, x) = \mathrm{sgn}(x)$ for $|x| \geqq \frac{1}{4}\log\varepsilon^{-1}$, we regard $\mathbb{Q}^\varepsilon_{m_0}$ as a probability measure on $C([0, \tau_0]; \mathcal{X})$, where $\mathcal{X} := \{m \in C(\mathbb{R}) : \lim_{x\to\pm\infty} m(x) = \pm 1\}$ endowed with the topology of uniform convergence. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_\tau, P)$ be a standard filtered probability space equipped with a Brownian motion B with diffusion coefficient $\frac{3}{4}$. Given $z \in \mathbb{R}$, let Ξ^z be the unique strong solution to the stochastic differential equation

$$\begin{cases} \mathrm{d}\,\Xi(\tau) = -24\,\mathrm{sh}\,(4\,\Xi(\tau))\,\mathrm{d}\tau + \mathrm{d}B(\tau), \\ \Xi(0) = z, \end{cases} \tag{2.9}$$

We finally denote by $\mathbb{Q}_z$ the probability measure on $C([0,\tau_0];\mathcal{X})$ defined by $\mathbb{Q}_z(A) := P\left(\bar{m}_{\Xi^z(\cdot)} \in A\right)$. In this setting, the analogous of the convergence to the soft wall in Theorem 2.1 is the weak convergence of $\mathbb{Q}^\varepsilon_{m_0}$ to $\mathbb{Q}_{z_0}$; here m_0 satisfies $\|m_0 - \bar{m}_{z_0}\|_\infty \leqq \varepsilon^{\frac12-\eta}$ for some η small enough. In [2] we also need such convergence to hold uniformly for z_0 in compacts; this is the content of the following theorem. Below we denote by $\mathbb{Q}(F)$ the expectation of the function F with respect to the measure $\mathbb{Q}$.

Theorem 2.2. *Let $\tau_0 > 0$. There exists $\eta_1 > 0$ such that for any $\eta \in [0,\eta_1]$ the following holds. For each $L > 0$ and each uniformly continuous and bounded function $F : C([0,\tau_0];\mathcal{X}) \to \mathbb{R}$ we have*

$$\lim_{\varepsilon\to 0} \sup_{z\in[-L,L]} \sup_{m_0\in\mathcal{N}^\varepsilon_\eta(z)} \left|\mathbb{Q}^\varepsilon_{m_0}(F) - \mathbb{Q}_z(F)\right| = 0, \tag{2.10}$$

where $\mathcal{N}^\varepsilon_\eta(z) := \left\{m \in \mathcal{X}_\varepsilon \,:\, \|m - \bar{m}_z\|_\infty \leqq \varepsilon^{\frac12-\eta}\right\}$.

Outline and basic strategy. The proof of Theorem 2.1 relies on an iterative scheme, in which we linearize (2.4) around $\bar{m}_\zeta$ for a suitable ζ recursively defined. From a geometrical point of view, we approximate the flow induced by (2.4) with a piecewise linear one, which stays close to the quasi-invariant manifold $\mathcal{M}_{a,b}$, and allows one to compute the motion along the manifold itself. More precisely, following [3,7,8], we split the time axis into intervals of length T, taking T diverging as $\varepsilon \to 0$, yet very small as compared to the macroscopic time ε^{-1}. For the piecewise linear flow, we compute the displacement of the center, effectively tracking the motion along the quasi-invariant manifold. To this end, sharp estimates on the linear flow are needed. We emphasize that, even if the linearization of (1.1) on the whole line around the standing wave $\bar{m}_\zeta$ is very well understood [11], for our purposes the finite size corrections are crucial, the non-linear drift in (2.8) being indeed due to them. Moreover, to control the difference between the true flow and the piecewise linear one, we need a priori bounds which allow us to neglect the non-linear terms. Finally, the convergence to the hard wall stated in item (iii) is proven by showing that the interface motion is accurately described by (2.8) also on the time scale $\lambda\varepsilon^{-1}$. The proof then follows by showing that the diffusive scaling of the latter converges to a reflected Brownian motion. The proof of Theorem 2.2 requires only minor modifications and it is sketched in Appendix A.

3. The iterative scheme

The notion of the "center" of a function plays an important role in our analysis. Following [7,8], given a function $f \in C([-a,b])$ we define its center ζ as a point in $(-a,b)$ such that

$$\langle f - \bar{m}_\zeta\,,\, \bar{m}'_\zeta\rangle = \int_{-a}^{b} \mathrm{d}x\, \left[f(x) - \bar{m}_\zeta(x)\right] \bar{m}'_\zeta(x) = 0. \tag{3.1}$$

Referring to [8] for an interpretation of the above definition in terms of the dynamics given by the linearization of (2.3) around $\bar{m}_\zeta$, here we simply note that ζ minimizes the L_2 norm of $f - \bar{m}_z$ as a function of z.

Given $\delta, \ell > 0$ we define

$$\Upsilon(\delta, \ell) := \left\{ f \in C([-a, b]) \, : \, \| f - \bar{m}_z \|_\infty < \delta \text{ for some } z \in (-a + \ell, b - \ell) \right\}.$$

The existence and uniqueness of the center holds for functions in $\Upsilon(\delta, \ell)$ for ε, δ small enough and ℓ large enough, as precisely stated in the next proposition. Recall that we have chosen $a = \frac{1}{4} \log \varepsilon^{-1}$, $b = \varepsilon^{-\beta}$. The result is analogous to [7, Proposition 3.2] where the whole line is considered, and the proof follows by standard implicit function arguments [7,8].

Proposition 3.1. *There are reals $\delta_0, \ell_0 > 0$ such that, for any ε small enough, if $f \in \Upsilon(\delta_0, \ell_0)$ then f has a unique center $\zeta \in (-a, b)$. Moreover there is a constant $C_0 > 0$ so that if $z \in (-a + \ell_0, b - \ell_0)$ is such that $\| f - \bar{m}_z \|_\infty < \delta_0$, we have*

$$|\zeta - z| \leqq C_0 \, \| f - \bar{m}_z \|_\infty$$

and

$$\zeta = z - \frac{3}{4} \langle \bar{m}_z', f - \bar{m}_z \rangle - \frac{9}{16} \langle \bar{m}_z', f - \bar{m}_z \rangle \langle \bar{m}_z'', f - \bar{m}_z \rangle + R(z, f),$$
$$|R(z, f)| \leqq C_0 \left\{ \| f - \bar{m}_z \|_\infty^3 + \left(\mathrm{e}^{-2(b-z)} + \mathrm{e}^{-2(a+z)} \right) \| f - \bar{m}_z \|_\infty \right\}.$$

In the sequel, given $f \in \Upsilon(\delta, \ell)$ with $\delta < \delta_0$ and $\ell > \ell_0$, we denote by $X(f)$ the center of f, which is well defined for ε sufficiently small. From now on we drop, however, the explicit dependence on ε from the notation. Let $m(t)$ be the solution to (2.4) with m_0 satisfying (2.6) and $\alpha \in (0, 1)$; we define the stopping times

$$S_{\delta,\ell} := \inf \{ t \in \mathbb{R}_+ \, : \, m(t) \notin \Upsilon(\delta, \ell) \}, \tag{3.2}$$
$$S_{\delta,\ell,\alpha} := S_{\delta,\ell} \wedge \inf \left\{ t \, : \, |X(m(t))| \geqq \alpha \, a \right\}. \tag{3.3}$$

We analyze $m(t)$ as long as it stays in $\Upsilon(\delta, \ell)$ and its center is not too far from the origin, namely we stop the evolution at the time $S_{\delta,\ell,\alpha}$ by considering $m(t \wedge S_{\delta,\ell,\alpha})$. We are going to introduce an iterative procedure in which we linearize (2.4) around $\bar{m}_x$ for a suitable x recursively defined. To do so, we need a few definitions.

Given $\zeta \in (-a, b)$, let $\varphi_\zeta \in C^2([-a, b])$ be the solution to

$$\begin{cases} \frac{1}{2} \varphi_\zeta''(x) - V''(\bar{m}_\zeta(x)) \varphi_\zeta(x) = 0, \\ \varphi_\zeta(-a) = -1 - \bar{m}_\zeta(-a), \\ \varphi_\zeta(b) = 1 - \bar{m}_\zeta(b). \end{cases} \tag{3.4}$$

An explicit computation yields

$$\varphi_\zeta(x) = \bar{m}_\zeta'(x) \left[c_\zeta q_\zeta(x) + d_\zeta \right], \quad q_\zeta(x) := \frac{h_\zeta(x) - h_\zeta(-a)}{h_\zeta(b) - h_\zeta(-a)}, \tag{3.5}$$

where

$$h_\zeta(x) := \int_\zeta^x \mathrm{d}y\, \frac{1}{\bar m'_\zeta(y)^2} = \frac{3}{8}(x-\zeta) + \frac{3}{8}\frac{\bar m_\zeta(x)}{\bar m'_\zeta(x)} + \frac{1}{4}\frac{\bar m_\zeta(x)}{\bar m'_\zeta(x)^2}, \tag{3.6}$$

$$c_\zeta := \frac{1}{1-\bar m_\zeta(-a)} + \frac{1}{1+\bar m_\zeta(b)}, \quad d_\zeta := -\frac{1}{1-\bar m_\zeta(-a)}. \tag{3.7}$$

We also introduce the operator H_ζ on $C_0([-a,b])$, the space of continuous functions vanishing at the endpoints, defined on $C^2_K([-a,b])$, the space of twice differentiable functions compactly supported in $(-a,b)$, by

$$H_\zeta f(x) := -\frac{1}{2} f''(x) + V''(\bar m_\zeta(x)) f(x) \tag{3.8}$$

and denote by $g_t^{(\zeta)} := \exp\{-tH_\zeta\}$ the corresponding semigroup.

Let $t_0 \in \mathbb{R}_+$, and $m(t)$, $t \geqq t_0$ be the mild solution to (2.3) with initial condition $m(t_0) = \bar m_\zeta + \vartheta$, for some $\zeta \in (-a+\ell_0, b-\ell_0)$ and $\vartheta \in C([-a,b])$ such that $\|\vartheta\|_\infty < \delta_0$ (δ_0, ℓ_0 as in Proposition 3.1). By writing $m(t) = \bar m_\zeta + v(t)$ and expanding $V'(\bar m_\zeta + v) = V'(\bar m_\zeta) + V''(\bar m_\zeta)v + 3\bar m_\zeta v^2 + v^3$, it is easy to check that $v(t)$ satisfies the integral equation

$$v(t) = \varphi_\zeta + g^{(\zeta)}_{t-t_0}\left(\vartheta - \varphi_\zeta\right) - \int_{t_0}^t \mathrm{d}s\, g^{(\zeta)}_{t-s}\left[3\bar m_\zeta v(s)^2 + v(s)^3\right]$$
$$+ \sqrt{\varepsilon}\int_{t_0}^t g^{(\zeta)}_{t-s}\mathrm{d}W(s). \tag{3.9}$$

Let $m(t)$, $t \geqq 0$, now be the solution to (2.4) and consider the partition $\mathbb{R}_+ = \bigcup_{n\geqq 0}[T_n, T_{n+1})$, where $T_n = nT$, $n \in \mathbb{N}$ and $T = \varepsilon^{-\gamma}$, $\gamma \in \left(0, \frac{1}{8}\right)$. We next define, by induction on $n \geqq 0$, reals x_n and functions $v_n(t) \equiv \{v_n(t,x),\ x \in [-a,b]\}$, $t \in [T_n, T_{n+1}]$. They will have the property that for any $t \in [T_n, T_{n+1}]$

$$m\left(t \wedge S_{\delta,\ell,\alpha}\right) = \bar m_{x_n} + v_n(t). \tag{3.10}$$

Set $x_0 := X(m_0)$, that is the center of m_0, and let $v_0(t)$, $t \in [0,T]$, be the solution to (3.9) with $t_0 = 0$, $\zeta = x_0$, and $\vartheta = m_0 - \bar m_{x_0}$, stopped at $S_{\delta,\ell,\alpha}$. Suppose now, by induction, that we have defined x_{n-1} and v_{n-1}. We then define x_n as the center of $m(T_n \wedge S_{\delta,\ell,\alpha}) = \bar m_{x_{n-1}} + v_{n-1}(T_n)$ (which exists by the definition of the stopping time $S_{\delta,\ell,\alpha}$) and $v_n(t)$, $t \in [T_n, T_{n+1}]$, as the solution to (3.9) with $t_0 = T_n$, $\zeta = x_n$, and $\vartheta = m(T_n \wedge S_{\delta,\ell,\alpha}) - \bar m_{x_n}$, stopped at $S_{\delta,\ell,\alpha}$. We emphasize that in this construction the initial condition $v_n(T_n)$ for the evolution in the interval $[T_n, T_{n+1}]$ is related to the final condition $v_{n-1}(T_n)$ of the previous interval by

$$v_n(T_n) = -\bar m_{x_n} + \bar m_{x_{n-1}} + v_{n-1}(T_n). \tag{3.11}$$

We consider the operator H_ζ defined in (3.8) also as an operator on $L_2([-a,b], \mathrm{d}x)$ self-adjoint with domain $W^{2,2}([-a,b],\mathrm{d}x) \cap W^{1,2}_0([-a,b],\mathrm{d}x)$. The bottom of its spectrum is an isolated eigenvalue $\lambda_0^{(\zeta)} > 0$ of multiplicity one. The corresponding eigenfunction, which we denote by $\Psi_0^{(\zeta)}$, is chosen positive. We also introduce the *spectral gap* of H_ζ, which is defined as $\text{gap}(H_\zeta) := \inf \text{spec}(H_\zeta \restriction (\Psi_0^{(\zeta)})^\perp)$, where $H_\zeta \restriction (\Psi_0^{(\zeta)})^\perp$ denotes the restriction of H_ζ to the subspace orthogonal to $\Psi_0^{(\zeta)}$. Recalling $g_t^{(\zeta)} = \mathrm{e}^{-tH_\zeta}$, we then define

$$g_t^{(\zeta,\perp)} f := g_t^{(\zeta)} f - \mathrm{e}^{-\lambda_0^{(\zeta)} t} \left\langle \Psi_0^{(\zeta)}, f \right\rangle \Psi_0^{(\zeta)}, \tag{3.12}$$

$$G^{(\zeta,\perp)} := \int_0^\infty \mathrm{d}t\, g_t^{(\zeta,\perp)}. \tag{3.13}$$

Note that $G^{(\zeta,\perp)}$ is well defined as $\lambda_0^{(\zeta)} > 0$. We denote by $g_t^{(\zeta,\perp)}(x,y)$, $t > 0$, and $G^{(\zeta,\perp)}(x,y)$, $x, y \in [-a,b]$, the corresponding integral kernels. We shall use the same notation for the semigroups acting on $C([-a,b])$.

Let $\overline{H}_\zeta$ be the same operator as in (3.8), but defined on the whole line $\mathbb{R}$, that is as an operator on $C_\mathrm{b}(\mathbb{R})$, the space of bounded continuous functions, or on $L_2(\mathbb{R}, \mathrm{d}x)$. It is well known that $\overline{H}_\zeta$ has a zero eigenvalue with eigenfunction $\bar{m}'_\zeta$ and a strictly positive spectral gap [11]. These properties play a crucial role in the analysis of the interface fluctuations for a stochastic reaction diffusion equation on the whole line or, in any case, with the interface sufficiently far from the boundary, see [3,6–8,14]. Analogously, we need sharp bounds on the convergence, in a suitable sense, of H_ζ to $\overline{H}_\zeta$ as $\varepsilon \to 0$, which are stated below and proved in Section 8. Note that, since H_ζ and $\overline{H}_\zeta$ are defined in different spaces, these bounds do not follow directly from standard perturbation theory. We introduce

$$\phi^{(\zeta)}(x) := \frac{\bar{m}'_\zeta(x)}{\|\bar{m}'_\zeta\|_2}, \quad x \in [-a,b]. \tag{3.14}$$

Theorem 3.2. *Let a and b as in the statement of Theorem 2.1. Then, for each $\alpha \in (0,1)$ there exist reals $\varepsilon_1, \delta_1, C_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1]$, $|\zeta| < \alpha\, a$, and $f \in C([-a,b])$*

$$\|g_t^{(\zeta)} f\|_\infty \leqq C_1 \|f\|_\infty \quad \textit{for any} \quad t \geqq 0, \tag{3.15}$$

$$\text{gap}(H_\zeta) \geqq \delta_1, \tag{3.16}$$

$$\|g_t^{(\zeta,\perp)} f\|_\infty \leqq C_1\, \mathrm{e}^{-\delta_1 t}\, \|f\|_2^{2/3} \|f\|_\infty^{1/3} \quad \textit{for any} \quad t \geqq 1. \tag{3.17}$$

Moreover, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{3}{2}(1-\alpha)+\eta} \left| \lambda_0^{(\zeta)} - 24\,\varepsilon\, \mathrm{e}^{-4\zeta} \right| = 0, \tag{3.18}$$

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{1-\alpha}{2}+\eta} \left\| \Psi_0^{(\zeta)} - \phi^{(\zeta)} \right\|_\infty = 0, \tag{3.19}$$

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{1-\alpha}{2}+\eta} \left\| \Psi_0^{(\zeta)} - \phi^{(\zeta)} \right\|_1 = 0, \tag{3.20}$$

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-(1-\alpha)+\eta} \left\langle \left| \Psi_0^{(\zeta)} - \phi^{(\zeta)} \right|, \phi^{(\zeta)} \right\rangle = 0, \tag{3.21}$$

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \left| \int_{-a}^{b} \mathrm{d}x\, \bar{m}'_\zeta(x)\, \bar{m}_\zeta(x)\, G^{(\zeta,\perp)}(x,x) \right| = 0. \tag{3.22}$$

4. A priori bounds

The following lemma captures the correct asymptotic behavior of the first terms on the right-hand side of (3.9). Recall that $T = \varepsilon^{-\gamma}$, $\gamma \in \big(0, \frac{1}{8}\big)$ and that φ_ζ is defined in (3.5).

Lemma 4.1. *Let $\alpha \in (0, \gamma)$. Then for each $\eta > 0$*

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \sup_{t\in[0,T]} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \left\| \varphi_\zeta - g_t^{(\zeta)}\varphi_\zeta \right\|_\infty = 0. \tag{4.1}$$

Proof. Recalling (3.5)–(3.7) and (3.14), we write

$$\varphi_\zeta - g_t^{(\zeta)}\varphi_\zeta = c_\zeta \left[\bar m'_\zeta q_\zeta - g_t^{(\zeta)}(\bar m'_\zeta q_\zeta) \right] + d_\zeta \left\| \bar m'_\zeta \right\|_2 \left[\phi^{(\zeta)} - g_t^{(\zeta)}\phi^{(\zeta)} \right].$$

Note that $\|\bar m'_\zeta\|_2^2 \leqq \int_{-\infty}^{\infty} \mathrm{d}x\, \bar m'_0(x)^2 = \frac{4}{3}$, $|c_\zeta| \leqq 2$, and $|d_\zeta| \leqq 1$ so from (3.15) we get

$$\left\| \varphi_\zeta - g_t^{(\zeta)}\varphi_\zeta \right\|_\infty \leqq (2 + C_1) \left\| \bar m'_\zeta\, q_\zeta \right\|_\infty + \sqrt{\frac{4}{3}} \left\| \phi^{(\zeta)} - g_t^{(\zeta)}\phi^{(\zeta)} \right\|_\infty .$$

By using $g_t^{(\zeta)}\Psi_0^{(\zeta)} = \mathrm{e}^{-\lambda_0^{(\zeta)}t}\Psi_0^{(\zeta)}$ and again (3.15),

$$\left\| \phi^{(\zeta)} - g_t^{(\zeta)}\phi^{(\zeta)} \right\|_\infty \leqq (1 + C_1) \left\| \Psi_0^{(\zeta)} - \phi^{(\zeta)} \right\|_\infty + \left(1 - \mathrm{e}^{-\lambda_0^{(\zeta)}t}\right) \left\| \Psi_0^{(\zeta)} \right\|_\infty .$$

By (3.18), for each $\eta > 0$, we have $1 - \mathrm{e}^{-\lambda_0^{(\zeta)}t} \leqq \varepsilon^{1-\alpha-\eta}T$ for any $t \in [0, T]$, $|\zeta| < \alpha a$, and ε small enough. Then, using (3.19),

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \sup_{t\in[0,T]} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \left\| \phi^{(\zeta)} - g_t^{(\zeta)}\phi^{(\zeta)} \right\|_\infty = 0. \tag{4.2}$$

We next note that, by (3.6), there exists $C_2 > 0$ such that

$$\sup_{|\zeta|<\alpha a} \sup_{x\in[-a,b]} \bar m'_\zeta(x)^2 |h_\zeta(x)| \leqq C_2, \tag{4.3}$$

whence there is $C_3 > 0$ such that, for $|\zeta| < \alpha a$ and ε small enough,

$$\left| \bar m'_\zeta(x) q_\zeta(x) \right| \leqq C_3 \frac{\bar m'_\zeta(x)^{-1} - h_\zeta(-a)}{h_\zeta(b) - h_\zeta(-a)} \leqq C\varepsilon^{-\frac{\alpha}{2}} \exp\left\{-2\,\varepsilon^{-\beta}\right\}, \tag{4.4}$$

where we used that for ε small enough and $|\zeta| < \alpha a$, $\bar m'_\zeta(x)^{-1}$ achieves its maximum at $x = b$. The estimate (4.1) follows. □

To simplify the notation let us introduce, for $n \in \mathbb{N}$ and $t \in [T_n, T_{n+1}]$,

$$z_n(t) := \int_{T_n}^{t} g_{t-s}^{(x_n)}\, \mathrm{d}W(s), \tag{4.5}$$

which is the last term that appears in the integral equation for v_n, see (3.9). Given $\tau \in \mathbb{R}_+$ we let $n_\varepsilon(\tau) := [\varepsilon^{-1}\tau/T]$ and $n_{\varepsilon,\delta,\ell,\alpha}(\tau) := \left[\left(\varepsilon^{-1}\tau \wedge S_{\delta,\ell,\alpha}\right)/T\right]$. Given $\eta > 0, \theta \in \mathbb{R}_+$, we define the event

$$\mathcal{B}^{(1)}_{\varepsilon,\theta,\eta} := \left\{ \sup_{0 \leqq n \leqq n_\varepsilon(\lambda\theta)} \sup_{t \in [T_n, T_{n+1}]} \|z_n(t)\|_\infty \leqq \varepsilon^{-\eta}\sqrt{T} \right\}. \tag{4.6}$$

Let also

$$z_n^\perp(t) := z_n(t) - \left\langle \Psi_0^{(x_n)}, z_n(t) \right\rangle \Psi_0^{(x_n)} = \int_{T_n}^t g_{t-s}^{(x_n,\perp)} \, \mathrm{d}W(s), \tag{4.7}$$

$$v_n^\perp(t) := v_n(t) - \left\langle \Psi_0^{(x_n)}, v_n(t) \right\rangle \Psi_0^{(x_n)} \tag{4.8}$$

be the component of $z_n(t)$, respectively, $v_n(t)$, orthogonal to $\Psi_0^{(x_n)}$. We define

$$\mathcal{B}^{(2)}_{\varepsilon,\theta,\eta} := \left\{ \sup_{0 \leqq n \leqq n_\varepsilon(\lambda\theta)} \sup_{t \in [T_n, T_{n+1}]} \|z_n^\perp(t)\|_\infty \leqq \varepsilon^{-\eta} \right\} \tag{4.9}$$

and set $\mathcal{B}_{\varepsilon,\theta,\eta} := \mathcal{B}^{(1)}_{\varepsilon,\theta,\eta} \cap \mathcal{B}^{(2)}_{\varepsilon,\theta,\eta}$. By standard Gaussian estimates, see [3, Appendix B], we have that for each $\theta, \eta, q > 0$

$$\mathbb{P}\left(\mathcal{B}_{\varepsilon,\theta,\eta}\right) \geqq 1 - \varepsilon^q \tag{4.10}$$

for any ε small enough.

Theorem 4.2. *Let $\alpha \in (0,\gamma)$; then there exists $\eta_0 > 0$ such that, for any $\theta \in \mathbb{R}_+$ and $\eta \in (0,\eta_0)$, on the event $\mathcal{B}_{\varepsilon,\theta,\eta}$ we have*

$$\sup_{0 \leqq n \leqq n_\varepsilon(\lambda\theta)} \sup_{t \in [T_n, T_{n+1}]} \|v_n(t)\|_\infty \leqq \sqrt{\varepsilon T}\, \varepsilon^{-2\eta}, \tag{4.11}$$

$$\sup_{0 \leqq n < n_{\varepsilon,\delta,\ell,\alpha}(\lambda\theta)} \left\{ \|v_n^\perp(T_{n+1})\|_\infty + \|v_n(T_n)\|_\infty \right\} \leqq \varepsilon^{\frac{1}{2}(1-\alpha)-2\eta}, \tag{4.12}$$

$$\sup_{0 \leqq n < n_{\varepsilon,\delta,\ell,\alpha}(\lambda\theta)} \sup_{t \in [T_n, T_{n+1}]} \|v_n(t) - \sqrt{\varepsilon} z_n(t)\|_\infty \leqq \varepsilon^{\frac{1}{2}(1-\alpha)-2\eta}, \tag{4.13}$$

$$\sup_{0 \leqq n < n_{\varepsilon,\delta,\ell,\alpha}(\lambda\theta)} \left| x_{n+1} - \left(x_0 - \frac{3}{4} \sum_{k=0}^n \left\langle \bar{m}'_{x_k}, v_k(T_{k+1}) \right\rangle \right) \right| \leqq \varepsilon^{-\frac{1}{2}\alpha - 3\eta}\, \lambda\, T^{-\frac{1}{2}}, \tag{4.14}$$

for any ε small enough.

Proof. By the recursive definition of $v_n(t)$, see in particular (3.11) and (3.15), the following holds. On the event $\mathcal{B}^{(1)}_{\varepsilon,\theta,\eta}$, for $t \leqq \varepsilon^{-1}\lambda\theta \wedge S_{\delta,\ell,\alpha}$ and $n = [t/T]$ we have (where we understand $v_{-1}(0) = m_0 - \bar{m}_{x_0}$)

$$\begin{aligned}\|v_n(t)\|_\infty &\leqq \sqrt{\varepsilon T}\varepsilon^{-\eta} + \left\|\varphi_{x_n} - g^{(x_n)}_{t-T_n}\varphi_{x_n}\right\|_\infty + C_1\left\|\bar{m}_{x_{n-1}} - \bar{m}_{x_n}\right\|_\infty \\ &\quad + C_1\|v_{n-1}(T_n)\|_\infty + 3\,C_1\int_{T_n}^t \mathrm{d}s\,\|v_n(s)\|_\infty^2\,[1+\|v_n(s)\|_\infty] \\ &\leqq 2\sqrt{\varepsilon T}\varepsilon^{-\eta} + C_1(C_0+1)\,\|v_{n-1}(T_n)\|_\infty \\ &\quad + 3\,C_1\int_{T_n}^t \mathrm{d}s\,\|v_n(s)\|_\infty^2\,[1+\|v_n(s)\|_\infty]\,.\end{aligned}$$

Above we used Proposition 3.1 and Lemma 4.1, note $\alpha \in (0,\gamma)$ implies $\varepsilon^{-\alpha} < T$. On the other hand, for $t \in (\varepsilon^{-1}\lambda\theta \wedge S_{\delta,\ell,\alpha}\,,\ \varepsilon^{-1}\lambda\theta]$ we clearly have $v_n(t) = v_{[S_{\delta,\ell,\alpha}/T]}(S_{\delta,\ell,\alpha})$. Recalling (2.6) and (3.17), the proof of (4.11) is now completed by a standard bootstrap argument, see [3, Proposition 4.1].

By the recursive definition of $v_n(t)$, Theorem 3.2 and (4.11), for $n<n_{\varepsilon,\delta,\ell,\alpha}(\lambda\theta)$, on the event $\mathcal{B}_{\varepsilon,\theta,\eta}$, we have

$$\begin{aligned}\|v_n^\perp(T_{n+1})\|_\infty &\leqq C\|\varphi_{x_n} - g^{(x_n)}_T\varphi_{x_n}\|_\infty + \varepsilon^{\frac12-\eta} \\ &\quad + C\mathrm{e}^{-\delta_1 T}\,\|v_n(T_n)\|_\infty^{1/3}\,\|v_n(T_n)\|_2^{2/3} + 4\,\varepsilon^{1-4\eta}T^2.\end{aligned}$$

Using Lemma 4.1 and again (4.11), the right-hand side is bounded by $\frac13\varepsilon^{\frac12(1-\alpha)-2\eta}$. Recalling (3.11) we have

$$v_{n+1}(T_{n+1}) = -\bar{m}_{x_{n+1}} + \bar{m}_{x_n} + \left\langle \phi^{(x_n)}, v_n(T_{n+1})\right\rangle \phi^{(x_n)} + D_n + v_n^\perp(T_{n+1}),$$

where

$$D_n := \left\langle \Psi_0^{(x_n)}, v_n(T_{n+1})\right\rangle \Psi_0^{(x_n)} - \left\langle \phi^{(x_n)}, v_n(T_{n+1})\right\rangle \phi^{(x_n)}.$$

From Theorem 3.2 and (4.11) it is straightforward to deduce $\|D_n\|_\infty \leqq \frac16\varepsilon^{\frac12(1-\alpha)-2\eta}$. To complete the proof of (4.12), it is then enough to show that

$$\left\|-\bar{m}_{x_{n+1}} + \bar{m}_{x_n} + \left\langle \phi^{(x_n)}, v_n(T_{n+1})\right\rangle \phi^{(x_n)}\right\|_\infty \leqq \frac16\varepsilon^{\frac12(1-\alpha)-2\eta},$$

which follows, by elementary computations, from Proposition 3.1 and (4.11), using the fact that there exists $C>0$ such that, for any $\varepsilon>0$,

$$\sup_{|\zeta|\leqq\alpha a}\left[\int_{-\infty}^{\infty}\mathrm{d}x\,\bar{m}'_\zeta(x)^2 - \|\bar{m}'_\zeta\|_2^2\right] = \sup_{|\zeta|\leqq\alpha a}\left[\frac43 - \|\bar{m}'_\zeta\|_2^2\right] \leqq C\varepsilon^{1-\alpha}. \tag{4.15}$$

The bound (4.13) follows, by (3.9), from (4.11), (4.12), and Lemma 4.1.

To prove (4.14), we first note that, by Proposition 3.1, the recursive definition of the center, and (4.11), for $n < n_{\varepsilon,\delta,\ell,\alpha}(\lambda\theta)$, we have

$$\begin{aligned}&\left|x_{n+1} - x_n + \frac34\left\langle \bar{m}'_{x_n}, v_n(T_{n+1})\right\rangle + \frac{9}{16}\left\langle \bar{m}'_{x_n}, v_n(T_{n+1})\right\rangle\left\langle \bar{m}''_{x_n}, v_n(T_{n+1})\right\rangle\right| \\ &\quad\leqq C_0\left[\varepsilon^{\frac32-6\eta}T^{\frac32} + 2\varepsilon^{\frac12(1-\alpha)}\varepsilon^{\frac12-2\eta}\sqrt{T}\right].\end{aligned}$$

By writing $v_n(T_{n+1}) = \left\langle \Psi_0^{(x_n)}, v_n(T_{n+1}) \right\rangle \Psi_0^{(x_n)} + v_n^{\perp}(T_{n+1})$ and using (3.20), the bound $\left|\langle \bar{m}''_{x_n}, \phi^{(x_n)} \rangle\right| \leqq \varepsilon^{1-\alpha}$ together with (4.11) and (4.12) we get

$$\left| \langle \bar{m}'_{x_n}, v_n(T_{n+1}) \rangle \langle \bar{m}''_{x_n}, v_n(T_{n+1}) \rangle \right| \leqq \varepsilon^{1-\frac{\alpha}{2}-4\eta} \sqrt{T}.$$

Putting together the above estimates, we get the bound (4.14). □

5. Recursive equation for the center and stability

Let x_0 be the center of the initial condition m_0 in (2.3), set $\xi_0 = x_0$ and

$$\begin{aligned} \xi_{n+1} &:= x_0 - \frac{3}{4} \sum_{k=0}^{n \wedge [S_{\delta,\ell,\alpha}/T]} \langle \bar{m}'_{x_k}, v_k(T_{k+1}) \rangle, \\ \sigma_n &:= -\frac{3}{4}\sqrt{\varepsilon}\, \langle \bar{m}'_{x_n}, z_n(T_{n+1}) \rangle \\ &= -\frac{3}{4}\sqrt{\varepsilon} \int_{T_n}^{T_{n+1}} \left\langle \bar{m}'_{x_n}, g^{(x_n)}_{T_{n+1}-t}\, \mathrm{d}W(t) \right\rangle, \\ F_n &:= \frac{3}{4}\, \varepsilon \int_{T_n}^{T_{n+1}} \mathrm{d}t \left\langle \bar{m}'_{x_n}, 3\bar{m}_{x_n} z_n(t)^2 \right\rangle. \end{aligned} \tag{5.1}$$

Notice that, by the bound (4.14), ξ_{n+1} is an approximation to the center x_{n+1} for $n < [S_{\delta,\ell,\alpha}/T]$. Also, conditionally on the centers $x_0, x_1, \ldots, x_n$, the random variables $\sigma_0, \ldots, \sigma_n$ are independent Gaussians with mean zero and variance $\frac{3}{4}\, \varepsilon T[1+o(1)]$. The next theorem identifies a recursive equation satisfied by ξ_n.

Theorem 5.1. *For each $n < [S_{\delta,\ell,\alpha}/T]$, we have*

$$\xi_{n+1} - \xi_n = \sigma_n + 12\varepsilon T \mathrm{e}^{-4\xi_n} + F_n + R_n, \tag{5.2}$$

where the remainder R_n can be bounded as follows. There exist $q, \alpha_0, \eta_0 > 0$ such that for any $\alpha \in (0, \alpha_0)$, $\eta \in (0, \eta_0)$ and $\theta \in \mathbb{R}_+$, in the event $\mathcal{B}_{\varepsilon,\theta,\eta}$, we have

$$\sup_{0 \leqq n < [S_{\delta,\ell,\alpha}/T]} |R_n| \leqq \varepsilon \lambda^{-1} T\, \varepsilon^q \tag{5.3}$$

for any ε small enough. Moreover, for each $\theta \in \mathbb{R}_+$ there exists $q > 0$ such that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{0 \leqq n < n_\varepsilon(\lambda\theta)} \left| \sum_{k=0}^{n} F_k \right| > \varepsilon^q \right) = 0. \tag{5.4}$$

We remark that, while the remainder R_n is deterministically small on the event $\mathcal{B}_{\varepsilon,\theta,\eta}$, the non-linear term F_n becomes negligible in the limit $\varepsilon \to 0$ only in probability. This is due to a cancellation in which we exploit a martingale structure of F_n. In other words F_n gives no contribution to the limit equation not because of its magnitude, which would instead give a finite contribution, but because its expected value vanishes in the limit. The same mechanism, which depends on the symmetry of V, was already exploited for the stochastic reaction diffusion equation with the interface far from the boundary [3,7,8].

Before proving Theorem 5.1, we state a lemma that identifies the leading corrections in Lemma 4.1 for $t = T$, which will be responsible for the non-linear drift in (5.2).

Lemma 5.2. *Let $\alpha \in (0, \frac{\gamma}{3})$. Then, for each $\eta > 0$,*

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-(1-\alpha)+\eta} \left| \left\langle \bar{m}'_\zeta, \varphi_\zeta - g_T^{(\zeta)}\varphi_\zeta \right\rangle + \frac{4}{3}\, 12\,\varepsilon T\, \mathrm{e}^{-4\zeta} \right| = 0. \tag{5.5}$$

Proof. Recalling (3.14), we write

$$\begin{aligned}
&\left\langle \bar{m}'_\zeta, \varphi_\zeta - g_T^{(\zeta)}\varphi_\zeta \right\rangle \\
&= \left\langle \bar{m}'_\zeta, \varphi_\zeta \right\rangle - \mathrm{e}^{-\lambda_0^{(\zeta)}T} \left\langle \Psi_0^{(\zeta)}, \varphi_\zeta \right\rangle \left\langle \bar{m}'_\zeta, \Psi_0^{(\zeta)} \right\rangle - \left\langle \bar{m}'_\zeta, g_T^{(\zeta,\perp)}\varphi_\zeta \right\rangle \\
&= \left\langle \bar{m}'_\zeta, \varphi_\zeta \right\rangle \left[1 - \mathrm{e}^{-\lambda_0^{(\zeta)}T} \left\langle \Psi_0^{(\zeta)}, \phi^{(\zeta)} \right\rangle \right] \\
&\quad - \mathrm{e}^{-\lambda_0^{(\zeta)}T} \left\langle \varphi_\zeta, \Psi_0^{(\zeta)} - \phi^{(\zeta)} \right\rangle \left\langle \bar{m}'_\zeta, \Psi_0^{(\zeta)} \right\rangle - \left\langle \bar{m}'_\zeta, g_T^{(\zeta,\perp)}\varphi_\zeta \right\rangle.
\end{aligned}$$

The last term above is easily bounded by using (3.5) and (3.17). Again by (3.5) and (3.21), one can easily show, see Lemma 4.1 for analogous computations, that

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-(1-\alpha)+\eta} \left\langle \Psi_0^{(\zeta)} - \phi^{(\zeta)}\,,\ \varphi_\zeta \right\rangle = 0.$$

From (4.15) and since $\sup_{|\zeta|<\alpha a} \left| d_\zeta + \frac{1}{2} \right| \leqq C\varepsilon^{\frac{1}{2}(1-\alpha)}$, again by (3.5) we have

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \left| \left\langle \bar{m}'_\zeta, \varphi_\zeta \right\rangle + \frac{2}{3} \right| = 0.$$

Finally, by (3.18) and (3.21),

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-(1-\alpha)+\eta} \sup_{|\zeta|<\alpha a} \left\{ \left| 1 - \left\langle \Psi_0^{(\zeta)}, \phi^{(\zeta)} \right\rangle \right| + \left| 1 - \mathrm{e}^{-\lambda_0^{(\zeta)}T} - 24\varepsilon T \mathrm{e}^{-4\zeta} \right| \right\} = 0,$$

which concludes the proof. □

Proof of Theorem 5.1. To simplify notation, we choose a suitable $\varepsilon_0 > 0$ and hereafter assume $\varepsilon \in (0, \varepsilon_0]$. By the recursive definition of v_n, we see that (5.2) holds with $R_n = -\frac{3}{4}\left[R_n^{(1)} + R_n^{(2)} + R_n^{(3)} + R_n^{(4)} \right]$ where

$$\begin{aligned}
R_n^{(1)} &:= \left\langle \bar{m}'_{x_n}, \varphi_{x_n} - g_T^{(x_n)}\varphi_{x_n} \right\rangle + \frac{4}{3}\, 12\varepsilon T \mathrm{e}^{-4\xi_n}, \\
R_n^{(2)} &:= \left\langle \bar{m}'_{x_n}, g_T^{(x_n)} v_n(T_n) \right\rangle, \\
R_n^{(3)} &:= -\int_{T_n}^{T_{n+1}} \mathrm{d}t\, \left\{ \left\langle \bar{m}'_{x_n}, g_{T_{n+1}-t}^{(x_n)} \left[3\bar{m}_{x_n} v_n(t)^2 \right] \right\rangle - \varepsilon \left\langle \bar{m}'_{x_n}, 3\bar{m}_{x_n} z_n(t)^2 \right\rangle \right\}, \\
R_n^{(4)} &:= -\int_{T_n}^{T_{n+1}} \mathrm{d}t\, \left\langle \bar{m}'_{x_n}, g_{T_{n+1}-t}^{(x_n)} \left[v_n(t)^3 \right] \right\rangle.
\end{aligned}$$

The error term $R_n^{(1)}$ is easily bounded by using Lemma 5.2 and (4.14). The bound for the terms $R_n^{(2)}$ and $R_n^{(4)}$ follows from Theorem 4.2. We next bound $R_n^{(3)}$. By Theorem 3.2 for each $\eta > 0$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \sup_{t\in[0,T]} \sup_{|\zeta|\leqq \alpha a} \left\| \bar{m}'_\zeta - g_t^{(\zeta)} \bar{m}'_\zeta \right\|_1 = 0;$$

so that, by Theorem 4.2 it is enough to prove (5.3) for

$$\widetilde{R}_n^{(3)} := \int_{T_n}^{T_{n+1}} \mathrm{d}t \,\left\langle \bar{m}'_{x_n}, \bar{m}_{x_n} \left[v_n(t) - \sqrt{\varepsilon} z_n(t)\right] \left[v_n(t) + \sqrt{\varepsilon} z_n(t)\right]\right\rangle.$$

We decompose $[T_n, T_{n+1}] = [T_n, T_n + \log^2 T] \cup [T_n + \log^2 T, T_{n+1}]$ and estimate separately the two time integrals. For the first one it is enough to notice that, by (4.11) and (4.13), we have

$$\left| \int_{T_n}^{T_n+\log^2 T} \mathrm{d}t \,\left\langle \bar{m}'_{x_n}, \bar{m}_{x_n} \left[v_n(t) - \sqrt{\varepsilon} z_n(t)\right] \left[v_n(t) + \sqrt{\varepsilon} z_n(t)\right]\right\rangle \right| \\ \leqq \varepsilon^{1-\frac{1}{2}\alpha-4\eta} \sqrt{T} \log^2 T.$$

To bound the second integral we write, by using the integral equation (3.9) for v_n and the iterative definition of v_n,

$$v_n(t) - \sqrt{\varepsilon} z_n(t) = \varphi_{x_n} - \mathrm{e}^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle \varphi_{x_n}, \Psi_0^{(x_n)} \right\rangle \Psi_0^{(x_n)} - g_{t-T_n}^{(x_n,\perp)} \varphi_{x_n} \\ + \mathrm{e}^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle v_n(T_n), \Psi_0^{(x_n)} \right\rangle \Psi_0^{(x_n)} + g_{t-T_n}^{(x_n,\perp)} v_n(T_n) + D_n(t),$$

where, by Theorem 4.2, $\sup_{t\in[T_n,T_{n+1}]} \|D_n(t)\|_\infty \leqq 4T^2\varepsilon^{1-4\eta}$. By the explicit expression (3.5), the bound (4.4) and Theorem 3.2, for each $\eta > 0$ we have

$$\left\langle \bar{m}'_{x_n}, \left| \varphi_{x_n} - \mathrm{e}^{-\lambda_0^{(x_n)}(t-T_n)} \left\langle \varphi_{x_n}, \Psi_0^{(x_n)} \right\rangle \Psi_0^{(x_n)} \right| \right\rangle \leqq \varepsilon^{1-\alpha-\eta} T.$$

Since, by the recursive definition of the centers x_n, $\langle \bar{m}'_{x_n}, v_n(T_n)\rangle = 0$, by (3.20) and (4.12), we have

$$\left| \left\langle v_n(T_n), \Psi_0^{(x_n)} \right\rangle \right| \leqq \varepsilon^{1-\alpha-3\eta}.$$

Finally, by (3.17), from (3.5) and (4.4), since $b = \varepsilon^{-\beta}$, there is $C > 0$ such that

$$\sup_{t\in[T_n+\log^2 T, T_{n+1}]} \left\{ \left\| g_{t-T_n}^{(x_n,\perp)} \varphi_{x_n} \right\|_\infty + \left\| g_{t-T_n}^{(x_n,\perp)} v_n(T_n) \right\|_\infty \right\} \leqq C \mathrm{e}^{-\delta_1 \log^2 T} \varepsilon^{-\frac{\beta}{3}}.$$

Putting all the above bounds together and using Theorem 4.2 to bound $\|v_n(t) + \sqrt{\varepsilon} z_n(t)\|_\infty$, we finally get

$$\left| \int_{T_n+\log^2 T}^{T_{n+1}} \mathrm{d}t \,\left\langle \bar{m}'_{x_n}, \bar{m}_{x_n} \left[v_n(t) - \sqrt{\varepsilon} z_n(t)\right] \left[v_n(t) + \sqrt{\varepsilon} z_n(t)\right]\right\rangle \right| \leqq \varepsilon^{\frac{3}{2}-\alpha-6\eta} T^{\frac{7}{2}},$$

which concludes the proof of (5.3).

We next prove (5.4). By the Doob decomposition,

$$\sum_{k=0}^{n-1} F_k = M_n + \sum_{k=0}^{n-1} \gamma_k, \tag{5.6}$$

where

$$\gamma_k := \mathbb{E}\left(F_k \,\middle|\, \mathcal{F}_{T_k}\right) \tag{5.7}$$

and M_n is an $\mathcal{F}_{T_n}$-martingale with bracket

$$\langle M \rangle_n = \sum_{k=0}^{n-1} \left\{ \mathbb{E}\left(F_k^2 \,\middle|\, \mathcal{F}_{T_k}\right) - \gamma_k^2 \right\}. \tag{5.8}$$

Since for $(t, x) \in [T_k, T_{k+1}] \times [-a, b]$

$$\mathbb{E}\left(z_k(t, x)^2 \,\middle|\, \mathcal{F}_{T_k}\right) = \int_{T_k}^{t} \mathrm{d}s\, g_{2(t-s)}^{(x_k)}(x, x),$$

we have

$$\begin{aligned} \gamma_k &:= -\frac{9}{4}\varepsilon \int_{T_k}^{T_{k+1}} \mathrm{d}t \int_{T_k}^{t} \mathrm{d}s \int_{-a}^{b} \mathrm{d}x\, \bar{m}'_{x_k}(x)\, \bar{m}_{x_k}(x)\, g_{2(t-s)}^{(x_k)}(x, x) \\ &= -\frac{9}{4}\varepsilon \int_0^T \mathrm{d}t\, (T - t) \int_{-a}^{b} \mathrm{d}x\, \bar{m}'_{x_k}(x)\, \bar{m}_{x_k}(x)\, g_{2t}^{(x_k, \perp)}(x, x) + r_k, \end{aligned}$$

where

$$r_k := -\frac{9}{4}\varepsilon \int_0^T \mathrm{d}t\, (T - t) \int_{-a}^{b} \mathrm{d}x\, \bar{m}'_{x_k}(x)\, \bar{m}_{x_k}(x)\, \exp\left\{-2\lambda_0^{(x_k)} t\right\} \Psi_0^{(x_k)}(x)^2.$$

Since $\left|\left\langle \bar{m}'_{x_k}, \bar{m}_{x_k} \left(\bar{m}'_{x_k}\right)^2 \right\rangle\right| \leqq \varepsilon^{\frac{3}{2}(1-\alpha)}$, by (3.18) and (3.20) we have that $|r_k| \leqq \varepsilon^{\frac{3}{2}(1-\alpha)} T^2$.

Recall that $G^{(\zeta, \perp)}$ has been defined in (3.13). We claim that

$$\sup_{|\zeta| < \alpha a} \sup_{x \in [-a, b]} \left| \int_0^T \mathrm{d}t\, \frac{T - t}{T}\, g_{2t}^{(\zeta, \perp)}(x, x) - \frac{1}{2} G^{(\zeta, \perp)}(x, x) \right| \leqq \frac{C}{T}. \tag{5.9}$$

To prove it, we write

$$g_{2t}^{(\zeta, \perp)}(x, x) = \sum_{i=1}^{\infty} \exp\left\{-2\, t\, \lambda_i^{(\zeta)}\right\} \Psi_i^{(\zeta)}(x)^2,$$

where $\lambda_i^{(\zeta)}$, respectively, $\Psi_i^{(\zeta)}$, $i \geqq 0$, are the eigenvalues, respectively, the eigenfunctions, of H_ζ. A straightforward computation yields

$$\frac{1}{T} \int_0^T \mathrm{d}t\, (T - t)\, g_{2t}^{(\zeta, \perp)}(x, x) = \sum_{i=1}^{\infty} \frac{\Psi_i^{(\zeta)}(x)^2}{2\lambda_i^{(\zeta)}} \left[1 - \frac{1 - \exp\left\{-2\lambda_i^{(\zeta)} T\right\}}{2\lambda_i^{(\zeta)} T} \right].$$

As $G^{(\zeta,\perp)}(x,x) = \sum_{i=1}^{\infty} \Psi_i^{(\zeta)}(x)^2/\lambda_i^{(\zeta)}$ the bound (5.9) follows from (3.16) and Remark 1 at the end of Section 8. By (3.22) and the previous bounds we finally get that there exists $q > 0$ such that

$$\sum_{k=0}^{n_\varepsilon(\lambda\theta)} |\gamma_k| \leqq n_\varepsilon(\lambda\theta) \sup_{0\leqq n\leqq n_\varepsilon(\lambda\theta)} |\gamma_n| \leqq \varepsilon^q .$$

We are left with the bound of the martingale part M_n. Given $q > 0$, by Doob's inequality, recalling (5.8),

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqq n\leqq n_\varepsilon(\lambda\theta)} |M_n| \geqq \varepsilon^q\right) &\leqq \varepsilon^{-2q}\, \mathbb{E}\left(\langle M\rangle_{n_\varepsilon(\lambda\theta)}\right)\\ &\leqq \varepsilon^{-2q} \sum_{k=0}^{n_\varepsilon(\lambda\theta)} \mathbb{E}\left[\mathbb{E}\left(F_k^2 \,\middle|\mathcal{F}_{T_k}\right)\right] \leqq C^2\varepsilon^{-2q}\left[n_\varepsilon(\lambda\theta)+1\right]\varepsilon^2 T^4, \quad (5.10)\end{aligned}$$

where we used that there exists $C > 0$ such that, for any $\varepsilon > 0$ and $k \leqq n_\varepsilon(\lambda\theta)$, we have

$$\sqrt{\mathbb{E}\left(F_k^2 \,\middle|\mathcal{F}_{T_k}\right)} \leqq C\,\varepsilon \int_{T_k}^{T_{k+1}} \mathrm{d}t \int_{-a}^{b} \mathrm{d}x\, \bar{m}'_{x_k}(x) \sqrt{\mathbb{E}\left(z_k(t,x)^4 \middle| \mathcal{F}_{T_k}\right)} \leqq C\varepsilon T^2,$$

which concludes the proof. □

In the following lemma we prove that ξ_n is bounded with probability close to one. In proving item (ii) in Theorem 2.1 we need such control for $n \leqq (\varepsilon T)^{-1}$, while for item (iii) we need that ξ_n grows at most as $\sqrt{\lambda}$ for $n \leqq \lambda(\varepsilon T)^{-1}$.

Lemma 5.3. *For each $\theta \in \mathbb{R}_+$ we have*

$$\lim_{L\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \mathbb{P}\left(\sup_{0\leqq n\leqq n_\varepsilon(\mu\theta)} |\xi_n| > L\sqrt{\mu}\right) = 0, \quad \mu = 1, \lambda. \qquad (5.11)$$

Proof. Since for $n \geqq [S_{\delta,\ell,\alpha}/T]$, by definition (5.1), $\xi_n = \xi_{[S_{\delta,\ell,\alpha}/T]}$, it is enough to prove the statement for $n < n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)$. Recall (5.2) and let

$$S_n := \sum_{k=0}^{n-1} \sigma_k, \quad A_n := S_n + x_0 + \sum_{k=0}^{n-1} \left[F_k + R_k\right]. \qquad (5.12)$$

By (2.6) and Proposition 3.1, for each $\eta > 0$ we have that, for any ε small enough,

$$|x_0| \leqq \varepsilon^{\frac{1}{2}-\eta}. \qquad (5.13)$$

Recalling definition (5.1), one can easyily show that there exists a real $C > 0$ such that, for any $\varepsilon > 0$,

$$\mathbb{E}\left(\sigma_k \,\middle|\mathcal{F}_{T_k}\right) = 0, \quad \mathbb{E}\left(\sigma_k^2 |\mathcal{F}_{T_k}\right) \leqq C\varepsilon T. \qquad (5.14)$$

Given $\theta \in \mathbb{R}_+$, an application of Doob inequality then yields

$$\lim_{L\to\infty} \varlimsup_{\varepsilon\to 0} \mathbb{P}\left(\sup_{0\leqq n \leqq n_\varepsilon(\mu\theta)} |S_n| > L\sqrt{\mu} \right) = 0. \tag{5.15}$$

By Theorem 5.1, (4.10), and (5.15), we have

$$\xi_n = \sum_{k=0}^{n-1} 12\varepsilon T \mathrm{e}^{-4\xi_k} + A_n, \tag{5.16}$$

with

$$\lim_{L\to\infty} \varlimsup_{\varepsilon\to 0} \mathbb{P}\left(\sup_{0\leqq n \leqq n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)} |A_n| > L\sqrt{\mu} \right) = 0. \tag{5.17}$$

Let $\bar{L} := \sup_{n=0,\dots,n_\varepsilon(\mu\theta)} |A_n|/\sqrt{\mu}$ and set $L_1 := 2(\bar{L}+1) + 12\theta + 1$. To prove (5.11) we may suppose, and we do now, that $\varepsilon T \mathrm{e}^{8L_1\sqrt{\mu}} \leqq 1$. We shall then prove that $\sup_{0\leqq n\leqq n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)} |\xi_n| \leqq L_1\sqrt{\mu}$. Indeed, for any $n \leqq n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)$ from (5.16) it is clear that $\xi_n \geqq -|A_n| \geqq -\bar{L}\sqrt{\mu}$. To prove the upper bound, by setting $\bar{n}(L_1) := \inf\{n \geqq 0 : \xi_n > 2L_1\sqrt{\mu}\} \wedge n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)$, we shall prove $\xi_n \leqq L_1\sqrt{\mu}$ for $n \leqq \bar{n}(L_1)$, which gives $\bar{n}(L_1) = n_{\varepsilon,\delta,\ell,\alpha}(\mu\theta)$ and concludes the proof. Given $n \leqq \bar{n}(L_1)$, let n_* the last up-crossing of $\sqrt{\mu}$, namely $n_* = \sup\{k \leqq n : \xi_k \leqq \sqrt{\mu}\}$. If $n = n_*$ there is nothing to prove, otherwise from (5.16) we get

$$\begin{aligned}
\xi_n &= \xi_{n_*} + A_n - A_{n_*} + 12\varepsilon T \mathrm{e}^{-4\xi_{n_*}} + \sum_{k=n_*+1}^{n-1} 12\varepsilon T \mathrm{e}^{-4\xi_k} \\
&\leqq (2\bar{L}+1)\sqrt{\mu} + 12\varepsilon T \mathrm{e}^{8L_1\sqrt{\mu}} + 12(n-n_*)\varepsilon T \mathrm{e}^{-4\sqrt{\mu}} \\
&\leqq (2\bar{L}+1)\sqrt{\mu} + 12\theta + 1 \leqq L_1\sqrt{\mu},
\end{aligned}$$

which concludes the proof. □

Proof of Theorem 2.1, item (i). Let us first prove that for each $\theta \in \mathbb{R}_+$, $\delta \in (0, \delta_0)$, $\ell \in (\ell_0, \infty)$ and $\alpha \in (0, 1/8)$ we have

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(S_{\delta,\ell,\alpha} \leqq \lambda\varepsilon^{-1}\theta \right) = 0. \tag{5.18}$$

Indeed, recalling (3.2) and (5.1), by (4.11), (4.14), (4.10), and Lemma 5.3 it follows that

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(S_{\delta,\ell} \leqq \lambda\varepsilon^{-1}\theta \right) = 0.$$

By using Proposition 3.1 and again (4.11), we then get also (5.18).

We now prove item (i) of Theorem 2.1 with $X_\varepsilon(t) := X\left(m(t \wedge S_{\delta,\ell,\alpha})\right)$, that is $X_\varepsilon(t)$ is the center of the solution to (2.4) stopped at $S_{\delta,\ell,\alpha}$. Note X_ε is a continuous $\mathcal{F}_t$-adapted process. Thanks to (5.18), it is enough to show that, for each $\theta, \eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{t\leqq \lambda\varepsilon^{-1}\theta \wedge S_{\delta,\ell,\alpha}} \left\| m(t) - \bar{m}_{X_\varepsilon(t)} \right\|_\infty > \varepsilon^{\frac{1}{2}-\eta} \right) = 0, \tag{5.19}$$

which follows, by taking γ small enough, from Proposition 3.1 and (4.11). □

6. Convergence to the soft wall

Recalling that $n_\varepsilon(\tau) = [\varepsilon^{-1}\tau/T]$, $T = \varepsilon^{-\gamma}$, and that ξ_n has been defined in (5.1), we define the continuous process $\xi_\varepsilon(\tau)$, $\tau \in \mathbb{R}_+$, as the piecewise linear interpolation of ξ_n namely, we set

$$\xi_\varepsilon(\tau) := \xi_{n_\varepsilon(\tau)} + [\tau - \varepsilon T n_\varepsilon(\tau)]\left[\xi_{n_\varepsilon(\tau)+1} - \xi_{n_\varepsilon(\tau)}\right]. \tag{6.1}$$

Recalling $X_\varepsilon(t)$ is the center of $m(t \wedge S_{\delta,\ell,\alpha})$, by (5.18), (4.14), and (4.11), we have that for each $\theta \in \mathbb{R}_+$ there exists a $q > 0$ such that

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{\tau\in[0,\lambda\theta]} \left|X_\varepsilon(\varepsilon^{-1}\tau) - \xi_\varepsilon(\tau)\right| > \varepsilon^q\right) = 0. \tag{6.2}$$

To prove item (ii) of Theorem 2.1, we shall identify the limiting equation satisfied by ξ_ε. To this end we need a few lemmata. Recalling the definition (5.12) of S_n, we denote by $S_\varepsilon(\tau)$ the continuous process defined, as in (6.1), by the linear interpolation of S_n.

The first lemma relies on standard martingale arguments to show the weak convergence of S_n to a Brownian motion. For completeness, we, however, detail the proof.

Lemma 6.1. *As $\varepsilon \to 0$, the process S_ε converges weakly in $C(\mathbb{R}_+)$ to a Brownian motion with diffusion coefficient $\frac{3}{4}$.*

Proof. Recalling (5.14), an application of the Doob inequality yields, for any $\tau \in \mathbb{R}_+$, $\eta > 0$

$$\lim_{\delta\to 0}\ \overline{\lim_{\varepsilon\to 0}}\ \mathbb{P}\left(\sup_{\substack{\tau_1,\tau_2\in[0,\tau]\\ |\tau_2-\tau_1|<\delta}} |S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1)| > \eta\right) = 0. \tag{6.3}$$

Since $S_\varepsilon(0) = 0$, by [4, Theorem 8.2], $\{S_\varepsilon\}$ is tight.

Let S be a weak limit of S_ε, we shall prove that $S(\tau)$ and $S(\tau)^2 - \frac{3}{4}\tau$ are martingales. By Levy's characterization theorem we then get the result. By (5.14) we have that, for each $\tau \in \mathbb{R}_+$, $\mathbb{E}\left(S_\varepsilon(\tau)^2\right)$ is bounded uniformly as $\varepsilon \to 0$. Let $0 \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_n \leqq \tau_1 < \tau_2$, F be a bounded continuous function on $\mathbb{R}^n$, and consider a subsequence, still denoted by ε, converging to zero such that S_ε converges weakly to S. We then have, by the boundedness of F and the uniform integrability of $S_\varepsilon(\tau)$,

$$\begin{aligned}&\mathbb{E}\left([S(\tau_2) - S(\tau_1)]\, F\left(S(s_1), \ldots, S(s_n)\right)\right)\\ &\quad= \lim_{\varepsilon\to 0} \mathbb{E}\left([S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1)]\, F\left(S_\varepsilon(s_1), \ldots, S_\varepsilon(s_n)\right)\right) = 0,\end{aligned}$$

where we used

$$S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1) = \sum_{k=n_\varepsilon(\tau_1)}^{n_\varepsilon(\tau_2)-1} \sigma_k + (\tau_2 - \varepsilon T n_\varepsilon(\tau_2))\, \sigma_{n_\varepsilon(\tau_2)} - (\tau_1 - \varepsilon T n_\varepsilon(\tau_1))\, \sigma_{n_\varepsilon(\tau_1)},$$

so that $\mathbb{E}\left(S_\varepsilon(\tau_2) - S_\varepsilon(\tau_1)\,\middle|\,\mathcal{F}_{T_{n_\varepsilon(\tau_1)}}\right) = 0$. As F, τ_1 and τ_2 were arbitrary, we obtain the fact that $S(\tau)$ is a martingale.

To show the second martingale relationship, we first prove the uniform integrability of $S_\varepsilon(\tau)^2$. It is enough to show that, for each $\tau \in \mathbb{R}_+$,

$$\varlimsup_{\varepsilon\to 0} \mathbb{E}\left[\sum_{k=0}^{n_\varepsilon(\tau)} \sigma_k\right]^4 < \infty,$$

which is proven as follows. By (5.12), S_n is a $\mathcal{F}_{T_n}$-martingale with quadratic variation $[S]_n = \sum_{k=0}^{n-1} \sigma_k^2$. By the BDG inequality, see, for example [18, VII, Section 3], (5.14), and the uniform bound $\mathbb{E}\left(\sigma_k^4 | \mathcal{F}_{T_k}\right) \leqq C(\varepsilon T)^2$ for some $C > 0$, which follows by a Gaussian computation, we get the above bound.

By (4.2) we have that, for each $\tau \in \mathbb{R}_+$,

$$\lim_{\varepsilon\to 0} \sup_{0\leqq n\leqq n_\varepsilon(\tau)} \frac{1}{\varepsilon T}\left|\mathbb{E}\left(\sigma_n^2 | \mathcal{F}_{T_n}\right) - \frac{3}{4}\varepsilon T\right| = 0,$$

which implies

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\left[\sum_{k=n_\varepsilon(\tau_1)}^{n_\varepsilon(\tau_2)-1} \sigma_k\right]^2 - \frac{3}{4}(\tau_2 - \tau_1)\,\middle|\,\mathcal{F}_{T_{n_\varepsilon(\tau_1)}}\right) = 0.$$

Thanks to the uniform integrability of $S_\varepsilon(\tau)^2$, we conclude that $S(\tau)^2 - \frac{3}{4}\tau$ is a martingale. □

Lemma 6.2. *For each sequence $\varepsilon \to 0$ the sequence ξ_ε is tight in $C(\mathbb{R}_+)$.*

Proof. From (2.6) and Proposition 2.1, $\xi_\varepsilon(0) \to 0$, so by [4, Theorem 8.2], it is enough to show that for each $\tau \in \mathbb{R}_+$, $\eta > 0$ we have

$$\lim_{\delta\to 0} \varlimsup_{\varepsilon\to 0} \mathbb{P}\left(\sup_{\substack{\tau_1,\tau_2\in[0,\tau]\\ |\tau_2-\tau_1|<\delta}} |\xi_\varepsilon(\tau_2) - \xi_\varepsilon(\tau_1)| > \eta\right) = 0. \tag{6.4}$$

By (6.1) and (5.11), to prove (6.4) it is enough to show that, for each $L < \infty$

$$\lim_{\delta\to 0} \varlimsup_{\varepsilon\to 0} \mathbb{P}\left(\sup_{\substack{\tau_1,\tau_2\in[0,\tau]\\ |\tau_2-\tau_1|<\delta}} \left|\xi_{n_\varepsilon(\tau_2)} - \xi_{n_\varepsilon(\tau_1)}\right| > \eta\,,\ \sup_{0\leqq n\leqq n_\varepsilon(\tau)} |\xi_n| \leqq L\right) = 0. \tag{6.5}$$

By Theorem 5.1, (4.10), and (5.18), for $\tau_1 < \tau_2$,

$$\xi_{n_\varepsilon(\tau_2)} - \xi_{n_\varepsilon(\tau_1)} = \sum_{k=n_\varepsilon(\tau_1)}^{n_\varepsilon(\tau_2)-1} \left(12\,\varepsilon T\, \mathrm{e}^{-4\xi_k} + \sigma_k\right) + R_\varepsilon(\tau_1, \tau_2),$$

where for each $\tau \in \mathbb{R}_+$ there exists $q > 0$ so that

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{\tau_1, \tau_2 \in [0,\tau]} |R_\varepsilon(\tau_1, \tau_2)| > \varepsilon^q \right) = 0.$$

By (6.3) it is now straightforward to conclude the proof of (6.5). □

Lemma 6.3. *For each $\delta > 0$, $\theta \in \mathbb{R}_+$, and $\mu = 1, \lambda$,*

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{s \in [0,\mu\theta]} \left| \xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s \mathrm{d}u \, 12 \exp\{-4\, \xi_\varepsilon(u)\} \right| > \delta\sqrt{\mu} \right) = 0.$$

Remark. In this section, the above lemma is used for $\mu = 1$; we shall use it with $\mu = \lambda$ in proving item (iii) in Theorem 2.1.

Proof of Lemma 6.3. By Lemma 5.3 it is enough to show that, for each $L > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{s \in [0,\mu\theta]} \left| \xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s \mathrm{d}u \, 12 \exp\{-4\, \xi_\varepsilon(u)\} \right| > \delta\sqrt{\mu}, \right.$$
$$\left. \sup_{0 \leqq n \leqq n_\varepsilon(\mu\theta)+1} |\xi_n| \leqq L\sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda. \tag{6.6}$$

Recalling the definition of ξ_n in (5.1), we see that the bound (4.11) and Proposition 3.1 yield $|\xi_{n+1} - \xi_n| \leqq C\varepsilon^{\frac{1}{2}-\eta}\sqrt{T}$ for $n \leqq n_\varepsilon(\lambda\theta)$ on a set of probability converging to 1 as $\varepsilon \to 0$ by (4.10). By definition (6.1), for each $\theta \in \mathbb{R}_+$, $\delta > 0$, and $L > 0$, we have

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{s \in [0,\mu\theta]} \left| \sum_{n=0}^{n_\varepsilon(s)} \varepsilon T \mathrm{e}^{-4\xi_n} - \int_0^s \mathrm{d}u \, \mathrm{e}^{-4\xi_\varepsilon(u)} \right| > \delta\sqrt{\mu}, \right.$$
$$\left. \sup_{0 \leqq n \leqq n_\varepsilon(\mu\theta)} |\xi_n| \leqq L\sqrt{\mu} \right) = 0, \quad \mu = 1, \lambda, \tag{6.7}$$

as it can be easily shown by the change of variable $u = \varepsilon t$ in the integral and using $\left|\mathrm{e}^{-4\xi_{n+1}} - \mathrm{e}^{-4\xi_n}\right| \leqq 4\,\mathrm{e}^{4\max\{|\xi_n|;|\xi_{n+1}|\}}|\xi_{n+1} - \xi_n|$. The proof of (6.6) is now completed by using Theorem 5.1, (4.10), and (5.18). □

Proof of Theorem 2.1, item (ii). Thanks to (6.2) it is enough to prove the statement for ξ_ε in place of Y_ε. Let us denote by P_ε, a probability on $C(\mathbb{R}_+) \times C(\mathbb{R}_+)$, the law of the process $(S_\varepsilon, \xi_\varepsilon)$. By Lemmata 6.1 and 6.2 there exists a subsequent $\varepsilon \to 0$ and a probability P such that P_ε converges weakly to P. Moreover, again by Lemma 6.1, the second marginal of P is the law of a Brownian motion with diffusion coefficient $\frac{3}{4}$. Denoting by $(x(\cdot), y(\cdot))$ the canonical coordinates in $C(\mathbb{R}_+)\times C(\mathbb{R}_+)$, for each $\delta > 0$ and $\tau \in \mathbb{R}_+$, we have

$$P\left(\sup_{s\in[0,\tau]} \left| y(s) - x(s) - \int_0^s \mathrm{d}u\, 12\exp\{-4y(u)\} \right| > \delta \right)$$
$$\leqq \varliminf_{\varepsilon\to 0} P_\varepsilon \left(\sup_{s\in[0,\tau]} \left| y(s) - x(s) - \int_0^s \mathrm{d}u\, 12\exp\{-4y(u)\} \right| > \delta \right)$$
$$= \varliminf_{\varepsilon\to 0} \mathbb{P} \left(\sup_{s\in[0,\tau]} \left| \xi_\varepsilon(s) - S_\varepsilon(s) - \int_0^s \mathrm{d}u\, 12\exp\{-4\xi_\varepsilon(u)\} \right| > \delta \right) = 0,$$

where in the last step, we used Lemma 6.3 with $\mu = 1$. As δ and τ were arbitrary, it follows that any limit point solves (2.8). In fact this also proves the existence of a weak solution to (2.8). Since the real function $y \to 12\mathrm{e}^{-4y}$ is locally Lipschitz, by [17, Theorem 5.2.5] there is pathwise uniqueness of (2.8). By [17, Corollary 5.3.23] it follows that there is a strong solution to (2.8), which is unique in the sense of the probability law. We then conclude that Y_ε weakly converges to the unique strong solution of (2.8) □

7. Convergence to the hard wall

To prove item (iii) of Theorem 2.1, we first state and prove an analogous result for the diffusive scaling of the stochastic equation (2.8). To simplify the notation we introduce a probabilistic model not related with the one introduced in Section 2 and denote by t the macroscopic time variable. Let B be a Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ and γ a positive parameter that will eventually diverge. We suppose given a sequence of $\mathcal{F}_t$-adapted continuous processes B_γ such that $B_\gamma(0) = 0$ and satisfying that for each $T \geqq 0$,

$$\mathcal{P}\left(\lim_{\gamma\to\infty} \sup_{t\in[0,T]} \left| B_\gamma(t) - B(t) \right| = 0 \right) = 1. \tag{7.1}$$

We consider the sequence of processes that solve the equation

$$Y_\gamma(t) = \gamma \int_0^t \mathrm{d}s\, \left[Y_\gamma(s)\right]_- + B_\gamma(t), \tag{7.2}$$

where $[Y]_- = \max\{0, -Y\}$ is the negative part of Y. We shall prove that Y_γ converges to a Brownian motion reflected at the origin. The precise statement is the following.

Theorem 7.1. *Let*

$$Y(t) := B(t) + \sup_{s\in[0,t]} \{-B(s)\}. \tag{7.3}$$

Then, for any $T \geqq 0$*,*

$$\mathcal{P}\left(\lim_{\gamma\to\infty}\sup_{t\in[0,T]}\left|Y_\gamma(t)-Y(t)\right|=0\right)=1.$$

Note that, by, for example [17, Theorem 6.17], Y has the law of a Brownian motion reflected at the origin.

Proof. Let

$$r_\gamma(T) := \sup_{t\in[0,T]}\sup_{\gamma'>\gamma}\left|B_{\gamma'}(t)-B_\gamma(t)\right| \tag{7.4}$$

and note that by (7.1), for each $T \in \mathbb{R}_+$ we have $r_\gamma(T) \to 0$ $\mathcal{P}$-a.s. as $\gamma \to \infty$. We claim that for $\gamma_1 < \gamma_2$, $t \in [0, T]$, we have

$$Y_{\gamma_2}(t) \geqq Y_{\gamma_1}(t) - 2\,r_{\gamma_1}(T). \tag{7.5}$$

Indeed, if $Y_{\gamma_2}(t) \geqq Y_{\gamma_1}(t)$ there is nothing to prove, otherwise let $\tau = \sup\{s \in [0,t] : Y_{\gamma_2}(s) \geqq Y_{\gamma_1}(s)\}$ which exists because $Y_{\gamma_2}(0) = Y_{\gamma_1}(0)$. By definition, $Y_{\gamma_2}(s) \leqq Y_{\gamma_1}(s)$ for $s \in [\tau, t]$; by writing the equation (7.2) in this interval and using the monotonicity of $x \mapsto [x]_-$ the bound (7.5) follows easily.

We next claim that

$$Y_\gamma(t) \leqq B_\gamma(t) + \sup_{s\in[0,t]}\{-B_\gamma(s)\} =: w_\gamma(t). \tag{7.6}$$

This can be proved as follows. We first note that $w_\gamma \geqq 0$. Let $t \geqq 0$, if $Y_\gamma(t) \leqq 0$ there is nothing to prove, otherwise, setting $\tau = \sup\{s \in [0,t] : Y_\gamma(s) = 0\}$ we have

$$\begin{aligned} Y_\gamma(t) &= Y_\gamma(t) - Y_\gamma(\tau) \\ &= B_\gamma(t) - B_\gamma(\tau) + \int_\tau^t \mathrm{d}s\,\gamma\left[Y_\gamma(s)\right]_- = B_\gamma(t) - B_\gamma(\tau) \leqq w_\gamma(t), \end{aligned}$$

where we used that $\left[Y_\gamma(s)\right]_- = 0$ for $s \in [\tau, t]$.

Let $Z(t) := \underline{\lim}_{\gamma\to\infty} Y_\gamma(t)$. By (7.1) and (7.6) we have $Z(t) \leqq Y(t)$. It is easy to show, by (7.5), that $\mathcal{P}$-a.s. $\lim_{\gamma\to\infty} Y_\gamma(t) = Z(t)$. To complete the proof of the theorem we shall prove: Z is a.s. continuous, $Z \geqq 0$, there exists a continuous increasing process ℓ so that $Z = B + \ell$ and $\int_0^\infty \mathrm{d}\ell(t)\,Z(t) = 0$. Then from the Skorohod Lemma, see, for example [17, Lemma 6.14], it follows $Z = Y$.

For $f \in C(\mathbb{R}_+)$, $\delta > 0$, and $T > 0$, we let $\omega_{\delta,T}(f)$ be the modulus of continuity of the function f on $[0, T]$, that is

$$\omega_{\delta,T}(f) := \sup_{\substack{s,t\in[0,T]\\|t-s|<\delta}} |f(t) - f(s)|.$$

We first show the a priori bound:

$$\inf_{t\in[0,T]} Y_\gamma(t) \geqq -2\,\omega_{\delta,T}(B_\gamma) - 4\,\mathrm{e}^{-\delta\gamma}\sup_{t\in[0,T]}|B_\gamma(t)|. \tag{7.7}$$

Indeed, pick $\tau \in [0,T]$ such that $\inf_{t\in[0,T]} Y_\gamma(t) = Y_\gamma(\tau)$. If $Y_\gamma(\tau) = 0$ there is nothing to prove, otherwise let $\sigma = \sup\{t \in [0,\tau] : Y_\gamma(t) = 0\}$. For $t \in [\sigma,\tau]$ we can integrate the Equation (7.2) getting:

$$\begin{aligned} Y_\gamma(\tau) &= B_\gamma(\tau) - B_\gamma(\sigma) - \int_\sigma^\tau \mathrm{d}s\, \gamma \mathrm{e}^{-(\tau-s)\gamma}\, [B_\gamma(s) - B_\gamma(\sigma)] \\ &= \mathrm{e}^{-(\tau-\sigma)\gamma}\, [B_\gamma(\tau) - B_\gamma(\sigma)] + \int_\sigma^\tau \mathrm{d}s\, \gamma \mathrm{e}^{-(\tau-s)\gamma} [B_\gamma(\tau) - B_\gamma(s)] \\ &= \mathrm{e}^{-(\tau-\sigma)\gamma}\, [B_\gamma(\tau) - B_\gamma(\sigma)] + \int_\sigma^{\sigma\vee(\tau-\delta)} \mathrm{d}s\, \gamma \mathrm{e}^{-(\tau-s)\gamma}\, [B_\gamma(\tau) - B_\gamma(s)] \\ &\quad + \int_{\sigma\vee(\tau-\delta)}^\tau \mathrm{d}s\, \gamma \mathrm{e}^{-(\tau-s)\gamma}\, [B_\gamma(\tau) - B_\gamma(s)] \\ &\geqq -4\,\mathrm{e}^{-\delta\gamma} \sup_{t\in[0,T]} |B_\gamma(t)| - 2\,\omega_{\delta,T}(B_\gamma). \end{aligned}$$

We next bound the modulus of continuity of Y_γ. We claim that

$$\omega_{\delta,T}(Y_\gamma) \leqq 8 \left[\omega_{\delta,T}(B_\gamma) + \mathrm{e}^{-\delta\gamma} \sup_{t\in[0,T]} |B_\gamma(t)| \right]. \tag{7.8}$$

Let us fix $t, s \in [0,T]$ with $|t-s| < \delta$. We consider first the case in which $Y_\gamma(u) \leqq 0$ for any $u \in [s,t]$. Solving equation (7.2) in this time interval, we get

$$\begin{aligned} Y_\gamma(t) - Y_\gamma(s) = \left(\mathrm{e}^{-(t-s)\gamma} - 1\right) Y_\gamma(s) + B_\gamma(t) - B_\gamma(s) \\ - \int_s^t \mathrm{d}u\, \gamma \mathrm{e}^{-(t-u)\gamma}\, \left[B_\gamma(u) - B)_\gamma(s)\right], \end{aligned}$$

so that, by (7.7),

$$\begin{aligned} \left|Y_\gamma(t) - Y_\gamma(s)\right| &\leqq \left|Y_\gamma(s)\right| + 2\,\omega_{\delta,T}(B_\gamma) \\ &\leqq 4 \left[\omega_{\delta,T}(B_\gamma) + \mathrm{e}^{-\delta\gamma} \sup_{t\in[0,T]} |B_\gamma(t)| \right]. \end{aligned} \tag{7.9}$$

The case in which $Y_\gamma(u) \geqq 0$ for any $u \in [s,t]$ we clearly have $\left|Y_\gamma(t) - Y_\gamma(s)\right| \leqq \omega_{\delta,T}(B_\gamma)$. The other cases can be reduced to the previous ones. We discuss only the case $Y_\gamma(s) < 0$, $Y_\gamma(t) < 0$. Let $\sigma = \inf\{u > s : Y_\gamma(u) = 0\}$ and $\tau = \sup\{u < t : Y_\gamma(u) = 0\}$. We then write $\left|Y_\gamma(t) - Y_\gamma(s)\right| = \left|Y_\gamma(t) - Y_\gamma(\tau)\right| + \left|Y_\gamma(\sigma) - Y_\gamma(s)\right|$ and use the bound (7.9) in the intervals $[s,\sigma]$ and $[\tau,t]$ to get (7.8).

By taking the limit as $\gamma \to \infty$ in (7.8), we get that the limiting process Z is continuous. Let $\bar{Y}_\gamma(t) := \inf_{\gamma'\geqq\gamma} Y_{\gamma'}(t)$ so that $\bar{Y}_\gamma(t) \uparrow Z(t)$. By the continuity of Z, the previous convergence is in fact uniform for t on compacts. By using (7.5) we get that $Y_\gamma(t) - \bar{Y}_\gamma(t)$ converges, $\mathcal{P}$-a.s., to zero uniformly for $t \in [0,T]$. Hence Y_γ converges to Z uniformly on compacts.

To show that $Z \geqq 0$, we note that

$$\int_0^t \mathrm{d}s\, \left[Y_\gamma(s)\right]_- = \frac{1}{\gamma}\left[Y_\gamma(t) - B_\gamma(t)\right].$$

By taking the limits $\gamma \to \infty$ and then $t \to \infty$, we get $\int_0^\infty \mathrm{d}s\, [Z(s)]_- = 0$, whence $Z \geqq 0$ by the continuity of Z.

Let us introduce the increasing process

$$\ell_\gamma(t) := \int_0^t \mathrm{d}s\, \gamma \left[Y_\gamma(s)\right]_- = Y_\gamma(t) - B_\gamma(t).$$

By the convergence of Y_γ to the continuous process Z,

$$\ell(t) := \lim_{\gamma\to\infty} \ell_\gamma(t) = Z(t) - B(t)$$

is a continuous increasing process. In particular the Lebesgue–Stieltjes measure $\mathrm{d}\ell_\gamma$ weakly converges to $\mathrm{d}\ell$ as $\gamma \to \infty$. To finally show $\int_0^\infty \mathrm{d}\ell(t)\, Z(t) = 0$ we note that the support of the measure $\mathrm{d}\ell_\gamma$ is a subset of $\{t \geqq 0 : Y_\gamma(t) \leqq 0\}$. By the uniform convergence of Y_γ to Z and the weak convergence of $\mathrm{d}\ell_\gamma$ to $\mathrm{d}\ell$, we have, for each $T \in \mathbb{R}_+$,

$$\int_0^T \mathrm{d}\ell(t)\, Z(t) = \lim_{\gamma\to\infty} \int_0^T \mathrm{d}\ell_\gamma(t)\, Y_\gamma(t) \leqq 0,$$

and we are done since $Z \geqq 0$. □

Given $\gamma > 0$, let X_γ be the solution of the equation

$$X_\gamma(t) = \gamma \int_0^t \mathrm{d}s\, 12 \exp\{-4\gamma X_\gamma(s)\} + B_\gamma(t). \tag{7.10}$$

Note that if $Y(\tau)$ solves (2.8) then $X_\lambda(t) := \lambda^{-1/2} Y(\lambda t)$ solves (7.10) in law with $\gamma = \sqrt{\lambda}$ and B_γ a Brownian motion for each γ.

Corollary 7.2. *As $\gamma \to \infty$ the process X_γ converges $\mathcal{P}$ almost surely to the continuous process Y defined by* (7.3).

Proof. For given $\delta > 0$, set $c_{\delta,\gamma} := 12\gamma \mathrm{e}^{-4\gamma\delta}$ and define the continuous process $Z_{\delta,\gamma}$ as

$$Z_{\delta,\gamma}(t) := \delta + B_\gamma(t) + c_{\delta,\gamma} t + \sup_{s\in[0,t]} \left[-B_\gamma(s) - c_{\delta,\gamma} s\right]. \tag{7.11}$$

Note that $Z_{\delta,\gamma}(0) = \delta$. Recall that Y_γ is the solution of (7.2). By arguing as in the proof of Theorem 7.1, we find that the following comparison holds. For each $\delta > 0$ and $\gamma > 1$, we have, $\mathcal{P}$ almost surely,

$$Y_\gamma \leqq X_\gamma \leqq Z_{\delta,\gamma}, \tag{7.12}$$

from which, by using Theorem 7.1, we find that the statement follows by taking first the limit as $\gamma \to \infty$ and then as $\delta \to 0$. □

We are now ready to conclude the proof of our main result. We next denote by θ the macroscopic time variable and recall $\lambda = \log \varepsilon^{-1}$. Recalling ξ_ε is defined in (6.1), let ζ_ε be the continuous process defined as $\zeta_\varepsilon(\theta) := \lambda^{-1/2} \xi_\varepsilon(\lambda\theta)$.

Lemma 7.3. *Let*

$$B_\varepsilon(\theta) := \zeta_\varepsilon(\theta) - \sqrt{\lambda}\int_0^\theta \mathrm{d}s\; 12\, \exp\{-4\sqrt{\lambda}\,\zeta_\varepsilon(s)\}. \tag{7.13}$$

The process B_ε weakly converges in $C(\mathbb{R}_+)$ to a Brownian motion with diffusion coefficient $\frac{3}{4}$.

Proof. Recalling S_ε is the linear interpolation of the sequence S_n defined in (5.12), let $\bar{S}_\varepsilon(\theta) := \lambda^{-1/2} S_\varepsilon(\lambda\theta)$. By arguing exactly as in Lemma 6.1, one shows that the process $\bar{S}_\varepsilon$ weakly converge in $C(\mathbb{R}_+)$ to a Brownian motion with diffusion $\frac{3}{4}$. Moreover, by Lemma 6.3 with $\mu = \lambda$, for each $\delta > 0$, $\theta \in \mathbb{R}_+$, we have

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{s\in[0,\theta]} \left| \zeta_\varepsilon(s) - \bar{S}_\varepsilon(s) - \sqrt{\lambda}\int_0^s \mathrm{d}u\; 12\, \mathrm{e}^{-4\sqrt{\lambda}\,\zeta_\varepsilon(u)} \right| > \delta \right) = 0,$$

which concludes the proof. □

Proof of Theorem 2.1, item (iii). Thanks to (6.2) it is enough to prove the statement for ζ_ε in place of Z_ε. By Lemma 7.3 and [18, Theorem III.8.1], there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and random elements B^*, B^*_ε, with values in $C(\mathbb{R}_+)$ such that B^* is a Brownian motion with diffusion coefficient $\frac{3}{4}$, the law of B^*_ε equals the one of B_ε defined in (7.13), and B^*_ε converges, $\mathbb{P}^*$ almost surely, to B^*. We now define ζ^*_ε as the solution to the equation

$$\zeta^*_\varepsilon(\theta) = B^*_\varepsilon(\theta) - \sqrt{\lambda}\int_0^\theta \mathrm{d}s\; 12\, \exp\{-4\sqrt{\lambda}\,\zeta^*_\varepsilon(s)\}.$$

By uniqueness of its solution, the law of ζ^*_ε equals the one of ζ_ε. By Corollary 7.2 $\zeta^*_\varepsilon(\theta)$ converges, $\mathbb{P}^*$ almost surely, to $B^*(\theta) + \sup_{s\leqq\theta}\{-B^*(s)\}$, whose law is that of a Brownian motion with diffusion coefficient $\frac{3}{4}$ reflected at the origin. □

8. Spectral analysis

In this section, we prove Theorem 3.2. To keep the notation simple we shall define the operator

$$H = -\frac{1}{2}\Delta + V''(\bar{m}), \quad \bar{m}(x) := \mathrm{th}(x), \tag{8.1}$$

acting on $L_2([-a, b])$ with Dirichlet boundary conditions. We denote by $\lambda_0 < \lambda_1 < \cdots < \lambda_i < \cdots$, respectively, Ψ_i (recall Ψ_0 is chosen positive), $i \geqq 0$, the eigenvalues, respectively, the eigenfunctions, of H and by $g_t := \exp\{-tH\}$ the corresponding semigroup. The operators $g_t^\perp$ and $G^\perp$ are defined as in (3.12) and (3.13).

By the standard techniques, it is not difficult to compute the Green operator $G = H^{-1}$ for the quartic double well potential V in (2.2) obtaining that its integral kernel is given by:

$$G(x, y) = \frac{2\bar{m}'(x)\bar{m}'(y)}{h(b) + h(a)} \times \begin{cases} [h(x) + h(a)]\,[h(b) - h(y)] & \text{if } -a \leqq x \leqq y \leqq b, \\ [h(y) + h(a)]\,[h(b) - h(x)] & \text{if } -a \leqq y < x \leqq b, \end{cases} \tag{8.2}$$

where, recalling (3.6),

$$h(x) := h_0(x) = \frac{3}{8}x + \frac{3}{8}\frac{\bar{m}(x)}{\bar{m}'(x)} + \frac{1}{4}\frac{\bar{m}(x)}{\bar{m}'(x)^2}. \tag{8.3}$$

Notation warning. In the sequel we will denote by C a generic positive constant, independent of a, b, whose numerical value may change from line to line and from one side to the other in an inequality.

We first obtain some rougher estimates by following the approach in [10, Lemma 2.1] where analogous bounds are proven in the case of Neumann boundary conditions.

Lemma 8.1. *There exists $K > 0$ and $a_* > 0$ such that, for any $b \geqq a \geqq a_*$,*

$$0 \leqq \lambda_0 \leqq K\,\mathrm{e}^{-4a}, \tag{8.4}$$

$$\langle \Psi_0, \bar{m}' \rangle \geqq \frac{1}{K}, \tag{8.5}$$

$$\lambda_1 - \lambda_0 \geqq \frac{1}{K}, \tag{8.6}$$

$$\|\Psi_0\|_\infty + \|\Psi_0'\|_\infty \leqq K. \tag{8.7}$$

Sketch of the proof. Step 1. An elementary computation shows that, for each $f \in C_0^2([-a, b])$,

$$\langle f, Hf \rangle = \frac{1}{2}\int_{-a}^{b} \mathrm{d}x\, \bar{m}'(x)^2 \left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{\bar{m}'(x)}\right]^2 \geqq 0,$$

which in particular implies $\lambda_0 \geqq 0$. On the other hand, by using $G\bar{m}'$ as test function in the variational characterization of the smallest eigenvalue,

$$\lambda_0 \leqq \frac{\langle G\bar{m}', HG\bar{m}' \rangle}{\|G\bar{m}'\|_2^2} = \frac{\langle \bar{m}', G\bar{m}' \rangle}{\|G\bar{m}'\|_2^2}. \tag{8.8}$$

From (8.2) we now get

$$G\bar{m}'(x) = 2h(a)\,\|\bar{m}'\|_2^2\,\bar{m}'(x) + 2\bar{m}'(x)B(x) + A(x), \tag{8.9}$$

where

$$A(x) = \frac{-2\bar{m}'(x)\,[h(x)+h(a)]}{h(b)+h(a)} \int_{-a}^{b} \mathrm{d}y\, \bar{m}'(y)^2\,[h(y)+h(a)], \tag{8.10}$$

$$B(x) = \int_{-a}^{x} \mathrm{d}y\, \bar{m}'(y)^2 h(y) + h(x) \int_{x}^{b} \mathrm{d}y\, \bar{m}'(y)^2. \tag{8.11}$$

Then

$$\langle \bar{m}', G\bar{m}' \rangle = 2h(a)\,\|\bar{m}'\|_2^4 + 2\langle \bar{m}', \bar{m}'B \rangle + \langle \bar{m}', A \rangle, \tag{8.12}$$

$$\begin{aligned} \|G\bar{m}'\|_2^2 &= 4h^2(a)\,\|\bar{m}'\|_2^6 + 8h(a)\,\|\bar{m}'\|_2^2\,\langle \bar{m}', \bar{m}'B \rangle + 4\|\bar{m}'B\|_2^2 \\ &\quad + 4h(a)\,\|\bar{m}'\|_2^2\,\langle \bar{m}', A \rangle + 4\langle \bar{m}'B, A \rangle + \|A\|_2^2. \end{aligned} \tag{8.13}$$

From (8.10) and (4.3), we get

$$\|A\|_\infty \leqq C\,b\,\frac{h^2(a)}{h(b)^{1/2}}, \quad \|A\|_2 \leqq C\,b\,\frac{h^2(a)}{h(b)^{1/2}}, \tag{8.14}$$

and, from (8.11) and (4.3),

$$|\bar{m}'(x)\,B(x)| \leqq \bar{m}'(x)(x+a) + C\,\mathrm{e}^{-2x}, \tag{8.15}$$

so that, after integrating,

$$\langle \bar{m}', \bar{m}'B \rangle \leqq C\,a, \quad \|\bar{m}'B\|_2^2 \leqq C\,\mathrm{e}^{4a}. \tag{8.16}$$

Substituting (8.12) and (8.13) in (8.8) the bound (8.4) follows.
Step 2. Let ψ be an eigenfunction associated to an eigenvalue $\lambda \leqq 1/2$ and choose a real ℓ_0 such that $\inf_{|x|\geqq \ell_0} V''(\bar{m}(x)) \geqq 3/2$. By comparison principle, we get

$$\begin{aligned} |\psi(x)| &\leqq |\psi(\ell)|\,\frac{\mathrm{sh}(\sqrt{2}(b-x))}{\mathrm{sh}(\sqrt{2}(b-\ell))}, \quad \forall\, \ell_0 \leqq \ell \leqq x \leqq b, \\ |\psi(x)| &\leqq |\psi(-\ell)|\,\frac{\mathrm{sh}(\sqrt{2}(x+a))}{\mathrm{sh}(\sqrt{2}(-\ell+a))}, \quad \forall\, -a \leqq x \leqq -\ell \leqq -\ell_0. \end{aligned} \tag{8.17}$$

Since ψ is normalized there exist reals $\ell_+ \in [\ell_0, \ell_0+1]$ and $\ell_- \in [-\ell_0-1, -\ell_0]$ such that $|\psi(\ell_\pm)| \leqq 1$. Hence, for any $b \geqq a > \ell_0 + 1$,

$$|\psi(x)| \leqq C\exp\{-\sqrt{2}|x|\} \quad \forall\, |x| \geqq \ell_0 + 1. \tag{8.18}$$

Step 3. By (8.18) there exist reals $\ell^*, a^* > 0$ such that $\int_{-\ell^*}^{\ell^*} \mathrm{d}x\, \Psi_0(x)^2 \geqq 1/2$ for any $a > a^*$. Since λ_0 is uniformly bounded by (8.4), by the Harnack inequality applied to the equation $[H - \lambda_0]\Psi_0 = 0$ in the interval $[-\ell^*-1, \ell^*+1]$ we get that, for any $b \geqq a \geqq a^*$, we have

$$\inf_{|x|\leqq \ell^*} \Psi_0(x) \geqq C \sup_{|x|\leqq \ell^*} \Psi_0(x) \geqq C \left[\frac{1}{2\ell^*} \int_{-\ell^*}^{\ell^*} \mathrm{d}x\, \Psi_0(x)^2 \right]^{1/2} \geqq \frac{C}{2\sqrt{\ell^*}}.$$

The above bound and $\bar{m}'(x) = \mathrm{ch}(x)^{-2}$ yield (8.5).

Step 4. We can assume $\lambda_1 \leqq 1/2$. As it is well known, the corresponding eigenfunction Ψ_1 has a unique zero x_0 in the open interval $(-a, b)$; moreover, by (8.17), $|x_0| < \ell_0$. Integration by parts and $H\bar{m}' = 0$ yields

$$\lambda_1 \geqq \lambda_1 \left| \int_{x_0}^{b} \mathrm{d}x\, \Psi_1(x)\, \bar{m}'(x) \right| = \frac{1}{2} \left| (\Psi_1'\bar{m}')(x_0) - (\Psi_1'\bar{m}')(b) \right| \geqq \frac{1}{2} \left| (\Psi_1'\bar{m}')(x_0) \right|$$

since $\mathrm{sgn}[(\Psi_1)'(x_0)\,(\Psi_1)'(b)] = -1$.

By the same argument as in Step 3, we have that either $\int_{x_0}^{\ell^*} \mathrm{d}x\, \Psi_1(x)^2 \geqq 1/4$ or $\int_{-\ell^*}^{x_0} \mathrm{d}x\, \Psi_1(x)^2 \geqq 1/4$. By using the Hopf maximum principle, we then deduce a lower bound on $|\Psi_1'(x_0)|$ which is uniform in $b \geqq a \geqq a^*$. The estimate (8.6) follows.

Step 5. A uniform bound for $\|\Psi_0\|_\infty$ follows from (8.18) and a comparison argument in the interval $[-\ell_0-1, \ell_0+1]$. Finally, since $H\Psi_0 = \lambda_0\Psi_0$ and $|V''(\bar{m})| \leqq 2$, we have $|\Psi_0''(x)| \leqq C\,\Psi_0(x)$. The bound (8.7) follows. □

Proof of Theorem 3.2. We observe that, given $\alpha \in (0, 1)$, it is equivalent to prove (3.15)–(3.22) for the operator H with

$$a = \frac{1}{4} \log \varepsilon^{-1} + \zeta, \quad b = \varepsilon^{-\beta} - \zeta, \quad |\zeta| \leqq \frac{1}{4} \alpha \log \varepsilon^{-1}, \tag{8.19}$$

and that Lemma 8.1 clearly holds for these values of a and b.

Proof of (3.15). By the Feynman–Kac formula, see, for example [13, Theorem 2.3], we have that, for any $f \in C([-a, b])$, $t > 0$, and $x \in (-a, b)$,

$$(g_t f)(x) = \mathbb{E}\left(f(B_t^{(x)})\, \mathbb{1}_{\{\tau_x > t\}} \exp\left\{ \int_0^t \mathrm{d}s\, V''(\bar{m}(B_s^{(x)})) \right\} \right), \tag{8.20}$$

where $\{B_t^{(x)},\ t \geqq 0\}$ is a Brownian motion starting at x and $\tau_x := \inf\{t \geqq 0 : B_t^{(x)} \notin (-a, b)\}$. The above representation permits one to compare g_t with the semigroup $\exp\{-t\bar{H}_0\}$, defined on the whole line $\mathbb{R}$. For the latter the analogous estimate has been proven in [5, Proposition A.8], whence

$$|(g_t f)(x)| \leqq (g_t|f|)(x) \leqq \left(\exp\{-t\bar{H}_0\}|f|\right)(x) \leqq C\|f\|_\infty.$$

Proof of (3.16). It is a restatement of (8.6).

Proof of (3.17). We will use an interpolation inequality, see [11, Lemma 5.1], that holds for each $F \in C^1([-a, b])$ such that $F(a) = F(b) = 0$,

$$\|F\|_\infty^3 \leqq \frac{3}{2} \|\nabla F\|_\infty \|F\|_2^2. \tag{8.21}$$

Recalling p_t^0 denotes the heat semigroup with zero boundary conditions at the endpoints of $[-a, b]$, we have the following:

$$\nabla g_t f = \nabla p_t^0 f - \int_0^t \mathrm{d}s\, \nabla p_{t-s}^0 V''(\bar{m}) g_s f.$$

Since $\|\nabla p_t^0 f\|_\infty \leqq C t^{-\frac{1}{2}} \|f\|_\infty$, by (3.15) and the above identity we conclude that $\|\nabla g_t f\|_\infty \leqq C\sqrt{t}\|f\|_\infty$ for any $t \geqq 1$. By choosing $F = g_t^\perp f$ in (8.21), the estimate (3.17) follows from (3.12), (8.7) and (3.16). □

To prove the estimates (3.18)–(3.22), we will use the Kellogg method, see, for example [19], to obtain successive approximations of the eigenvalues and eigenvectors by iterations of the Green operator G applied to the function $\phi(x) := \phi^{(0)}(x) = \|\bar{m}'\|_2^{-1}\bar{m}'(x)$, $x \in [-a,b]$. Let $f_0 := \phi$, $f_1 := Gf_0$, $f_2 := Gf_1$ and $e_1 := f_1/\|f_1\|_2$, $e_2 := f_2/\|f_2\|_2$ be their L_2-normalizations. Also, let

$$\mu := \frac{\|f_1\|_2}{\|f_2\|_2}, \quad R^2 := \sup_{x\in[-a,b]}\int_{-a}^{b} \mathrm{d}y\, G(x,y)^2, \quad c := \langle \Psi_0, \phi\rangle. \tag{8.22}$$

Then, by [19, Section 28.1], we have the estimates

$$\begin{aligned} &0 \leqq \mu - \lambda_0 \leqq \frac{\lambda_0}{2}\left[\frac{\lambda_0}{\lambda_1}\right]^2 \frac{1-c^2}{c^2}, \\ &\|\Psi_0 - e_1\|_2 \leqq \frac{\lambda_0}{\lambda_1}\frac{\sqrt{1-c^2}}{c}, \\ &\|\Psi_0 - e_2\|_\infty \leqq R\,\lambda_1\left[\frac{\lambda_0}{\lambda_1}\right]^2 \frac{\sqrt{1-c^2}}{c}. \end{aligned} \tag{8.23}$$

To use the above estimates, we will need expressions for e_i, $i = 1, 2$, and μ. They are given in terms of the following formulae. From (8.2) and (8.9), we have the following:

$$G^2\bar{m}'(x) = P(x) + U(x), \tag{8.24}$$

where

$$\begin{aligned} &P(x) = 4h^2(a)\,\|\bar{m}'\|_2^4\,\bar{m}'(x) + 4h(a)\,\|\bar{m}'\|_2^2\,\bar{m}'(x)B(x) + 2G(\bar{m}'B)(x), \\ &U(x) = 2h(a)\,\|\bar{m}'\|_2^2\,A(x) + GA(x). \end{aligned} \tag{8.25}$$

Also,

$$\begin{aligned} \|G^2\bar{m}'\|_2^2 &= 16h^4(a)\,\|\bar{m}'\|_2^{10} + 32h^3(a)\,\|\bar{m}'\|_2^6\,\langle\bar{m}', \bar{m}'B\rangle \\ &\quad + 16h^2(a)\,\|\bar{m}'\|_2^4\,\|\bar{m}'B\|_2^2 + 16h^2(a)\,\|\bar{m}'\|_2^4\,\langle G\bar{m}', \bar{m}'B\rangle \\ &\quad + 16h(a)\,\|\bar{m}'\|_2^2\,\langle\bar{m}'B, G(\bar{m}'B)\rangle \\ &\quad + 4\|G(\bar{m}'B)\|_2^2 + \|U\|_2^2 + 2\langle P, U\rangle. \end{aligned} \tag{8.26}$$

We finally remark that, by (8.5), $c = \|\bar{m}'\|_2^{-1}\langle\Psi_0, \bar{m}'\rangle$ is uniformly bounded from below by some positive constant.

Proof of (3.18). By (8.4) and (8.19), we find that, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-(1-\alpha)+\eta}\,\lambda_0 = 0. \tag{8.27}$$

From (8.23), (3.16), and (8.27), to prove (3.18) it is enough to show that

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-\frac{3}{2}(1-\alpha)+\eta}\left|\mu - 24\,\varepsilon\,\mathrm{e}^{-4\zeta}\right| = 0. \tag{8.28}$$

From (8.13), the estimates (8.14), (8.16), and (8.19), it follows that

$$\begin{aligned} &\|G\bar{m}'\|_2 = 2h(a)\|\bar{m}'\|_2^3\,(1+\Delta_1),\\ &\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac12(1-\alpha)+\eta}\Delta_1 = 0. \end{aligned} \tag{8.29}$$

Analogously, from (8.26) and the estimate $\|G(\bar{m}'\,B)\|_2^2 \leqq C\,a^4\,h^2(a)$ (that follows from (8.2) and (8.15)), together with (8.14) and (8.16),

$$\begin{aligned} &\|G^2\bar{m}'\|_2 = 4h^2(a)\|\bar{m}'\|_2^5\,(1+\Delta_2),\\ &\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac12(1-\alpha)+\eta}\Delta_2 = 0. \end{aligned} \tag{8.30}$$

Substitution of the previous expressions in the definition of μ yields

$$\mu = \frac{1}{2h(a)\,\|\bar{m}'\|_2^2}\,\frac{1+\Delta_1}{1+\Delta_2}, \tag{8.31}$$

from which (8.28) follows since, by (8.19), $\varepsilon\,\mathrm{e}^{-4\zeta} = \mathrm{e}^{-4a}$ and, by (8.3),

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac32(1-\alpha)+\eta}\left|\frac{1}{2h(a)\,\|\bar{m}'\|_2^2} - 24\,\mathrm{e}^{-4a}\right| = 0. \tag{8.32}$$

Proof of (3.19). By (8.22) and (8.2), we have $R^2 = \sup_{x\in[-a,b]} G(x,x) \leqq C\,h(a)$. From (8.23), (8.27), and (3.16), to prove (3.19) it is then enough to show that, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac12(1-\alpha)+\eta}\ \|e_2-\phi\|_\infty = 0. \tag{8.33}$$

From the definition of e_2, (8.24), (8.25), and (8.30), we have

$$\begin{aligned} e_2(x)-\phi(x) &= \frac{G^2\bar{m}'(x)}{\|G^2\bar{m}'\|_2} - \phi(x)\\ &= \frac{1}{1+\Delta_2}\left[\frac{\bar{m}'(x)B(x)}{h(a)\|\bar{m}'\|_2^3} + \frac{G(\bar{m}'B)(x)}{2h^2(a)\|\bar{m}'\|_2^5} + \frac{U(x)}{4h^2(a)\|\bar{m}'\|_2^5} - \Delta_2\,\phi(x)\right]. \end{aligned} \tag{8.34}$$

Now, by (8.14), (8.15), (8.25), and using the definition (8.2), it is easy to show that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac12(1-\alpha)+\eta}\ \frac{\|\bar{m}'B\|_\infty}{h(a)} = 0,\\ &\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac14\alpha\log\varepsilon^{-1}}\ \varepsilon^{-(1-\alpha)+\eta}\ \frac{\|U\|_\infty + \|G(\bar{m}'B)\|_\infty}{h^2(a)} = 0, \end{aligned} \tag{8.35}$$

from which (8.33) follows.

Proof of (3.20). For any $\ell < a$ we have

$$\|\Psi_0 - \phi\|_1 \leqq 2\ell\ \|\Psi_0 - \phi\|_\infty + \int_{[-a,b]\setminus[-\ell,\ell]} \mathrm{d}x\ (\Psi_0(x) + \phi(x))\,.$$

Then, by (3.19), (8.18), and recalling $\phi(x) \leqq \mathrm{e}^{-2|x|}$, we get (3.20) by choosing, for example $\ell = \log^4 \varepsilon^{-1}$.

Proof of (3.21). To prove (3.21) recall that, from (8.23), (8.27), and (3.16), it is sufficient to show that, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-(1-\alpha)+\eta}\ \langle |e_2 - \phi|\,, \phi\rangle = 0.$$

Substituting in $\langle |e_2 - \phi|\,, \phi\rangle$ the expression (8.34), since $\|\phi\|_1 \leqq C$, the limit above follows from (8.35) and the first estimate in (8.16).

Proof of (3.22). From the definition (8.2) and (3.6), we get

$$G^\perp(x,x) = 2\bar{m}'(x)^2\,[h(x)+h(a)]\left[1 - \frac{h(x)+h(a)}{h(b)+h(a)}\right] - \frac{\Psi_0^2(x)}{\lambda_0}. \tag{8.36}$$

Since $h(x)\bar{m}'(x)^2 \leqq C$ (see (4.3)),

$$\int_{-a}^{b} \mathrm{d}x\ \bar{m}'(x)^3\,|\bar{m}(x)|\ \frac{[h(x)+h(a)]^2}{h(b)+h(a)} \leqq C\,\frac{\sqrt{h(b)}+h^2(a)}{h(b)+h(a)}. \tag{8.37}$$

We next notice that, by (3.21) and the definition (8.22) of c, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta}\ \sqrt{1-c^2} = 0,$$

so that, by (8.23), (8.27), and (3.16),

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta} \int_{-a}^{b} \mathrm{d}x\ \bar{m}'(x)\,|\bar{m}(x)| \left|\frac{\Psi_0^2(x)}{\lambda_0} - \frac{e_1^2(x)}{\mu}\right| = 0. \tag{8.38}$$

On the other hand, from (8.29), (8.30), and (8.31),

$$\begin{aligned} &\frac{e_1(x)^2}{\mu} = \frac{1+\Delta_3}{2h(a)\|\bar{m}'\|_2^4}\left[G\bar{m}'(x)\right]^2, \\ &\lim_{\varepsilon\to 0} \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}} \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta}\Delta_3 = 0. \end{aligned} \tag{8.39}$$

Taking the square in (8.9) and substituting into (8.39), from (8.14), (8.15), (8.16) and (8.39), we find that it follows that

$$\frac{e_1(x)^2}{\mu} = 2h(a)(1+\Delta_3)\bar{m}'(x)^2 + \frac{4B(x)\bar{m}'(x)^2}{\|\bar{m}'\|_2^2} + W(x),$$

with

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta}\ \left\|(\bar{m}')^2\,\bar{m}\,W\right\|_1 = 0.$$

By (8.36), (8.37), (8.38), and the above limit, we are reduced to prove that, for each $\eta > 0$,

$$\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\frac{1}{4}\alpha\log\varepsilon^{-1}}\ \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta}\ Q(\zeta,\varepsilon) = 0, \tag{8.40}$$

where

$$Q(\zeta,\varepsilon) := \left|\int_{-a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\,\bar{m}(x)\left(2\,h(x) - \frac{4B(x)}{\|\bar{m}'\|_2^2} - 2\Delta_3 h(a)\right)\right|. \tag{8.41}$$

Now, since $(\bar{m}')^3\bar{m}$ is an odd function, we have

$$\left|\int_{-a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\bar{m}(x)\right| = \left|\int_{a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\bar{m}(x)\right| \leqq C\,\mathrm{e}^{-4a}. \tag{8.42}$$

From (8.11),

$$h(x) - \frac{2B(x)}{\|\bar{m}'\|_2^2} = h(x)\left[1 - 2\int_x^b \mathrm{d}y\,\frac{\bar{m}'(y)^2}{\|\bar{m}'\|_2^2}\right] - 2\int_{-a}^{x}\mathrm{d}y\,\frac{\bar{m}'(y)^2}{\|\bar{m}'\|_2^2}\,h(y).$$

Since $\bar{m}(x) = \mathrm{th}(x)$,

$$2\int_x^b \mathrm{d}y\,\frac{\bar{m}'(y)^2}{\|\bar{m}'\|_2^2} = \frac{3}{2}\int_x^\infty \mathrm{d}y\,\bar{m}'(y)^2 + D(x) = 1 + \frac{\bar{m}^3(x) - 3\bar{m}(x)}{2} + D(x),$$

with $|D(x)| \leqq C\mathrm{e}^{-4a}$. Then, recalling $h(x)\bar{m}'(x)^2 \leqq C$,

$$\begin{aligned}
&\left|\int_{-a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\bar{m}(x)\,h(x)\left[1 - 2\int_x^b \mathrm{d}y\,\frac{\bar{m}'(y)^2}{\|\bar{m}'\|_2^2}\right]\right| \\
&\quad\leqq C\,\mathrm{e}^{-4a} + \left|\int_{-a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\bar{m}(x)\,h(x)\,\frac{\bar{m}^3(x) - 3\bar{m}(x)}{2}\right| \\
&\quad\leqq C\,\mathrm{e}^{-2a},
\end{aligned} \tag{8.43}$$

where we have used that the integrand in the last integral is an odd function, see (8.3). Finally, observing that $-6(\bar{m}')^3\bar{m} = [(1-\bar{m}^2)^3]'$, integration by parts in the remaining integral yields

$$\begin{aligned}
&\left|\int_{-a}^{b}\mathrm{d}x\,\bar{m}'(x)^3\bar{m}(x)\int_{-a}^{x}\mathrm{d}y\,\bar{m}'(y)^2\,h(y)\right| \\
&\quad\leqq \left|\frac{\left(1-\bar{m}^2(b)\right)^3}{6}\int_{-a}^{b}\mathrm{d}y\,\bar{m}'(y)^2\,h(y)\right| \\
&\qquad + \left|\int_{-a}^{b}\mathrm{d}x\,\frac{\left(1-\bar{m}^2(x)\right)^3}{6}\bar{m}'(x)^2\,h(x)\right| \leqq C\,\mathrm{e}^{-6a}.
\end{aligned} \tag{8.44}$$

To estimate the last integral, we have used again $h(x)\bar{m}'(x)^2 \leqq C$ and that the integrand is an odd function. From (8.42), (8.43), and (8.44) the limit (8.40) follows. □

Remark 1. Proceeding as in the proof of (3.22), from (3.18), (3.19), and (8.23) it can be shown that $\sup_{x\in[-a,b]} G^{\perp}(x,x) < \infty$.

Remark 2. From the previous computations, it follows that $G^{\perp}(x,y)$ converges pointwise, as $\varepsilon \to 0$, to the kernel of the generalized Green function $\bar{G}$ which inverts $\bar{H}_0$ on the subspace orthogonal to $\bar{m}'$. This kernel is

$$\bar{G}(x,y) := \begin{cases} \dfrac{3}{4}\bar{m}'(x)\bar{m}'(y)\left[u(x) + u(-y) + \dfrac{5}{12}\right] & \text{if } x \leqq y, \\ \\ \dfrac{3}{4}\bar{m}'(x)\bar{m}'(y)\left[u(-x) + u(y) + \dfrac{5}{12}\right] & \text{if } x > y, \end{cases} \tag{8.45}$$

where

$$u(x) := \frac{1}{24}\mathrm{e}^{4x} + \frac{1}{3}\mathrm{e}^{2x} + \frac{1}{2}x - \frac{3}{8}. \tag{8.46}$$

This expression has been obtained in [6, Proposition 3.3], where, however, the constant $\frac{5}{2}$ should read $\frac{5}{12}$.

Appendix A: Fluctuations of a localized interface

In this section, we sketch the proof of Theorem 2.2, which describes the asymptotic behavior of the interface when $a = b = \frac{1}{4}\log\varepsilon^{-1}$, by pointing out the relevant differences with respect to the case $a = \frac{1}{4}\log\varepsilon^{-1}$, $b \gg a$. We then explain how to get the uniformity with respect to the initial condition.

Sketch of the proof of Theorem 2.2. Fix $\tau_0 > 0$. Throughout this section we denote by $m(t;m_0)$, $t \in [0,\varepsilon^{-1}\tau_0]$, the solution to (2.3) with $a = b = \frac{1}{4}\log\varepsilon^{-1}$, to emphasize its dependence on the initial condition $m_0 \in \mathcal{X}_\varepsilon$. Accordingly, we let $X(m_0)$, respectively, $X(t;m_0)$, be the center of m_0, respectively, $m(t \wedge S_{\delta,\ell,\alpha}; m_0)$, see (3.3). Recalling the set $\mathcal{N}_\eta^\varepsilon(z)$ is defined in the statement of the theorem, for each $L > 0$, we define $\mathcal{N}_\eta^{\varepsilon,L} := \bigcup_{z\in[-L,L]} \mathcal{N}_\eta^\varepsilon(z)$. The iterative scheme of Section 3 is repeated with no changes in the present setting.

Step 1. Spectral analysis. We claim that Theorem 3.2 holds with the only change that the asymptotic (3.18) for the smallest eigenvalue has to be replaced by

$$\lim_{\varepsilon\to 0} \sup_{|\zeta|<\alpha a} \varepsilon^{-\frac{3}{2}(1-\alpha)+\eta} \left|\lambda_0^{(\zeta)} - 48\,\varepsilon\,\mathrm{ch}(4\zeta)\right| = 0. \tag{A.1}$$

As in Section 8, we fix the center at the origin and study the operator (8.1) in the interval $[-\ell-\zeta, \ell-\zeta]$. The asymptotic of the eigenvalue $\lambda_0^{(\zeta)}$ can be obtained as in (8.23). The asymptotic of μ, as defined (8.22), is obtained as follows. Instead of (8.9) we decompose here

$$
\begin{aligned}
G\bar{m}'(x) &= 2\,\|\bar{m}'\|_2^2\,\frac{h(\ell+\zeta)\,h(\ell-\zeta)}{h(\ell+\zeta)+h(\ell-\zeta)}\,\bar{m}'(x)\\
&+\frac{2\,h(\ell-\zeta)\,\bar{m}'(x)}{h(\ell+\zeta)+h(\ell-\zeta)}\left[\int_{-\ell-\zeta}^{x}\mathrm{d}y\,\bar{m}'(y)^2h(y)+h(x)\int_{x}^{\ell-\zeta}\mathrm{d}y\,\bar{m}'(y)^2\right]\\
&-\frac{2\,h(\ell+\zeta)\,\bar{m}'(x)}{h(\ell+\zeta)+h(\ell-\zeta)}\left[h(x)\int_{-\ell-\zeta}^{x}\mathrm{d}y\,\bar{m}'(y)^2+\int_{x}^{\ell-\zeta}\mathrm{d}y\,\bar{m}'(y)^2\,h(y)\right]\\
&-\frac{2\,h(x)\,\bar{m}'(x)}{h(\ell+\zeta)+h(\ell-\zeta)}\int_{-\ell-\zeta}^{\ell-\zeta}\mathrm{d}y\,\bar{m}'(y)^2\,h(y)
\end{aligned}
$$

and get

$$
\begin{aligned}
\|G\bar{m}'\|_2 &= 2\,\frac{h(\ell+\zeta)h(\ell-\zeta)}{h(\ell+\zeta)+h(\ell-\zeta)}\,\|\bar{m}'\|_2^3\,(1+\widetilde{\Delta}_1),\\
\|G^2\bar{m}'\|_2 &= 4\left[\frac{h(\ell+\zeta)h(\ell-\zeta)}{h(\ell+\zeta)+h(\ell-\zeta)}\right]^2\,\|\bar{m}'\|_2^5\,(1+\widetilde{\Delta}_2),
\end{aligned}
$$

where $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ satisfy the estimates stated in (8.29) and (8.30) for Δ_1 and Δ_2. The bound (A.1) now follows by direct computations, see (8.31) and (8.32).
Step 2. A priori bounds and recursive equation for the center. The a priori bounds of Section 4 depend only on $b \geqq a$ and therefore hold also in the present setting. Moreover, there exists $\eta_1 > 0$ such that the following holds. For each $L > 0$ and $\eta \in [0, \eta_1]$ there exists $\eta_0 > 0$ such that the bounds stated in Theorem 4.2 hold for $\eta \in (0, \eta_0)$ uniformly with respect to m_0 in the set $\mathcal{N}_{\eta'}^{\varepsilon,L}$, $\eta' \in [0, \eta_1]$.

The key estimate (5.5) in Lemma 5.2 for the identification of the non-linear drift is here replaced by

$$
\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\alpha a}\ \varepsilon^{-(1-\alpha)+\eta}\ \left|\left\langle \bar{m}'_\zeta,\varphi_\zeta - g_T^{(\zeta)}\varphi_\zeta\right\rangle - \frac{4}{3}\,24\,\varepsilon T\ \mathrm{sh}(4\zeta)\right| = 0, \qquad \text{(A.2)}
$$

which is proven as follows. Recalling (3.5), we have

$$
\begin{aligned}
&\sup_{|\zeta|<\alpha a}\left|d_\zeta+\frac{1}{2}\right| \leqq C\varepsilon^{\frac{1}{2}(1-\alpha)},\\
&\sup_{|\zeta|<\alpha a}\left|c_\zeta-1\right| \leqq C\varepsilon^{\frac{1}{2}(1-\alpha)},\\
&\left|\frac{h_\zeta(\ell)+h_\zeta(-\ell)}{h_\zeta(-\ell)-h_\zeta(\ell)}-\mathrm{th}(4\zeta)\right| \leqq C\varepsilon^{\frac{1}{2}(1-\alpha)},
\end{aligned}
$$

whence

$$
\lim_{\varepsilon\to 0}\ \sup_{|\zeta|<\alpha a}\ \varepsilon^{-\frac{1}{2}(1-\alpha)+\eta}\ \left|\left\langle \bar{m}'_\zeta,\varphi_\zeta\right\rangle - \frac{2}{3}\,\mathrm{th}(4\zeta)\right| = 0.
$$

In view of this bound and (A.1), we can repeat the computations in Lemma 5.2 and get (A.2).

Let ξ_n and σ_n be defined as in (5.1). We emphasize that $\xi_0 = x_0 = X(m_0)$ so that the whole sequence ξ_n depends on the initial condition m_0. By using (A.2) and following the same steps as in Theorem 5.1, it is easy to prove its analog in the present setting with a uniform control on $m_0 \in \mathcal{N}_\eta^{\varepsilon,L}$, $\eta \in [0, \eta_1]$. Set $b(x) := -24\,\mathrm{sh}(4x)$, then

$$\xi_{n+1} - \xi_n = \sigma_n + \varepsilon T b(\xi_n) + \Theta_n, \tag{A.3}$$

where, for each $L \in \mathbb{R}_+$, there exists $q > 0$ such that

$$\lim_{\varepsilon\to 0} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \mathbb{P}\left(\sup_{0\leqq n<\varepsilon^{-1}\tau_0/T} \left| \sum_{k=0}^{n} \Theta_k \right| > \varepsilon^q \right) = 0. \tag{A.4}$$

Moreover, by the same argument as in Lemma 5.3, the above statement implies that, for each $L \in \mathbb{R}_+$, we have

$$\lim_{K\to\infty} \overline{\lim_{\varepsilon\to 0}} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \mathbb{P}\left(\sup_{0\leqq n\leqq n_\varepsilon(\tau_0)} |\xi_n| > K \right) = 0, \tag{A.5}$$

which yields, see the end of Section 5,

$$\lim_{\varepsilon\to 0} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \mathbb{P}\left(S_{\delta,\ell,\alpha} \leqq \varepsilon^{-1}\tau_0 \right) = 0 \tag{A.6}$$

and

$$\lim_{\varepsilon\to 0} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-1}\tau_0]} \left\| m(t;m_0) - \bar{m}_{X(t;m_0)} \right\|_\infty > \varepsilon^{\frac{1}{2}-\eta} \right) = 0. \tag{A.7}$$

Step 3. A coupling argument. Recall that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is the filtered probability space where the cylindrical Wiener process W appearing in (2.3) is defined and that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_\tau, P)$ is the filtered probability space where the Brownian motion B appearing in (2.9) is defined. The expectation with respect to $\mathbb{P}$, respectively, P, is denoted by $\mathbb{E}$, respectively, E.

By (A.7), the uniform convergence (2.10) follows once we show there exists $\eta_1 > 0$ such that for each $\eta \in [0, \eta_1]$, $L > 0$, and each uniformly continuous and bounded function $F : C([0, \tau_0]; \mathcal{X}) \to \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0} \sup_{z_0\in[-L,L]} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon}(z_0)} \left| \mathbb{E}\, F\left(\bar{m}_{Y(\cdot;m_0)}\right) - E\, F\left(\bar{m}_{\Xi^{z_0}(\cdot)}\right) \right| = 0, \tag{A.8}$$

where $Y(\tau; m_0) := X(\varepsilon^{-1}\tau; m_0)$, $\tau \in [0, \tau_0]$. Let $\xi_\varepsilon(\cdot; m_0)$ be as defined in (6.1). The estimate (6.2) holds uniformly, namely

$$\lim_{\varepsilon\to 0} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \mathbb{P}\left(\sup_{\tau\in[0,\tau_0]} \left| X_\varepsilon(\varepsilon^{-1}\tau; m_0) - \xi_\varepsilon(\tau; m_0) \right| > \varepsilon^q \right) = 0. \tag{A.9}$$

Let $\zeta_n := \Xi^{z_0}(\varepsilon T_n)$ and denote by $\zeta_\varepsilon(\cdot; z_0)$ its piecewise linear interpolation as in (6.1). By (A.9) and the continuity of Ξ^{z_0}, (A.8) is proven once we show

$$\lim_{\varepsilon\to 0} \sup_{z_0\in[-L,L]} \sup_{m_0\in\mathcal{N}_\eta^\varepsilon(z_0)} \left|\mathbb{E}\, F\left(\bar{m}_{\xi_\varepsilon(\cdot;m_0)}\right) - E\, F\left(\bar{m}_{\zeta_\varepsilon(\cdot;z_0)}\right)\right| = 0, \quad \text{(A.10)}$$

Given the random variables $\sigma_0, \dots, \sigma_n$, we define the sequence β_n by the recursive relation $\beta_{n+1} = \beta_n + \varepsilon T\, b(\beta_n) + \sigma_n$, with $\beta_0 = \xi_0 = X(m_0)$. Recall $b(x) = -24\,\mathrm{sh}(4x)$. The recursive relation (A.3), the bounds (A.4) and (A.5) imply, by a standard Gronwall argument,

$$\lim_{\varepsilon\to 0} \sup_{m_0\in\mathcal{N}_\eta^{\varepsilon,L}} \left|\mathbb{E}\, F\left(\bar{m}_{\xi_\varepsilon(\cdot;m_0)}\right) - \mathbb{E}\, F\left(\bar{m}_{\beta_\varepsilon(\cdot;m_0)}\right)\right| = 0, \qquad \text{(A.11)}$$

where $\beta_\varepsilon(\cdot; m_0)$ is the piecewise linear interpolation of the sequence β_n.

We now set $\widetilde{\Omega} := \Omega \times \hat{\Omega}$, $\widetilde{\mathcal{F}} := \mathcal{F} \times \hat{\mathcal{F}}$, $\widetilde{\mathcal{F}}_t := \mathcal{F}_t \times \hat{\mathcal{F}}_{\varepsilon t}$, $\widetilde{\mathbb{P}} := \mathbb{P} \times P$. On this probability space we define the sequence $\tilde{\beta}_n$ as

$$\begin{cases} \tilde{\beta}_{n+1} = \tilde{\beta}_n + \varepsilon T\, b(\tilde{\beta}_n) + \sqrt{\dfrac{s_n}{\varepsilon T}}\, \big[B(\varepsilon T_{n+1}) - B(\varepsilon T_n)\big], \\ \tilde{\beta}_0 = \beta_0 = \xi_0 = X(m_0), \end{cases} \qquad \text{(A.12)}$$

where

$$s_n \equiv s_n(x_n) := \frac{4}{3}\,\mathbb{E}\left[\sigma_n^2 | x_n\right] = \frac{3}{4}\varepsilon \int_0^T \mathrm{d}t\, \langle \bar{m}'_{x_n}, g_{2t}^{(x_n)} \bar{m}'_{x_n}\rangle.$$

We next observe that, conditionally on the centers $x_0, \dots, x_n$, the random variables $\sigma_0, \dots, \sigma_n$ are independent Gaussians with variances $\frac{3}{4}s_0, \dots, \frac{3}{4}s_n$, so that the sequence β_n and $\tilde{\beta}_n$ have the same law. By (A.11), to prove (A.10) it is enough to show that

$$\lim_{\varepsilon\to 0} \sup_{z_0\in[-L,L]} \sup_{m_0\in\mathcal{N}_\eta^\varepsilon(z_0)} \tilde{\mathbb{E}} \left|F\left(\bar{m}_{\tilde{\beta}_\varepsilon(\cdot;m_0)}\right) - F\left(\bar{m}_{\zeta_\varepsilon(\cdot;z_0)}\right)\right| = 0, \quad \text{(A.13)}$$

where $\tilde{\beta}_\varepsilon(\cdot; m_0)$ is the piecewise linear interpolation of the sequence $\tilde{\beta}_n$. Set $\varrho_n := \tilde{\beta}_n - \zeta_n$; it satisfies the recursive equation

$$\varrho_{n+1} = \varrho_n + \varepsilon T\left[b(\tilde{\beta}_n)] - b(\zeta_n)\right] + R_n^{(1)} + R_n^{(2)},$$

where

$$R_n^{(1)} = \varepsilon T\, b(\zeta_n) - \int_{\varepsilon T_n}^{\varepsilon T_{n+1}} \mathrm{d}\tau\, b(\Xi^{z_0}(\tau)),$$

$$R_n^{(2)} = \left(\sqrt{\frac{s_n}{\varepsilon T}} - 1\right)\big[B(\varepsilon T_{n+1}) - B(\varepsilon T_n)\big].$$

Finally, since $\varrho_0 = X(m_0) - z_0$, for each $L > 0$ we have

$$\lim_{\varepsilon\to 0} \sup_{z_0\in[-L,L]} \sup_{m_0\in\mathcal{N}_\eta^\varepsilon(z_0)} |\varrho_0| = 0.$$

By simple estimates on $R_n^{(i)}$, $i = 1, 2$ and Doob's inequality, a Gronwall argument shows that, for each $\delta > 0$,

$$\lim_{\varepsilon \to 0} \sup_{z_0 \in [-L,L]} \sup_{m_0 \in \mathcal{N}_\eta^\varepsilon(z_0)} \tilde{\mathbb{P}}\left(\sup_{k \leqq n_\varepsilon(\tau_0)} |\varrho_k| > \delta \right) = 0,$$

which yields (A.13). □

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