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Perturbative Analysis of Disordered Ising Models Close to Criticality

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We consider a two-dimensional Ising model with random i.i.d. nearest-neighbor ferromagnetic couplings and no external magnetic field. We show that, if the probability of supercritical couplings is small enough, the system admits a convergent cluster expansion with probability one. The associated polymers are defined on a sequence of increasing scales; in particular the convergence of the above expansion is compatible with the infinite differentiability of the free energy but does not imply its analyticity. The basic tools in the proof are a general theory of graded cluster expansions and a stochastic domination of the disorder.

KEY WORDS: ising models, disordered systems, cluster expansion, griffiths' singularity

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1. INTRODUCTION AND MAIN RESULT

In Ref. 3, we developed a general theory concerning a *graded* perturbative expansion for a class of lattice spin systems. This theory is useful when the system deserves a multi-scale description namely, when a recursive analysis is needed on increasing length scales. The typical example is provided by a disordered system, like a quenched spin glass, having a *good* behavior in average but with the possibility of arbitrarily large *bad* regions. Here by good behavior we mean the one of a

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weakly coupled random field. In the bad (i.e. not good) regions the system can be instead strongly correlated. If those bad regions are suitably sparse then the good ones become dominant, allowing an analysis based on an iterative procedure.

Consider Ising systems described by the following formal Hamiltonian which includes the inverse temperature

$$\mathcal{H}(\sigma) = - \sum_{\substack{x,y\in\mathbb{Z}^2 \\ |x-y|=1}} J_{x,y}\sigma_x\sigma_y - h\sum_x \sigma_x \tag{1.1}$$

where $\sigma_x \in \{-1, +1\}$, $h \in \mathbb{R}$, and $J_{x,y}$ are i.i.d. random variables. A well-known example is the Edwards–Anderson model[10] defined by choosing $J_{x,y}$ centered Gaussian random variables with variance s^2. Let $h = 0$; if s^2 is small enough, so that the probability of subcritical couplings is close to one, we expect a weak coupling regime. However, in the infinite lattice $\mathbb{Z}^2$ there are, with probability one, arbitrarily large regions where the random couplings take large positive values giving rise, inside these regions, to the behavior of a low-temperature ferromagnetic Ising system with long-range order.

A simpler system is the so called *diluted* Ising model, defined by choosing $J_{x,y}$ equal to $K > 0$ with probability q and to zero with probability $1 - q$. In this case it is possible to show that for q sufficiently small and K sufficiently large the infinite-volume free energy is infinitely differentiable but not analytical in h.[13,22] This is a sort of infinite-order phase transition called Griffiths singularity. A similar behavior is conjectured for a general distribution of the random couplings whenever the probability of supercritical values is sufficiently small but strictly positive. More precisely, in such a situation, we expect an exponential decay of correlations with a non-random decay rate but with a random unbounded prefactor. This should imply infinite differentiability of the limiting quenched free energy, but the presence of arbitrarily large regions with supercritical couplings should cause the breaking of analyticity.

A complete analysis of disordered systems in the Griffiths phase is given in Ref. 11, where a powerful and widely applicable perturbative expansion has been introduced. The typical applications are high-temperature spin glasses and random-field Ising models with large variance. From now on we focus however on *ferromagnetic* random Ising systems with *bounded* couplings $J_{x,y}$ and $h = 0$. Let K_c be the critical coupling for the standard two-dimensional Ising model and $K_1 < K_c$ be such that for coupling constants $J_{x,y} \in [0, K_1]$ the standard high temperature cluster expansion is convergent, see e.g. §20.5.(i) in Ref. 12. In the context of Ref. 11 a bond $\{x, y\}$ is to be considered *bad* if the corresponding coupling $J_{x,y}$ exceeds the value K_1. The theory developed in Ref. 11 is based on a multi-scale classification of the bad bonds yielding that, with high probability, larger and larger bad regions are farther and farther apart. In particular there exists a constant $q_1 \in (0, 1)$ such that if $\mathrm{Prob}(J_{x,y} > K_1) \le q_1$ then the system admits a convergent graded cluster expansion implying the exponential decay of correlations as stated above.

The aim of the present paper is to analyze disordered systems that are weakly coupled only on a sufficiently large scale depending on the thermodynamic parameters. In particular we consider random ferromagnetic Ising models allowing typical values of the coupling constants arbitrarily close to the critical value K_c. In this case we need a graded cluster expansion based on a *scale-adapted* approach. To introduce this notion let us take for a while the case of the deterministic ferromagnetic Ising model with coupling K smaller than the critical value K_c. The standard high temperature expansions, converging for coupling smaller than K_1, involve perturbations around a universal reference system consisting of independent spins. In Refs. 18, 19 another perturbative expansion has been introduced, around a non-trivial model-dependent reference system, that can be called *scale-adapted.* Its use is necessary if we want to treat perturbatively the system at *any* $K < K_c$ since the correlation length can become arbitrarily large close to criticality. The geometrical objects (polymers) involved in the scale-adapted expansion live on a suitable length scale ℓ whereas in the usual high-temperature expansions they live on scale one. The small parameter is no more K but rather the ratio between the correlation length at the given K and the length scale ℓ at which we analyze the system. Of course the smaller is $K_c - K$ the larger has to be taken the length ℓ.

In the context of the random ferromagnetic Ising model with bounded interaction, letting $q(b) = \text{Prob}(J_{x,y} > b)$, $b \in \mathbb{R}_+$, we prove that there exists a real function q_0 such that the following holds. If for some $b \in [0, K_c)$ we have $q(b) < q_0(b)$, then the system admits, with probability one w.r.t. the disorder, a convergent graded cluster expansion implying, in particular, the exponential decay of correlations with a random prefactor[(4)]; such a decay can be proven in a simpler way by using the methods in Ref. 9 or in Ref. 2, however to get the expansion (2.20) a graded cluster expansion is needed. The results in Ref. 11 can thus be seen as a special case of the above statement. We emphasize that since we consider situations arbitrarily close to criticality, the first step of our procedure, consisting in the integration over the good region, is a scale-adapted expansion. In other words, the minimal length scale involved in the perturbative expansion developed in the present paper, is not one as in Ref. 11, but rather depends on the thermodynamic parameters and diverges when approaching the critical point. The multi-scale analysis of the bad regions, simpler than the one in Ref. 11, is achieved by exploiting the peculiarities of the model. In particular, the basic probability estimates are deduced via a stochastic domination by a Bernoulli random field.

For $x = (x_1, x_2) \in \mathbb{R}^2$ we let $|x| := |x_1| + |x_2|$. The spatial structure is modeled by the two-dimensional lattice $\mathcal{L} := \mathbb{Z}^2$ endowed with the distance $D(x, y) := |x - y|$. We let e_1 and e_2 be the coordinate unit vectors. As usual for Λ, $\Delta \subset \mathcal{L}$ we set $D(\Lambda, \Delta) := \inf\{D(x, y), x \in \Lambda, y \in \Delta\}$ and $\text{Diam}(\Lambda) :=$

$\sup\{D(x,y), x, y \in \Lambda\}$. The notation $\Lambda \subset\subset \mathbb{L}$ means that Λ is a finite subset of $\mathcal{L}$. We let $\mathcal{E} := \{\{x,y\} \subset \mathcal{L} : D(x,y) = 1\}$ be the collection of *bonds* in $\mathcal{L}$. Given a positive integer m we let $\mathbb{F}_m$ be the collection of all the finite subsets of $\mathcal{L}$ which can be written as disjoint unions of squares with sides of length m parallel to the coordinate axes.

The single-spin state space is $\mathcal{X}_0 := \{-1, +1\}$ which we consider endowed with the discrete topology, the associated Borel σ-algebra is denoted by $\mathcal{F}_0$. The configuration space in $\Lambda \subset \mathcal{L}$ is defined as $\mathcal{X}_\Lambda := \mathcal{X}_0^\Lambda$ and considered equipped with the product topology and the corresponding Borel σ-algebra $\mathcal{F}_\Lambda$. We let $\mathcal{X}_\mathcal{L} =: \mathcal{X}$ and $\mathcal{F}_\mathcal{L} =: \mathcal{F}$. Given $\Delta \subset \Lambda \subset \mathcal{L}$ and $\sigma := \{\sigma_x \in \mathcal{X}_{\{x\}}, x \in \Lambda\} \in \mathcal{X}_\Lambda$, we denote by σ_Δ the *restriction* of σ to Δ namely, $\sigma_\Delta := \{\sigma_x, x \in \Delta\}$. Let m be a positive integer and let $\Lambda_1, \ldots, \Lambda_m \subset \mathcal{L}$ be pairwise disjoint subsets of $\mathcal{L}$; for $\sigma_k \in \mathcal{X}_{\Lambda_k}$, with $k = 1, \ldots, m$, we denote by $\sigma_1\sigma_2\cdots\sigma_m$ the configuration in $\mathcal{X}_{\Lambda_1\cup\cdots\cup\Lambda_m}$ such that $(\sigma_1\sigma_2\cdots\sigma_m)_{\Lambda_k} = \sigma_k$ for all $k \in \{1, \ldots, m\}$.

A function $f : \mathcal{X} \to \mathbb{R}$ is called *local* iff there exists $\Lambda \subset\subset \mathcal{L}$ such that $f \in \mathcal{F}_\Lambda$ namely, f is $\mathcal{F}_\Lambda$-measurable for some bounded set Λ. If $f \in \mathcal{F}_\Lambda$ we shall sometimes misuse the notation by writing $f(\sigma_\Lambda)$ for $f(\sigma)$. We also introduce $C(\mathcal{X})$, the space of continuous functions on $\mathcal{X}$, which becomes a Banach space under the norm $||f||_\infty := \sup_{\sigma\in\mathcal{X}} |f(\sigma)|$; note that the local functions are dense in $C(\mathcal{X})$.

We let $\mathcal{J} := \mathbb{R}^\mathcal{E}$, which we consider equipped with its Borel σ-algebra $\mathcal{B}$. We denote by J_e, $e \in \mathcal{E}$, the canonical coordinates on $\mathcal{J}$. Let $\mathbb{P}_0$ be a probability measure on $\mathbb{R}$ with compact support in $\mathbb{R}_+$ namely, there exists a real $M > 0$ such that $\mathbb{P}_0([0, +M]) = 1$. We define on $(\mathcal{J}, \mathcal{B})$ the product measure $\mathbb{P} := \mathbb{P}_0^\mathcal{E}$.

Given $\Lambda \subset\subset \mathcal{J}$, the *disordered finite-volume Hamiltonian* is the function $H_\Lambda : \mathcal{X} \times \mathcal{J} \to \mathbb{R}$ defined as

$$H_\Lambda(\sigma, J) := \sum_{\substack{\{x,y\}\in\mathcal{E};\\ \{x,y\}\cap\Lambda\neq\emptyset}} J_{\{x,y\}}\sigma_x\sigma_y \tag{1.2}$$

Given $J \in \mathcal{J}$, we define the quenched (finite volume) *Gibbs measure* in Λ, with boundary condition $\tau \in \mathcal{X}$, as the following probability measure on $\mathcal{X}_\Lambda$. Given $\sigma \in \mathcal{X}_\Lambda$ we set

$$\mu^\tau_{\Lambda,J}(\sigma) := \frac{1}{Z_\Lambda(\tau, J)} \exp\{+H_\Lambda(\sigma\tau_{\Lambda^c}, J)\} \tag{1.3}$$

where

$$Z_\Lambda(\tau, J) := \sum_{\sigma\in\mathcal{X}_\Lambda} \exp\{+H_\Lambda(\sigma\tau_{\Lambda^c}, J)\} \tag{1.4}$$

Note that, for notational convenience, we changed the signs in the definition of the Hamiltonian (1.2) and in the Gibbs measure (1.3).

Theorem 1.1. *Let $K_c := (1/2)\log(1+\sqrt{2})$ be the critical coupling of the standard two-dimensional Ising model. There exists a function $q_0 : [0, K_c) \to (0, 1]$ such that the following holds. Suppose that for some $b \in [0, K_c)$*

$$q \equiv q(b) := 1 - \mathbb{P}_0([0, b] \leq q_0(b)) \tag{1.5}$$

then there exists a positive integer $\ell = \ell(b)$ and a set $\bar{\mathcal{J}} \in \mathcal{B}$, with $\mathbb{P}(\bar{\mathcal{J}}) = 1$, such that the following statements hold. There exist two families of local functions $\{\Psi_{X,\Lambda}, \Phi_{X,\Lambda} : \mathcal{X} \times \bar{\mathcal{J}} \to \mathbb{R}, X \subset\subset \mathcal{L}, \Lambda \in \mathbb{F}_\ell\}$, called effective potential, such that: $\Psi_{X,\Lambda}, \Phi_{X,\Lambda} \in \mathcal{F}_{X\cap\Lambda^c} \times \mathcal{B}$ and for each $\Lambda, \Lambda' \in \mathbb{F}_\ell$, $X \subset\subset \mathcal{L}$ such that $X \cap \Lambda = X \cap \Lambda'$ one has that $\Psi_{X,\Lambda} = \Psi_{X,\Lambda'}$ and $\Phi_{X,\Lambda} = \Phi_{X,\Lambda'}$. Moreover for each $\Lambda \in \mathbb{F}_\ell$

1. *for each $(\tau, J) \in \mathcal{X} \times \bar{\mathcal{J}}$ we have the convergent expansion*

$$\log Z_\Lambda(\tau, J) = \sum_{X\cap\Lambda\neq\emptyset} [\psi_{X,\Lambda}(\tau, J) + \Phi_{X,\Lambda}(\tau, J)] \tag{1.6}$$

2. *for each $x \in \mathcal{L}$ there exists a function $r_x : \bar{\mathcal{J}} \to \mathbb{N}$ such that for each $J \in \bar{\mathcal{J}}$ we have $\Psi_{X,\Lambda}(\cdot, J) = 0$ for $X \subset\subset \mathcal{L}$ such that* $\mathrm{diam}(X) > r_x(J)$ *and $X \ni x$;*
3. *there exist reals $\alpha > 0$ and $C < \infty$ such that for any $J \in \bar{\mathcal{J}}$*

$$\sup_{s\in\mathcal{L}} \sum_{X\ni x} e^{\alpha \mathrm{Diam}(X)} \sup_{\Lambda\in\mathbb{F}_\ell} ||\Phi_{X,\Lambda}(\cdot, J)||_\infty \leq C \tag{1.7}$$

In the deterministic case, $J_{x,y} = K$ with $K \in [0, K_c)$, the expansion (1.6) holds with $\Psi = 0$.[18] In such a case (1.7) implies one of the Dobrushin–Shlosman equivalent conditions for (*restricted*) *complete analyticity*, see Condition IVa in Ref. 8 and Eq. (2.15) in Ref. 4. On the other hand, in our disordered setting the family Ψ does not vanish due to the presence of arbitrarily large regions of strong couplings. Nevertheless in item 2 we state that the *range* of the effective potential Ψ, although unbounded, is finite with probability one. We emphasize that in general the statements in items 1–3 of Theorem 1.1 are not sufficient to deduce complete analiticity.

In Ref. 6 we shall prove a similar result in a more general context and, by using the combinatorial approach in Ref. 4, deduce an exponential decay of correlations from the convergence of the graded cluster expansion.

2. GRADED CLUSTER EXPANSION

In this section we prove, relying on some probability estimates on the multiscale geometry of the disorder which are discussed in Sec. 3, Theorem 1.1. We follow a classical strategy in disordered systems. Let us fix a realization $J \in \mathcal{J}$ of the random couplings. We first perform a cluster expansion in the regions where the model satisfies a strong mixing condition implying an effective weak interaction on a proper scale. We are then left with an effective residual interaction between the regions with strong couplings. Since large values of coupling constants have small probability, the regions of strong couplings are well separated on the lattice; we can thus use the graded cluster expansion developed in Ref. 3 to treat the residual interaction.

2.1. Good and Bad Events

Given a positive integer ℓ, we consider the ℓ-rescaled lattice $\mathcal{L}^{(\ell)} := (\ell\mathbb{Z})^2$, which is embedded in $\mathcal{L}$ namely, points in $\mathcal{L}^{(\ell)}$ are also points in $\mathcal{L}$, and for each $i \in \mathcal{L}^{(\ell)}$ we set

$$Q_\ell(i) := \{x \in \mathcal{L} : i_1 \le x_1 \le i_1 + \ell - 1 \text{ and } i_2 \le x_2 \le i_2 + \ell - 1\} \tag{2.1}$$

For $i \in \mathcal{L}^{(\ell)}$ and $b \in [0, K_c)$, we introduce the bad event

$$E_i \equiv E_i^{(\ell),b} := \bigcup_{\substack{e\in\mathcal{E};\\ e\cap Q_\ell(i)\neq\emptyset}} \{J_e > b\} \tag{2.2}$$

Note that E_i occurs iff in the square $Q_\ell(i)$ there exists a coupling, taking into account also the boundary bonds, larger than b. We then define the binary random variable $\omega_i \equiv \omega_i^{(\ell),b} : \mathcal{J} \to \{0, 1\}$ as

$$\omega_i \equiv \omega_i^{(\ell),b} := 1\!\!1_{E_i^{(\ell),b}} \tag{2.3}$$

Given $J \in \mathcal{J}$ we say that a site $i \in \mathcal{L}^{(\ell)}$ is *good* (resp. *bad*) if and only if $\omega_i(J) = 0$ (resp. $\omega_i(J) = 1$) and we set $\mathcal{L}_b^{(\ell)}(J) := \{i \in \mathcal{L}^{(\ell)} : \omega_i(J) = 0\}$.

2.2. On goodness

In this subsection we clarify to which extent the good sites in $\mathcal{L}^{(\ell)}$ are good. We shall show that for ℓ large enough, given $b \in [0, K_c)$ and $J \in \mathcal{J}$, on the good part of the lattice $\mathcal{L}_b^{(\ell)}(J)$ the quenched disordered model satisfies a strong mixing condition allowing a nice cluster expansion.

A few more definitions are needed; let $i \in \mathcal{L}^{(\ell)}$ and $k \in \{1, 2\}$, we denote by $P^{i,k}$ the family of all non-empty subsets $I \subset \mathcal{L}^{(\ell)}$ such that for each $j \in I$ we have $j_k = i_k$ and $j_h \in \{i_h - \ell, i_h, i_h + \ell\}$ for $h = 1, 2$ and $h \neq k$. We set

$$I_\pm := \partial^{(\ell)} I \cap \{j \in \mathcal{L}^{(\ell)} : j_k = i_k \pm \ell\}$$

where for any $I \subset \mathcal{L}^{(\ell)}$ we have set $\partial^{(\ell)} I = \{j \in \mathcal{L}^{(\ell)} \setminus I : D(j, I) = \ell\}$. For $\sigma \in \mathcal{X}$ we set $\sigma_\pm := \sigma_{\cup_{i\in I_\pm} Q_\ell(i)}$ and $\sigma_0 := \sigma_{(\cup_{i\in I_+\cup I_-} Q_\ell(i))^c}$. Moreover for each $I \subset \mathcal{L}^{(\ell)}$ we set $\mathcal{O}_\ell I := \bigcup_{i\in I} Q_\ell(i)$ and for each $X \subset \mathcal{L}$ we set $\mathcal{O}^\ell X := \{i \in \mathcal{L}^{(\ell)} : X \cap Q_\ell(i) \neq \emptyset\}$.

Lemma 2.1. *Let $b \in [0, K_c)$, there exists an integer $\ell_0 = \ell_0(b)$ and a real $m_0 = m_0(b) > 0$ such that for each ℓ multiple of ℓ_0, $J \in \mathcal{J}$, and $i \in \mathcal{L}^{(\ell)}$ we have*

$$\sup_{k=1,2} \sup_{I \in P^{i,k}} \sup_{\sigma,\zeta,\tau \in \mathcal{X}} \left| \frac{Z_{\mathcal{O}_\ell\left(I \cap \mathcal{L}_b^{(\ell)}\right)}(\sigma_+\sigma_-\tau_0, J) Z_{\mathcal{O}_\ell\left(I \cap \mathcal{L}_b^{(\ell)}\right)}(\zeta_+\zeta_-\tau_0, J)}{Z_{\mathcal{O}_\ell\left(I \cap \mathcal{L}_b^{(\ell)}\right)}(\sigma_+\zeta_-\tau_0, J) Z_{\mathcal{O}_\ell\left(I \cap \mathcal{L}_b^{(\ell)}\right)}(\zeta_+\sigma_-\tau_0, J)} - 1 \right| \leq e^{-m_0 \ell} \quad (2.4)$$

Proof: The proof, which is based on classical results on (non-disordered) two-dimensional Ising models adapted to the present non-translation-invariant interaction, is organized in three steps. Given $b \in [0, K_c)$ we let $\mathcal{J}_b := [0, b]^{\mathcal{E}} \subset \mathcal{J}$. We first prove that for each $J \in \mathcal{J}_b$ there exists a unique infinite-volume Gibbs measure w.r.t. the *local Gibbs specification* $Q_{\Lambda,J}(\cdot|\tau) := \mu^\tau_{\Lambda,J}(\cdot)$ [see Ref. 7, p350 and Eq. (1.3) above], $\Lambda \subset\subset \mathcal{L}$, $\tau \in \mathcal{X}$ Then we show that the corresponding infinite-volume two-point correlations decay exponentially with the distance. From this we finally derive the bound (2.4).

We consider $\mathcal{X}$ endowed with the natural partial ordering $\sigma \leq \sigma'$ iff for any $x \in \mathcal{L}$ we have $\sigma_x \leq \sigma'_x$. Given two probabilities μ, ν on $\mathcal{X}$ we write $\mu \leq \nu$ iff for any continuous increasing (w.r.t. the previous partial ordering) function f we have $\mu(f) \leq \nu(f)$. Here $\mu(f)$ denotes the expectation of f w.r.t. the measure μ.

Step 1. Let us denote by $+$ (resp. $-$) the configuration with all the spins equal to $+1$ (resp. -1). By monotonicity, which is a consequence of the FKG inequalities, see e.g. Theorem 4.4.1 in Ref. 12, we get that for each $J \in \mathcal{J}$ and $A \subset \mathcal{F}$

$$\exists \lim_{\Lambda \uparrow \mathcal{L}} \mu^\pm_{\Lambda,J}(A) =: \mu^\pm_J(A) \quad (2.5)$$

where the limit is taken along an increasing sequence invading $\mathcal{L}$. Moreover, again by the FKG inequalities, we have that any infinite-volume Gibbs measure μ_J satisfies the inequalities

$$\mu^-_J \leq \mu_J \leq \mu^+_J \quad (2.6)$$

We now notice that for each $J \in \mathcal{J}_b$ and $x \in \mathcal{L}$

$$\lim_{\Lambda \uparrow \mathcal{L}} \mu^\pm_{\Lambda,J}(\sigma_x) = 0 \quad (2.7)$$

indeed if we let $B \in \mathcal{J}$ be such that $B_e = b$, $e \in \mathcal{E}$, we have that for each $J \in \mathcal{J}_b$, $x \in \mathcal{L}$ and $\Lambda \subset\subset \mathcal{L}$

$$\mu^-_{\Lambda,B}(\sigma_x) \leq \mu^-_{\Lambda,J}(\sigma_x) \leq 0 \leq \mu^+_{\Lambda,J}(\sigma_x) \leq \mu^+_{\Lambda,B}(\sigma_x) \quad (2.8)$$

where we used the Griffiths inequalities, see e.g. Theorem 4.1.3 in Ref. 12. By using Refs. 15, 20, 21 and the FKG inequalities we have that (2.8) implies (2.7) since $0 \leq b \leq K_c$.

Again by FKG, Eqs. (2.5) and (2.7) imply that for each $J \in \mathcal{J}_b$ the infinite volume Gibbs measure w.r.t. the local specification $Q_{\Lambda,J}(\cdot|\tau)$, $\Lambda \subset\subset \mathcal{L}$, $\tau \in \mathcal{X}$ is unique; we denote this measure by μ_J.

Step 2. Let $x, y \in \mathcal{L}$, by the Griffiths' inequalities we have that for each $J \in \mathcal{J}_b$

$$0 \leq \mu_J(\sigma_x;\sigma_y) = \mu_J(\sigma_x\sigma_y) \leq \mu_B(\sigma_x\sigma_y) \tag{2.9}$$

where we recall B has been defined above (2.8). By using (2.9) and classical exact results on the two-dimensional Ising model, see e.g., Ref. 1 we have that there exists a positive real $C_3(b) < \infty$ such that for any $x, y \in \mathcal{L}$

$$\mu_J(\sigma_x;\sigma_y) \leq \mu_B(\sigma_x\sigma_y) \leq C_3(b)e^{-D(x,y)/C_3(b)} \tag{2.10}$$

Step 3. We observe that the argument of the proof of Theorem 2.1 in Ref. 14 applies to the present not translationally invariant setting. Indeed, it depends only on the Lebowitz inequalities, which hold true, and the bound (2.10). We thus obtain that there exists a positive real $C_2(b) < \infty$ such that for each $\tau, \tau' \in \mathcal{X}$, $J \in \mathcal{J}_b$, $\Lambda \subset\subset \mathcal{L}$, $\Delta \subset \Lambda$, and $A \in \mathcal{F}_\Delta$

$$|\mu^{\tau}_{\Lambda,J}(A) - \mu^{\tau'}_{\Lambda,J}(A)| \leq C_2(b)e^{-D(\Lambda^c,\Delta)/C_2(b)} \tag{2.11}$$

which is, in the terminology introduced in Ref. 16, the *weak mixing* condition for the local specification $Q_{\Lambda,J}(\cdot|\tau)$. By exploiting the two-dimensionality of the model and by using the result in Ref. 17, we get that there exist an integer $\ell_1(b)$ and a positive real $C_1(b) < \infty$ such that for any $\tau \in \mathcal{X}$, $J \in \mathcal{J}_b$, $\Lambda \in \mathbb{F}_{\ell_1(b)}$, $\Delta \subset \Lambda$, $x \in \mathcal{L} \setminus \Lambda$, and $A \subset \mathcal{F}_\Delta$

$$|\mu^{\tau}_{\Lambda,J}(A) - \mu^{\tau^x}_{\Lambda,J}(A)| \leq C_1(b)e^{-D(x,\Delta)/C_1(b)} \tag{2.12}$$

where $\tau^x \in \mathcal{X}$ is given by $\tau^x_x = -\tau_x$ and $\tau^x_y = \tau_y$ for all $y \neq x$, and we recall that the collection of volumes $\mathbb{F}_\ell$ has been defined in Sec. 1. Again in the terminology of Ref. 16, the bound (2.12) is called *strong mixing* condition. By Corollary 3.2, Eqs. (3.9) and (3.14) in Ref. 18, see also Eq. (2.5.32) in Ref. 19, it implies the statement of the Lemma. □

2.3. Cluster Expansion in the Good Region

In this subsection we cluster expand the partition function over the good part of the lattice. Consider a positive integer ℓ and the ℓ-rescaled lattice $\mathcal{L}^{(\ell)} = (\ell\mathbb{Z})^2$. We denote by $D_\ell(i,j) = (1/\ell)D(i,j)$ the natural distance in $\mathcal{L}^{(\ell)}$ and by $\text{Diam}_\ell(I) := \sup_{i,j\in I} D_\ell(i,j)$ the diameter of a subset $I \subset \mathcal{L}^{(\ell)}$ Given $I \subset \mathcal{L}^{(\ell)}$ and a real $r > 0$, we denote by $B^{(\ell)}_r(I) := \{j \in \mathcal{L}^{(\ell)} : D_\ell(I,J) \leq r\}$ the r-neighborhood of I.

We associate with each site $i \in \mathcal{L}^{(\ell)}$ the *single-site block-spin configuration space* $\mathcal{X}^{(\ell)}_i := \mathcal{X}_{Q_\ell(i)}$. Given $I \subset \mathcal{L}^{(\ell)}$ we consider the *block-spin configuration* *space* $\mathcal{X}_I^{(\ell)} := \otimes_{i\in I}\mathcal{X}_i^{(\ell)}$, $I \subset \mathcal{L}^{(\ell)}$ equipped with the product topology and the corresponding Borel σ-algebra $\mathcal{F}_I^{(\ell)}$. As before we set $\mathcal{X}^{(\ell)} := \mathcal{X}_{\mathcal{L}^{(\ell)}}^{(\ell)}$ and $\mathcal{F}^{(\ell)} := \mathcal{F}_{\mathcal{L}^{(\ell)}}^{(\ell)}$.

As for the lattices, see the definition just above the Lemma 2.1, we introduce operators which allow to pack spins and unpack block spins. We define the *packing* operator $\mathcal{O}^\ell : \mathcal{X} \to \mathcal{X}^{(\ell)}$ associating to each spin configuration $\sigma \in \mathcal{X}$ the block-spin configuration $\mathcal{O}^\ell\sigma \in \mathcal{X}^{(\ell)}$ given by $(\mathcal{O}^\ell\sigma)_i := \{\sigma_x, x \in Q_\ell(i)\}$, $i \in \mathcal{L}^{(\ell)}$. The *unpacking* operator $\mathcal{O}_\ell : \mathcal{X}^{(\ell)} \to \mathcal{X}$ associates to each block-spin configuration $\zeta \in \mathcal{L}^{(\ell)}$ the unique spin configuration $\mathcal{O}_\ell\zeta \in \mathcal{X}$ such that $\zeta_i = \{(\mathcal{O}_\ell\zeta)_x, x \in Q_\ell(i)\}$ for all $i \in \mathcal{L}^{(\ell)}$. We remark also that the two operators allow the packing of the spin σ-algebra and the unpacking of the block-spin one namely, for each $I \subset \mathcal{L}^{(\ell)}$ and $A \subset \mathcal{L}$ we have

$$\mathcal{O}_\ell\big(\mathcal{F}_I^{(\ell)}\big) = \mathcal{F}_{\mathcal{O}_\ell I} \quad \text{and} \quad \mathcal{O}(\mathcal{F}_\Lambda) \subset \mathcal{F}_{\mathcal{O}^\ell\Lambda}^{(\ell)}. \tag{2.13}$$

Where in the last relation the equality between the two σ-algebras stands if and only if $\mathcal{O}_\ell\mathcal{O}^\ell\Lambda = \Lambda$.

Given $\Delta \subset\subset \mathcal{L}^{(\ell)}$ we define the *block-spin Hamiltonian* $H_\Delta^{(\ell)} : \mathcal{X}^{(\ell)} \times \mathcal{J} \to \mathbb{R}$ as $H_\Delta^{(\ell)}(\zeta, J) := H_{\mathcal{O}_\ell\Delta}(\mathcal{O}_\ell\zeta, J)$ for $\zeta \in \mathcal{X}^{(\ell)}$ and $J \in \mathcal{J}$. The corresponding finite-volume Gibbs measure, with boundary condition $\xi \in \mathcal{X}^{(\ell)}$, is denoted by $\mu_{\Delta,J}^{(\ell),\xi}$, the partition function by $Z_\Delta^{(\ell)}(\xi, J)$ namely,

$$Z_\Delta^{(\ell)}(\xi, J) = Z_{\mathcal{O}_\ell\Delta}(\mathcal{O}_\ell\xi, J) \tag{2.14}$$

Let $J \in \mathcal{J}$, $\xi \in \mathcal{X}^{(\ell)}$, $\Delta \subset\subset \mathcal{L}^{(\ell)}$ and recall $\mathcal{L}_b^{(\ell)}(J)$ has been defined below (2.3). In the following Proposition 2.2 we cluster expand $Z_{\Delta\cap\mathcal{L}_b^{(\ell)}(J)}^{(\ell)}(\xi, J)$ and show, in particular, that Condition 2.1 in Ref. 3 is satisfied. To state the result we need few more definitions.

Let $\mathcal{E}^{(\ell)} := \{\{x, y\} \subset \mathcal{L}^{(\ell)} : D_\ell(x, y) = 1\}$ the collection of edges in $\mathcal{L}^{(\ell)}$. We say that two edges $e, e' \in \mathcal{E}^{(\ell)}$ are connected if and only if $e \cap e' \neq \emptyset$. A subset $(V, E) \subset (\mathcal{L}^{(\ell)}, \mathcal{E}^{(\ell)})$ is said to be connected iff for each pair $x, y \in V$, with $x \neq y$, there exists in E a path of connected edges joining them. We agree that if $|V| = 1$ then $(V, \emptyset)$ is connected. For $X \subset \mathcal{L}^{(\ell)}$ finite we then set

$$\mathbb{T}_\ell(X) := \inf\{|E|, (V, E) \subset (\mathcal{L}^{(\ell)}, \mathcal{E}^{(\ell)}) \text{ is connected and } V \supset X\} \tag{2.15}$$

Note that $\mathbb{T}_\ell(X) = 0$ if $|X| = 1$ and for $x, y \in \mathcal{L}^{(\ell)}$ we have $\mathbb{T}_\ell(\{x, y\}) = D_\ell(x, y)$.

Proposition 2.2. *Let $b \in [0, K_c)$ and $\ell_0 = \ell_0(b)$ as in Lemma 2.1. Then for all integer ℓ multiple of ℓ_0, $J \in \mathcal{J}$, $\xi \in \mathcal{X}^{(\ell)}$, and $\Delta \subset\subset \mathcal{L}^{(\ell)}$ we have*

$$\log Z_{\Delta\cap\mathcal{L}_b^{(\ell)}(J)}^{(\ell)}(\xi, J) = \sum_{I\cap\Delta\neq\emptyset} V_{I,\Delta}^{(\ell)}(\xi, J) \tag{2.16}$$

for a suitable collection of local functions $V_\Delta^{(\ell)} := \{V_{I,\Delta}^{(\ell)} : \mathcal{X}^{(\ell)} \times \mathcal{J} \to \mathbb{R}, I \cap \Delta \neq \emptyset\}$ *satisfying:*

1. *given* $\Delta, \Delta' \subset\subset \mathcal{L}^{(\ell)}$ *if* $I \cap \Delta = I \cap \Delta'$ *then* $V_{I,\Delta}^{(\ell)}(\cdot, J) = V_{I,\Delta'}^{(\ell)}(\cdot, J)$ *for any* $J \in \mathcal{J}$*;*
2. $V_{I,\Delta}^{(\ell)}(\cdot, J) \in \mathcal{F}^{(\ell)}_{I \cap (\Delta \cap \mathcal{L}_b^{(\ell)}(J))^c}$ *for any* $J \in \mathcal{J}$*;*
3. *if* $I \cap (B_6^{(\ell)}(\Delta))^c \neq \emptyset$ *then* $V_{I,\Delta}^{(\ell)} = 0$.

Moreover, the effective potential $V_\Delta^{(\ell)}$ *can be bounded as follows. There exist reals* $\alpha_1 = \alpha_1(b) > 0$, $A_1 = A_1(b) < \infty$, *and* $n_1 = n_1(b) < \infty$ *such that for any* $J \in \mathcal{J}$

$$\sup_{i \in \mathcal{L}^{(\ell)}} \sum_{I \ni i} e^{\alpha_1 \ell \mathbb{T}_\ell(I)} \sup_{\substack{\Delta \subset\subset \mathcal{L}^{(\ell)}; \\ \Delta \cap I \neq \emptyset}} ||V_{I,\Delta}^{(\ell)}(\cdot, J)||_\infty \leq A_1 \ell^{n_1} \tag{2.17}$$

Proof of Proposition 2.2. The proof can be achieved by applying the arguments in Ref. 18, 19, where this result is proven with periodic boundary conditions. We refer to Theorem 5.1 in Ref. 5 for the modifications needed to cover the case of arbitrary boundary conditions and for the stated ℓ-dependence of the bound (2.17). Item 3 follows from Figs. 2 and 3 in Ref. 5. □

2.4. Geometry of Badness

To characterize the sparseness of the bad region we follow the ideas developed in Refs. 3, 5, 11.

Definition 2.3. *We say that two strictly increasing sequences* $\Gamma = \{\Gamma_k\}_{k \geq 1}$ *and* $\gamma = \{\gamma_k\}_{k \geq 1}$ *are* moderately steep scales *iff they satisfy the following conditions:*

1. $\Gamma_1 \geq 2$, *and* $\Gamma_k < \gamma_k/2$ *for any* $k \geq 1$;
2. *for* $k \geq 1$ *set* $\vartheta_k := \sum_{h=1}^{k}(\Gamma_h + \gamma_h)$ *and* $\lambda := \inf_{k \geq 1}(\Gamma_{k+1}/\vartheta_k)$, *then* $\lambda \geq 5$;
3. $\sum_{k=1}^{\infty} \frac{\Gamma_k}{\gamma_k} \leq \frac{1}{2}$;
4. $a_0 := \sum_{k=0}^{\infty} 2^{-k} \log[2(\Gamma_{k+1} + \gamma_{k+1}) + 1]^2 < \infty$;
5. *for each* $a > 0$ *we have* $\sum_{k=1}^{\infty}[2(\vartheta_k + \Gamma_k) + 1]^2 \exp\{-a2^{k-1}\} < \infty$.

We remark that items 4 and 5 differ slightly from the corresponding ones in Definition 3.1 in Ref. 5. This is due to the fact that the analysis of the geometry of bad sets given in the present paper is based on a stochastic domination argument while the one in §3 in Ref. 5 depends on mixing properties of the disorder.

Definition 2.4. *We say that $\mathcal{G} := \{\mathcal{G}_k\}_{k\geq 0}$, where each $\mathcal{G}_k$ is a collection of finite subsets of $\mathcal{L}^{(\ell)}$, is a graded disintegration of $\mathcal{L}^{(\ell)}$ iff*

1. *for each $g \in \bigcup_{k\geq 0} \mathcal{G}_k$ there exists a unique $k \geq 0$, which is called the* grade *of g, such that $g \in \mathcal{G}_k$;*
2. *the collection $\bigcup_{k\geq 0} \mathcal{G}_k$ of finite subsets of $\mathcal{L}^{(\ell)}$ is a partition of the lattice $\mathcal{L}^{(\ell)}$ namely, it is a collection of non-empty pairwise disjoint finite subsets of $\mathcal{L}^{(\ell)}$ such that*

$$\bigcup_{k\geq 0} \bigcup_{g\in\mathcal{G}_k} g = \mathcal{L}^{(\ell)}. \tag{2.18}$$

Given $\mathbb{G}_0 \subset \mathcal{L}^{(\ell)}$ and Γ, γ moderately steep scales, we say that a graded disintegration $\mathcal{G}$ is a gentle disintegration *of $\mathcal{L}^{(\ell)}$ with respect to $\mathbb{G}_0, \Gamma, \gamma$ iff the following recursive conditions hold:*

3. *$\mathcal{G}_0 = \{\{i\}, i \in \mathbb{G}_0\}$;*
4. *if $g \in \mathcal{G}_k$ then $\mathrm{Diam}_\ell(g) \leq \Gamma_k$ for any $k \geq 1$;*
5. *set $\mathbb{G}_k := \bigcup_{g\in\mathcal{G}_k} g \subset \mathcal{L}^{(\ell)}$, $\mathbb{B}_0 := \mathcal{L}^{(\ell)} \setminus \mathbb{G}_0$ and $\mathbb{B}_k := \mathbb{B}_{k-1} \setminus \mathbb{G}_k$, then for any $g \in \mathcal{G}_k$ we have $D_\ell(g, \mathbb{B}_{k-1} \setminus g) > \gamma_k$ for any $k \geq 1$;*
6. *given $g \in \mathcal{G}_k$, let $Y_0(g) := \{j \in \mathcal{L}^{(\ell)} : \inf_{i\in\mathcal{Q}(g)}[|i_1 - j_1| \vee |i_2 - j_2|] \leq \vartheta_k\}$ where $\mathcal{Q}(g) \subset\subset \mathcal{L}^{(\ell)}$ is the smallest rectangle, with axes parallel to the coordinate directions, that contains g; then for each $i \in \mathcal{L}^{(\ell)}$ we have*

$$\varkappa_i := \sup\{k \geq 1 : \exists g \in \mathcal{G}_k \text{ such that } Y_0(g) \ni i\} < \infty$$

We call k-gentle, resp. k-bad, the sites in $\mathbb{G}_k$, resp. $\mathbb{B}_k$. The elements of $\mathcal{G}_k$, with $k \leq 1$, are called k-gentle atoms. Finally, we set $\mathcal{G}_{\geq k} := \bigcup_{h\geq k} \mathcal{G}_h$.

We next state a proposition, whose proof is the topic of Sec. 3, which will ensure that for q small enough the bad sites of the lattice $\mathcal{L}^{(\ell)}$ can be classified according to the notion of gentle disintegration for suitable scales. We note that items 2 and 3 in Definition 2.3 force a super-exponential growth of the sequences Γ and γ. It is easy to show that, given $\beta \geq 8$, the sequences

$$\Gamma_k := e^{(\beta+1)(3/2)^k} \quad \text{and} \quad \gamma_k := \frac{1}{8} e^{\beta(3/2)^{k+1}} \quad \text{for } k \geq 1 \tag{2.19}$$

are moderately steep scales in the sense of Definition 2.3. Given $b \in [0, K_c)$, let $\alpha_1(b)$, $A_1(b)$, and $n_1(b)$ as in Proposition 2.2. It is easy to show that there exists $\beta_0 = \beta_0(b)$ such that the scales Γ, γ in (2.19) with $\beta = \beta_0$ satisfy the conditions stated in items 1–4 in the hypotheses of Theorem 2.5 in Ref. 3 for any ℓ large enough. We understand that the constants α and A in those items are to be replaced by $\alpha_1\ell$ and $A_1\ell^{n_1}$ respectively.

Proposition 2.5. *Given $b \in [0, K_c)$ let Γ, γ as in (2.19) with $\beta = \beta_0(b)$. There exist $q_0(b) \in [0, 1)$ and a multiple $\bar{\ell} = \bar{\ell}(b)$ of $\ell_0(b)$, see Lemma 2.1, such that if $q < q_0(b)$, recall (1.5), then there exists a $\mathcal{B}$-measurable set $\bar{\mathcal{J}} \subset J$, with $\mathbb{P}(\bar{\mathcal{J}}) = 1$, such that for each $J \in \bar{\mathcal{J}}$ there exists a gentle disintegration $\mathcal{G}(J)$, see Definition 2.4, of $\mathcal{L}^{(\bar{\ell})}$ with respect to $\mathcal{L}_b^{(\bar{\ell})}(J)$ and Γ, γ.*

2.5. Cluster Expansion in the Bad Region

In this subsection we sum over the configurations on the bad sites in $\mathcal{L}^{(\ell)} \setminus \mathcal{L}_b^{(\ell)}(J)$. We show that provided J is chosen in the full $\mathbb{P}$-measure set $\bar{\mathcal{J}} \subset \mathcal{J}$, see Proposition 2.5, it is possible to organize the sum iteratively using a hierarchy of sparse bad regions of the lattice.

Proof of Theorem 1.1. We apply Proposition 2.5 to construct the set $\bar{\mathcal{J}}$ and, for each $J \in \bar{\mathcal{J}}$, the gentle disintegration $\mathcal{G}(J)$ of $\mathcal{L}^{(\bar{\ell})}$ with respect to $\mathcal{L}_b^{(\bar{\ell})}(J)$ and Γ, γ as in (2.19) with $\beta = \beta_0(b)$. For the sake of simplicity we set $\ell - \bar{\ell}(b)$ in the sequel of the proof.

Pick $J \in \bar{\mathcal{J}}$, $\xi \in \mathcal{X}^{(\ell)}$, and $\Delta \subset\subset \mathcal{L}^{(\ell)}$; by applying Proposition 2.2 we cluster expand $\log Z^{(\ell)}_{\Delta\cap\mathcal{L}_b^{(\ell)}(J)}(\xi, J)$. We get that Condition 2.1 in Ref. 3 holds with effective potential $V_\Delta^{(\ell)}(\cdot, J)$, $\alpha = \alpha_1\ell$, $A = A_1\ell^{n_1}$ (here α_1, A_1, and n_1 are the constants appearing in (2.17)), and $r = 6$. By applying Theorem 2.5 in Ref. 3 to the lattice $\mathbb{L} = \mathcal{L}^{(\ell)}$ and the gentle disintegration $\mathcal{G}(J)$ w.r.t. $\mathcal{L}_b^{(\ell)}(J), \Gamma, \gamma$, it follows that there exist functions $\Psi_{I,\Delta}^{(\ell)}(\cdot, J), \Phi_{I,\Delta}^{(\ell)}(\cdot, J) \in \mathcal{F}_{I\cap\Delta^c}^{(\ell)}$, with $I \subset\subset \mathcal{L}^{(\ell)}$, such that we have the totally convergent expansion

$$\log Z_\Delta^{(\ell)}(\xi, J) = \sum_{I\cap\Delta\neq\emptyset} \left[\Psi_{I,\Delta}^{(\ell)}(\xi, J) + \Phi_{I,\Delta}^{(\ell)}(\xi, J)\right] \tag{2.20}$$

Moreover: (i) for each $\Delta, \Delta' \subset\subset \mathcal{L}^{(\ell)}$ and each $I \subset\subset \mathcal{L}^{(\ell)}$ such that $I \cap \Delta = I \cap \Delta'$ we have that $\Psi_{I,\Delta}^{(\ell)} = \Psi_{I,\Delta'}^{(\ell)}$ and $\Phi_{I,\Delta}^{(\ell)} = \Phi_{I,\Delta'}^{(\ell)}$; (ii) for $J \in \bar{\mathcal{J}}$, for $I, \Delta \subset\subset \mathcal{L}^{(\ell)}$, if $\mathrm{Diam}_\ell(I) > 6$ and there exists no $g \in \mathcal{G}_{\geq 1}(J)$ such that $Y_0(g) = X$ then $\Psi_{I,\Delta}^{(\ell)}(\cdot, J) = 0$; (iii) for each $J \in \bar{\mathcal{J}}$ we have

$$\sup_{i\in\mathcal{L}^{(\ell)}} \sum_{I\ni i} e^{c\alpha_1\ell\mathrm{Diam}_\ell(I)} \sup_{\Delta\subset\subset\mathcal{L}^{(\ell)}} ||\Phi_{I,\Delta}^{(\ell)}(\cdot, J)||_\infty \leq 1 + e^{-c\alpha_1\ell\gamma_1}\left[\frac{1+e^{-c\alpha_1\ell/4}}{1-e^{-c\alpha_1\ell/4}}\right]. \tag{2.21}$$

where $c = 2^{-6}3^{-2}$.

To get the expansion (1.6) we next pull back the $\Psi^{(\ell)}$ and $\Phi^{(\ell)}$ to the original scale. We define the family $\{\Psi_{X,\Lambda}, \Phi_{X,\Lambda} : \mathcal{X} \times \bar{\mathcal{J}} \to \mathbb{R}, X \subset\subset \mathcal{L}, \Lambda \in \mathbb{F}_\ell\}$ as follows: for each $\tau \in \mathcal{X}$, $X \subset\subset \mathcal{L}$, and $\Lambda \in \mathbb{F}_\ell$ we set

$$\Psi_{X,\Lambda}(\tau, J) := \begin{cases} \Psi^{(\ell)}_{I,\mathcal{O}^\ell\Lambda}(\mathcal{O}^\ell\tau, J) & \text{if } \exists I \subset \mathcal{L}^{(\ell)} : \mathcal{O}_\ell I = X \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

and

$$\Phi_{X,\Lambda}(\tau, J) := \begin{cases} \Phi^{(\ell)}_{I,\mathcal{O}^\ell\Lambda}(\mathcal{O}^\ell\tau, J) & \text{if } \exists I \subset \mathcal{L}^{(\ell)} : \mathcal{O}_\ell I = X \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

Now, the expansion (1.6) follows from (2.20), (2.14), (2.22), and (2.23). The measurability properties of the functions Ψ and Φ follow from (2.13) and the analogous properties of the functions $\Psi^{(\ell)}$ and $\Phi^{(\ell)}$. Item 2 follows from ii) above and item 6 in Definition 2.4 once we set $r_x(J) := \ell[(\Gamma_{\varkappa_x} + 2\theta_{\varkappa_x}) \vee 6]$ for each $x \in \mathcal{L}$ and $J \in \bar{\mathcal{J}}$ where $\varkappa_x$ is defined in item 6 of Definition 2.4. We finally prove item 3. Let $J \in \bar{\mathcal{J}}$ and set $\alpha := c\alpha_1$, we have

$$\begin{aligned} &\sup_{x\in\mathcal{L}} \sum_{X\ni x} e^{\alpha \mathrm{Diam}(X)} \sup_{\Lambda\in\mathbb{F}_\ell} ||\Phi_{X,\Lambda}(\cdot, J)||_\infty \\ &\quad = \sup_{x\in\mathcal{L}} \sum_{X\ni x} e^{c\alpha_1 \ell \mathrm{Diam}(X)/\ell} \sup_{\Lambda\in\mathbb{F}_\ell} ||\Phi_{X,\Lambda}(\cdot, J)||_\infty \\ &\quad = \sup_{x\in\mathcal{L}} \sum_{\substack{I\subset\mathcal{L}^{(\ell)};\\ \mathcal{O}_\ell I\ni x}} e^{c\alpha_1 \ell \mathrm{Diam}_\ell(I)} \sup_{\Delta\subset\subset\mathcal{L}^{(\ell)}} ||\Phi^{(\ell)}_{I,\Delta}(\cdot, J)||_\infty \end{aligned} \tag{2.24}$$

By setting

$$C := 1 + e^{-4\alpha} \left[\frac{1 + e^{-\alpha/4}}{1 - e^{-\alpha/4}} \right]^d$$

the bound (1.7) follows by using (2.21) and $\gamma_1 \geq 4$, see item 1 in Definition 2.3. □

3. GRADED GEOMETRY

In this section we prove Proposition 2.5. Recalling the random field ω has been introduced in (2.3), we define $\Omega := \{0, 1\}^{\mathcal{L}^{(\ell)}}$ and let $\mathcal{A}$ be the corresponding Borel σ-algebra. We denote by $\mathbb{Q} = \mathbb{Q}^{(\ell),b}$, a probability on Ω, the distribution of the random field ω. Given $I \subset \mathcal{L}^{(\ell)}$ we set $\mathcal{A}_I := \sigma\{\omega_i, i \in I\} \subset \mathcal{A}$.

Since we assumed the coupling J_e, $e \in \mathcal{E}$, to be i.i.d. random variables, the measure $\mathbb{Q}$ is translationally invariant. Let us introduce the parameter p which measures the strength of the disorder

$$p := \operatorname*{ess\,sup}_{\omega\in\Omega} \mathbb{Q}(\omega_0 = 1|\mathcal{A}_{\{0\}^c})(\omega) \tag{3.1}$$

where the essential supremum is taken w.r.t. $\mathbb{Q}$.

We shall first prove that if p is small enough, depending on the parameter a_0 appearing in item 4 in the Definition 2.3, then we can construct a gentle disintegration in the sense of Definition 2.4. We finally show that the above condition is met if $q_0(b)$ in (1.5) is properly chosen.

Let us first describe an algorithm to construct the family $\mathcal{G}$ introduced in Definition 2.4. Given a configuration $\omega \in \Omega$ and Γ, γ moderately steep scales, we define the following inductive procedure in a finite volume $\Lambda \subset\subset \mathcal{L}^{(\ell)}$ which finds the k-gentle sites in Λ. Set $\mathbb{G}_0 := \mathcal{L}_b^{(\ell)}(\omega)$, $\mathcal{G}_0 := \{\{i\}, i \in \mathbb{G}\}$, and $\mathbb{B}_0 := \mathcal{L}^{(\ell)} \setminus \mathbb{G}_0$. At step $k \geq 1$ do the following:

1. $i = 1$ and $V = \emptyset$;
2. if $(\mathbb{B}_{k-1} \cap \Lambda) \setminus V = \emptyset$ then goto 6;
3. pick a point $x \in (\mathbb{B}_{k-1} \cap \Lambda) \setminus V$. Set $A = B_{k-1}^{(\ell)}(x) \cap \mathbb{B}_{k-1}$ and $V = V \cap A$;
4. if $\mathrm{Diam}_\ell(A) \leq \Gamma_k$ and $D_\ell(A, \mathbb{B}_{k-1} \setminus A) > \gamma_k$ then $g_k^i = A$ and $i = i + 1$;
5. goto 2;
6. set $\mathcal{G}_k := \{g_k^m, m = 1, \ldots, i-1\}$ with the convention $\mathcal{G}_k = \emptyset$ if $i = 1$, $\mathbb{G}_k := \bigcup_{m=1}^{i-1} g_k^m$, and $\mathbb{B}_k : \mathbb{B}_{k-1} \setminus \mathbb{G}_k$.

Set now $k = k + 1$ and repeat the algorithm until $\Gamma_k > \mathrm{Diam}_\ell \Lambda$.

Let us briefly describe what the above algorithm does. At step k we have inductively constructed $\mathbb{B}_{k-1}$, the set of $(k-1)$-bad sites; we stress that sites in $\mathcal{L}^{(\ell)} \setminus \Lambda$ may belong to $\mathbb{B}_{k-1}$. Among the sites in $\mathbb{B}_{k-1} \cap \Lambda$ we are now looking for the k-gentle ones. The set V is used to keep track of the sites tested against k-gentleness. At step 3 we pick a new site $x \in \mathbb{B}_{k-1} \cap \Lambda$ and test it, at step 4, for k-gentleness against $\mathbb{B}_{k-1}$, i.e. including also bad sites in $\mathcal{L}^{(\ell)} \setminus \Lambda$. Note that the families $\mathcal{G}_k$ for any $k \geq 1$ are independent on the way in which x is chosen at step 3 of the algorithm. Suppose, indeed, to choose $x \in (\mathbb{B}_{k-1} \cap \Lambda) \setminus V$ at step 3 and to find that $A = B_{\Gamma_k}^{(\ell)}(x) \cap \mathbb{B}_{k-1}$ is a k-gentle cluster. Consider $x' \in A$ such that $x' \neq x$ and set $A' := B_{\Gamma_k}^{(\ell)}(x') \cap \mathbb{B}_{k-1}$: since A passes the test against k-gentleness at step 4 of the algorithm, we have $A \subset A'$. By changing the role of x and x' we get $A = A'$.

After a finite number of operations (bounded by a function of $|\Lambda|$), the algorithm stops and outputs the family $\mathcal{G}_k(\Lambda)$ (note we wrote explicitly the dependence on Λ) with the following property. If $g \in \mathcal{G}_k(\Lambda)$ then $\mathrm{Diam}_\ell(g) \leq \Gamma_k$ and $D_\ell(g, \mathbb{B}_{k-1}(\Lambda) \setminus g) > \gamma_k$. Note that g is not necessarily connected.

We finally take an increasing sequence of sets $\Lambda_i \subset\subset \mathcal{L}^{(\ell)}$ invading $\mathcal{L}^{(\ell)}$ and we sequentially perform the above algorithm. This means the algorithm for Λ_i, is performed independently of the outputs previously obtained. It is easy to show that if $g \in \mathcal{G}_k(\Lambda_i)$ then $g \in \mathcal{G}(\Lambda_{i+1})$ therefore $\mathcal{G}_k(\Lambda_i)$ is increasing in $i \geq 1$, so that we can define $\mathcal{G}_k := \lim_{i\to\infty} \mathcal{G}_k(\Lambda_i) = \cup_i \mathcal{G}_k(\Lambda_i)$ and $\mathbb{G}_k := \lim_{i\to\infty} \mathbb{G}_k(\Lambda_i) = \bigcup_{g\in\mathcal{G}_k} g$. Hence, $\mathbb{B}_k(\Lambda_i) = \mathbb{B}_{k-1}(\Lambda_i) \setminus \mathbb{G}_k(\Lambda_i) = \mathcal{L}^{(\ell)} \setminus \bigcup_{j=0}^{k-1} \mathbb{G}_j(\Lambda_i)$ is decreasing in $i \geq 1$, so that $\mathbb{B}_k := \lim_{i\to\infty} \mathbb{B}_k(\Lambda_i) = \bigcap_i \mathbb{B}_k(\Lambda_i)$. We also remark that, by construction, $\{\mathbb{B}_k, k \geq 0\}$ is a decreasing sequence. Note that from the construction it follows that it is possible to decide whether a site x is k-gentle by looking only at the ω's inside a cube centered at x of radius ϑ_k, as defined in item 2 of Definition 2.3. Hence, see Lemma 3.4 in Ref. 5, we have the following lemma.

Lemma 3.1. *Let $\mathbb{G}_k$ and $\mathcal{G}_k$, $k = 0, 1, \ldots$, as constructed above. Then for each $x \in \mathcal{L}^{(\ell)}$*

$$\{\omega : x \in \mathbb{G}_k(\omega)\} \in \mathcal{A}_{B^{(\ell)}_{\vartheta_k}(x)} \tag{3.2}$$

Theorem 3.2. *Let the sequences Γ, γ, satisfy the conditions in items 1, 2, and 4 in Definition 2.3. Let also $p < \exp\{-a_0/2\}$ and set $a := -\log p - a_0/2 > 0$. Then*

$$\mathbb{Q}(x \in \mathbb{B}_k) \leq \exp\{-a2^k\} \tag{3.3}$$

Remark. From the previous bound and item 5 in Definition 2.3, via a straightforward application of Borel–Cantelli lemma, see the proof of Theorem 3.3 in Ref. 5 for the details, we deduce the following. There exists an $\mathcal{A}$-measurable set $\bar{\Omega} \subset \Omega$, with $\mathbb{Q}(\bar{\Omega}) = 1$, such that for each $\omega \in \bar{\Omega}$ there exists a gentle disintegration $\mathcal{G}(\omega)$, see Definition 2.4, of $\mathcal{L}^{(\ell)}$ with respect to $\mathcal{L}_b^{(\ell)}(\omega)$ and and Γ, γ.

The first step in proving Theorem 3.2 consists in replacing the non-product measure $\mathbb{Q}$ by a Bernoulli product measure with parameter p. This is a standard argument which we report for completeness. We consider Ω endowed with the natural partial ordering $\omega \leq \omega'$ iff for any $x \in \mathcal{L}^{(\ell)}$ we have $\omega_x \leq \omega'_x$. Given two probabilities Q, P on Ω we write $Q \leq P$ iff for any continuous increasing (w.r.t. the previous partial ordering) function f we have $Q(f) \leq P(f)$.

Lemma 3.3. *Let Q_p be the Bernoulli measure on Ω with marginals $Q_p(\omega_x = 1) = p$ and recall the parameter p has been defined in (3.1). Then $\mathbb{Q} \leq Q_p$.*

Proof: For $\Lambda \subset \mathcal{L}^{(\ell)}$ we denote by $\mathbb{Q}_\Lambda$ the marginal of $\mathbb{Q}$ on $\Omega_\Lambda = \{0, 1\}^\Lambda$ by Q_p the Bernoulli measure on $\{0, 1\}$, $Q_p(\{1\}) = p$. The lemma follows by induction from

$$\mathbb{Q}_\Lambda(d\omega_\Lambda) \leq Q_p(d\omega_x)\mathbb{Q}_{\Lambda\setminus\{x\}}(d\omega_{\Lambda\setminus\{x\}}) \quad \forall x \in \Lambda, \quad \forall \Lambda \subset \mathcal{L}^{(\ell)} \tag{3.4}$$

It is easy to show

$$\mathbb{Q}_\Lambda(\omega_x = 1|\mathcal{A}_{\Lambda\setminus\{x\}}) = \mathbb{Q}_{\Lambda^c}(\mathbb{Q}(\omega_x = 1|\mathcal{A}_{\{x\}^c})) \quad \mathbb{Q}_\Lambda\text{-a.s.}$$

therefore (3.1) and the translation invariance of $\mathbb{Q}$ imply

$$\operatorname*{ess\,sup}_{\omega_{\Lambda\setminus\{x\}}} \mathbb{Q}_\Lambda(\omega_x = 1|\mathcal{A}_{\Lambda\setminus\{x\}})(\omega_{\Lambda\setminus\{x\}}) \le p \tag{3.5}$$

We next prove (3.4). Let f be a continuous and increasing function on Ω_Λ; by taking conditional expectation we have

$$\begin{aligned}
\mathbb{Q}_\Lambda(f) &= \mathbb{Q}_{\Lambda\setminus\{x\}}(\mathbb{Q}_\Lambda(f[1\!\!1_{\omega_x=1} + 1\!\!1_{\omega_x=0}])|\mathcal{A}_{\Lambda\setminus\{x\}}) \\
&= \int \mathbb{Q}_{\Lambda\setminus\{x\}}(d\omega_{\Lambda\setminus\{x\}})\big\{f(\omega_{\Lambda\setminus\{x\}}0_{\{x\}}) + \mathbb{Q}_\Lambda\big(\omega_x = 1|\mathcal{F}^{(\omega)}_{\Lambda\setminus\{x\}}\big) \\
&\quad \times [f(\omega_{\Lambda\setminus\{x\}}1_{\{x\}}) - f(\omega_{\Lambda\setminus\{x\}}0_{\{x\}})]\big\} \\
&\le \int \mathbb{Q}_{\Lambda\setminus\{x\}}(d\omega_{\Lambda\setminus\{x\}})\{p[f(\omega_{\Lambda\setminus\{x\}}1_{\{x\}}) - f(\omega_{\Lambda\setminus\{x\}}0_{\{x\}})] + f(\omega_{\Lambda\setminus\{x\}}0_{\{x\}})\} \\
&= \int Q_{\Lambda\setminus\{x\}}(d\omega_{\Lambda\setminus\{x\}})\mathbb{Q}_p(d\omega_x)f(\omega_\Lambda)
\end{aligned} \tag{3.6}$$

where we used that f is increasing and (3.5) in the inequality. □

Lemma 3.4. *For each $x \in \mathcal{L}^{(\ell)}$ and $k \ge 0$ the event $\{\omega : x \in \mathbb{B}_k(\omega)\}$ is increasing namely,*

$$\omega \le \omega' \Rightarrow \mathbb{B}_k(\omega) \subset \mathbb{B}_k(\omega') \tag{3.7}$$

Proof: We prove (3.7) by induction on k. First of all we note that by definition of the natural partial order on Ω it holds for $k = 0$. Let us prove that

$$\mathbb{G}_{k+1}(\omega') \subset \bigcup_{j=0}^{k+1} \mathbb{G}_j(\omega) \tag{3.8}$$

Let $x \in \mathbb{G}_{k+1}(\omega')$, then either $x \in \bigcup_{j=0}^{k} \mathcal{G}_j(\omega)$ or $x \in \mathbb{B}_k(\omega)$. In the former case we are done, in the latter we have that, since $x \in \mathbb{G}_{k+1}(\omega')$, there exists a set $g' \subset \mathbb{B}_k(\omega')$ such that: (i) $x \in g'$; (ii) $\mathrm{Diam}_\ell(g') \le \Gamma_k$; (iii) $D_\ell(g', \mathbb{B}_k(\omega') \setminus g') > \gamma_k$. Set now $g := g' \cap \mathbb{B}_k(\omega)$, by the inductive hypotheses it is easy to verify that g satisfies the three properties above with ω' replaced by ω. Hence $x \in \mathbb{G}_{k+1}(\omega)$. From (3.8) and the induction hypotheses we get $\bigcup_{j=0}^{k+1} \mathbb{G}_j(\omega') \subset \bigcup_{j=0}^{k+1} \mathbb{G}_j(\omega)$. Since $\mathbb{B}_{k+1}(\omega) = \mathbb{B}_0(\omega) \setminus \bigcup_{j=0}^{k+1} \mathbb{G}_j(\omega)$, we have proven (3.7) with k replaced by $k + 1$. □

The key step in proving Theorem 3.2 is the following recursive estimate on the degree of badness.

Lemma 3.5. *Let Γ, γ satisfy the conditions in items 1, 2, and 4 in Definition 2.3 and set $\psi_k := \mathcal{Q}_p(x \in \mathbb{B}_k)$, note ψ_k is independent of x by translational invariance, and $A_k(x) := B^{(\ell)}_{\gamma_k+\Gamma_k}(x) \setminus B_{(\Gamma_k-1)/2}(x)$. Then*

$$\psi_{k+1} \le |A_{k+1}|\psi_k^2 \tag{3.9}$$

where $|A_k| = |A_k(x)|$ does not depend on x.

Proof: By recalling the definition of the k-bad set $\mathbb{B}_k$ we have

$$\{x \in \mathbb{B}_{k+1}\} = \{x \in \mathbb{B}_k\} \cap \{x \notin \mathbb{G}_{k+1}\} \tag{3.10}$$

On the other hand, by the construction of the $(k+1)$-gentle sites,

$$\{x \in \mathbb{B}_k\} \cap \{x \notin \mathbb{G}_{k+1}\} \subset \{x \in \mathbb{B}_k\} \cap \{\exists y \in A_{k+1}(x) : y \in \mathbb{B}_k\} \tag{3.11}$$

indeed, given $\mathbb{B}_k$, if there were no k-bad site in the annulus $A_{k+1}(x)$ then x would have been $(k+1)$-gentle. From (3.10) and (3.11)

$$\begin{aligned}\psi_{k+1} &= \mathcal{Q}_p\{x \in \mathbb{B}_{k+1}\} \le \mathcal{Q}_p\left(\bigcup_{y\in A_{k+1}(x)} \{x \in \mathbb{B}_k\} \cap (y \in \mathbb{B}_k)\right)\\ &\le \sum_{y\in A_{k+1}(x)} \mathcal{Q}_p(\{x \in \mathbb{B}_k\} \cap \{y \in \mathbb{B}_k\})\\ &= \sum_{y\in A_{k+1}(x)} \mathcal{Q}_p(\{x \in \mathbb{B}_k\})\mathcal{Q}_p(\{y \in \mathbb{B}_k\}) = |A_{k+1}|\psi_k^2 \end{aligned}\tag{3.12}$$

where in the last step, we used (3.2), the definition of $A_k(x)$, item 2 in Definition 2.3, the product structure of the measure $\mathcal{Q}_p$ and its translation invariance. □

Proof of Theorem 3.2. By Lemmata 3.3 and 3.4 it is enough to prove the bound (3.3) for the Bernoulli measure $\mathcal{Q}_p$.

Let $f_k := -\log \psi_k$ and $b_k := \log |A_{k+1}|$, where ψ_k and A_k have been defined in Lemma 3.5. Then by iterating (3.9) and using item 4 in Definition 2.3, we get

$$f_{k+1} \ge 2f_k - b_k \ge \cdots \ge 2^{k+1} f_0 - 2^k \sum_{j=0}^{k} 2^{-j} b_j \ge 2^{k+1} f_0 - 2^k a_0 = 2^{k+1} a \tag{3.13}$$

where we recall that $a = -\log\, p - a_0/2 = f_0 - a_0/2 > 0$. □

Recall q has been defined in (1.5). To gently disintegrate the lattice $\mathcal{L}^{(\ell)}$ by means of Theorem 3.2, we need a bound on the badness parameter p, see (3.1), in terms of q.

Lemma 3.6. *Recall p has been defined in (3.1) and q in (1.5); then*

$$p \leq 1 - (1-q)^{2\ell(\ell-1)} + 4\frac{1-(1-q)^{\ell}}{1-(1-q)^{2\ell(\ell+1)}} \tag{3.14}$$

Proof: For each $i \in \mathcal{L}^{(\ell)}$ we define the five events E_i^0, $E_i^{1,\pm}$, and $E_i^{2,\pm}$:

$$E_i^0 := \bigcup_{\substack{e\in\mathcal{E};\\ e\subset\mathcal{Q}_\ell(i)}} \{J_e > b\} \quad \text{and} \quad E_i^{s,\pm} := \bigcup_{\substack{e\in\mathcal{E};e\cap\mathcal{Q}_\ell(i)\neq\emptyset,\\ e\cap\mathcal{Q}_\ell(i\pm\ell e_s)\neq\emptyset}} \{J_e > b\} \tag{3.15}$$

where $s = 1, 2$ and we recall e_1 and e_2 are the coordinate unit vectors in $\mathcal{L}$. By using the equality

$$E_i = E_i^0 \cup E_i^{1,-} \cup E_i^{2,+} \cup E_i^{1,+} \cup E_i^{2,-}$$

and the product nature $\mathbb{P}$, we have that

$$\begin{aligned}\mathbb{P}(E_i|\{\omega_j = a_j\}_{j\neq i}) &\leq \mathbb{P}\big(E_i^0|\{\omega_j = a_j\}_{j\neq i}\big)\\ &\quad+ \sum_{s=1}^{2}\mathbb{P}\big(E_i^{s,+}|\{\omega_j = a_j\}_{j\neq i}\big) + \sum_{s=1}^{2}\mathbb{P}\big(E_i^{s,-}|\{\omega_j = a_j\}_{j\neq i}\big)\\ &= \mathbb{P}\big(E_i^0\big) + 4\mathbb{P}\big(E_i^{1,-}|\omega_{i-\ell e_1} = a_{i-\ell e_1}|\big)\end{aligned} \tag{3.16}$$

Since $E_i^{1,-} \cap \{\omega_{i-\ell_1} = 0\} = \emptyset$, the l.h.s. of (3.16) can be bounded uniformly in $\{a_j\}_{j\neq i}$ by $\mathbb{P}(E_i^0) + 4\mathbb{P}(E_i^{1,-})/\mathbb{P}(\omega_{i-\ell e_1} = 1)$. The lemma then follows by a straightforward computation. □

Proof of Proposition 2.5. Let the functions $q_0 : [0, K_c) \to (0, 1]$ and $\bar{\ell} : [0, K_c) \to [0, \infty)$ be constructed as follows.

1. By Lemma 2.1, given $b \in [0, K_c)$, we find $\ell_0(b)$ and $m_0(b)$ such that (2.4) holds.
2. By Proposition 2.2 we find $\alpha_1(b)$, $A_1(b)$, and $n_1(b)$ such that the bound (2.17) holds.
3. Choose $\beta_0(b)$ as below (2.19) and let the scales Γ, γ, be as in (2.19) with $\beta = \beta_0(b)$.
4. Compute $a_0(b)$ in item 4 in Definition 2.3.

5. Let $q_0(b)$ and $\bar{\ell}(b)$, with $\bar{\ell}(b)$ multiple of $\ell_0(b)$ such that

$$1-(1-q)^{2\bar{\ell}(b)(\bar{\ell}(b)-1)} + 4\frac{1-(1-q)^{\bar{\ell}(b)}}{1-(1-q)^{2\bar{\ell}(b)(\bar{\ell}(b)+1)}} < \exp\{-a_0(b)/2\}$$

for any $q \leq q_0(b)$.

Step 5 is possible because for each $\bar{\ell}(b)$ fixed the l.h.s. of the inequality above converges to $2/(\ell+1)$ as $q \to 0$.

By applying Lemma 3.6, Theorem 3.2, and the remark following it we conclude the proof of the proposition. □

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