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LARGE DEVIATIONS OF THE EMPIRICAL CURRENT IN INTERACTING PARTICLE SYSTEMS*

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Abstract. We study current fluctuations in lattice gases in the hydrodynamic scaling limit. More precisely, we prove a large deviation principle for the empirical current in the symmetric simple exclusion process with rate functional I. We then estimate the asymptotic probability of a fluctuation of the average current over a large time interval and show that the corresponding rate function can be obtained by solving a variational problem for the functional I. For the symmetric simple exclusion process the minimizer is time independent so that this variational problem can be reduced to a time-independent one. On the other hand, for other models the minimizer is time dependent. This phenomenon is naturally interpreted as a dynamical phase transition.

Key words. interacting particle systems, large deviations, hydrodynamic limit

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1. Introduction.

In the last 20 years, interacting particle systems have become a main subject of research in physics for the insight they provide on the dynamical aspects of statistical physics. On the mathematical side they provide a source of new, interesting problems in probability theory. Particularly relevant are the results obtained on their hydrodynamical limits and the associated large deviations, since a microscopic derivation of phenomelogical macroscopic laws can be rigorously established. More precisely, for symmetric conservative interacting particle systems, it has been shown that the empirical density satisfies a parabolic evolution equation. The associated dynamical large deviations rate function measures the asymptotic probability, as the number of particles diverges, of fluctuations from the hydrodynamical evolution. As discussed in [1], this rate function provides a new approach to the analysis of stationary nonequilibrium states. These states describe a physical situation in which there is a macroscopic flow through the system and the Gibbsian description is not applicable. Rigorous proofs of the dynamical large deviation principle have been obtained for some equilibrium models (see, e.g., [9], [12]) and for the nonequilibrium simple exclusion process (see [2]).

Besides the empirical density, a very important observable is the current, which measures the flux of particle. This quantity gives information that cannot be recovered from the density because from a density trajectory we can determine the current trajectory only up to a divergence-free vector field. In [3], [4] we have introduced, at a heuristic level, the large deviation principle for the empirical current. In the present

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paper we prove it in the case of the symmetric simple exclusion and illustrate some relevant applications.

The simple exclusion process is a lattice gas with an exclusion principle: a particle can move to a neighboring site only if this is empty. The particle dynamics is given by a Markov process on the state space $\{0,1\}^{\mathbf{T}_N^d}$, where $\mathbf{T}_N^d = (\mathbf{Z}/N\mathbf{Z})^d$ is the discrete d-dimensional torus with N^d points. We denote by $\eta = \{\eta(x),\, x \in \mathbf{T}_N^d\}$ a configuration, so that $\eta(x) = 1$ when the site x is occupied, and $\eta(x) = 0$ otherwise. Let π^N be the empirical density of particles. The hydrodynamic limit for this model is particularly simple: in the limit $N \to \infty$ the empirical density π^N satisfies the heat equation. To discuss the large deviation asymptotics we need to introduce the *mobility* $\chi(\pi)$, which describes the response to an external field; for the symmetric simple exclusion we have $\chi(\pi) = \pi(1-\pi)$. We introduce the integrated empirical current $\mathbf{W}_t^N$, which measures the total net flow of particles in the time interval $[0,t]$, associated with a trajectory η. We shall prove a large deviation principle which can be informally written as follows. Fix a possible path $\mathbf{W}_t$, $t \in [0,T]$, of the integrated empirical current; then

$$\mathbf{P}^N\big\{\mathbf{W}_t^N \approx \mathbf{W}_t,\ t \in [0,T]\big\} \sim \exp\big\{-N^d\, I_{[0,T]}(\mathbf{W})\big\}, \tag{1.1}$$

where the rate functional is

$$I_{[0,T]}(\mathbf{W}) = \frac{1}{2}\int_0^T dt \left\langle \left[\dot{\mathbf{W}}_t + \frac{1}{2}\nabla\pi_t\right],\ \frac{1}{\pi_t(1-\pi_t)}\left[\dot{\mathbf{W}}_t + \frac{1}{2}\nabla\pi_t\right]\right\rangle. \tag{1.2}$$

In the above formula $\dot{\mathbf{W}}_t$ is the instantaneous current at time t. Moreover π_t, which represents the associated fluctuation of the empirical density, is obtained from $\dot{\mathbf{W}}$ by solving the continuity equation $\partial_t\pi + \nabla\cdot\dot{\mathbf{W}} = 0$. Finally, $\langle\cdot,\cdot\rangle$ denotes integration with respect to the space variables. Note that (1.2) can be interpreted, in analogy to the classical Ohm's law, as the total energy dissipated in the time interval $[0,T]$ by the extra current $\dot{\mathbf{W}} + \frac{1}{2}\nabla\pi$. The large deviation principle of the empirical density [12], as we show, can also easily be deduced from (1.1), (1.2).

Using (1.1), (1.2) we then analyze the fluctuations properties of the mean empirical current $\mathbf{W}_T^N/T = T^{-1}\int_0^T dt\,\dot{\mathbf{W}}_t$ over a large time interval $[0,T]$. This is the question addressed in [6] in one space dimension by postulating an "additivity principle" which relates the fluctuation of the current in the whole system to the fluctuations in subsystems. We show that the probability of observing a given, divergence-free, time averaged fluctuation $\mathbf{J}$ can be described by a rate functional $\Phi(\mathbf{J})$; i.e., as $N \to \infty$ and $T \to \infty$ we have

$$\mathbf{P}^N\left\{\frac{\mathbf{W}_T^N}{T} \approx \mathbf{J}\right\} \sim \exp\big\{-N^d\, T\,\Phi(\mathbf{J})\big\}.$$

The functional Φ is characterized by a variational problem for the functional $I_{[0,T]}$,

$$\Phi(\mathbf{J}) = \lim_{T\to\infty}\inf_{\mathbf{W}}\frac{1}{T}\, I_{[0,T]}(\mathbf{W}), \tag{1.3}$$

where the infimum is carried over all paths $\mathbf{W}_t$ such that $\mathbf{W}_T = T\mathbf{J}$.

Let us denote by U the functional obtained by restricting the infimum in (1.3) to the paths $\mathbf{W}$ such that $\dot{\mathbf{W}}_t$ is divergence free for any $t \in [0,T]$. The associated density profile π_t does not evolve. We get

$$U(\mathbf{J}) = \inf_\rho \frac{1}{2}\left\langle \left[\mathbf{J} + \frac{1}{2}\nabla\rho\right],\ \frac{1}{\rho(1-\rho)}\left[\mathbf{J} + \frac{1}{2}\nabla\rho\right]\right\rangle, \tag{1.4}$$

where the infimum is carried out over all the density profiles ρ. This is the functional introduced in [6] in one space dimension. For the symmetric simple exclusion process we prove that the additivity principle postulated in [6] gives the correct answer, that is, $\Phi = U$. On the other hand, while Φ is always convex, the functional U may be nonconvex. In general (see [3], [4]) we interpret the strict inequality $\Phi < U$ as a dynamical phase transition. In such a case the minimizers for (1.3) become in fact time dependent and the invariance under time shifts is broken. In [4] we have shown that, for the one-dimensional Kipnis–Marchioro–Presutti (KMP) model (see [5], [11]), which is defined by a harmonic chain with random exchange of energy between neighboring oscillators, the following holds when it is considered with a periodic boundary condition. The functional U is given by $U(J) = \frac{1}{2} J^2/\chi(m) = \frac{1}{2} J^2/m^2$, where m is the (conserved) total energy. Moreover, for J large enough, $\Phi(J) < U(J)$. This inequality is obtained by constructing a suitable traveling wave current path whose cost is less than $U(J)$. In the present paper we give a more formal presentation of these results. We finally mention that the strict inequality $\Phi < U$ also occurs for the weakly asymmetric exclusion process for a sufficiently large external field [7].

2. Notation and results. For $N \geqq 1$, let $\mathbf{T}_N^d = (\mathbf{Z}/N\mathbf{Z})^d$ be the discrete d-dimensional torus with N^d points. Consider the symmetric simple exclusion process on $\mathbf{T}_N^d$. This is the Markov process on the state space $\mathcal{X}_N := \{0,1\}^{\mathbf{T}_N^d}$ whose generator L_N is given by

$$(L_N f)(\eta) = \frac{N^2}{2} \sum_{x,y \in \mathbf{T}_N^d,\, |x-y|=1} \big\{ f(\sigma^{x,y}\eta) - f(\eta) \big\} .$$

In this formula $\eta = \{\eta(x),\ x \in \mathbf{T}_N^d\} \in \mathcal{X}_N$ is a configuration of particles, so that $\eta(x) = 0$ (respectively, $\eta(x) = 1$), if and only if site x is vacant (respectively, occupied) for η, and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, y, \\ \eta(y) & \text{if} \quad z = x, \\ \eta(x) & \text{if} \quad z = y. \end{cases}$$

Notice that we speed up time by N^2. Denote by $\{\eta_t \colon t \geqq 0\}$ the Markov process with generator L_N and by $\mathbf{P}^N_{\mu^N}$ the probability measure on the Skorokhod space $D(\mathbf{R}_+, \mathcal{X}_N)$ induced by the Markov process η_t and a probability measure μ^N on $\mathcal{X}_N$, standing for the initial distribution. When μ^N is a Dirac measure concentrated on a configuration η^N, we denote $\mathbf{P}^N_{\mu^N}$ by $\mathbf{P}^N_{\eta^N}$. An elementary computation shows that this process is reversible with respect to any Bernoulli product measure on $\{0,1\}^{\mathbf{T}_N^d}$ with parameter $m \in [0,1]$.

Denote by $\mathcal{M}(\mathbf{T}^d)$ the space of finite signed measures on $\mathbf{T}^d$, the d-dimensional torus of side 1, endowed with the weak topology, and by $\mathcal{F} = \mathcal{F}_{+,1}(\mathbf{T}^d)$ the set of positive, measurable functions bounded by 1 endowed with the same weak topology. For a finite signed measure m we let $\langle m, F\rangle$ be the integral of a continuous function $F \colon \mathbf{T}^d \to \mathbf{R}$ with respect to m. Likewise for a profile $\pi \in \mathcal{F}$ and $F \in C(\mathbf{T}^d)$ we denote by $\langle \pi, F\rangle$ the integral of πF.

For $N \geqq 1$ and a configuration $\eta \in \mathcal{X}_N$, denote by $\pi^N = \pi^N(\eta^N)$ the empirical density of particles. It is defined as

$$\pi^N(u) = \sum_{x \in \mathbf{T}_N^d} \mathbb{1}\{uN \in B(x)\}\, \eta(x),$$

where, for $x = (x_1, \dots, x_d) \in \mathbf{T}_N^d$, $B(x)$ is the cube $[x_1, x_1 + 1) \times \cdots \times [x_d, x_d + 1)$. Set $\pi_t^N = \pi^N(\eta_t)$ and notice that π_t^N belongs to $\mathcal{F}$ for each $t \geqq 0$.

For $t \geqq 0$ and two neighboring sites $x, y \in \mathbf{T}_N^d$, denote by $J_t^{x,y}$ the total number of particles that jumped from x to y in the macroscopic time interval $[0, t]$. Let $\{e_k\colon 1 \leqq k \leqq d\}$ be the canonical basis of $\mathbf{R}^d$; the difference $W_t^{x,x+e_j} = J_t^{x,x+e_j} - J_t^{x+e_j,x}$ represents the net flow of particles across the bond $\{x, x+e_j\}$ in the time interval $[0, t]$.

For $t \geqq 0$, we define the *empirical integrated current* $\mathbf{W}_t^N = (W_{1,t}^N, \dots, W_{d,t}^N) \in \mathcal{M}_d = \{\mathcal{M}(\mathbf{T}^d)\}^d$ as the vector-valued finite signed measure on $\mathbf{T}^d$ induced by the net flow of particles in the time interval $[0, t]$:

$$W_{j,t}^N = N^{-(d+1)} \sum_{x \in \mathbf{T}_N^d} W_t^{x,x+e_j}\, \delta_{x/N}, \qquad j = 1, \dots, d,$$

where δ_u stands for the Dirac measure concentrated on u. Notice the extra factor N^{-1} in the normalizing constant which corresponds to the diffusive rescaling of time. In particular, for a continuous vector field $\mathbf{F} = (F_1, \dots, F_d) \in C(\mathbf{T}^d; \mathbf{R}^d)$ the integral of $\mathbf{F}$ with respect to $\mathbf{W}_t^N$, also denoted by $\langle \mathbf{W}_t^N, \mathbf{F} \rangle$, is given by

$$\langle \mathbf{W}_t^N, \mathbf{F} \rangle = N^{-(d+1)} \sum_{j=1}^d \sum_{x \in \mathbf{T}_N^d} F_j\left(\frac{x}{N}\right) W_t^{x,x+e_j}. \tag{2.1}$$

The purpose of this article is to prove a large deviation principle for the empirical integrated current $\mathbf{W}_t^N$ and discuss the asymptotic behavior as $t \to \infty$. We start with the law of large numbers. Fix a profile $\lambda \in \mathcal{F}$ and let $\{\mu^N\colon N \geqq 1\}$ be a sequence of measures on $\mathcal{X}_N$ associated with λ in the sense that the empirical density converges to λ in probability with respect to μ^N. Namely, for each $F \in C(\mathbf{T}^d)$ and $\delta > 0$, we have

$$\lim_{N \to \infty} \mu^N\left\{\left|\langle \pi^N, F \rangle - \int_{\mathbf{T}^d} \lambda(u)\, F(u)\, du\right| > \delta\right\} = 0. \tag{2.2}$$

It is well known (see, e.g., [10]) that in such a case the empirical density π_t^N converges in probability to $\rho = \rho(t, u)$ which solves the heat equation

$$\begin{cases} \partial_t \rho = \dfrac{1}{2}\, \Delta \rho, \\ \rho(0, \cdot) = \lambda(\cdot), \end{cases} \tag{2.3}$$

where $\Delta = \nabla \cdot \nabla$ stands for the Laplacian and ∇ for the gradient. We claim that the empirical current converges to the time integral of $-\frac{1}{2}\, \nabla \rho$. This is the content of the next result, which is proved in subsection 3.1 in a more general context.

PROPOSITION 2.1. *Fix a profile $\lambda \in \mathcal{F}$ and consider a sequence of probability measures μ^N associated with λ in the sense of* (2.2). *Let ρ be the solution of the heat equation* (2.3). *Then, for each $T > 0$, $\delta > 0$, and $\mathbf{F} \in C(\mathbf{T}^d; \mathbf{R}^d)$,*

$$\lim_{N \to \infty} \mathbf{P}_{\mu^N}^N\left\{\left|\langle \mathbf{W}_T^N, \mathbf{F} \rangle + \frac{1}{2} \int_0^T dt \int_{\mathbf{T}^d} \mathbf{F}(u) \cdot \nabla \rho(t, u)\, du\right| > \delta\right\} = 0.$$

We turn now to the large deviations of the pair $(\mathbf{W}^N, \pi^N)$. Fix $T > 0$ and denote by $D([0,T], \mathcal{M}_d \times \mathcal{F})$ the space of càdlàg trajectories with values in $\mathcal{M}_d \times \mathcal{F}$ endowed with the Skorokhod topology. Fix a profile $\gamma \in C^2(\mathbf{T}^d)$ bounded away from 0 and 1: there exists $\delta > 0$ such that $\delta \leqq \gamma \leqq 1 - \delta$. To focus on the dynamical fluctuations, we assume that the process starts from a deterministic initial condition η^N which is associated with γ in the sense that $\pi^N(\eta^N) \to \gamma$ in $\mathcal{F}$.

Let $\mathfrak{A}_\gamma$ be the set of trajectories $(\mathbf{W}, \pi)$ in $C([0,T], \mathcal{M}_d \times \mathcal{F})$ such that for any $t \in [0,T]$ and any $F \in C^1(\mathbf{T}^d)$

$$\langle \pi_t, F\rangle - \langle \gamma, F\rangle = \langle \mathbf{W}_t, \nabla F\rangle, \qquad \mathbf{W}_0 = 0. \tag{2.4}$$

Note that $\mathfrak{A}_\gamma$ is a *closed and convex* subset of $C([0,T], \mathcal{M}_d \times \mathcal{F})$. Equation (2.4) is the weak formulation of the continuity equation $\dot\pi_t + \nabla\cdot\dot{\mathbf{W}}_t = 0$, $\pi_0 = \gamma$, $\mathbf{W}_0 = 0$. For each $\mathbf{F} \in C^{1,1}([0,T] \times \mathbf{T}^d; \mathbf{R}^d)$, define the *convex and lower semicontinuous* functional $J_{\mathbf{F}}$ as follows. If $(\mathbf{W}, \pi) \in C([0,T]; \mathcal{M}_d \times \mathcal{F})$, we set

$$\begin{aligned} J_{\mathbf{F}}(\mathbf{W}, \pi) &= \langle \mathbf{W}_T, \mathbf{F}_T\rangle - \int_0^T dt\, \langle \mathbf{W}_t, \partial_t \mathbf{F}_t\rangle \\ &\quad - \frac{1}{2}\int_0^T dt\, \big\langle \pi_t, \nabla\cdot \mathbf{F}_t\big\rangle - \frac{1}{2}\int_0^T dt\, \big\langle \chi(\pi_t),\ |\mathbf{F}_t|^2\big\rangle, \end{aligned} \tag{2.5}$$

where, here and in what follows, $\chi(a) = a(1-a)$ is the mobility. We set $J_{\mathbf{F}}(\mathbf{W}, \pi) = +\infty$ if $(\mathbf{W}, \pi) \notin C([0,T]; \mathcal{M}_d \times \mathcal{F})$. Since $C([0,T]; \mathcal{M}_d \times \mathcal{F})$ is a closed subset of $D([0,T]; \mathcal{M}_d \times \mathcal{F})$, the argument in [10, section 10.1] proves the convexity and lower semicontinuity of $J_{\mathbf{F}}$.

Let finally

$$J(\mathbf{W}, \pi) = \sup_{\mathbf{F}\in C^{1,1}} J_{\mathbf{F}}(\mathbf{W}, \pi), \quad I(\mathbf{W}, \pi) = \begin{cases} J(\mathbf{W}, \pi) & \text{if} \quad (\mathbf{W}, \pi) \in \mathfrak{A}_\gamma, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Notice that the functional J is convex and lower semicontinuous, properties which are inherited by I because $\mathfrak{A}_\gamma$ is a closed and convex subset of $C([0,T], \mathcal{M}_d \times \mathcal{F})$. Notice furthermore that the continuity equation (2.4) determines the trajectory π as a function of $\mathbf{W}$ and the initial condition γ. Therefore, the rate function $I(\mathbf{W}, \pi)$ can be thought of as a function of $\mathbf{W}$ and the initial density profile. In subsection 3.3 we derive a more explicit formula for the rate function (2.6). If $I(\mathbf{W}, \pi) < \infty$, we have

$$I(\mathbf{W}, \pi) = \frac{1}{2}\int_0^T dt\, \big\langle \chi(\pi_t),\ |\mathbf{F}_t|^2\big\rangle,$$

where the vector-valued function $\mathbf{F}_t$ is the solution of

$$\partial_t \mathbf{W}_t + \frac{1}{2}\nabla \pi_t = \chi(\pi_t)\, \mathbf{F}_t.$$

Thus, formally,

$$I(\mathbf{W}, \pi) = \frac{1}{2}\int_0^T dt\, \Big\langle \frac{1}{\chi(\pi_t)}\, \big|\dot{\mathbf{W}}_t - \dot{\mathbf{W}}_t(\pi)\big|^2 \Big\rangle,$$

where $\dot{\mathbf{W}}_t = \partial_t \mathbf{W}_t$ is the instantaneous current at time t for the path $(\mathbf{W}, \pi)$ and $\dot{\mathbf{W}}_t(\pi) = -\frac{1}{2}\nabla \pi_t$ is the typical instantaneous current at time t associated with the density profile π_t. Recall in fact that the hydrodynamic equation (2.3) is in our case the heat equation.

THEOREM 2.1. *Consider a profile $\gamma \in C^2(\mathbf{T}^d)$ bounded away from 0 and 1 and a sequence $\{\eta^N \colon N \geqq 1\}$ such that $\pi^N(\eta^N) \to \gamma$ in $\mathcal{F}$. Then, for each closed set F and each open set G of $D([0,T], \mathcal{M}_d \times \mathcal{F})$, we have*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\big\{\big(\mathbf{W}^N,\ \pi^N\big) \in F\big\} \leqq - \inf_{(\mathbf{W},\pi)\in F} I(\mathbf{W},\pi),$$
$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\big\{\big(\mathbf{W}^N,\ \pi^N\big) \in G\big\} \geqq - \inf_{(\mathbf{W},\pi)\in G} I(\mathbf{W},\pi).$$

We next state a "large deviation principle" for the mean empirical current $\mathbf{W}^N_T/T$ in the interval $[0,T]$ as we let *first* $N \to \infty$ and *then* $T \to \infty$.

Let us denote by $\mathcal{B} \subset \mathcal{M}_d$ the set of divergence-free measures, i.e.,

$$\mathcal{B} := \Big\{\mathbf{J} \in \mathcal{M}_d \colon \langle \mathbf{J}, \nabla f\rangle = 0 \text{ for any } f \in C^1(\mathbf{T}^d)\Big\} \tag{2.7}$$

which is a closed subspace of $\mathcal{M}_d$. Given $m \in (0,1)$, we introduce the set of profile with mass m, i.e., we set $\mathcal{F}_m := \{\rho \in \mathcal{F} \colon \int_{\mathbf{T}^d} du\, \rho(u) = m\}$. We finally define $U_m \colon \mathcal{M}_d \to [0,+\infty]$ by

$$U_m(\mathbf{J}) := \inf_{\substack{\rho\in\mathcal{F}_m\cap C^2(\mathbf{T}^d)\\ 0<\rho<1}} \frac{1}{2}\left\langle \left[\mathbf{j} + \frac{1}{2}\nabla\rho\right],\ \frac{1}{\rho(1-\rho)}\left[\mathbf{j} + \frac{1}{2}\nabla\rho\right]\right\rangle \tag{2.8}$$

if $\mathbf{J} \in \mathcal{B}$, $\mathbf{J}(du) = \mathbf{j}\, du$, and $U_m(\mathbf{J}) = +\infty$ otherwise. In section 5.1 we show that U_m is a lower semicontinuous convex functional.

In section 5.1 we also prove that in the one-dimensional case, where $J \in \mathcal{B}$ and $J(du) = j\, du$ imply that j is constant du-a.e., we simply have

$$U_m(J) := \begin{cases} \dfrac{1}{2}\dfrac{j^2}{m(1-m)} & \text{if } J = j\, du \text{ for some } j \text{ which is } du\text{-a.e. constant,}\\ +\infty & \text{otherwise.}\end{cases} \tag{2.9}$$

THEOREM 2.2. *Let $m \in (0,1)$, let $\gamma \in C^2(\mathbf{T}^d) \cap \mathcal{F}_m$ be bounded away from 0 and 1, and let $\eta^N \in \mathcal{X}_N$ be a sequence such that $\pi^N(\eta^N) \to \gamma$ in $\mathcal{F}$. Then, for each closed set C and each open set G of $\mathcal{M}_d$, we have*

$$\limsup_{T\to\infty}\limsup_{N\to\infty} \frac{1}{TN^d} \log \mathbf{P}^N_{\eta^N}\left\{\frac{1}{T}\,\mathbf{W}^N_T \in C\right\} \leqq - \inf_{\mathbf{J}\in C} U_m(\mathbf{J}),$$
$$\liminf_{T\to\infty}\liminf_{N\to\infty} \frac{1}{TN^d} \log \mathbf{P}^N_{\eta^N}\left\{\frac{1}{T}\,\mathbf{W}^N_T \in G\right\} \geqq - \inf_{\mathbf{J}\in G} U_m(\mathbf{J}).$$

3. Large deviation for the empirical current on a fixed time interval. In this section we prove Theorem 2.1. The proof is similar to that of the large deviation principle for the empirical density; see [12] or [10, Chap. 10]. We therefore present only the main modifications.

3.1. Weakly asymmetric exclusion processes. In this subsection we prove the law of large numbers for the empirical current of weakly asymmetric exclusion processes. Proposition 2.1 follows as a particular case.

Fix $T > 0$ and a time-dependent vector-valued function $\mathbf{F} = (F_1, \dots, F_d) \in C^{1,1}([0,T] \times \mathbf{T}^d;\, \mathbf{R}^d)$. Denote by $L_{\mathbf{F},N}$ the time-dependent generator on $\mathcal{X}_N$ given by

$$(L_{\mathbf{F},N} f)(\eta) = \frac{N^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbf{T}_N^d} c^{\mathbf{F}}_{x,x+e_j}(t,\eta) \big\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \big\},$$

where the rate $c^{\mathbf{F}}_{x,x+e_j}(t,\eta)$ is given by

$$\eta(x)\,\big[1-\eta(x+e_j)\big]\, e^{N^{-1}F_j(t,x/N)} + \eta(x+e_j)\,\big[1-\eta(x)\big]\, e^{-N^{-1}F_j(t,x/N)}.$$

Hence, for N large, instead of jumping from x to $x+e_j$ (respectively, from $x+e_j$ to x) with rate $\frac{1}{2}$, at a macroscopic time t particles jump with rate $\frac{1}{2}\{1+N^{-1}F_j(t,x/N)\}$ (respectively, $\frac{1}{2}\{1-N^{-1}F_j(t,x/N)\}$) and a small drift appears due to the external field $\mathbf{F}$. For a probability measure μ^N on $\mathcal{X}_N$, denote by $\mathbf{P}^N_{\mathbf{F},\mu^N}$ the measure on the path space $D(\mathbf{R}_+, \mathcal{X}_N)$ induced by the Markov process η_t with generator $L_{\mathbf{F},N}$ and initial distribution μ^N.

Let $\rho^{\mathbf{F},\lambda}$ be a unique weak solution of the parabolic equation

$$\begin{cases} \partial_t \rho = \dfrac{1}{2}\,\Delta\rho - \nabla\cdot\big\{\chi(\rho)\,\mathbf{F}\big\}, \\ \rho(0,\cdot) = \lambda(\cdot). \end{cases} \tag{3.1}$$

Write the previous differential equation as

$$\dot\rho_t + \nabla\cdot\dot{\mathbf{W}}_{\mathbf{F}}(\rho_t) = 0,$$

where $\dot{\mathbf{W}}_{\mathbf{F}}(\rho)$ is the instantaneous current associated with the profile ρ and is given by

$$\dot{\mathbf{W}}_{\mathbf{F}}(\rho) = -\frac{1}{2}\,\nabla\rho + \chi(\rho)\,\mathbf{F}.$$

The main result of this section states that $\mathbf{W}^N_t$ converges in probability to the time integral of $\dot{\mathbf{W}}_{\mathbf{F}}(\rho)$.

LEMMA 3.1. *Fix a profile $\lambda\colon \mathbf{T}^d \to [0,1]$ and consider a sequence of probability measures $\{\mu^N\colon N \geqq 1\}$ on $\mathcal{X}_N$ associated with λ in the sense of* (2.2). *For each $t > 0$, $\delta > 0$, $G \in C(\mathbf{T}^d)$, and $\mathbf{H} \in C^1(\mathbf{T}^d;\, \mathbf{R}^d)$,*

$$\lim_{N\to\infty} \mathbf{P}^N_{\mathbf{F},\mu^N}\Big\{ \Big| \big\langle \pi^N_t,\, G\big\rangle - \big\langle \rho^{\mathbf{F},\lambda}_t,\ G\big\rangle \Big| > \delta \Big\} = 0,$$

$$\lim_{N\to\infty} \mathbf{P}^N_{\mathbf{F},\mu^N}\Big\{ \Big| \big\langle \mathbf{W}^N_t,\, \mathbf{H}\big\rangle - \int_0^t ds\, \big\langle \dot{\mathbf{W}}_{\mathbf{F}}(\rho^{\mathbf{F},\lambda}_s),\ \mathbf{H}\big\rangle \Big| > \delta \Big\} = 0,$$

where $\langle \dot{\mathbf{W}}_{\mathbf{F}}(\rho^{\mathbf{F},\lambda}_s), \mathbf{H}\rangle$ stands for

$$\frac{1}{2}\,\big\langle \rho^{\mathbf{F},\lambda}_s,\, \nabla\cdot\mathbf{H}\big\rangle + \big\langle \chi\big(\rho^{\mathbf{F},\lambda}_s\big),\ \mathbf{F}_s\cdot\mathbf{H}\big\rangle.$$

Proof. The law of large numbers for the empirical density follows from the usual entropy method; see, e.g., [10, Chap. 6]. For each $t \geqq 0$ the empirical density π_t^N converges in probability to $\rho^{\mathbf{F},\lambda}(t,\cdot)$.

To derive the hydrodynamic equation for the current, fix $t > 0$ and a smooth vector field $\mathbf{H}\colon \mathbf{T}^d \to \mathbf{R}^d$. Let $\widetilde{\mathbf{W}}_t^{N,\mathbf{H}}$ be the martingale defined by

$$\widetilde{\mathbf{W}}_t^{N,\mathbf{H}} = \langle \mathbf{W}_t^N, \mathbf{H}\rangle - \int_0^t ds\, \frac{N^2}{2N^{d+1}} \sum_{j,x} H_j\Big(\frac{x}{N}\Big)\, \eta_s(x)[1-\eta_s(x+e_j)]\, e^{F_j(s,x/N)/N}$$
$$+ \int_0^t ds\, \frac{N^2}{2N^{d+1}} \sum_{j,x} H_j\Big(\frac{x}{N}\Big)\, \eta_s(x+e_j)\big[1-\eta_s(x)\big]\, e^{-F_j(s,x/N)/N}.$$

An elementary computation shows that the quadratic variation of this martingale vanishes in $L^1(\mathbf{P}^N_{\mathbf{F},\mu^N})$ as $N\uparrow\infty$. On the other hand, after a Taylor expansion and few summations by parts, the time integral can be rewritten as

$$\int_0^t ds \left\langle \pi_s^N,\ \frac{1}{2}\nabla\cdot\mathbf{H} + \mathbf{F}_s\cdot\mathbf{H}\right\rangle + O_{\mathbf{F},\mathbf{H}}(N^{-1})$$
$$- \int_0^t ds\, \frac{1}{N^d} \sum_{j=1}^d \sum_{x\in\mathbf{T}_N^d} H_j\Big(\frac{x}{N}\Big)\, F_j\Big(s,\frac{x}{N}\Big)\, \eta_s(x)\,\eta_s(x+e_j),$$

where $O_{\mathbf{F},\mathbf{H}}(N^{-1})$ is an expression whose absolute value is bounded by CN^{-1} for some constant depending only on $\mathbf{F}$ and $\mathbf{H}$. By the two block estimates and the law of large numbers for the empirical density, as $N\uparrow\infty$, the previous expression converges in $\mathbf{P}^N_{\mathbf{F},\mu^N}$-probability to

$$\frac{1}{2}\int_0^t ds\, \langle \rho_s^{\mathbf{F},\lambda},\ \nabla\cdot\mathbf{H}\rangle + \int_0^t ds\, \langle \chi(\rho_s^{\mathbf{F},\lambda}),\ \mathbf{F}_s\cdot\mathbf{H}\rangle.$$

Since the martingale $\widetilde{\mathbf{W}}_t^{N,\mathbf{H}}$ vanishes in L^2 as $N\uparrow\infty$, the lemma is proved.

The same result holds for the generator $\widetilde{L}_{\mathbf{F},N}$ defined by

$$\big(\widetilde{L}_{\mathbf{F},N} f\big)(\eta) = \frac{N^2}{2} \sum_{\substack{x,y\in\mathbf{T}_N^d\\ |x-y|=1}} \eta(x)\big[1-\eta(y)\big]\, e^{N^{-1}\mathbf{F}(t,x/N)\cdot(y-x)}\, \big\{f(\sigma^{x,y}\eta) - f(\eta)\big\}.$$

However, the computations of the exponential martingales in the next subsection are slightly more complicated if we use this expression instead of $L_{\mathbf{F},N}$.

3.2. Large deviations upper bound. We first remark that Lemma 3.2 below implies that the probability of the event $(\mathbf{W}^N,\pi^N) \notin C([0,T];\mathcal{M}_d\times\mathcal{F})$ is super-exponentially small as $N\to\infty$. Recalling the definition (2.6) of the rate function I, it is therefore enough to prove the upper bound for closed subsets of $C([0,T];\ \mathcal{M}_d\times\mathcal{F})$. We shall first prove it for compacts and then show the exponential tightness.

We start by recalling the superexponential estimate of [9], [12]. For a positive integer ℓ and x in $\mathbf{Z}^d$, denote by $\eta^\ell(x)$ the empirical density of particles on a box of size $2\ell+1$ centered at x: $\eta^\ell(x) = (2\ell+1)^{-d}\sum_{|y-x|\leqq\ell}\eta(y)$. Moreover, for $1\leqq j\leqq d$, $\varepsilon>0$, let

$$V_{j,N,\varepsilon}(\eta) = \frac{1}{N^d}\sum_{x\in\mathbf{T}_N^d} \left| \frac{1}{(2\varepsilon N+1)^d} \sum_{|y-x|\leqq\varepsilon N} \eta(y)\,\eta(y+e_j) - \big[\eta^{N\varepsilon}(x)\big]^2 \right|.$$

THEOREM 3.1. *For each* $1 \leqq j \leqq d$, $T > 0$, *each sequence of measures* $\{\mu^N : N \geqq 1\}$, *and each* $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\mu^N} \left\{ \left| \int_0^T V_{j,N,\varepsilon}(\eta_t)\, dt \right| > \delta \right\} = -\infty.$$

Fix a vector-valued function $\mathbf{F}\colon [0,T]\times \mathbf{T}^d \to \mathbf{R}^d$ in $C^{1,1}$, and by $d\mathbf{P}^N_{\mathbf{F},\mu^N}/d\mathbf{P}^N_{\mu^N}(T)$ denote the Radon–Nikodým derivative of $\mathbf{P}^N_{\mathbf{F},\mu^N}$ with respect to $\mathbf{P}^N_{\mu^N}$ restricted to the time interval $[0,T]$. A long but elementary computation gives that

$$\begin{aligned}\frac{1}{N^d} \log \frac{d\mathbf{P}^N_{\mathbf{F},\mu^N}}{d\mathbf{P}^N_{\mu^N}}(T) &= \frac{1}{N^{d+1}} \sum_{j=1}^d \sum_{x\in \mathbf{T}^d_N} \int_0^T F_j\Big(t,\ \frac{x}{N}\Big)\, dW_t^{x,x+e_j} \\ &\quad - \frac{1}{2}\int_0^T dt\, \big\langle \pi^N_t,\ \nabla\cdot \mathbf{F}_t \big\rangle \\ &\quad - \frac{1}{4}\int_0^T dt\, \frac{1}{N^d} \sum_{j=1}^d \sum_{x\in \mathbf{T}^d_N} F_j\Big(t,\ \frac{x}{N}\Big)^2 \tau_x h_j(\eta_t) + O_{\mathbf{F}}(N^{-1}),\end{aligned}$$

where $h_j(\eta) = \eta(0) + \eta(e_j) - 2\eta(0)\,\eta(e_j)$ and τ_x denotes the translation of x. In particular, on the set

$$\sum_{j=1}^d \left| \int_0^T V_{j,N,\varepsilon}(\eta_t)\, dt \right| \leqq \delta,$$

integrating by parts the first term on the right-hand side of the penultimate formula, we obtain that $N^{-d}\log\{d\mathbf{P}^N_{\mathbf{F},\mu^N}/d\mathbf{P}^N_{\mu^N}\}(T)$ is bounded below by $J_{\mathbf{F},\varepsilon,\delta}(\mathbf{W}^N,\,\pi^N) - C(\mathbf{F})\times\{\varepsilon+\delta\}$ for every $\varepsilon > 0$. Here

$$\begin{aligned}J_{\mathbf{F},\varepsilon,\delta}(\mathbf{W},\pi) &= \langle \mathbf{W}_T, \mathbf{F}_T\rangle - \int_0^T dt\, \langle \mathbf{W}_t, \partial_t \mathbf{F}_t\rangle \\ &\quad - \frac{1}{2}\int_0^T dt \langle \pi_t, \nabla\cdot \mathbf{F}_t\rangle - \frac{1}{2}\int_0^T dt\, \big\langle \chi(\pi^\varepsilon_t),\ |\mathbf{F}_t|^2 \big\rangle,\end{aligned}$$

$C(\mathbf{F})$ is a finite constant depending only on $\mathbf{F}$, and π^ε_t is the function defined by $\pi^\varepsilon_t(u) = (2\varepsilon)^{-d}\int_{[u-\varepsilon\mathbf{1},u+\varepsilon\mathbf{1}]} dv\, \pi_t(v)$, where $[u-\varepsilon\mathbf{1}, u+\varepsilon\mathbf{1}]$ is the hypercube $[u_1-\varepsilon,\ u_1+\varepsilon]\times\cdots\times[u_d-\varepsilon,\ u_d+\varepsilon]$.

For each $\varepsilon > 0$, $\delta > 0$, and $\mathbf{F}$ of class $C^{1,1}$, the functional $J_{\mathbf{F},\varepsilon,\delta}$ is continuous in $C([0,T];\mathcal{M}_d\times\mathcal{F})$. By repeating the arguments presented in [10, section 10.4], we then obtain that for each compact set K of $C([0,T],\,\mathcal{M}_d\times\mathcal{F})$,

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\big\{\big(\mathbf{W}^N,\ \pi^N\big)\in K\big\} \leqq - \inf_{(\mathbf{W},\pi)\in K} J(\mathbf{W},\pi),$$

where J is defined in (2.6).

To extend the upper bound from compact to closed sets, we next prove the exponential tightness of the sequence $(\mathbf{W}^N, \pi^N)$. As stated before, the following lemma also implies that the probability of the event $(\mathbf{W}^N,\pi^N)\notin C([0,T];\mathcal{M}_d\times\mathcal{F})$ is super-exponentially small.

LEMMA 3.2. *Fix a sequence of measures* $\{\mu^N\colon N \geqq 1\}$, *a continuous function* $G\colon \mathbf{T}^d \to \mathbf{R}$, *a vector-valued function* $\mathbf{H}\colon \mathbf{T}^d \to \mathbf{R}^d$ *in* C^1, *and* $\varepsilon > 0$. *Then,*

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\,\log \mathbf{P}^N_{\mu^N}\Big\{\sup_{|t-s|\leqq\delta}\big|\langle\pi^N_t,\,G\rangle - \langle\pi^N_s,\,G\rangle\big| > \varepsilon\Big\} = -\infty,$$

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^d}\,\log \mathbf{P}^N_{\mu^N}\Big\{\sup_{|t-s|\leqq\delta}\big|\langle\mathbf{W}^N_t,\,\mathbf{H}\rangle - \langle\mathbf{W}^N_s,\,\mathbf{H}\rangle\big| > \varepsilon\Big\} = -\infty.$$

Proof. The proof of the first estimate is similar to that of the exponential tightness of the empirical measure presented in [10, section 10.4]. In our context the initial configuration is not, however, the invariant measure. The necessary modifications are worked out in what follows in the proof of the second statement of the lemma.

To prove the second estimate, first observe that by a triangular inequality and since

$$\limsup_{N\to\infty} N^{-d}\log\{a_N + b_N\} \leqq \max\Big\{\limsup_{N\to\infty} N^{-d}\log a_N,\ \limsup_{N\to\infty} N^{-d}\log b_N\Big\}, \tag{3.2}$$

it is enough to estimate

$$\max_{0\leqq k\leqq T\delta^{-1}}\limsup_{N\to\infty}\frac{1}{N^d}\,\log\mathbf{P}^N_{\mu^N}\Big\{\sup_{t_k\leqq t\leqq t_{k+1}}\big|\langle\mathbf{W}^N_t,\,\mathbf{H}\rangle - \langle\mathbf{W}^N_{t_k},\,\mathbf{H}\rangle\big| > \frac{\varepsilon}{3}\Big\}, \tag{3.3}$$

where $t_k = k\delta$. By (3.2), we may also disregard the absolute value in the previous expression provided we estimate the same term with $-\mathbf{H}$ in place of $\mathbf{H}$. Fix $a > 0$ and denote by $\mathcal{M}_t = \mathcal{M}_t(a,\mathbf{H})$ the mean one exponential martingale whose logarithm is given by

$$\begin{aligned}
&\frac{a}{N}\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^t H_j\Big(s,\ \frac{x}{N}\Big)\,dW^{x,x+e_j}_s\\
&\quad - N^2\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^t ds\,\eta_s(x)\,\big[1-\eta_s(x+e_j)\big]\big\{e^{aN^{-1}H_j(s,x/N)} - 1\big\}\\
&\quad - N^2\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^t ds\,\eta_s(x+e_j)\big[1-\eta_s(x)\big]\big\{e^{-aN^{-1}H_j(s,x/N)} - 1\big\}.
\end{aligned}$$

Since H_j are C^1 functions, a Taylor expansion and a summation by parts show that the expressions inside the integrals in the last two terms are bounded by $C_{\mathbf{H}}a(1+a)\,N^d$, where $C_{\mathbf{H}}$ is a finite constant depending only on $\mathbf{H}$. Therefore, by multiplying by aN^d, adding and subtracting the appropriate integrals and exponentiating, we get that

$$\begin{aligned}
&\mathbf{P}^N_{\mu^N}\Big\{\sup_{t_k\leqq t\leqq t_{k+1}}\langle\mathbf{W}^N_t,\,\mathbf{H}\rangle - \langle\mathbf{W}^N_{t_k},\,\mathbf{H}\rangle > \frac{\varepsilon}{3}\Big\}\\
&\qquad\leqq \mathbf{P}^N_{\mu^N}\Big\{\sup_{t_k\leqq t\leqq t_{k+1}}\frac{\mathcal{M}_t}{\mathcal{M}_{t_k}} > \exp\Big\{\frac{1}{6}\,aN^d\varepsilon\Big\}\Big\}
\end{aligned}$$

provided $C_{\mathbf{H}}(1+a)\,\delta \leqq \varepsilon/6$. Since $\mathcal{M}_t/\mathcal{M}_{t_k}$ is a positive martingale equal to 1 at time t_k, by Doob's inequality, the last expression is bounded by $\exp\{-\frac{1}{6}\,aN^d\varepsilon\}$.

Therefore, (3.3) is less than or equal to $-a\varepsilon/6$ for all δ small enough. This shows that the second expression in the statement of the lemma is bounded by $-a\varepsilon/6$. Letting $a \uparrow \infty$, we conclude the proof.

Standard arguments, presented in [10, section 10.4], together with Lemma 3.2 permit us to extend the upper bound for compact sets to closed sets.

We conclude this subsection, proving that we may set $J(\mathbf{W}, \pi) = +\infty$ on the set of paths $(\mathbf{W}, \pi)$ which do not belong to $\mathfrak{A}_\gamma$.

LEMMA 3.3. *Fix a sequence of measures* $\{\mu^N :\ N \geqq 1\}$ *and a function* $H\colon\ [0,T]\times \mathbf{T}^d \to \mathbf{R}$ *in* $C^{1,2}$. *Let*

$$\begin{aligned} L_T(\pi^N, H) &= \langle \pi^N_T,\, H_T\rangle - \langle \pi^N_0,\, H_0\rangle - \int_0^T dt\, \langle \pi^N_t,\, \partial_t H_t\rangle, \\ V_T\big(\pi^N,\, \mathbf{W}^N, H\big) &= L_T\big(\pi^N, H\big) \\ &\quad - \frac{1}{N^{d+1}} \sum_{j=1}^d \sum_{x\in\mathbf{T}^d_N} \int_0^T (\partial_{u_j} H)\Big(t,\ \frac{x}{N}\Big)\, dW^{x,x+e_j}_t. \end{aligned} \tag{3.4}$$

Then, for any $\delta > 0$,

$$\limsup_{N\to\infty} \frac{1}{N^d}\ \log \mathbf{P}^N_{\mu^N} \Big\{ \big| V_T\big(\pi,\, \mathbf{W}^N, H\big)\big| > \delta \Big\} = -\infty.$$

Proof. Fix a function $H\colon\ [0,T]\times\mathbf{T}^d \to \mathbf{R}$ in $C^{1,2}$. A summation by parts shows that

$$\begin{aligned} &\frac{1}{N^{d+1}} \sum_{j=1}^d \sum_{x\in\mathbf{T}^d_N} \int_0^T N\Big\{ H\Big(t,\ x + \frac{e_j}{N}\Big) - H\Big(t,\ \frac{x}{N}\Big)\Big\}\, dW^{x,x+e_j}_t \\ &\quad = \frac{1}{N^d} \sum_{x\in\mathbf{T}^d_N} \int_0^T H\Big(t,\ \frac{x}{N}\Big) \sum_{j=1}^d d\{W^{x-e_j,x}_t - W^{x,x+e_j}_t\}. \end{aligned}$$

Since $\sum_{j=1}^d \{W^{x-e_j,x}_s - W^{x,x+e_j}_s\}$ increases by one each time a particle jumps to x and decreases by one each time a particle leaves x, this sum is equal to $\eta_s(x) - \eta_0(x)$. In particular, an integration by parts gives that the previous integral is equal to $L_T(\pi^N, H)$ defined in (3.4). Therefore, by a second order Taylor expansion,

$$\begin{aligned} V_T(\pi^N,\, \mathbf{W}^N, H) &= \frac{1}{N^{d+2}} \sum_{j=1}^d \sum_{x\in\mathbf{T}^d_N} \int_0^T (\partial^2_{u_j} H)\Big(t,\ \frac{x}{N}\Big)\, dW^{x,x+e_j}_t \\ &\quad + \frac{o_H(1)}{N^{d+2}} \sum_{j=1}^d \sum_{x\in\mathbf{T}^d_N} \{J^{x,x+e_j}_t + J^{x+e_j,x}_t\}, \end{aligned}$$

where $o_H(1)$ depends on H and vanishes as $N \uparrow \infty$.

We prove that the first expression on the right-hand side is superexponentially small, with the argument for the second one being similar. For simplicity, set

$F_j = \partial^2_{u_j} H$. By Chebyshev's inequality, for every $a > 0$,

$$\mathbf{P}^N_{\mu^N}\Big\{\Big|\frac{1}{N^{d+2}}\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T F_j\Big(t,\ \frac{x}{N}\Big)\,dW^{x,x+e_j}_t\Big| > \delta\Big\}$$

$$\leqq e^{-a\delta N^d}\mathbf{E}^N_{\mu^N}\Big[\exp\Big\{aN^{-2}\Big|\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T F_j\Big(t,\ \frac{x}{N}\Big)\,dW^{x,x+e_j}_t\Big|\Big\}\Big]. \tag{3.5}$$

Since $e^{|x|} \leqq e^x + e^{-x}$, we estimate this last expectation without the absolute value. Denote by $\mathbf{M}_T$ the mean one exponential martingale whose logarithm is given by

$$\begin{aligned}&\frac{a}{N^2}\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T F_j\Big(t,\ \frac{x}{N}\Big)\,dW^{x,x+e_j}_t\\ &\quad - N^2\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T dt\,\eta_t(x)\big[1-\eta_t(x+e_j)\big]\big\{e^{aN^{-2}F_j(t,x/N)}-1\big\}\\ &\quad - N^2\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T dt\,\eta_t(x+e_j)\big[1-\eta_t(x)\big]\big\{e^{-aN^{-2}F_j(t,x/N)}-1\big\}.\end{aligned}$$

Since F_j are continuous functions, a Taylor expansion and a summation by parts show that the last two integrals can be written as

$$\Big\{a o_{\mathbf{F}}(1) + \frac{C(\mathbf{F})\,a^2}{N^2}\Big\}\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T dt\,\eta_t(x) \leqq \Big\{a o_{\mathbf{F}}(1) + \frac{C(\mathbf{F})\,a^2}{N^2}\Big\}\,dN^dT,$$

where $o_{\mathbf{F}}(1)$ is an expression depending on $\mathbf{F}$ which vanishes as $N\uparrow\infty$.

Since $\mathbf{M}_T$ is a mean one exponential martingale, the right-hand side of (3.5) is bounded above by

$$\exp\Big\{aN^d\big(-\delta + o_{\mathbf{F}}(1)\,dT + C(\mathbf{F})\,adN^{-2}T\big)\Big\}.$$

In particular,

$$\limsup_{N\to\infty}\frac{1}{N^d}\,\log\mathbf{P}^N_{\mu^N}\Big\{\Big|\frac{1}{N^{d+2}}\sum_{j=1}^d\sum_{x\in\mathbf{T}^d_N}\int_0^T F_j\Big(t,\ \frac{x}{N}\Big)\,dW^{x,x+e_j}_t\Big| > \delta\Big\} \leqq -a\delta$$

for every $a > 0$. This proves the lemma.

From this lemma, (3.2), and Lemma 3.2 it follows that for every closed set F, every $\delta > 0$, and every finite family $\{H_j,\ 1\leqq j\leqq \ell\}$ of functions in $C^{1,2}$

$$\limsup_{N\to\infty}\frac{1}{N^d}\,\log\mathbf{P}^N_{\mu^N}\big\{\big(\mathbf{W}^N,\pi^N\big)\in F\big\} \leqq -\inf_{(\mathbf{W},\pi)\in F\cap\mathfrak{A}_\ell} J(\mathbf{W},\pi),$$

where

$$\mathfrak{A}_\ell = \bigcap_{j=1}^{\ell}\Big\{(\mathbf{W},\pi):\ \big|V_T(\pi,\mathbf{W},H_j)\big|\leqq\delta\Big\}.$$

Since this inequality holds for every $\delta > 0$ and every finite sequence H_j, letting $\delta \downarrow 0$ and considering a dense family of functions H_j, we obtain that

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\mu^N}\big\{\big(\mathbf{W}^N, \pi^N\big) \in F\big\} \leqq - \inf_{(\mathbf{W},\pi)\in F\cap\mathfrak{A}} J(\mathbf{W}, \pi),$$

where $\mathfrak{A}$ is the set of paths $(\mathbf{W}, \pi)$ such that $\dot\pi_t + \nabla\cdot\dot{\mathbf{W}}_t = 0$. Up to this point, we did not need any assumption on the sequence of initial measures μ^N; but the hypothesis that we are starting from a deterministic profile now plays a role in replacing the set $\mathfrak{A}$ in the previous formula with the set $\mathfrak{A}_\gamma$, proving the upper bound of the large deviation principle.

3.3. The rate function. To prove the lower bound of the large deviation principle in Theorem 2.1, we first obtain an explicit representation of the functional I on the paths with a finite rate function.

Given a path $\pi \in D([0, T]; \mathcal{F})$, we denote by $L^2(\pi)$ the Hilbert space of vector-valued functions $\mathbf{G}\colon [0, T] \times \mathbf{T}^d \to \mathbf{R}^d$ endowed with the inner product $\langle\cdot,\cdot\rangle_\pi$ defined by

$$\langle \mathbf{H}, \mathbf{G}\rangle_\pi = \int_0^T dt \int_{\mathbf{T}^d} du\, \chi\big(\pi(t, u)\big)\, \mathbf{H}(t, u)\cdot \mathbf{G}(t, u).$$

Fix a pair $(\mathbf{W}, \pi)$ such that $I(\mathbf{W}, \pi) < \infty$. In particular, $(\mathbf{W}, \pi) \in C([0, T]; \mathcal{M}_d \times \mathcal{F})$. Following the arguments in [10, section 10.5], from Riesz's representation theorem, we derive the existence of a function $\mathbf{G}$ in $L^2(\pi)$ such that

$$\begin{cases} I(\mathbf{W}, \pi) = \dfrac{1}{2} \displaystyle\int_0^T dt\, \big\langle \chi(\pi_t),\ |\mathbf{G}_t|^2 \big\rangle, \\ \partial_t \mathbf{W}_t + \dfrac{1}{2}\nabla \pi_t = \chi(\pi_t)\, \mathbf{G}_t, \end{cases} \tag{3.6}$$

where the last equation has to be understood in the weak sense: For each $\mathbf{H} \in C^1(\mathbf{T}^d; \mathbf{R}^d)$ and each $0 \leqq s \leqq t \leqq T$, we have

$$\langle \mathbf{W}_t, \mathbf{H}\rangle - \langle \mathbf{W}_s, \mathbf{H}\rangle = \frac{1}{2}\int_s^t dr\, \langle \pi_r,\ \nabla\cdot\mathbf{H}\rangle + \int_s^t dr\, \langle \chi(\pi_r),\ \mathbf{G}_r\cdot\mathbf{H}\rangle.$$

3.4. The lower bound. In this subsection we prove the lower bound in Theorem 2.1. Denote by $\mathcal{S}$ the set of trajectories $(\mathbf{W}, \pi)$ in $\mathfrak{A}_\gamma$ for which there exists a vector-valued function $\mathbf{G}$ in $C^{1,1}([0, T] \times \mathbf{T}^d)$ such that $(\mathbf{W}, \pi)$ is the solution of (3.6).

For paths $(\mathbf{W}, \pi)$ in $\mathcal{S}$, we may repeat the arguments presented in the proof of the lower bound in the large deviation principle for the empirical density in [10, section 10.5] to conclude that for each open set G

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\big\{\big(\mathbf{W}^N, \pi^N\big) \in G\big\} \geqq - \inf_{(\mathbf{W},\pi)\in G\cap\mathcal{S}} I(\mathbf{W},\, \pi).$$

To conclude the proof, it remains to show that for all pairs $(\mathbf{W}, \pi)$ with finite rate function, $I(\mathbf{W}, \pi) < \infty$, there exists a sequence $(\mathbf{W}_k, \pi_k)$ in $\mathcal{S}$ converging to $(\mathbf{W}, \pi)$ and such that $\lim_{k\to\infty} I(\mathbf{W}_k, \pi_k) = I(\mathbf{W}, \pi)$. When this occurs, we shall say that the sequence $(\mathbf{W}_k, \pi_k)$ I-converges to $(\mathbf{W}, \pi)$. In the context of the symmetric exclusion process, the argument is not too difficult because the rate function is convex. We follow the proof of the lower bound presented in [2].

The proof is divided into two steps. We first show that there exists a sequence $(\mathbf{W}_k, \pi_k)$ which I-converges to $(\mathbf{W}, \pi)$ and such that, for each k, π_k is bounded away from 0 and 1 uniformly in $[0,T] \times \mathbf{T}^d$. To do this, following [2] we consider a convex combination of $(\mathbf{W}, \pi)$ with the solution of the hydrodynamic equation (3.1) with external field $\mathbf{F} = 0$ and initial condition $\pi_0 = \gamma$, $\mathbf{W}_0 = 0$.

Consider now a pair $(\mathbf{W}, \pi)$ whose empirical density π is bounded away from 0 and 1. Since $I(\mathbf{W}, \pi)$ is finite, by subsection 3.3, there exists a vector-valued function $\mathbf{G}$ in $L^2(\pi)$ satisfying (3.6). Since π is bounded away from 0 and 1, $L^2(\pi)$ coincides with the usual L^2 space associated with the Lebesgue measure on $[0,T] \times \mathbf{T}^d$. Consider a sequence of smooth vector-valued functions $\mathbf{G}_n \colon [0,T] \times \mathbf{T}^d \to \mathbf{R}^d$ converging in L^2 to $\mathbf{G}$ and denote by $(\mathbf{W}^n, \pi^n)$ the pair in $\mathfrak{A}_\gamma$ which solves (3.6) with $\mathbf{G}_n$ instead of $\mathbf{G}$. Repeating the arguments presented in [2, section 3.6], one can prove that $(\mathbf{W}^n, \pi^n)$ I-converges to $(\mathbf{W}, \pi)$. This concludes the proof of the lower bound.

3.5. Large deviations for the empirical density. In this subsection we show that the large deviation principle for the empirical density, proved in [12], follows from Theorem 2.1. Indeed, the large deviation principle for the empirical density can be recovered from that for the current density by the contraction principle. The rate function $\mathcal{I}$ is given by the variational formula

$$\mathcal{I}(\pi) = \inf_{\mathbf{W} \in \mathfrak{W}_\pi} I(\mathbf{W}, \pi), \tag{3.7}$$

where $\mathfrak{W}_\pi$ stands for a set of currents $\mathbf{W}$ satisfying $\dot{\pi}_t + \nabla \cdot \dot{\mathbf{W}}_t = 0$, as formulated in (2.4).

This variational problem is simple to solve. Let us first assume that π is smooth and bounded away from 0 and 1. Fix a current $\mathbf{W}$ in $\mathfrak{W}_\pi$ and denote by $\mathbf{G}$ the external field associated with $\mathbf{W}$ through (3.6). For $0 \leqq t \leqq T$, let H_t be the solution of the elliptic equation

$$\nabla \cdot \big(\chi(\pi_t)\, \mathbf{G}_t\big) = \nabla \cdot \big(\chi(\pi_t)\, \nabla H_t\big)$$

and set $\mathbf{F}_t = \chi(\pi_t)\{\mathbf{G}_t - \nabla H_t\}$. By definition, $\nabla \cdot \mathbf{F}_t = 0$. Let $\mathbf{w}$ be the current defined by

$$\dot{\mathbf{w}}_t + \frac{1}{2} \nabla \pi_t = \chi(\pi_t)\, \nabla H_t. \tag{3.8}$$

$\mathbf{w}$ belongs to $\mathfrak{W}_\pi$ because by construction $\nabla \cdot \dot{\mathbf{w}} = \nabla \cdot \dot{\mathbf{W}}$. Moreover, by the explicit formula (3.6) for the rate function and by definition of $\mathbf{F}$,

$$\begin{aligned} I(\mathbf{W}, \pi) &= \frac{1}{2} \int_0^T dt\, \big\langle \chi(\pi_t),\ |\mathbf{G}_t|^2 \big\rangle \\ &= \frac{1}{2} \int_0^T dt\, \Big\{ \big\langle \chi(\pi_t),\ |\nabla H_t|^2 \big\rangle + 2 \langle \mathbf{F}_t \cdot \nabla H_t \rangle + \langle \chi(\pi_t)^{-1} |\mathbf{F}_t|^2 \rangle \Big\}. \end{aligned}$$

Since $\nabla \cdot \mathbf{F} = 0$, an integration by parts shows that the cross term vanishes. On the other hand, by the explicit formula (3.6) for the rate function and by (3.8), the first term on the right-hand side is $I(\mathbf{w}, \pi)$. Thus, $I(\mathbf{W}, \pi) \geqq I(\mathbf{w}, \pi)$. In particular, in the variational problem (3.7), we can restrict our attention to currents $\mathbf{W}$ for which the associated external fields $\mathbf{G}$ are in gradient form.

Now, consider two currents $\mathbf{W}^1$, $\mathbf{W}^2$ in $\mathfrak{W}_\pi$ and assume that both external fields $\mathbf{G}_1$, $\mathbf{G}_2$ associated with these currents through (3.6) are in gradient form:

$\mathbf{G}_j = \nabla H^j$, $j = 1, 2$. Taking the divergence of (3.6) and recalling that $\dot\pi_t + \nabla\cdot\dot{\mathbf{W}}_t = 0$, we obtain that

$$\dot\pi_t = \frac{1}{2}\,\Delta\pi_t - \nabla\cdot\big(\chi(\pi_t)\,\nabla H_t^j\big)$$

for $j = 1, 2$ and each $0 \leqq t \leqq T$. In particular, $\nabla\cdot\big(\chi(\pi_t)\,\nabla[H_t^1 - H_t^2]\big) = 0$. Taking the inner product with respect to $H_t^1 - H_t^2$ and integrating by parts, we get that

$$\int_{\mathbf{T}^d} du\,\chi(\pi_t)\,|\nabla H_t^1 - \nabla H_t^2|^2 = 0$$

for every $0 \leqq t \leqq T$. In particular, $I(\mathbf{W}^1, \pi) = I(\mathbf{W}^2, \pi)$. This proves that the variational problem (3.7) is attained on currents for which the associated external field is in gradient form:

$$\mathcal{I}(\pi) = \inf_{\mathbf{W}\in\mathfrak{W}_\pi} I(\mathbf{W}, \pi) = \frac{1}{2}\int_0^T dt\,\big\langle\chi(\pi_t)\,|\nabla H_t|^2\big\rangle,$$

where H_t is given by

$$\dot\pi_t = \frac{1}{2}\,\Delta\pi_t - \nabla\cdot\big(\chi(\pi_t)\,\nabla H_t\big).$$

This is exactly the large deviations rate function for the empirical density obtained in [12]. This identity has been obtained for smooth paths π bounded away from 0 and 1. However, by the arguments of the previous subsection, we can extend it to all paths π.

4. Large deviations of the mean current on a long time interval. In this section we investigate the large deviations properties of the mean empirical current $\mathbf{W}_T^N/T$ as we let *first* $N\to\infty$ and *then* $T\to\infty$. We emphasize that the analysis carried out in this section does not depend on the details of the symmetric simple exclusion process so that it holds in a general setting.

Given a profile $\gamma\in C^2(\mathbf{T}^d)$, $T>0$, and $\mathbf{W}\in D([0,T];\mathcal{M}_d)$, let $\pi\in D([0,T];\mathcal{F})$ be the solution of (2.4) and denote by $I_{[0,T]}(\mathbf{W}\,|\,\gamma)$ the functional defined in (2.6), in which we made explicit the dependence on the time interval $[0,T]$ and on the initial profile γ. We define $\Phi_T(\cdot\,|\,\gamma)\colon\ \mathcal{M}_d\to[0,+\infty]$ as the functional

$$\Phi_T(\mathbf{J}\mid\gamma) = T^{-1}\inf_{\mathbf{W}\in\mathcal{A}_{T,\mathbf{J}}} I_{[0,T]}(\mathbf{W}\mid\gamma), \tag{4.1}$$

where

$$\mathcal{A}_{T,\mathbf{J}} := \big\{\mathbf{W}\in D\big([0,T];\mathcal{M}_d\big)\colon\ \mathbf{W}_T = T\mathbf{J}\big\}.$$

Recalling that the set $\mathcal{B}$ of divergence-free measures has been defined in (2.7), we also define

$$\widetilde\Phi(\mathbf{J}\mid\gamma) := \begin{cases}\inf_{T>0}\Phi_T(\mathbf{J}\mid\gamma) & \text{if } \mathbf{J}\in\mathcal{B},\\ +\infty & \text{otherwise.}\end{cases} \tag{4.2}$$

Finally, denote by $\Phi(\mathbf{J}\,|\,\gamma) := \sup_{U\ni\mathbf{J}}\inf_{\mathbf{J}'\in U}\widetilde\Phi(\mathbf{J}'\,|\,\gamma)$, where $U\subset\mathcal{M}_d$ is open, the lower semicontinuous envelope of $\widetilde\Phi(\cdot\,|\,\gamma)$.

Remark 4.1. The functional $\Phi(\cdot\,|\,\gamma)$ depends on the initial condition γ only through its total mass $m = \int du\, \gamma(u)$. This holds in the present setting of periodic boundary conditions; in the case of Dirichlet boundary conditions, when the density is fixed at the boundary, $\Phi(\cdot\,|\,\gamma)$ would be completely independent on γ. Furthermore the functional $\Phi(\cdot\,|\,\gamma)$ is convex.

THEOREM 4.1. *Let $\gamma \in C^2(\mathbf{T}^d)$ be bounded away from 0 and 1 and let $\eta^N \in \mathcal{X}_N$ be a sequence such that $\pi^N(\eta^N) \to \gamma$ in $\mathcal{F}$. Then, for each closed set C and every open set U of $\mathcal{M}_d$, we have*

$$\limsup_{T\to\infty}\limsup_{N\to\infty} \frac{1}{TN^d}\, \log \mathbf{P}^N_{\eta^N}\left\{\frac{1}{T}\mathbf{W}^N_T \in C\right\} \leqq -\inf_{\mathbf{J}\in C} \Phi(\mathbf{J}\mid\gamma),$$
$$\liminf_{T\to\infty}\liminf_{N\to\infty} \frac{1}{TN^d}\, \log \mathbf{P}^N_{\eta^N}\left\{\frac{1}{T}\mathbf{W}^N_T \in U\right\} \geqq -\inf_{\mathbf{J}\in U} \Phi(\mathbf{J}\mid\gamma).$$

If T were fixed, the above large deviation principle would directly follow from Theorem 2.1. The asymptotic $T \to \infty$ is related to the so-called Γ-convergence of the rate functions. We first discuss this issue in a general setting. Let $\mathcal{X}$ be a metric space; we recall that a sequence $F_T\colon \mathcal{X} \to [0,+\infty]$ of functions Γ-converges to $F\colon \mathcal{X} \to [0,+\infty]$ as $T \to \infty$ if and only if for each $x \in \mathcal{X}$ the following holds:

$$\text{For any sequence} \quad x_T \to x \quad \text{we have} \quad F(x) \leqq \liminf_{T\to\infty} F_T(x_T), \tag{4.3}$$
$$\text{there exists a sequence} \quad x_T \to x \quad \text{such that} \quad F(x) \geqq \limsup_{T\to\infty} F_T(x_T). \tag{4.4}$$

LEMMA 4.1. *Let $P_{N,T}$ be a two parameter family of probabilities on $\mathcal{X}$ endowed with its Borel σ-algebra. Assume that for each fixed $T > 0$ the family $\{P_{N,T}\}_{N\in\mathbf{N}}$ satisfies the weak large deviation principle with rate function TF_T; that is, for each K compact and U open in $\mathcal{X}$, we have*

$$\limsup_{N\to\infty} \frac{1}{N}\, \log P_{N,T}(K) \leqq -T \inf_{x\in K} F_T(x), \tag{4.5}$$
$$\liminf_{N\to\infty} \frac{1}{N}\, \log P_{N,T}(U) \geqq -T \inf_{x\in U} F_T(x). \tag{4.6}$$

Assume also that the sequence F_T Γ-converges to F as $T \to \infty$. Then for each K compact and U open in $\mathcal{X}$, we have

$$\limsup_{T\to\infty}\limsup_{N\to\infty} \frac{1}{NT}\, \log P_{N,T}(K) \leqq -\inf_{x\in K} F(x), \tag{4.7}$$
$$\liminf_{T\to\infty}\liminf_{N\to\infty} \frac{1}{NT} \log P_{N,T}(U) \geqq -\inf_{x\in U} F(x). \tag{4.8}$$

Proof. To deduce (4.7), (4.8) from (4.5), (4.6) we need to show that for each K compact and U open in $\mathcal{X}$ we have

$$\liminf_{T\to\infty} \inf_{x\in K} F_T(x) \geqq \inf_{x\in K} F(x), \qquad \limsup_{T\to\infty} \inf_{x\in U} F_T(x) \leqq \inf_{x\in U} F(x).$$

These bounds are a direct consequence of (4.3), (4.4); see, e.g., [8, Proposition 1.18].

The following lemma follows from Theorem 2.1 by the contraction principle.

LEMMA 4.2. *Consider a profile $\gamma \in C^2(\mathbf{T}^d)$ bounded away from 0 and 1 and a sequence $\{\eta^N\colon N \geqq 1\}$ such that $\pi^N(\eta^N) \to \gamma$ in $\mathcal{F}$. Then for each $T > 0$ we have* *the following large deviation principle for the mean empirical current* $\mathbf{W}_T^N/T$. *For each* C *closed and each* U *open in* $\mathcal{M}_d$

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\Big\{\frac{\mathbf{W}_T^N}{T}\in C\Big\} \leqq -T \inf_{\mathbf{J}\in C} \Phi_T(\mathbf{J}\mid\gamma),$$
$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}^N_{\eta^N}\Big\{\frac{\mathbf{W}_T^N}{T}\in U\Big\} \geqq -T \inf_{\mathbf{J}\in U} \Phi_T(\mathbf{J}\mid\gamma),$$

where we recall the functional Φ_T *is defined in* (4.1).

PROPOSITION 4.1. *Let* $\gamma\in C^2(\mathbf{T}^d)$ *be bounded away from* 0 *and* 1. *The sequence of functionals* $\Phi_T(\cdot\,|\,\gamma)$ Γ*-converges to the functional* $\Phi(\cdot\,|\,\gamma)$ *defined after* (4.2).

The previous proposition, together with Lemmas 4.1 and 4.2, proves the "large deviation principle" stated in Theorem 4.1 for compact sets. For its proof, we need a few preliminary lemmas. For each $\mathbf{J}\notin\mathcal{B}$, since the empirical density is bounded, by the continuity equation (2.4), we have that $\Phi_T(\mathbf{J}\,|\,\gamma)=+\infty$ if T is sufficiently large. We next show that this holds uniformly for all $\mathbf{J}$ whose distance from $\mathcal{B}$ is uniformly bounded below. To this end we introduce the following metric on $\mathcal{M}_d$. Pick a sequence of smooth vector fields $\mathbf{G}_k\in C^1(\mathbf{T}^d;\,\mathbf{R}^d)$, $k\geqq 1$, dense in the unit ball of $C(\mathbf{T}^d;\,\mathbf{R}^d)$; for $\mathbf{J},\mathbf{J}'\in\mathcal{M}_d$ we then define

$$\varrho(\mathbf{J},\mathbf{J}')=\sum_{k=1}^{\infty}\frac{1}{2^k}\,1\wedge\big|\langle\mathbf{J}-\mathbf{J}',\mathbf{G}_k\rangle\big|.$$

It is easy to show that ϱ is a metric inducing the weak topology of $\mathcal{M}_d$.

LEMMA 4.3. *For each* $\delta\in(0,1)$ *there exists* $T_0=T_0(\delta)\in\mathbf{R}_+$ *such that for any* $T\geqq T_0$ *we have* $\Phi_T(\mathbf{J}\,|\,\gamma)=+\infty$ *for any* $\mathbf{J}\in\mathcal{M}_d$ *such that* $\varrho(\mathbf{J},\mathcal{B})\geqq\delta$.

Proof. Given $\mathbf{J}\in\mathcal{M}_d$ let us denote by $\widehat{\mathbf{J}}\in\mathcal{B}$ its projection on the subspace $\mathcal{B}$. If $\mathbf{J}(du)=\mathbf{j}\,du$ for some $\mathbf{j}\in L_2(\mathbf{T}^d;\,\mathbf{R}^d)$, this is simply the orthogonal projection of $\mathbf{j}$ to $\mathcal{B}$. In general $\widehat{\mathbf{J}}\in\mathcal{M}_d$ is defined by $\langle\widehat{\mathbf{J}},\nabla F\rangle=0$ for any $F\in C^1(\mathbf{T}^d)$ and $\langle\widehat{\mathbf{J}},\mathbf{B}\rangle=\langle\mathbf{J},\mathbf{B}\rangle$ for any $\mathbf{B}\in C^1(\mathbf{T}^d;\mathbf{R}^d)$ such that $\nabla\cdot\mathbf{B}=0$. It is easy to verify that $\widehat{\mathbf{J}}$ is uniquely defined by the above requirements. We then have

$$\delta\leqq\varrho(\mathbf{J},\mathcal{B})\leqq\varrho(\mathbf{J},\widehat{\mathbf{J}})=\sum_{k=1}^{\infty}\frac{1}{2^k}\big(1\wedge\big|\langle\mathbf{J}-\widehat{\mathbf{J}},\mathbf{G}_k\rangle\big|\big)=\sum_{k=1}^{\infty}\frac{1}{2^k}\big(1\wedge\big|\langle\mathbf{J},\nabla F_k\rangle\big|\big),$$

where $F_k\in C^1\big(\mathbf{T}^d\big)$ is obtained from $\mathbf{G}_k$ by solving the Poisson equation $\Delta F_k=\nabla\cdot\mathbf{G}_k$ so that $\mathbf{G}_k=\nabla F_k+\mathbf{B}_k$ with $\nabla\cdot\mathbf{B}_k=0$. Note that $\|F_k\|_{L_2}\leqq C_0\|\mathbf{G}_k\|_{L_2}\leqq C_0$ for some constant C_0 not depending on k.

From the previous inequality we get that $\bar{k}=\bar{k}(\mathbf{J})$ exists such that $|\langle\mathbf{J},\nabla F_{\bar{k}}\rangle|\geqq\delta$. Let $\mathbf{W}\in\mathcal{A}_{T,\mathbf{J}}$ and denote by π_t, $t\in[0,T]$, the corresponding solution of the continuity equation (2.4). By choosing $F=F_{\bar{k}}$ and using that $\mathbf{W}_T=T\mathbf{J}$, from (2.4) we get

$$\langle\pi_T-\gamma,F_{\bar{k}}\rangle=T\,\langle\mathbf{J},\nabla F_{\bar{k}}\rangle.$$

Since $-1\leqq\pi_T-\gamma\leqq 1$, the absolute value of the left-hand side above is bounded above by C_0. On the other hand, since $\varrho(\mathbf{J},\mathcal{B})\geqq\delta$, the absolute value of the right-hand side above is bounded below by δT. By taking $T_0>C_0\,\delta^{-1}$, the lemma follows.

LEMMA 4.4. *Consider a profile* $\gamma\in C^2(\mathbf{T}^d)$ *bounded away from* 0 *and* 1 *and let* $\mathbf{J}\in\mathcal{B}$, $T>0$. *We then have* $\Phi_T(\mathbf{J}\,|\,\gamma)<+\infty$ *if and only if* $\mathbf{J}(du)=\mathbf{j}\,du$ *for some*

$\mathbf{j} \in L_2(\mathbf{T}^d; \mathbf{R}^d)$. *Moreover there exists a constant* $C_1 \in (0,\infty)$ (*depending on* γ) *such that for any* $T > 0$ *and any* $\mathbf{J}(du) = \mathbf{j}\, du$ *we have*

$$\frac{1}{C_1}\,\langle \mathbf{j},\, \mathbf{j}\,\rangle \leqq \Phi_T(\mathbf{J}|\gamma) \leqq C_1\big[\langle \mathbf{j},\, \mathbf{j}\rangle + 1\big]. \tag{4.9}$$

Proof. Let $\mathbf{W} \in \mathcal{A}_{T,\mathbf{J}}$; by choosing the vector field $\mathbf{F}$ in the variation expression (2.6) constant in time and divergence-free we get

$$\frac{1}{T}\, I_{[0,T]}(\mathbf{W} \,|\, \gamma) \geqq \langle \mathbf{J}, \mathbf{F}\rangle - \frac{1}{2}\,\frac{1}{4}\,\langle \mathbf{F},\, \mathbf{F}\rangle,$$

where we used that $\chi(\pi) \leqq \frac{1}{4}$. Recalling that $\mathbf{J} \in \mathcal{B}$, by optimizing over $\mathbf{F} \in \mathcal{B}$ we see that $\Phi_T(\mathbf{J}\,|\,\gamma) = +\infty$ unless $\mathbf{J}(du) = \mathbf{j}\, du$ for some $\mathbf{j} \in L_2(\mathbf{T}^d; \mathbf{R}^d)$. In fact this argument also proves the first inequality in (4.9). To prove the second inequality in (4.9) it is enough to construct an appropriate path $\mathbf{W} \in \mathcal{A}_{T,\mathbf{J}}$; we simply take $\mathbf{W}_t(du) = t\,\mathbf{j}\, du$. The solution of the continuity equation (2.4) is then given by $\pi_t = \gamma$ and, by (3.6),

$$\frac{1}{T}\, I_{[0,T]}(\mathbf{W} \mid \gamma) = \frac{1}{2}\left\langle \frac{1}{\chi(\gamma)},\ \left|\mathbf{j} + \frac{1}{2}\,\nabla\gamma\right|^2 \right\rangle.$$

Recalling that γ is bounded away from 0 and 1, $\chi(\gamma) = \gamma(1-\gamma)$, the result follows.

We next show that on divergence-free measures $\mathbf{J}$ the functional $T\,\Phi_T(\cdot\,|\,\gamma)$ is subadditive.

LEMMA 4.5. *Consider a profile* $\gamma \in C^2(\mathbf{T}^d)$ *bounded away from* 0 *and* 1 *and let* $\mathbf{J} \in \mathcal{B}$. *Then, for each* $T, S > 0$, *we have*

$$(T+S)\,\Phi_{T+S}(\mathbf{J} \mid \gamma) \leqq T\,\Phi_T(\mathbf{J} \mid \gamma) + S\,\Phi_S(\mathbf{J} \mid \gamma).$$

Proof. By (4.1), given $\varepsilon > 0$ there are $\mathbf{W}^1 \in \mathcal{A}_{T,\mathbf{J}}$ and $\mathbf{W}^2 \in \mathcal{A}_{S,\mathbf{J}}$ such that

$$\Phi_T(\mathbf{J} \mid \gamma) \geqq T\, I_{[0,T]}(\mathbf{W}^1 \mid \gamma) - \frac{1}{2}\,\varepsilon, \qquad \Phi_S(\mathbf{J} \mid \gamma) \geqq S\, I_{[0,S]}(\mathbf{W}^2 \,|\, \gamma) - \frac{1}{2}\,\varepsilon.$$

Let $\mathbf{W}_t$, $t \in [0, T+S]$, be the path obtained by gluing $\mathbf{W}^1$ with $\mathbf{W}^2$, i.e., we define $\mathbf{W}_t := \mathbf{W}^1_{t\wedge T} + 1\!\!\mathrm{I}_{[T,T+S]}(t)\,\mathbf{W}^2_{t-T}$ and denote by π_t the corresponding solution of the continuity equation (2.4). Then $\mathbf{W} \in \mathcal{A}_{T+S,\mathbf{J}}$ and, by the invariance of $I_{[0,T]}$ with respect to time shifts,

$$\begin{aligned}(T+S)\,\Phi_{T+S}(\mathbf{J} \mid \gamma) &\leqq I_{[0,T+S]}(\mathbf{W} \,|\, \gamma) = I_{[0,T]}(\mathbf{W}^1 \mid \gamma) + I_{[0,S]}(\mathbf{W}^2|\pi_T)\\ &\leqq T\,\Phi_T(\mathbf{J} \mid \gamma) + S\,\Phi_S(\mathbf{J} \mid \gamma) + \varepsilon,\end{aligned}$$

where we used that $\mathbf{J} \in \mathcal{B}$, $\mathbf{W}^1 \in \mathcal{A}_{T,\mathbf{J}}$ implies $\pi_T = \gamma$. The lemma is proved.

Recall that $\widetilde{\Phi}(\cdot\,|\,\gamma)$ is defined in (4.2) and note that, by Lemmas 4.3 and 4.4, $\widetilde{\Phi}(\mathbf{J}\,|\,\gamma)$ equals $+\infty$ unless $\mathbf{J}(du) = \mathbf{j}\, du$ for some divergence-free $\mathbf{j} \in L_2(\mathbf{T}^d; \mathbf{R}^d)$. By the subadditivity proved above we have that $\widetilde{\Phi}(\mathbf{J}\,|\,\gamma) = \lim_{T\to\infty} \Phi_T(\mathbf{J}\,|\,\gamma)$ pointwise in $\mathbf{J}$. However, the pointwise convergence does not imply the Γ-convergence, and some more efforts are required.

LEMMA 4.6. *Consider a profile* $\gamma \in C^2(\mathbf{T}^d)$ *bounded away from* 0 *and* 1, *and let* $\overline{\mathbf{J}} \in \mathcal{B}$. *Then, for each open neighborhood* U *of* $\overline{\mathbf{J}}$, *we have*

$$\limsup_{T\to\infty}\, \inf_{\mathbf{J}\in U} \Phi_T(\mathbf{J} \mid \gamma) \leqq \inf_{\mathbf{J}\in U} \widetilde{\Phi}(\mathbf{J} \mid \gamma).$$

Proof. Thanks to Lemma 4.4 we can assume that the set U is a bounded subset of $L_2(\mathbf{T}^d;\mathbf{R}^d)$. Pick $S>0$ and let $k:=[T/S]$, $R:=T-kS\in[0,S)$; by Lemma 4.5 we get

$$\begin{aligned}\limsup_{T\to\infty}\inf_{\mathbf{J}\in U}\Phi_T(\mathbf{J}\mid\gamma) &\leqq \limsup_{k\to\infty}\inf_{\mathbf{J}\in U\cap\mathcal{B}}\left[\frac{kS}{kS+R}\,\Phi_S(\mathbf{J}\mid\gamma)+\frac{R}{kS+R}\,\Phi_R(\mathbf{J}\mid\gamma)\right]\\ &\leqq \limsup_{k\to\infty}\left[\frac{kS}{kS+R}\inf_{\mathbf{J}\in U\cap\mathcal{B}}\Phi_S(\mathbf{J}\mid\gamma)+\frac{R}{kS+R}\sup_{\mathbf{J}\in U\cap\mathcal{B}}\Phi_R(\mathbf{J}\mid\gamma)\right]\\ &=\inf_{\mathbf{J}\in U\cap\mathcal{B}}\Phi_S(\mathbf{J}\mid\gamma),\end{aligned}$$

where we used again Lemma 4.4 to get $\sup_{\mathbf{J}\in U\cap\mathcal{B}}\Phi_R(\mathbf{J}\,|\,\gamma)<\infty$. By taking the infimum over $S>0$ we get the result.

LEMMA 4.7. *Given $m\in(0,1)$, let $S_m\colon\,\mathcal{F}_m\to\mathbf{R}_+$ be the functional*

$$S_m(\rho):=\int_{\mathbf{T}^d}du\left\{\rho(u)\,\log\frac{\rho(u)}{m}+\big[1-\rho(u)\big]\,\log\frac{1-\rho(u)}{1-m}\right\}.$$

Then for each $\delta>0$ there exists $T_0=T_0(\delta)>0$ such that the following holds. For each $\gamma_1,\gamma_2\in\mathcal{F}_m$ there exists a path $(\mathbf{W},\pi)\in C([0,T_0];\,\mathcal{M}_d\times\mathcal{F})\cap\mathfrak{A}_{\gamma_1}$ such that $\pi_0=\gamma_1$, $\pi_{T_0}=\gamma_2$,

$$\big|\langle\mathbf{W}_{T_0},\,\mathbf{F}\rangle\big|\leqq\frac{1}{2}\,T_0\|\nabla\cdot\mathbf{F}\|_{L_2}+\delta\|\mathbf{F}\|_{L_2}\qquad\text{for any}\quad\mathbf{F}\in C^1(\mathbf{T}^d;\,\mathbf{R}^d),\tag{4.10}$$

and

$$I_{[0,T_0]}\big((\mathbf{W},\pi)\,\big|\,\gamma_1\big)\leqq S_m(\gamma_2)+\delta.\tag{4.11}$$

Proof. The strategy to construct the path π is the following. Starting from γ_1 we follow the hydrodynamic equation (2.3) until we reach a small neighborhood (in a strong topology) of the constant profile m, paying no cost, then we move "straight," paying only a small cost, to a suitable point in that small neighborhood which is chosen so that starting from it we can follow the time reversed hydrodynamic equation to get to γ_2; the cost of this portion of the path is $S_m(\gamma_2)$. The current $\mathbf{W}$ is chosen so that (2.4), (3.8) hold; i.e., it is the one whose cost is minimal among the ones compatible with the density path π.

Let $\lambda\in\mathcal{F}_m$ and denote by $P_t\lambda$ the solution of the Cauchy problem (2.3) (P_t is indeed the heat semigroup on $\mathbf{T}^d$). By the regularizing properties of the heat semigroup, given $\delta_1>0$ there exists a time T_1 such that $\|P_t\lambda\|_{H_1}+\|P_t\lambda-m\|_\infty\leqq\delta_1$, for any $t\geqq T_1$. Here $\|\varphi\|_{H_1}=\|\nabla\varphi\|_{L_2}$ is the standard Sobolev norm on $\mathbf{T}^d$ and the time T_1 is independent on λ because $0\leqq\lambda\leqq1$. We now choose $\delta_1<\frac{1}{2}\,[m\wedge(1-m)]$ and let $T_0:=2T_1+1$, $\overline{\gamma}_i=P_{T_1}\gamma_i$, $i=1,2$. The density path π is then constructed as

$$\pi_t:=\begin{cases}P_t\gamma_1 & \text{for}\quad t\in[0,T_1],\\ \overline{\gamma}_1[1-(t-T_1)]+\overline{\gamma}_2[t-T_1] & \text{for}\quad t\in(T_1,T_1+1),\\ P_{T_0-t}\gamma_2 & \text{for}\quad t\in[T_0-T_1,T_0]\end{cases}$$

while the associated current path $\mathbf{W}$ is such that

$$\dot{\mathbf{W}}_t=\begin{cases}-\frac{1}{2}\,\nabla P_t\gamma_1 & \text{for}\quad t\in[0,T_1],\\ -\frac{1}{2}\,\nabla\pi_t+\chi(\pi_t)\,\nabla H_t & \text{for}\quad t\in(T_1,T_1+1),\\ \frac{1}{2}\,\nabla P_{T_0-t}\gamma_2 & \text{for}\quad t\in[T_0-T_1,T_0],\end{cases}$$

where $H \in C^{1,2}((T_1, T_1+1) \times \mathbf{T}^d)$ solves

$$\nabla \cdot \big[\chi(\pi_t)\, \nabla H_t\big] = -\partial_t \pi_t + \frac{1}{2}\, \Delta \pi_t.$$

Note there exists a unique solution since the right-hand side is orthogonal to the constants (note that $\overline{\gamma}_i \in \mathcal{F}_m$, $i = 1, 2$).

It is straightforward to verify that $(\pi, \mathbf{W}) \in \mathfrak{A}_{\gamma_1}$; i.e., the continuity equation (2.4) holds. Thanks to the invariance of I with respect to time shifts, the cost of this path is

$$\begin{aligned} I_{[0,T_0]}\big((\mathbf{W}, \pi) \mid \gamma_1\big) &= I_{[0,T_1]}\big((\mathbf{W}, \pi) \mid \gamma_1\big) + I_{[0,1]}\big((\mathbf{W}_{\bullet - T_1}, \pi_{\bullet - T_1}) \mid \overline{\gamma}_1\big) \\ &\quad + I_{[0,T_1]}\big((\mathbf{W}_{\bullet - (T_1+1)}, \pi_{\bullet - (T_1+1)}) \mid \overline{\gamma}_2\big). \end{aligned} \tag{4.12}$$

Since in the time interval $[0, T_1]$ the path follows the hydrodynamic equation, the first term on the right-hand side of (4.12) vanishes. By considering the time reversal of the portion of the path in the interval $[T_0 - T_1, T_0]$, it is straightforward to verify (see [2, Lemma 5.4]) that

$$I_{[0,T_1]}\big((\mathbf{W}_{\bullet - (T_1+1)}, \pi_{\bullet - (T_1+1)})\big|\overline{\gamma}_2\big) = S_m(\gamma_2) - S_m(\overline{\gamma}_2) \leqq S_m(\gamma_2).$$

It remains to bound the second term on the right-hand side of (4.12). Note that, by construction, in the interval $(T_1, T_1 + 1)$ we have

$$\inf_u \overline{\gamma}_1(u) \wedge \inf_u \overline{\gamma}_2(u) \leqq \pi_t \leqq \sup_u \overline{\gamma}_1(u) \vee \sup_u \overline{\gamma}_2(u)$$

which implies, by the choice of δ_1,

$$\frac{1}{2}\, \big[m \wedge (1-m)\big] \leqq \pi_t \leqq 1 - \frac{1}{2}\, \big[m \wedge (1-m)\big],$$

i.e., the path π_t is uniformly bounded away from 0 and 1. By the same computations as in [2, Lemma 5.7], it is not difficult to show that there exists a constant $C > 0$ depending only on m such that

$$\begin{aligned} I_{[0,1]}\big(\mathbf{W}_{\bullet - T_1}, \pi_{\bullet - T_1}\big|\overline{\gamma}_1\big) &\leqq C \int_0^1 dt \left\| \partial_t \pi_{t+T_1} + \frac{1}{2}\, \Delta \pi_{t+T_1} \right\|_{H_{-1}}^2 \\ &\leqq C' \big[\|\overline{\gamma}_2\|_{H_1}^2 + \|\overline{\gamma}_1\|_{H_1}^2\big] \end{aligned}$$

which, by taking δ_1 small enough, concludes the proof of (4.11). The bound (4.10) follows easily from the construction of the path $\mathbf{W}$ by using the Cauchy–Schwarz inequality and what is proved above.

LEMMA 4.8. *Consider a profile $\gamma \in C^2(\mathbf{T}^d)$ bounded away from 0 and 1, and let $\overline{\mathbf{J}} \in \mathcal{B}$. Then, for each open neighborhood U of $\overline{\mathbf{J}}$, we have*

$$\liminf_{T \to \infty} \inf_{\mathbf{J} \in U} \Phi_T(\mathbf{J} \mid \gamma) \geqq \inf_{\mathbf{J} \in U} \widetilde{\Phi}(\mathbf{J} \mid \gamma).$$

Proof. Recalling definition (4.2), it is enough to show

$$\liminf_{T \to \infty} \inf_{\mathbf{J} \in U} \Phi_T(\mathbf{J} \mid \gamma) \geqq \liminf_{T \to \infty} \inf_{\mathbf{J} \in U \cap \mathcal{B}} \Phi_T(\mathbf{J} \mid \gamma). \tag{4.13}$$

Given $T > 0$ there exist $\mathbf{J}^1 \in U$ and $\mathbf{W}^1 \in \mathcal{A}_{T, \mathbf{J}^1}$ such that

$$\inf_{\mathbf{J} \in U} \Phi_T(\mathbf{J} \mid \gamma) \geqq \frac{1}{T} I_{[0,T]}(\mathbf{W}^1 \mid \gamma) - \frac{1}{T}.$$

Let π_t^1, $t \in [0,T]$, be the density path associated with $\mathbf{W}^1$, $\delta > 0$, and T_0 as in Lemma 4.7. We now define, on the time interval $[0, T+T_0]$, the path

$$\mathbf{W}_t := \mathbf{W}^1_{t\wedge T} + 1\!\!1_{[T,T+T_0]}(t)\, \mathbf{W}^2_{t-T},$$

where $\mathbf{W}^2_t$, $t \in [0,T_0]$, is the path constructed in Lemma 4.7 with $\gamma_1 = \pi^1_T$ and $\gamma_2 = \gamma$. Finally, let $\mathbf{J} = \mathbf{W}_{T+T_0}/(T+T_0) = T\mathbf{J}^1/(T+T_0) + \mathbf{W}^2_{T_0}/(T+T_0)$; note that, by construction, $\mathbf{J} \in \mathcal{B}$. From Lemma 4.7 it now follows that, for T large enough, $\mathbf{J} \in U$ and

$$\frac{1}{T+T_0}\, I_{[0,T+T_0]}(\mathbf{W}\,|\,\gamma) \leqq \frac{1}{T+T_0}\, I_{[0,T]}(\mathbf{W}^1\mid\gamma) + \frac{1}{T+T_0}\big[S_m(\gamma)+\delta\big].$$

By taking the limit $T\to\infty$, (4.13) follows. The lemma is proved.

Proofs of Proposition 4.1 *and Remark* 4.1. By Lemmas 4.3, 4.6, and 4.8 we have that, for each $\overline{\mathbf{J}} \in \mathcal{M}_d$ and any neighborhood U of $\overline{\mathbf{J}}$,

$$\lim_{T\to\infty} \inf_{\mathbf{J}\in U} \Phi_T(\mathbf{J}\mid\gamma) = \inf_{\mathbf{J}\in U} \widetilde{\Phi}(\mathbf{J}\mid\gamma).$$

The Γ-convergence of the sequence $\Phi_T(\cdot\,|\,\gamma)$ to the lower semicontinuous envelope of $\widetilde{\Phi}(\cdot\,|\,\gamma)$ now follows from the topological definition of Γ-convergence; see, e.g., [8, section 1.4].

We next prove Remark 4.1. Let $m \in (0,1)$; by using the path introduced in Lemma 4.7 it is straightforward to check that, for each $\gamma_1,\gamma_2 \in \mathcal{F}_m$ and $\mathbf{J}\in\mathcal{B}$, we have $\widetilde{\Phi}(\mathbf{J}\,|\,\gamma_1) = \widetilde{\Phi}(\mathbf{J}\mid\gamma_2)$, which proves the first statement.

Since $\Phi(\cdot\,|\,\gamma)$ is the lower semicontinuous envelope of $\widetilde{\Phi}(\cdot\,|\,\gamma)$, it is enough to prove the convexity of the latter. As $\mathcal{B}$ is a closed convex subset of $\mathcal{M}_d$, it is furthermore enough to show that for each $\mathbf{J}_1,\mathbf{J}_2 \in \mathcal{B}$, each $p\in(0,1)$, and each $\gamma \in C^2(\mathbf{T}^d)$ bounded away from 0 and 1, we have

$$\widetilde{\Phi}(p\,\mathbf{J}_1 + (1-p)\,\mathbf{J}_2\,|\,\gamma) \leqq p\,\widetilde{\Phi}(\mathbf{J}_1\,|\,\gamma) + (1-p)\,\widetilde{\Phi}(\mathbf{J}_2\,|\,\gamma). \tag{4.14}$$

Given $\varepsilon > 0$ we can find $T>0$, $\mathbf{W}^1 \in \mathcal{A}_{pT,\mathbf{J}_1}$, and $\mathbf{W}^2 \in \mathcal{A}_{(1-p)T,\mathbf{J}_2}$ so that

$$\widetilde{\Phi}(\mathbf{J}_1\,|\,\gamma) \geqq \frac{1}{pT}\, I_{[0,pT]}(\mathbf{W}^1\mid\gamma) - \varepsilon,$$

$$\widetilde{\Phi}(\mathbf{J}_2\,|\,\gamma) \geqq \frac{1}{(1-p)T}\, I_{[0,(1-p)T]}(\mathbf{W}^2\,|\,\gamma) - \varepsilon.$$

By the same arguments used in Lemma 4.5, the path obtained by gluing $\mathbf{W}^1$ to $\mathbf{W}^2$ is in the set $\mathcal{A}_{T,p\mathbf{J}_1+(1-p)\,\mathbf{J}_2}$. The bound (4.14) follows.

We conclude this section by proving the exponential tightness needed to complete the proof of Theorem 4.1.

LEMMA 4.9. *Fix a sequence $\eta^N \in \mathcal{X}_N$. There exists a sequence of compact sets $\{K_\ell\colon \ell \geqq 1\}$ of $\mathcal{M}_d$ such that*

$$\limsup_{T\to\infty}\limsup_{N\to\infty} \frac{1}{TN^d}\, \log \mathbf{P}^N_{\eta^N}\Big\{\frac{1}{T}\,\mathbf{W}^N_T \in K^c_\ell\Big\} \leqq -\ell.$$

Proof. Fix a vector field $\mathbf{H}\colon \mathbf{T}^d \to \mathbf{R}^d$. We claim that for every $A>0$,

$$\mathbf{P}^N_{\eta^N}\Big\{\big|\langle \mathbf{W}^N_T, \mathbf{H}\rangle\big| \geqq AT\Big\} \leqq 2\exp\big\{-TN^d[AC(\mathbf{H})^{-1} - 7]\big\}, \tag{4.15}$$

where

$$C(\mathbf{H}) = \max_j \big\{1 \vee \|H_j\|_\infty^2 \vee \|\partial_{u_j} H_j\|_\infty\big\}.$$

Indeed, by the Chebyshev exponential inequality,

$$\mathbf{P}_{\eta^N}^N \Big\{ \langle \mathbf{W}_T^N, \mathbf{H}\rangle \geqq AT \Big\} \leqq e^{-\theta A T N^d} \mathbb{E}_{\eta^N}^N \Big[\mathcal{M}_T(\theta, \mathbf{H})\, e^{f_N(T,\eta,\mathbf{H})}\Big]$$

for every $\theta > 0$. Here $\mathcal{M}_t(\theta, \mathbf{H})$ is the mean one exponential martingale defined just after (3.3), with a now time-independent vector field $\mathbf{H}$, and

$$\begin{aligned} f_N(T,\eta,\mathbf{H}) = N^2 \sum_{j=1}^d \sum_{x\in \mathbf{T}_N^d} \int_0^T ds\, \eta_s(x)\, \big[1-\eta_s(x+e_j)\big] \Big\{ e^{\theta N^{-1} H_j(x/N)} - 1\Big\} \\ + N^2 \sum_{j=1}^d \sum_{x\in \mathbf{T}_N^d} \int_0^T ds\, \eta_s(x+e_j)\big[1-\eta_s(x)\big]\big\{e^{-\theta N^{-1} H_j(x/N)} - 1\big\}. \end{aligned}$$

A Taylor expansion shows that the absolute value of $f_N(T,\eta,\mathbf{H})$ is bounded by $C(\mathbf{H})\, T N^d \{\theta + 2\theta^2 e^{\theta C(\mathbf{H})}\}$. To conclude the proof of the claim it remains to choose $\theta = C(\mathbf{H})^{-1}$, to remind us that $\mathcal{M}_t(\theta, \mathbf{H})$ has mean one and to repeat the same argument with $-\mathbf{H}$ in place of $\mathbf{H}$.

Recall the definition of the sequence $\{\mathbf{G}_k \colon k \geqq 1\}$ defined just after Proposition 4.1 and assume, without loss of generality, that $C(\mathbf{G}_k) \leqq k$. For each $\ell \geqq 1$, the set K_ℓ of measures defined by

$$K_\ell = \bigcap_{k\geqq 1} \big\{\mathbf{J} \colon \big|\langle \mathbf{J}, \mathbf{G}_k\rangle\big| \leqq (k+7)^2\, \ell\big\}$$

is compact. On the other hand, by (4.15),

$$\mathbf{P}_{\eta^N}^N \bigg\{ \frac{1}{T} \mathbf{W}_T^N \in K_\ell^c \bigg\} \leqq 4 e^{-T N^d \ell}$$

provided N is sufficiently large. This proves the lemma.

5. Dynamical phase transitions.

In this section we analyze the variational problem (4.2) defining the functional $\widetilde{\Phi}$. For the symmetric simple exclusion process we prove, in subsection 5.1, that $\widetilde{\Phi} = U$, where U is defined in (2.8). Therefore no dynamical phase transition occurs in this case. In subsection 5.2 we consider a system with general transport coefficients and show that, under suitable convexity assumptions, it is possible to construct a traveling wave whose cost is, for J large, strictly less than the constant profile. These convexity hypotheses are satisfied for the KMP model [5], [11], and therefore we prove that a dynamical phase transition takes place.

5.1. Symmetric simple exclusion process.

The following statement is essentially proved in [4]; for the reader's convenience we reproduce below its proof in a more formal setting. Together with Theorem 4.1 it concludes the proof of Theorem 2.2.

PROPOSITION 5.1. *For each $m \in (0,1)$ the functional $U_m \colon \mathcal{M}_d \to [0,\infty]$ defined in (2.8) is lower semicontinuous. Moreover if $\gamma \in C^2(\mathbf{T}^d) \cap \mathcal{F}_m$ is bounded away from 0 and 1, we have $U_m(\mathbf{J}) = \Phi(\mathbf{J} \,|\, \gamma)$ for any $\mathbf{J} \in \mathcal{M}_d$.*

Proof. We first prove the lower semicontinuity of U_m. Given $(\mathbf{J},\rho)\in\mathcal{M}_d\times\mathcal{F}_m$ we define

$$\mathcal{U}(\rho,\mathbf{J}):=\sup_{\mathbf{F}}\mathcal{V}_{\mathbf{F}}(\rho,\mathbf{J}),$$

where

$$\mathcal{V}_{\mathbf{F}}(\rho,\mathbf{J}):=\langle\mathbf{J},\mathbf{F}\rangle-\frac{1}{2}\,\langle\rho,\nabla\cdot\mathbf{F}\rangle-\frac{1}{2}\,\big\langle\mathbf{F},\chi(\rho)\,\mathbf{F}\big\rangle$$

and the supremum is carried over all smooth vector fields $\mathbf{F}\in C^1(\mathbf{T}^d;\mathbf{R}^d)$. Note that, if $\mathbf{J}(du)=\mathbf{j}\,du$ for some $\mathbf{j}\in L_2(\mathbf{T}^d;\mathbf{R}^d)$ and $\rho\in C^2(\mathbf{T}^d)$ is bounded away from 0 and 1, we have

$$\mathcal{U}(\rho,\mathbf{J})=\frac{1}{2}\left\langle\left[\mathbf{j}+\frac{1}{2}\,\nabla\rho\right],\ \frac{1}{\rho(1-\rho)}\,\left[\mathbf{j}+\frac{1}{2}\,\nabla\rho\right]\right\rangle.$$

Recalling definition (2.8), by the approximation arguments in subsection 3.4, we have $U_m(\mathbf{J})=\inf_{\rho\in\mathcal{F}_m}\mathcal{U}(\rho,\mathbf{J})$.

By the concavity of $\chi(\rho)$ we have that, for each fixed $\mathbf{F}$, the functional $\mathcal{V}_{\mathbf{F}}(\cdot,\cdot)\colon\mathcal{F}\times\mathcal{M}_d\to\mathbf{R}$ is convex and lower semicontinuous. The lower semicontinuity of $\mathcal{U}$, and hence of U_m, now follows easily. We also note that the previous argument shows that $\mathcal{U}$ is a convex functional on $\mathcal{F}\times\mathcal{M}_d$.

We next prove that for each $\gamma\in C^2(\mathbf{T}^d)\cap\mathcal{F}_m$ bounded away from 0 and 1 we have $U_m(\mathbf{J})=\widetilde{\Phi}(\mathbf{J}\,|\,\gamma)$. From the definitions (2.8), (4.2) and Lemma 4.4, we can assume that $\mathbf{J}(du)=\mathbf{j}\,du$ for some $\mathbf{j}\in L_2(\mathbf{T}^d;\mathbf{R}^d)$ divergence free.

We first show that for each $T>0$, and each path $(\mathbf{W},\pi)$ such that $\mathbf{W}\in\mathcal{A}_{T,\mathbf{J}}$, we have

$$\frac{1}{T}\,I_{[0,T]}\big((\mathbf{W},\pi)\,|\,\gamma\big)\geqq U_m(\mathbf{J}).$$

Indeed, thanks to the approximation constructed in section 3.4 we can assume that π is a smooth path bounded away from 0 and 1. For such a smooth path, (3.6) yields

$$\frac{1}{T}\,I_{[0,T]}\big((\mathbf{W},\pi)\mid\gamma\big)=\frac{1}{T}\int_0^T dt\,\mathcal{U}(\pi_t,\dot{\mathbf{W}}_t)\geqq\mathcal{U}\bigg(\frac{1}{T}\int_0^T dt\,\pi_t,\,\mathbf{J}\bigg)\geqq U_m(\mathbf{J}),$$

where we used, in the second step, the joint convexity of the functional $\mathcal{U}$ and Jensen's inequality. In the last step we finally used that by conservation of mass, $\pi_t\in\mathcal{F}_m$ for any $t\in[0,T]$.

To show the converse inequality it is enough to construct, for each T large enough, an appropriate path. Given $\varepsilon>0$ there exists $\rho\in C^2(\mathbf{T}^d)\cap\mathcal{F}_m$ bounded away from 0 and 1 such that

$$U_m(\mathbf{j}\,du)\geqq\mathcal{U}(\rho,\mathbf{j})-\varepsilon.$$

For $T>2$ we construct the path $(\mathbf{W},\pi)$ such that

$$\dot{\mathbf{W}}_t(du)=\begin{cases}\widehat{\mathbf{w}}\,du & \text{if}\quad t\in[0,1),\\[4pt] \dfrac{T}{T-2}\,\mathbf{j}\,du & \text{if}\quad t\in[1,T-1],\\[4pt] -\widehat{\mathbf{w}}\,du & \text{if}\quad t\in(T-1,T],\end{cases}$$

where $\widehat{\mathbf{w}}$ solves $\nabla \cdot \widehat{\mathbf{w}} = \gamma - \rho$. It exists because $\gamma, \rho \in \mathcal{F}_m$, i.e., they have the same mass. The density path π is the corresponding solution of (2.4), i.e.,

$$\pi_t = \begin{cases} \gamma(1-t) + \rho t & \text{if} \quad t \in [0,1), \\ \rho & \text{if} \quad t \in [1, T-1], \\ \rho(T-t) + \gamma(T+1-t) & \text{if} \quad t \in (T-1, T]. \end{cases}$$

It is straightforward to verify that $\mathbf{W} \in \mathcal{A}_{T,\mathbf{J}}$. Moreover

$$\lim_{T\to\infty} \frac{1}{T}\, I_{[0,T]}\big((\mathbf{W}, \pi) \mid \gamma\big) = \mathcal{U}(\rho, \mathbf{J}) \leqq U_m(\mathbf{J}) + \varepsilon,$$

which concludes the proof.

We conclude this section by showing that in the one-dimensional case the functional U_m is given by (2.9).

LEMMA 5.1. *Let $m \in (0,1)$, $d = 1$, and $J(du) = j\, du$ for some $j \in \mathbf{R}$. Then*

$$U_m(J) = \frac{1}{2}\, \frac{j^2}{\chi(m)}\,.$$

Remark. As it will be apparent from the proof, this lemma holds whenever the real function $\rho \mapsto 1/\chi(\rho)$ is convex.

Proof. Let $\rho \in C^2(\mathbf{T}) \cap F_m$ be bounded away from 0 and 1. We have

$$\int_{\mathbf{T}} du\, \frac{[j + \rho'(u)/2]^2}{\chi(\rho(u))} = \int_{\mathbf{T}} du\, \frac{j^2}{\chi(\rho(u))} + \int_{\mathbf{T}} du\, \frac{[\rho'(u)/2]^2}{\chi(\rho(u))}$$

because the cross term vanishes upon integration. By Jensen's inequality,

$$\int_{\mathbf{T}} du\, \frac{j^2}{\chi(\rho(u))} \geqq \frac{j^2}{\chi(m)}\,.$$

On the other hand, by considering the constant profile $\rho(u) = m$, we trivially have $U_m(j\, du) \leqq \frac{1}{2}\, j^2/\chi(m)$. The lemma is therefore proved.

5.2. Other models. The general structure of the hydrodynamic equation obtained for the scaling limit of the empirical density for stochastic lattice gases with a weak external field $\mathbf{B}$ has the form (see [10], [13])

$$\partial_t \rho + \nabla \cdot \dot{\mathbf{W}}(\rho) = 0, \qquad \dot{\mathbf{W}}(\rho) = -\frac{1}{2}\, D(\rho)\, \nabla\rho + \chi(\rho)\, \mathbf{B},$$

where $D(\rho)$ is the diffusion coefficient and $\chi(\rho)$ is the mobility.

In this general context, for smooth profiles, we set

$$\mathcal{U}(\rho, \mathbf{J}) = \frac{1}{2} \Big\langle \big[\mathbf{J} - \dot{\mathbf{W}}(\rho)\big],\ \frac{1}{\chi(\rho)}\, \big[\mathbf{J} - \dot{\mathbf{W}}(\rho)\big] \Big\rangle.$$

The integrated empirical current is expected to satisfy (see [4] for a heuristic derivation) a large deviation principle with rate function $I_{[0,T]}(\mathbf{W}) = \int_0^T dt\, \mathcal{U}(\pi_t, \dot{\mathbf{W}}_t)$ in which π is obtained from $\dot{\mathbf{W}}_t$ by solving the continuity equation $\partial_t \pi + \nabla \cdot \dot{\mathbf{W}} = 0$.

We analyze the variational problem (4.2) in this general setting and show that, under some assumptions on $D(\rho)$, $\chi(\rho)$, a time-dependent strategy is more convenient than taking a density path π constant in time, so that $\Phi < U$. For simplicity, we here discuss only the one-dimensional case $d = 1$ and assume that there is no external field, $\mathbf{B} = 0$; see [4, section 6.2] for more details.

Given a mass m and $v \in \mathbf{R}$, let $\Psi_v \colon \mathbf{R} \to \mathbf{R}_+$ be defined by

$$\Psi_v(J) = \inf_\rho \frac{1}{2} \int_0^1 du \, \frac{\{J + v\,[\rho(u) - m] - w(\rho(u))\}^2}{\chi(\rho(u))}, \tag{5.1}$$

where $w(\rho) = \dot{W}(\rho) = -\frac{1}{2} D(\rho) \nabla \rho$ and the infimum is carried over the profiles ρ of mass m, i.e., over $\mathcal{F}_m$. It will be convenient to write the term $D(\rho) \nabla \rho$ as $\nabla d(\rho)$, i.e., $d(\rho)$ is an antiderivative of $D(\rho)$.

We claim that for each $v \in \mathbf{R}$

$$\Phi \leqq \Psi_v. \tag{5.2}$$

Indeed, consider a profile ρ_0 in $\mathcal{F}_m$. Let $T = v^{-1}$ and set $\rho(t,u) = \rho_0(u - vt)$, $w(t,u) = J + v[\rho_0(u - tv) - m]$ in the time interval $[0,T]$. An elementary computation shows that the continuity equation holds and that the time average over the time interval $[0,T]$ of $w(\cdot, u)$ is equal to J. In particular,

$$\Phi(J) \leqq \frac{1}{T} \int_0^T dt \, \mathcal{U}\big(\rho(t), w(t)\big).$$

On the other hand, it is easy to show by periodicity that the right-hand side is equal to

$$\frac{1}{2} \int_0^1 du \, \frac{\{J + v[\rho_0(u) - m] - w(\rho_0)\}^2}{\chi(\rho_0(u))}.$$

Optimizing over the profile ρ_0, we conclude the proof of (5.2).

We next show that, if the real function $\rho \mapsto 1/\chi(\rho)$ is convex and $\chi''(m) > 0$ for some $0 < m < 1$, then $\Phi(J) < U(J)$ for $|J|$ sufficiently large.

To prove the previous statement, we first note that, in view of (5.2) and of Lemma 5.1, it is enough to show that there exists $\lambda \in \mathbf{R}$ such that

$$\limsup_{|J| \to \infty} \frac{\Psi_{\lambda J}(J)}{J^2} < \frac{1}{2\chi(m)}\,. \tag{5.3}$$

Fix a mass m, a current J, and take $v = \lambda J$. For $\rho \in \mathcal{F}_m$, by expanding the square we get that

(5.4)

$$\int_0^1 du \, \frac{\{J + \lambda J[\rho - m] + \frac{1}{2} \nabla d(\rho)\}^2}{\chi(\rho)} = J^2 \int_0^1 du \, \frac{\{1 + \lambda[\rho - m]\}^2}{\chi(\rho)} + \frac{1}{4} \int_0^1 du \, \frac{[\nabla d(\rho)]^2}{\chi(\rho)}$$

because the cross term vanishes. Expand the square on the first integral. Let $F(r) = F_{\lambda,m}(r)$ be the smooth function defined by $F(r) = \{1 + \lambda[r - m]\}^2/\chi(r)$. An elementary computation shows that

$$F''(m) = \frac{1}{\chi(m)^3} \Big\{ 2\chi(m)^2 \lambda^2 - 4\chi(m)\,\chi'(m)\,\lambda + 2\chi'(m)^2 - \chi(m)\,\chi''(m) \Big\}.$$

Let $\lambda = \chi'(m)/\chi(m)$. For this choice $F''(m) < 0$. In particular, we can choose a nonconstant profile $\rho(u)$ in $\mathcal{F}_m$ close to m such that $F''(\rho(u)) < 0$ for every u. Hence, by Jensen's inequality, the coefficient of J^2 in (5.4) is strictly less than $\chi(m)^{-1}$. This completes the proof of the claim.

For the KMP model [5], [11] we have $D(\rho) = 1$ and $\chi(\rho) = \rho^2$. In particular χ and $1/\chi$ are convex functions. Hence, by the above results, we have $\Phi(J) < U(J)$ for all sufficiently large currents J.

For the weakly asymmetric exclusion process with large external field, a similar phenomenon occurs. More precisely, as shown in [7], there exists a traveling wave whose cost is strictly less than the constant (in space and time) profile. A numerical computation [7] suggests also that the minimizer of the variational problem (4.2) is indeed a traveling wave.

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