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Renormalization Group in the Uniqueness Region: Weak Gibbsianity and Convergence*

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Abstract: We analyze the block averaging transformation applied to lattice gas models with short range interaction in the uniqueness region below the critical temperature. We prove weak Gibbsianity of the renormalized measure and convergence of the renormalized potential in a weak sense. Since we are arbitrarily close to the coexistence region we have a diverging characteristic length of the system: the correlation length or the critical length for metastability, or both. Thus, to perturbatively treat the problem we have to use a scale–adapted expansion. Moreover, such a model below the critical temperature resembles a disordered system in the presence of Griffiths' singularity. Then the cluster expansion that we use must be graded with its minimal scale length diverging when the coexistence line is approached.

1. Introduction

In this paper we analyze, from a rigorous point of view, the well known Renormalization Group (RG) map called Block Averaging Transformation (BAT). Following [15] we say that a stochastic field is *strongly*, resp. *weakly*, Gibbsian if its family of conditional probabilities has the Gibbsian form with respect to a potential absolutely *uniformly*, resp. *pointwise almost surely*, converging. Thus in both cases the DLR equations are satisfied but with different notions on the summability properties of the potential. We refer to [19] for a general description of the Gibbs formalism especially in connection with renormalization–group maps and to [5, 15, 26] for a discussion of the weak Gibbs property.

Under suitable strong mixing conditions, i.e. exponential decay of truncated expectations, for the original (*object*) system we establish the weak Gibbs property of the renormalized (*image*) measure and the convergence, in a suitable sense, of the renormalized potential under iteration of BAT. A relevant application will be the standard

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two–dimensional Ising model in the uniqueness region. For this case, when the temperature is higher than the critical value T_c, actually we have strong Gibbsianity of the renormalized measure for all large enough scales of the transformation as shown in [1]. On the other hand for $T < T_c$, violation of strong Gibbsianity is expected and actually proven for $T \ll T_c$, see [19]. In the present paper we prove the weak Gibbsianity of the renormalized measure.

Let us focus, for the moment, on the two–dimensional Ising model. A more general setup will be introduced in the following section. We give here some specific definitions. The state space of the object system is $\mathcal{S} = \otimes_{x\in\mathcal{L}}\mathcal{S}_x$, with $\mathcal{S}_x = \{-1, +1\}$, $\mathcal{L} = \mathbb{Z}^2$; for $\Lambda \subset \mathcal{L}$ we set $\mathcal{S}_\Lambda = \otimes_{x\in\Lambda}\mathcal{S}_x$. The (negative) Hamiltonian in a finite volume Λ with boundary condition $\tau \in \mathcal{S}_{\mathcal{L}\setminus\Lambda}$ is

$$H_\Lambda(\sigma\tau) = \beta \sum_{\substack{\{x,y\}\subset\Lambda:\\ |x-y|=1}} \sigma_x\sigma_y + \beta \sum_{\substack{x\in\Lambda,\, y\notin\Lambda:\\ |x-y|=1}} \sigma_x\tau_y + \beta h \sum_{x\in\Lambda} \sigma_x$$

with $\beta = 1/T > 0$ the inverse temperature, $h \in \mathbb{R}$ the magnetic field and $\sigma \in \mathcal{S}_\Lambda$. Notice that in this section we use the magnetic language whereas in the following we will use the equivalent lattice gas formulation. The corresponding finite volume Gibbs measure is

$$\mu^\tau_{\beta,h,\Lambda}(\sigma) = \frac{\exp\{H_\Lambda(\sigma\tau)\}}{\sum\limits_{\sigma'\in\mathcal{S}_\Lambda} \exp\{H_\Lambda(\sigma'\tau)\}}.$$

We denote by $\mu = \mu_{\beta,h}$ the unique infinite volume Gibbs measure in the uniqueness region deprived of the critical point given by $\{\beta < \beta_c\} \cup \{\beta > \beta_c,\, h \neq 0\}$, where $\beta_c = 1/T_c = \log(1+\sqrt{2})/2$ is the inverse critical temperature, see for instance [21].

Let $\mathcal{L}^{(\ell)} = (\ell\mathbb{Z})^2$, $\ell \in \mathbb{N}$ and partition $\mathcal{L}$ as the disjoint union of ℓ–block $Q_\ell(i) = Q_\ell(0) + i$, where $i \in \mathcal{L}^{(\ell)}$, and $Q_\ell(0)$ is the square of side ℓ with the origin the site with smallest coordinates. We associate with each $i \in \mathcal{L}^{(\ell)}$ a *renormalized spin* m_i taking values in

$$\mathcal{S}^{(\ell)}_i = \left\{ \frac{-\ell^d - \ell^d\bar{m}}{\sqrt{\ell^d\chi}}, \frac{-\ell^d + 2 - \ell^d\bar{m}}{\sqrt{\ell^d\chi}}, \dots, \frac{\ell^d - \ell^d\bar{m}}{\sqrt{\ell^d\chi}} \right\},$$

where $\bar{m} = \bar{m}_{\beta,h} = \mu_{\beta,h}(\sigma_0)$ is the equilibrium magnetization and $\chi = \chi(\beta,h) = \sum_{x\in\mathcal{L}}[\mu_{\beta,h}(\sigma_0\sigma_x) - \mu_{\beta,h}(\sigma_0)\mu_{\beta,h}(\sigma_x)]$ is the susceptibility. For $I \subset \mathcal{L}^{(\ell)}$ we write $\mathcal{S}^{(\ell)}_I = \otimes_{i\in I}\mathcal{S}^{(\ell)}_i$; we also set $\mathcal{S}^{(\ell)} = \otimes_{i\in\mathcal{L}^{(\ell)}}\mathcal{S}^{(\ell)}_i$.

The renormalized measure $\nu^{(\ell)} = \nu^{(\ell)}_{\beta,h}$ on the renormalized space $\mathcal{S}^{(\ell)}$ is defined via its finite dimensional distributions; let $I \subset\subset \mathcal{L}^{(\ell)}$, where $\subset\subset$ means finite subset, and pick $\widetilde{m} \in \mathcal{S}^{(\ell)}_I$, then set

$$\nu^{(\ell)}_{\beta,h}\big(\{m \in \mathcal{S}^{(\ell)} : m_I = \widetilde{m}\}\big) = \int_{\mathcal{S}} \mathrm{d}\mu_{\beta,h}(\sigma) \prod_{i\in I} \delta(M_i(\sigma_{Q_\ell(i)}) - \widetilde{m}_i), \qquad (1.1)$$

where for all $i \in \mathcal{L}^{(\ell)}$ and $\eta \in \mathcal{S}_{Q_\ell(i)}$ we have introduced the empirical magnetization, centered and normalized,

$$M_i(\eta) = \frac{1}{\sqrt{\ell^d\chi}} \sum_{x\in Q_\ell(i)} [\eta_x - \bar{m}]. \qquad (1.2)$$

We write $\nu_{\beta,h}^{(\ell)} = T^{(\ell)} \mu_{\beta,h}$ and note that the semigroup property holds namely, $T^{(\ell)}\, T^{(\ell')} = T^{(\ell\ell')}$. The image measure $\nu_{\beta,h}^{(\ell)}$ represents the distribution of the empirical block magnetization under the object measure $\mu_{\beta,h}$.

We would like to analyze the map on the potentials induced by the map $T^{(\ell)}$ that was defined on (infinite volume) measures. A preliminary condition for this program is that the renormalized measure is strongly or weakly Gibbsian with respect to the renormalized potential, see [19].

We introduce, now, the finite volume setup. Let $I \subset\subset \mathcal{L}^{(\ell)}$ be a finite box in $\mathcal{L}^{(\ell)}$ and consider the corresponding box $\Lambda = Q_\ell(I) \subset \mathcal{L}$. We introduce the *renormalized Hamiltonian* $\mathcal{H}_I^{(\ell),\tau}$ with boundary condition $\tau \in \mathcal{S}_{\mathcal{L}\setminus\Lambda}$ by setting

$$e^{\mathcal{H}_I^{(\ell),\tau}(m)} = \sum_{\sigma\in\mathcal{S}_\Lambda} e^{H_\Lambda^\tau(\sigma)} \prod_{i\in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i) \tag{1.3}$$

for each $m \in \mathcal{S}_I^{(\ell)}$. In the computation of the *renormalized potential* associated with the renormalized Hamiltonian $\mathcal{H}_I^{(\ell),\tau}$, a crucial role is played by the *constrained systems* obtained by conditioning the object system to a fixed renormalized spin configuration, see the pioneering paper [7]. More precisely, the equilibrium probability measure of the constrained model associated with the renormalized configuration $m \in \mathcal{S}_I^{(\ell)}$ on the finite volume $\Lambda = Q_\ell(I) \subset\subset \mathcal{L}$ is given by

$$\mu_{m,\Lambda}^{(\ell),\tau}(\sigma) = \frac{e^{H_\Lambda^\tau(\sigma)} \prod_{i\in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i)}{\sum_{\eta\in\mathcal{S}_\Lambda} e^{H_\Lambda^\tau(\eta)} \prod_{i\in I} \delta(M_i(\eta_{Q_\ell(i)}) - m_i)} \tag{1.4}$$

for all $\sigma \in \mathcal{S}_\Lambda$. Notice that from (1.3) it follows that the renormalized Hamiltonian $\mathcal{H}_I^{(\ell),\tau}(m)$ is equal to the logarithm of the partition function of the corresponding constrained system which is defined as

$$Z_{m,\Lambda}^{(\ell),\tau} = \sum_{\sigma\in\mathcal{S}_\Lambda} e^{H_\Lambda^\tau(\sigma)} \prod_{i\in I} \delta(M_i(\sigma_{Q_\ell(i)}) - m_i). \tag{1.5}$$

The measure $\mu_{m,I}^{(\ell),\tau}$ can be called *multi–canonical*, because it is nothing but the original measure *constrained* to the assigned magnetizations in the ℓ–blocks contained in Λ. Of course $\mu_{m,I}^{(\ell),\tau}$ does not depend at all on the magnetic field h.

It has been shown in [19] that for any $\ell \in \mathbb{N}$ even there exists $\beta_0 = \beta_0(\ell)$ such that the renormalized measure $\nu_{\beta,h}^{(\ell)}$, arising from the application of the BAT map to the Ising measure $\mu_{\beta,h}$, is non–Gibbsian at any h and $\beta > \beta_0$. This pathological behavior is a consequence of violation of *quasi–locality*, a continuity property of its conditional probabilities which constitutes a necessary condition for Gibbsianity, see [16, 17, 19]. This, in turn, is a consequence of a first order phase transition with long range order of a particular constrained model: the one corresponding to $m_i = 0$ for all $i \in \mathcal{L}^{(\ell)}$. It is clear that this pathology is completely independent of the value of the magnetic field h acting on the object system. On the other hand it is also clear that this "bad" configuration, inducing non–Gibbsianity, is very atypical with respect to $\nu_{\beta,h}^{(\ell)}$ for $h \neq 0$. Thus it is reasonable to expect the validity of a weaker property of Gibbsianity.

Before discussing this point let us recall the main result of [1] on strong Gibbsianity above T_c in two dimensions which will be useful for a comparison with the results obtained in the present paper for the case with $T < T_c$. The case $d > 2$ will be discussed later on.

Theorem 1.1. *Consider the two–dimensional Ising system with $\beta < \beta_c$ and $h \in \mathbb{R}$ given. Then there exists $\ell_0 \in \mathbb{N}$ such that for any ℓ large enough multiple of ℓ_0 the measure $\nu^{(\ell)} = \nu^{(\ell)}_{\beta,h}$ is Gibbsian in the sense that for each $Y \subset\subset \mathcal{L}^{(\ell)}$ and for each local function $f : \mathcal{S}^{(\ell)}_Y \to \mathbb{R}$ we have*

$$\nu^{(\ell)}(f) = \int_{\mathcal{S}^{(\ell)}} \nu^{(\ell)}(dm') \frac{1}{Z_Y(m')} \sum_{m \in \mathcal{S}^{(\ell)}_Y} f(m) \times \exp\Big\{ \sum_{X \cap Y \neq \emptyset} [\psi^{(\ell)}_X(m_{Y\cap X} m'_{Y^c \cap X}) + \phi^{(\ell)}_X(m_{Y\cap X} m'_{Y^c\cap X})]\Big\}, \tag{1.6}$$

where

$$Z_Y(m') = \sum_{m \in \mathcal{S}^{(\ell)}_Y} \exp\Big\{ \sum_{X\cap Y \neq \emptyset} [\psi_X(m_{Y\cap X} m'_{Y^c\cap X}) + \phi_X(m_{Y\cap X} m'_{Y^c\cap X})]\Big\}. \tag{1.7}$$

The family $\{\phi^{(\ell)}_X + \psi^{(\ell)}_X,\ X \subset\subset \mathcal{L}^{(\ell)}\}$, with $\phi^{(\ell)}_X, \psi^{(\ell)}_X : \mathcal{S}^{(\ell)}_X \to \mathbb{R}$, is translationally invariant and satisfies the uniform bound

$$\sum_{X \ni 0} e^{\alpha|X|} \sup_{m \in \mathcal{S}^{(\ell)}_X} \Big(\big|\psi^{(\ell)}_X(m)\big| + \big|\phi^{(\ell)}_X(m)\big| \Big) < \infty \tag{1.8}$$

for a suitable $\alpha > 0$. Moreover, there exists $\kappa \in \mathbb{N}$ such that $\Psi^{(\ell)}_X = 0$ if $\mathrm{diam}(X) \geq \kappa$. Finally we have that for the same α as in (1.8),

$$\lim_{\ell\to\infty} \sum_{X \ni 0} e^{\alpha|X|} \sup_{m \in \mathcal{S}^{(\ell)}_X} \big|\phi^{(\ell)}_X(m)\big| = 0, \tag{1.9}$$

$\psi^{(\ell)}_{\{i\}}(m_i) = -m_i^2/2$, for $i \in \mathcal{L}^{(\ell)}$, and there exists $a > 0$ such that

$$\lim_{\ell\to\infty} \sup_{\substack{m \in \mathcal{S}^{(\ell)}_X \\ |m_i| \leq \ell^a, i \in X}} \big|\psi^{(\ell)}_X(m)\big| = 0 \qquad \text{for } |X| \geq 2.$$

The crucial point to obtain the above result is the validity of a strong mixing condition for the object system *uniformly* in the magnetic field h. This fails below T_c, because of the phase transition at $h = 0$. By only assuming strong mixing of the object system, without uniformity in h, we can expect only weak Gibbsianity since, as we said before, for $T < T_c$ violation of strong Gibbsianity has been proven in [19]. Let us now state our main results on weak Gibbsianity and convergence of the renormalized potential as $\ell \to \infty$; this theorem is an immediate consequence of the more general results that will be stated in Theorems 2.1 and 2.2.

Theorem 1.2. *Consider the two–dimensional Ising model. Given* $(\beta, h) \in \{\beta < \beta_c\} \cup \{\beta > \beta_c,\ h \neq 0\}$*, there exists* ℓ_0 *such that for any large enough* ℓ *multiple of* ℓ_0*, the measure* $\nu^{(\ell)}$ *is weakly Gibbsian in the sense that it satisfies the DLR equations (1.6) with respect to a potential* $\{\psi_X^{(\ell)} + \phi_X^{(\ell)},\ X \subset\subset \mathcal{L}^{(\ell)}\}$*,* $\psi_X^{(\ell)}, \phi_X^{(\ell)} : \mathcal{S}_X^{(\ell)} \mapsto \mathbb{R}$*, satisfying the following.*

There exists a measurable set $\bar{\mathcal{S}}^{(\ell)} \subset \mathcal{S}^{(\ell)}$*, such that* $\nu^{(\ell)}(\bar{\mathcal{S}}^{(\ell)}) = 1$*, and functions* $r_i^{(\ell)} : \bar{\mathcal{S}}^{(\ell)} \mapsto \mathbb{N} \setminus \{0\}$*, for all* $i \in \mathcal{L}^{(\ell)}$*, such that for each* $m \in \bar{\mathcal{S}}^{(\ell)}$*, if* $X \ni i$ *and* $\mathrm{diam}(X) > r_i^{(\ell)}(m)$*, then* $\psi_X^{(\ell)}(m) = 0$*. Furthermore, for each* $i \in \mathcal{L}^{(\ell)}$ *and* $m \in \bar{\mathcal{S}}^{(\ell)}$ *there exists a real* $c_i^{(\ell)}(m) \in [0, \infty)$ *such that*

$$\sum_{X \ni i} |\psi_X^{(\ell)}(m_X)| \leq c_i^{(\ell)}(m). \tag{1.10}$$

There exists C independent of ℓ *such that*

$$\sup_{m \in \mathcal{S}^{(\ell)}} \sup_{i \in \mathcal{L}^{(\ell)}} \sum_{X \ni i} |\phi_X^{(\ell)}(m)| < C. \tag{1.11}$$

For each $i \in \mathcal{L}^{(\ell)}$ *we have* $\psi_{\{i\}}^{(\ell)}(m) = -m_i^2/2$ *and for each* $q \in [1, +\infty)$*,*

$$\lim_{\ell \to \infty} \sup_{i \in \mathcal{L}^{(\ell)}} \nu^{(\ell)}\Big(\Big| \sum_{X \ni i:\, |X| \geq 2} \psi_X^{(\ell)} \Big|^q \Big) = 0 \tag{1.12}$$

and

$$\lim_{\ell \to \infty} \sup_{m \in \mathcal{S}^{(\ell)}} \sup_{i \in \mathcal{L}^{(\ell)}} \sum_{X \ni i} |\phi_X^{(\ell)}(m)| = 0. \tag{1.13}$$

Remark. From the more general result stated in Theorems 2.1 and 2.2 below, we get that Theorem 1.2 extends to the case $d > 2$, $h \neq 0$, $\beta > \beta_0(d, |h|)$ for a suitable function $\beta_0 : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$. Indeed in this case the required strong mixing condition is satisfied.

In this low–temperature case we have a diverging scale even when we are far away from the critical point. It is not the correlation length but, rather, the “critical length for metastability”, diverging when $h \to 0$ as $1/h$, representing the scale for which the magnetic field decides the phase; this is given as the scale for which the boundary conditions are “screened” by the magnetic field h. Notice that in the region $\{\beta < \beta_c\} \cup \{\beta > \beta_c,\ h \neq 0\}$, where for $d = 2$ the strong mixing is satisfied [25, 30], both the critical length and the correlation length can diverge even simultaneously.

Let us discuss, now, the result of Theorem 1.2. As we said above it is sufficient that there exists one “bad” renormalized configuration giving rise to long range order for the corresponding constrained system, to induce violation of Gibbsianity. For BAT it has been shown in [19] that for any magnetic field h, a bad configuration is $m_i \approx 0$ for all $i \in \mathcal{L}^{(\ell)}$. For $h \neq 0$ this is a very atypical configuration; however with small but positive probability we have arbitrarily large bad regions with m_i close to zero. To be more precise we shall call “good” a block magnetization m_i belonging to a suitable interval such that: inside such an interval the system has a good behavior in the strong mixing sense and the probability to be bad (not good) is sufficiently small, see Subsect. 4.3. In order to prove weak Gibbsianity the key property is that bad regions are far apart: larger and larger bad regions are sparser and sparser.

As we discussed in [2] this situation is similar to that of disordered systems in the presence of the Griffiths' singularity. A multi–scale analysis is needed. The natural approach, quite complicated from the technical point of view, is to use a graded cluster expansion. For disordered systems there are clever methods, see [3,10], avoiding cluster expansion, that enable to prove results like exponential decay of correlations for almost all realizations of the disorder. In the case of BAT, in order to compute renormalized potentials in the weakly Gibbsian case, the use of the full theory of the graded cluster expansion (like the one in [20]) appears to be unavoidable. Since we want to study a region of parameters arbitrarily close to the critical point ($h \neq 0$, $0 < T \leq T_c$) the distinctive character of our graded cluster expansion is that the minimal scale may be chosen arbitrarily large and diverging as $T \to T_c$ and/or $h \to 0$. The minimal scale involved in our discussion being divergent, we need to use a scale–adapted cluster expansion, see [1,27,28], based on a finite size mixing condition.

In this case, contrary to low and high temperature expansion or high magnetic field expansion, the small parameter is the ratio between the diverging length and the suitably large finite size where the mixing condition holds. We want to stress again that in our approach, according to the general renormalization group ideology, we first fix the values of the thermodynamic parameters of the object system and, subsequently, the value of the scale of BAT. In other words we *take advantage* from choosing the scale ℓ of the transformation large enough. On the other side we cannot exclude that, for given values of β and h, if ℓ is not sufficiently large, weak Gibbsianity ceases to be valid. In [5,26] the authors study decimation transformation, see [19], at large β and *arbitrary h*. They first fix the scale of the transformation and, subsequently, choose the temperature below which they get weak Gibbsianity.

In the present paper, in the context of weak Gibbsianity, we give also results of convergence of renormalized potentials when iterating BAT, which, by the semigroup property, corresponds to taking the limit as $\ell \to \infty$. It appears clear that for that purpose one has to use a perturbative theory based on scale–adapted cluster expansion. Even far away from the critical point, in order to prove convergence directly, one needs to take advantage from choosing large ℓ. In [6] the author uses a high temperature expansion giving rise to a polymer system whose activity is small uniformly in ℓ; he then proves convergence by making use of the general result [23] according to which, to get convergence in a suitable sense, one needs only to prove uniform boundedness in a suitable norm. This situation is similar to the one of [29] where the author uses a low temperature expansion that converges uniformly in ℓ.

The paper is organized as follows: in Sect. 2 we introduce the basic notation and state our main results on the weak Gibbsianity and convergence of the renormalized potentials as $\ell \to \infty$. In Sect. 3 we prove the required probability estimates on the configuration of "bad" magnetizations. Then, in Sect. 4 we construct the full measure set where the conditional probabilities have the Gibbs form. In Sect. 5, following [1,27,28], we perform the scale–adapted cluster expansion on the "good" part of the lattice. In Sect. 6 we apply the theory of the graded expansion developed in [2] to prove the main results.

2. Notation and Results

In this section we give the basic definitions, introduce the general setup, and state our main results.

2.1. The lattice. For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ we let $|x| := \sup_{k=1,\cdots,d} |x_k|$. The spatial structure is modeled by the d–dimensional lattice $\mathcal{L} := \mathbb{Z}^d$, in which we let e_i, $i = 1, \ldots, d$, be the coordinate unit vectors. For each strictly positive integer s, we introduce the s–rescaled lattice $\mathcal{L}^{(s)} := (s\mathbb{Z})^d$ which is embedded in $\mathcal{L}$ namely, points in $\mathcal{L}^{(s)}$ are also points in $\mathcal{L}$, see Fig. 1. Given an integer $s \geq 1$ we next define some geometrical notions on the s–rescaled lattice $\mathcal{L}^{(s)}$. If $s = 1$ they refer to the original lattice $\mathcal{L}$ and in such a case we drop s from the notation.

We set $e_i^{(s)} := s\, e_i$, $i = 1, \ldots, d$ and use $\Lambda^{\mathrm{c}} := \mathcal{L}^{(s)} \setminus \Lambda$ to denote the complement of $\Lambda \subset \mathcal{L}^{(s)}$. For Λ a finite subset of $\mathcal{L}^{(s)}$ (we use $\Lambda \subset\subset \mathcal{L}^{(s)}$ to indicate that Λ is finite), $|\Lambda|$ denotes the cardinality of Λ. We consider $\mathcal{L}^{(s)}$ endowed with the distance $\mathrm{d}_s(x, y) := |x - y|/s$. As usual for $\Lambda, \Delta \subset \mathcal{L}^{(s)}$ we set $\mathrm{d}_s(\Lambda, \Delta) := \inf\{\mathrm{d}_s(x, y),\ x \in \Lambda,\ y \in \Delta\}$ and $\mathrm{diam}_s(\Lambda) := \sup\{\mathrm{d}_s(x, y),\ x, y \in \Lambda\}$. Moreover, for each $\Lambda \subset\subset \mathcal{L}^{(s)}$ we denote by $\mathcal{Q}^{(s)}(\Lambda) \subset\subset \mathcal{L}^{(s)}$ the smallest parallelepiped, with axes parallel to the coordinate directions, containing Λ. We say that $x, y \in \mathcal{L}^{(s)}$ are *nearest neighbors* iff $\mathrm{d}_s(x, y) = 1$; we say that $\Lambda \subset \mathcal{L}^{(s)}$ is *s–connected* iff for each $x, y \in \Lambda$ there exists a path of pairwise nearest neighbor sites of Λ joining x and y.

For $x \in \mathcal{L}^{(s)}$ and m a strictly positive real we set $Q_m^{(s)}(x) := \{y \in \mathcal{L}^{(s)} : x_k \leq y_k \leq x_k + s(m - 1),\ \forall k = 1, \ldots, d\}$. For $X \subset\subset \mathcal{L}^{(s)}$ and $m > 0$ we set $B_m^{(s)}(X) := \{y \in \mathcal{L}^{(s)} : \mathrm{d}_s(X, y) \leq m\}$; if $x \in \mathcal{L}^{(s)}$ we write $B_m^{(s)}(x)$ for $B_m^{(s)}(\{x\})$. Note that $Q_m^{(s)}(x)$ is the cube of s–side length $[m]$ with x the site with smallest coordinates, while $B_m^{(s)}(x)$ is the ball of s–radius $[m]$ centered at x, hence it is a cube of s–side length $2[m] + 1$. We remark, also, that $|Q_m^{(s)}(x)| = [m]^d$ and $|B_m^{(s)}(x)| = (2[m] + 1)^d$. We shall denote $Q_m^{(s)}(0)$, resp. $B_m^{(s)}(0)$, simply by $Q_m^{(s)}$, resp. $B_m^{(s)}$.

For $r > 0$ and $\Lambda \subset \mathcal{L}^{(s)}$ we set $\partial^{(s),r}\Lambda := \{x \in \Lambda^{\mathrm{c}} : \mathrm{d}_s(x, \Lambda) \leq r\}$; finally we set $\overline{\Lambda}^{(s),r} := \Lambda \cup \partial^{(s),r}\Lambda$. If $r = 1$ we drop it from the notation, i.e. $\partial^{(s)}\Lambda := \partial^{(s),1}\Lambda$ and $\overline{\Lambda}^{(s),1} =: \overline{\Lambda}^{(s)}$.

Let $\mathcal{E}^{(s)} := \big\{\{x, y\},\ x, y \in \mathcal{L}^{(s)} : \mathrm{d}_s(x, y) = 1\big\}$ be the collection of *edges* in $\mathcal{L}^{(s)}$. Note that, according to our definitions, the edges can also be diagonal. We say that two edges $e, e' \in \mathcal{E}^{(s)}$ are connected if and only if $e \cap e' \neq \emptyset$. A subset $(V, E) \subset (\mathcal{L}^{(s)}, \mathcal{E}^{(s)})$ is said to be connected iff for each pair $x, y \in V$, with $x \neq y$, there exists in E a path of connected edges joining them. For $\Lambda \subset\subset \mathcal{L}^{(s)}$ we then set

$$\mathbb{T}_s(\Lambda) := \inf \Big\{ |E|,\ (V, E) \subset (\mathcal{L}^{(s)}, \mathcal{E}^{(s)}) \text{ is connected and } V \supset \Lambda \Big\}. \tag{2.1}$$

We agree that $\mathbb{T}_s(\Lambda) = 0$ if $|\Lambda| = 1$ and remark that for each $x, y \in \mathcal{L}^{(s)}$ we have $\mathbb{T}_s(\{x, y\}) = \mathrm{d}_s(x, y)$.

Let u be a multiple of s, we define the *unpacking* and the *packing* operators which associate subsets of the u–rescaled lattice to subsets of the s–rescaled lattice and vice versa. More precisely, the *unpacking* operator $\mathcal{O}_u^s$ maps a set $\Lambda \subset \mathcal{L}^{(u)}$ to

$$\mathcal{O}_u^s \Lambda := \bigcup_{x \in \Lambda} Q_{u/s}^{(s)}(x).$$

Note that the cubes $Q_{u/s}^{(s)}(x)$ appearing above are disjoint namely, $Q_{u/s}^{(s)}(x) \cap Q_{u/s}^{(s)}(y) = \emptyset$ for any $x, y \in \Lambda$ such that $x \neq y$. The *packing* operator $\mathcal{O}_s^u$ maps a set $\Lambda \subset \mathcal{L}^{(s)}$ to $\mathcal{O}_s^u \Lambda := \{x \in \mathcal{L}^{(u)} : \Lambda \cap Q_{u/s}^{(s)}(x) \neq \emptyset\}$. We note that the restriction of $\mathcal{O}_s^u$ to the range

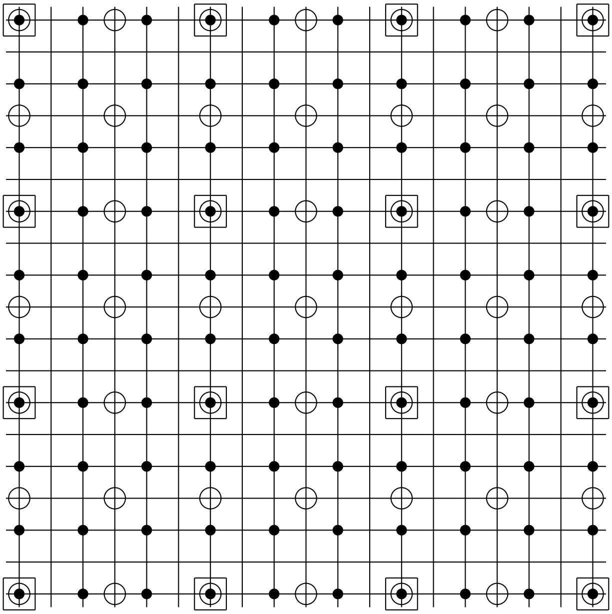

Fig. 1. The lattices $\mathcal{L}$, $\mathcal{L}^{(2)}$, $\mathcal{L}^{3)}$ and $\mathcal{L}^{(6)}$ are depicted in the two–dimensional case. Sites in $\mathcal{L}$ are represented by the intersections of the lines, solid circles represent sites belonging to $\mathcal{L}^{(2)}$, open circles represent sites belonging to $\mathcal{L}^{(3)}$, open squares represent sites of $\mathcal{L}^{(6)}$

of $\mathcal{O}_u^s$ is the inverse operator of $\mathcal{O}_u^s$ namely, $\mathcal{O}_s^u \mathcal{O}_u^s \Lambda = \Lambda$ for all $\Lambda \subset \mathcal{L}^{(u)}$. Note that, as mentioned before, we let $\mathcal{O}_u := \mathcal{O}_u^1$ and $\mathcal{O}^u := \mathcal{O}_1^u$.

2.2. The configuration space. We deal with lattice systems whose single spin space is not translationally invariant and labelled by points in the s–rescaled lattice $\mathcal{L}^{(s)}$. As usual for $s = 1$ we recover the notation for the original lattice and drop s from the notation. Given a collection of strictly positive integers $S_x^{(s)}$, $x \in \mathcal{L}^{(s)}$, such that $S^{(s)} := \sup_{x\in\mathcal{L}^{(s)}} S_x^{(s)} < +\infty$, the configuration space associated to $x \in \mathcal{L}^{(s)}$ is a finite set $\mathcal{S}_x^{(s)}$, with $|\mathcal{S}_x^{(s)}| = S_x^{(s)} + 1$ which we consider endowed with the discrete topology, the associated Borel σ–algebra is denoted by $\mathcal{F}_{\{x\}}^{(s)}$.

The configuration space in $\Lambda \subset \mathcal{L}^{(s)}$ is defined as $\mathcal{S}_\Lambda^{(s)} := \otimes_{x\in\Lambda}\mathcal{S}_x^{(s)}$ and equipped with the product topology; we will let $\mathcal{S}_{\mathcal{L}^{(s)}}^{(s)} =: \mathcal{S}^{(s)}$. We denote by $\mathcal{F}^{(s)}$ the Borel σ–algebra on $\mathcal{S}^{(s)}$ and for each $\Lambda \subset \mathcal{L}^{(s)}$ we set $\mathcal{F}_\Lambda^{(s)} := \otimes_{x\in\Lambda}\mathcal{F}_{\{x\}}^{(s)} \subset \mathcal{F}^{(s)}$.

Given $\Delta \subset \Lambda \subset \mathcal{L}^{(s)}$ and $\sigma := \{\sigma_x \in \mathcal{S}_x^{(s)},\ x \in \Lambda\} \in \mathcal{S}_\Lambda^{(s)}$, we denote by σ_Δ the *restriction* of σ to Δ namely, $\sigma_\Delta := \{\sigma_x,\ x \in \Delta\}$. Let m be a positive integer and let $\Lambda_1, \ldots, \Lambda_m \subset \mathcal{L}^{(s)}$ be pairwise disjoint subsets of $\mathcal{L}^{(s)}$; for each $\sigma_k \in \mathcal{S}_{\Lambda_k}^{(s)}$, with $k = 1, \ldots, m$, we denote by $\sigma_1\sigma_2\cdots\sigma_m$ the configuration in $\mathcal{S}_{\Lambda_1\cup\cdots\cup\Lambda_m}^{(s)}$ such that $(\sigma_1\sigma_2\cdots\sigma_m)_{\Lambda_k} = \sigma_k$ for all $k \in \{1, \ldots, m\}$. For $x \in \mathcal{L}^{(s)}$ we define the shift Θ_x acting on $\mathcal{S}^{(s)}$ by setting $(\Theta_x\sigma)_y := \sigma_{y+x}$, for all $y \in \mathcal{L}^{(s)}$ and $\sigma \in \mathcal{S}^{(s)}$.

A function $f : \mathcal{S}^{(s)} \to \mathbb{R}$ is called a *local* function if and only if there exists $\Lambda \subset\subset \mathcal{L}^{(s)}$ such that $f \in \mathcal{F}_\Lambda^{(s)}$ namely, f is $\mathcal{F}_\Lambda^{(s)}$–measurable for some bounded set Λ. For f a local function we shall denote by $\mathrm{supp}(f)$, the so–called support of f, the smallest $\Lambda \subset\subset \mathcal{L}^{(s)}$ such that $f \in \mathcal{F}_\Lambda^{(s)}$. If $f \in \mathcal{F}_\Lambda^{(s)}$ we shall sometimes misuse the notation by writing $f(\sigma_\Lambda)$ for $f(\sigma)$. We also introduce $C(\mathcal{S}^{(s)})$ the space of continuous functions on $\mathcal{S}^{(s)}$ which becomes a Banach space under the norm $\|f\|_\infty := \sup_{\sigma\in\mathcal{S}^{(s)}} |f(\sigma)|$; note that the local functions are dense in $C(\mathcal{S}^{(s)})$.

2.3. The potential. Consider the integer $s \geq 1$, a *potential* $\Phi^{(s)}$, for a lattice model on $\mathcal{L}^{(s)}$ with configuration space $\mathcal{S}^{(s)}$ as above, is a collection of local functions labelled by finite subsets of $\mathcal{L}^{(s)}$ namely, $\Phi^{(s)} := \{\Phi_X^{(s)} \in \mathcal{F}_X^{(s)},\ X \subset\subset \mathcal{L}^{(s)}\}$. We say that $\Phi^{(s)}$ is *finite range* iff there exists $r > 0$ such that $\Phi_X^{(s)} = 0$ if $\mathrm{diam}_s(X) > r$; we say it is *translationally invariant* iff for each $x \in \mathcal{L}^{(s)}$, $\Phi_X^{(s)}(\sigma) = \Phi_{X-x}^{(s)}(\Theta_x\sigma)$. We note that the potentials, which do not need to satisfy the conditions above, form a linear space in which, given $a \geq 0$, we introduce the norm $\|\cdot\|_a$ defined by

$$\|\Phi^{(s)}\|_a := \sup_{x\in\mathcal{L}^{(s)}} \sum_{X\ni x} e^{a\,\mathrm{diam}_s(X)} \|\Phi_X^{(s)}\|_\infty. \tag{2.2}$$

We also note that in the translation invariant case we can omit the supremum above. Note that the Banach space defined by the norm above is too large to have a satisfactory theory of high temperature phases. Indeed in [11, 12] Dobrushin and Martirosyan have shown the following: let $h : \mathbb{N} \to \mathbb{R}_+$ and set

$$\|\Phi^{(s)}\|_{\mathrm{DM}} := \sum_{X\ni 0} h(|X|)\, \|\Phi_X^{(s)}\|_\infty.$$

If $\exp\{-\gamma n\}h(n) \to 0$ in the limit $n \to \infty$ for all $\gamma > 0$, there exist complex interactions with arbitrarily small norm $\|\cdot\|_{\mathrm{DM}}$, giving rise to a phase transition in the sense that the corresponding partition function vanishes for a sequence of cubes $\Lambda_n \to \mathcal{L}^{(s)}$, see also [18].

Given $\Lambda \subset\subset \mathcal{L}^{(s)}$ and a potential $\Phi^{(s)}$ with bounded $\|\cdot\|_0$ norm, the *finite volume Hamiltonian* associated to a configuration $\sigma \in \mathcal{S}^{(s)}$ in Λ is given by:

$$H_\Lambda^{(s)}(\sigma) := \sum_{\substack{X\subset\subset\mathcal{L}^{(s)}\\ X\cap\Lambda\neq\emptyset}} \Phi_X^{(s)}(\sigma). \tag{2.3}$$

Note that the sum on the r.h.s. of (2.3) is absolutely convergent (uniformly in σ) by the boundedness of $\|\Phi^{(s)}\|_0$. We also remark that for a potential of range r the Hamiltonian depends only on $\sigma_{\overline{\Lambda}^{(s),r}}$, namely $H_\Lambda^{(s)} \in \mathcal{F}_{\overline{\Lambda}^{(s),r}}^{(s)}$. We also let $E_\Lambda^{(s)}(\sigma)$ be the self–interaction associated to the volume Λ, i.e. the Hamiltonian with free boundary conditions namely,

$$E_\Lambda^{(s)}(\sigma) := \sum_{X\subset\Lambda} \Phi_X^{(s)}(\sigma). \tag{2.4}$$

We have that the map $E_\Lambda^{(s)} : \mathcal{S}^{(s)} \to \mathbb{R}$ depends only on the spins inside Λ namely, $E_\Lambda^{(s)} \in \mathcal{F}_\Lambda^{(s)}$.

2.4. The Gibbs measures. Pick $s \geq 1$ and consider a potential $\Phi^{(s)}$ of bounded $\|\cdot\|_0$ norm. For each $\Lambda \subset\subset \mathcal{L}^{(s)}$ we define the (finite volume) Gibbs measure in Λ, with boundary condition $\tau \in \mathcal{S}^{(s)}$, as the following measure on $\mathcal{S}_\Lambda^{(s)}$:

$$\mu_\Lambda^{(s),\tau}(\sigma) := \frac{1}{Z_\Lambda^{(s)}(\tau)} \exp\left\{H_\Lambda^{(s)}(\sigma\tau_{\Lambda^c})\right\}$$

for any $\sigma \in \mathcal{S}_\Lambda^{(s)}$, where $Z_\Lambda^{(s)}(\tau)$, called *partition function*, is the normalization constant, i.e.

$$Z_\Lambda^{(s)}(\tau) := \sum_{\sigma\in\mathcal{S}_\Lambda^{(s)}} e^{H_\Lambda^{(s)}(\sigma\tau_{\Lambda^c})}. \tag{2.5}$$

Note that we defined the Gibbs measure with a sign convention opposite to the usual one and include the inverse temperature in the definition of the Hamiltonian; in fact it will be kept fixed in our analysis.

We regard $\mu_\Lambda^{(s),\tau}$ also as a measure on the whole $\mathcal{S}^{(s)}$ by giving zero mass to the configurations σ which do not agree with τ on Λ^c. The (infinite volume) Gibbs states associated to the potential $\Phi^{(s)}$ are then the probability measures $\mu^{(s)}$ on $\mathcal{S}^{(s)}$ which satisfy the DLR equations

$$\int \mu^{(s)}(d\tau)\, \mu_\Lambda^{(s),\tau}(f) = \mu^{(s)}(f) \qquad \text{for any } \Lambda \subset\subset \mathcal{L}^{(s)},\ f \in C(\mathcal{S}^{(s)}), \tag{2.6}$$

where $\mu^{(s)}(f)$ denotes the expectation of f w.r.t. $\mu^{(s)}$.

Given two local functions $f, g : \mathcal{S}^{(s)} \to \mathbb{R}$ we denote, finally, by $\mu^{(s)}(f; g) := \mu^{(s)}(fg) - \mu^{(s)}(f)\mu^{(s)}(g)$ the covariance between f and g.

Condition $\mathbf{SM}^{(s)}(\ell_0, b, B)$ (Strong Mixing). *Given a positive integer ℓ_0 and two strictly positive reals b, B we say that the potential $\Phi^{(s)}$ satisfies $SM^{(s)}(\ell_0, b, B)$ if and only if for any volume $I \subset\subset \mathcal{L}^{(s\ell_0)}$ by setting $\Lambda := \mathcal{O}_{s\ell_0}^s I = \bigcup_{i\in I} Q_{\ell_0}^{(s)}(i)$, the following bound holds. For each pair of local functions f, g such that* $\mathrm{supp}(f) \subset \Lambda$, $\mathrm{supp}(g) \subset \Lambda$, *and* $|\mathrm{supp}(f)| \wedge |\mathrm{supp}(g)| \exp\{-b\mathrm{d}_s(\mathrm{supp}(f), \mathrm{supp}(g))\} \le 1$ *we have*

$$\sup_{\tau\in\mathcal{S}^{(s)}} |\mu_\Lambda^{(s),\tau}(f; g)| \le B\|f\|_\infty\|g\|_\infty |\mathrm{supp}(f)| \wedge |\mathrm{supp}(g)|\, e^{-b\,\mathrm{d}_s(\mathrm{supp}(f),\mathrm{supp}(g))}. \tag{2.7}$$

It is a standard result that if there exist ℓ_0, b, B such that Condition $\mathrm{SM}^{(s)}(\ell_0, b, B)$ is satisfied then there exists a unique infinite volume Gibbs state namely, the DLR equations (2.6) admit a unique solution.

2.5. Lattice gas potential. A lattice gas is a translational invariant Gibbs field in the case $s = 1$, which is then dropped from the notation, and $S_x = 1$ for each $x \in \mathcal{L}$; in such a case the single site configuration space associated to $x \in \mathcal{L}$ is $\mathcal{X}_x := \{0, 1\}$. With the notation introduced in Subsect. 2.2, for each $\Lambda \subset \mathcal{L}$, we denote by $\mathcal{X}_\Lambda$ the configuration space on Λ equipped with the product topology and, as usual, we let $\mathcal{X} := \mathcal{X}_\mathcal{L}$. For $\eta \in \mathcal{X}$ the value $\eta_x \in \{0, 1\}$ is interpreted as the occupation number in $x \in \mathcal{L}$. Moreover, we denote by $\mathcal{F}$ the Borel σ–algebra on $\mathcal{X}$ and set $\mathcal{F}_\Lambda := \{\eta_x \in \mathcal{X}_x,\ x \in \mathcal{L}\} \subset \mathcal{F}$.

For a translationally invariant lattice gas we denote by U the potential and observe that $U_{\{x\}}(\eta) = \lambda\eta_x + a$ for some constants $\lambda, a \in \mathbb{R}$. We neglect the constant a, which does not affect the definition of the Gibbs measure, and note that λ is interpreted as the *chemical potential*. We also introduce the *activity* $z \in \mathbb{R}_+$ by $z := e^\lambda$ which we use to parameterize lattice gases with different chemical potentials. In such a case we write $U = (z, U_{>1})$, where $U_{>1} := \{U_X \in \mathcal{F}_X,\ X \subset\subset \mathcal{L}, |X| > 1\}$ and call $U_{>1}$ the interaction.

Coherently with the notation introduced in Subsect. 2.4 the infinite volume Gibbs measure is denoted by μ. We shall sometimes write μ_z for the infinite volume Gibbs measure, $\mu^\tau_{\Lambda,z}$ for the finite volume Gibbs measure on $\Lambda \subset\subset \mathcal{L}$, and $Z_{\Lambda,z}(\tau)$ for the partition function of the lattice gas in order to explicitly indicate the dependence on the activity z.

2.6. Block averaging transformation (BAT). Let μ be the (unique) infinite volume Gibbs measure of a finite range translationally invariant lattice gas satisfying Condition SM(ℓ_0, b, B). Let $\rho := \mu(\eta_0)$ be the equilibrium density and let us denote the compressibility by

$$\chi := \sum_{x\in\mathcal{L}} \mu(\eta_0; \eta_x). \tag{2.8}$$

Note that SM(ℓ_0, b, B) implies that there exists a real number $C \in (0, +\infty)$ such that $C^{-1} \le \chi \le C$.

We consider a positive integer ℓ and the *renormalized lattice* $\mathcal{L}^{(\ell)}$. For $I \subset \mathcal{L}^{(\ell)}$ we define the function $N^{(\ell)}_I : \mathcal{X}_I \to \{0, 1, \ldots, \ell^d\}^I$, which counts the total number of particles in each block $Q_\ell(i)$, as follows

$$\big(N^{(\ell)}(\eta)\big)_i := \sum_{x\in Q_\ell(i)} \eta_x \tag{2.9}$$

for all $i \in I$ and $\eta \in \mathcal{X}_I$. As usual we will let $N^{(\ell)}_{\mathcal{L}^{(\ell)}} =: N^{(\ell)}$ and $N^{(\ell)}_{\{i\}} =: N^{(\ell)}_i$ for all $i \in \mathcal{L}^{(\ell)}$.

For any $i \in \mathcal{L}^{(\ell)}$ we define, moreover, the set

$$\mathcal{M}^{(\ell)}_i := \left\{ \frac{-\rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}}, \frac{1-\rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}}, \ldots, \frac{|Q_\ell|(1-\rho)}{\sqrt{|Q_\ell|\chi}} \right\} \tag{2.10}$$

that we consider equipped with the discrete topology. For $I \subset \mathcal{L}^{(\ell)}$ we introduce, following the notation in Subsect. 2.2, the *renormalized configuration space* $\mathcal{M}^{(\ell)}_I := \otimes_{i\in I}\mathcal{M}^{(\ell)}_i$; we set $\mathcal{M}_{\mathcal{L}^{(\ell)}} =: \mathcal{M}^{(\ell)}$ and denote by $\mathcal{B}^{(\ell)}$ its Borel σ–algebra. For $I \subset \mathcal{L}^{(\ell)}$ we also set $\mathcal{B}^{(\ell)}_I := \sigma\{m_i \in \mathcal{M}^{(\ell)}_i,\ i \in I\} \subset \mathcal{B}^{(\ell)}$. Moreover we define the measurable function $M^{(\ell)}_I : (\mathcal{X}_{\mathcal{O}_\ell I}, \mathcal{F}_{\mathcal{O}_\ell I}) \longrightarrow (\mathcal{M}^{(\ell)}_I, \mathcal{B}^{(\ell)}_I)$ by setting

$$M^{(\ell)}_I(\eta) := \frac{N^{(\ell)}_I(\eta) - \rho|Q_\ell|}{\sqrt{|Q_\ell|\chi}} \tag{2.11}$$

for $\eta \in \mathcal{X}_{\mathcal{O}_\ell I}$. We also let $M^{(\ell)}_{\mathcal{L}^{(\ell)}} =: M^{(\ell)}$ and $M^{(\ell)}_{\{i\}} =: M^{(\ell)}_i$ for all $i \in \mathcal{L}^{(\ell)}$.

Finally, we define the *renormalized measure* $\nu^{(\ell)} := \mu \circ (M^{(\ell)})^{-1}$, which is naturally induced by $M^{(\ell)}$ on $\mathcal{M}^{(\ell)}$, and for $I \subset \mathcal{L}^{(\ell)}$ and $\tau \in \mathcal{X}$ we let $\nu_I^{(\ell),\tau} := \mu^\tau_{\mathcal{O}_\ell I} \circ (M_I^{(\ell)})^{-1}$. We avoid the troublesome issue of describing Gibbs measures on non–compact single spin space, see [19] for a discussion, and consider $\nu^{(\ell)}$ only for finite ℓ.

2.7. Main results. In this subsection we state the main theorems on the weak Gibbsianity of the renormalized measure and the corresponding convergence as $\ell \to \infty$.

Theorem 2.1. *Let U be a lattice gas potential satisfying SM(ℓ_0, b, B). Then for any large enough ℓ multiple of ℓ_0 there exists a family of functions $\{\psi_I^{(\ell)}, \phi_I^{(\ell)},\ I \subset\subset \mathcal{L}^{(\ell)}\}$, with $\psi_I^{(\ell)}, \phi_I^{(\ell)} : \mathcal{M}^{(\ell)} \mapsto \mathbb{R}$, such that*

1. *For each $I \subset\subset \mathcal{L}^{(\ell)}$ we have $\psi_I^{(\ell)}, \phi_I^{(\ell)} \in \mathcal{B}_I^{(\ell)}$.*
2. *For each $I \subset\subset \mathcal{L}^{(\ell)}$ we have $\psi_I^{(\ell)}, \phi_I^{(\ell)} \equiv 0$ if I is not ℓ–connected.*
3. *The functions $\psi_I^{(\ell)}, \phi_I^{(\ell)}$ are translationally invariant in the sense specified in Subsect. 2.3 namely, $\psi_I^{(\ell)}(m) = \psi_{I-i}^{(\ell)}(\Theta_i m)$ and $\phi_I^{(\ell)}(m) = \phi_{I-i}^{(\ell)}(\Theta_i m)$ for any $i \in \mathcal{L}^{(\ell)}$, $I \subset\subset \mathcal{L}^{(\ell)}$, and $m \in \mathcal{M}^{(\ell)}$.*
4. *There exist a measurable set $\bar{\mathcal{M}}^{(\ell)} \subset \mathcal{M}^{(\ell)}$, such that $\nu^{(\ell)}(\bar{\mathcal{M}}^{(\ell)}) = 1$, and functions $r_i^{(\ell)} : \bar{\mathcal{M}}^{(\ell)} \mapsto \mathbb{N} \setminus \{0\}$, for all $i \in \mathcal{L}^{(\ell)}$, such that for each $m \in \bar{\mathcal{M}}^{(\ell)}$ if $I \ni i$ and $\mathrm{diam}_\ell(I) > r_i^{(\ell)}(m)$ then $\psi_I^{(\ell)}(m) = 0$. In particular for each $i \in \mathcal{L}^{(\ell)}$ and $m \in \bar{\mathcal{M}}^{(\ell)}$ there exists a real $c_i^{(\ell)}(m) \in [0, \infty)$ such that*
$$\sum_{I \ni i} |\psi_I^{(\ell)}(m)| \le c_i^{(\ell)}(m). \tag{2.12}$$
5. *For each $q \in [1, \infty)$*
$$\sup_{i \in \mathcal{L}^{(\ell)}} \nu^{(\ell)}\Big(\Big|\sum_{I \ni i} \psi_I^{(\ell)}\Big|^q\Big) < \infty. \tag{2.13}$$
6. *There exists $\alpha' > 0$ such that*
$$\sup_{m \in \mathcal{M}^{(\ell)}} \sup_{i \in \mathcal{L}^{(\ell)}} \sum_{I \ni i} e^{\alpha' \mathrm{diam}_\ell(I)} |\phi_I^{(\ell)}(m)| < \infty. \tag{2.14}$$
7. *The DLR equations hold, namely for each $J \subset\subset \mathcal{L}^{(\ell)}$ and for each local function $f \in \mathcal{B}_J^{(\ell)}$ we have*
$$\nu^{(\ell)}(f) = \int_{\mathcal{M}^{(\ell)}} \nu^{(\ell)}(dm') \frac{1}{Z_J(m')} \times \sum_{m \in \mathcal{M}_J^{(\ell)}} f(m) \exp\Big\{ \sum_{I \cap J \neq \emptyset} [\psi_I^{(\ell)}(m m'_{J^c}) + \phi_I^{(\ell)}(m m'_{J^c})]\Big\}, \tag{2.15}$$
where
$$Z_J(m') := \sum_{m \in \mathcal{M}_J^{(\ell)}} \exp\Big\{ \sum_{I \cap J \neq \emptyset} [\psi_I^{(\ell)}(m m'_{J^c}) + \phi_I^{(\ell)}(m m'_{J^c})]\Big\}.$$

In the next theorem we state that in a suitable sense the renormalized potential converges to the one of independent harmonic oscillators.

Theorem 2.2. *In the same hypotheses as in Theorem 2.1, the family* $\{\psi_I^{(\ell)}, \phi_I^{(\ell)}, I \subset\subset \mathcal{L}^{(\ell)}\}$ *is such that*

1. *For each* $i \in \mathcal{L}^{(\ell)}$ *we have* $\psi_{\{i\}}^{(\ell)}(m) = -m_i^2/2$ *and for each* $q \in [1, +\infty)$
$$\lim_{\ell\to\infty} \sup_{i\in\mathcal{L}^{(\ell)}} \nu^{(\ell)}\Big(\Big|\sum_{I\ni i:\,|I|\geq 2} \psi_I^{(\ell)}\Big|^q\Big) = 0. \tag{2.16}$$
2. *There exists* $\alpha' > 0$ *such that*
$$\lim_{\ell\to\infty} \sup_{m\in\mathcal{M}^{(\ell)}} \sup_{i\in\mathcal{L}^{(\ell)}} \sum_{I\ni i} e^{\alpha' \operatorname{diam}_\ell(I)} |\phi_I^{(\ell)}(m)| = 0, \tag{2.17}$$

where the limits $\ell \to \infty$ *are taken along multiples of* ℓ_0.

In order to compute the renormalized potential we compute the partition function of the constrained system (1.5). Since we are in the low temperature regime, the constrained system will not have good mixing properties *for all* possible values of the image variable $m \in \mathcal{M}^{(\ell)}$. We thus look at the constrained model as a disordered system and look for properties which hold for $\nu^{(\ell)}$–*almost all* image variables, that is we have to construct the set $\bar{\mathcal{M}}^{(\ell)}$ properly. More precisely, we enclose the constrained models in a huge volume and we try to compute their partition function via a uniform convergent cluster expansion. We then face the typical problem of the Griffiths' phase in disordered systems: anomalous values of the image variables, which do occur somewhere in our volume, might produce arbitrarily large regions of strong interaction.

To overcome the above difficulty we follow a classical strategy in disordered systems. Let us fix a configuration of the image variables. We first perform a cluster expansion in the domains where the constrained model verifies a uniform strong mixing condition implying an effective weak interaction on a proper scale. We are then left with an effective residual interaction between the domains of strong interaction. Since anomalous values of the image variables have small probability, in the set $\bar{\mathcal{M}}^{(\ell)}$ the strong interacting domains are well separated on the lattice; we can thus use the graded cluster expansion developed in [2] to treat the effective interaction.

2.8. Synopsis. In Sect. 3 we construct the full measure set $\bar{\mathcal{M}}^{(\ell)}$ algorithmically. The required probability estimates are proven in a general setting, not necessarily Gibbsian, of the underlying distribution of the disorder. The analysis is based on the exponential decay of correlations and the key recursive estimate in Lemma 3.6 is inspired by the approach to the Anderson localization in [9].

In Sect. 4 we define the constrained models. We also introduce the condition, see (4.8), on the image variable m to identify the good part of the lattice where the constrained systems satisfy the uniform strong mixing condition. We finally prove, in Theorem 4.4, that the general theory developed in Sect. 3 can be applied.

We then fix a value $m \in \bar{\mathcal{M}}^{(\ell)}$ and compute the partition function of the corresponding constrained model for a sequence of volumes invading the whole lattice. More precisely in Sect. 5 we use a procedure similar to [1,27,28] to integrate over the good part of the lattice and get the expansion in Theorem 5.1. In Sect. 6 we feed the effective potential of Theorem 5.1 to the general theory developed in [2] to integrate over the bad part of the lattice.

To complete the proof of Theorem 2.1 we need to express the output of [2], see Theorem 6.1, as the sum of local functions. This is not a trivial point since, a priori, the partition function of the constrained models depends on the whole infinite volume image variable m. Nevertheless, the graded cluster expansion in [2] has been developed with a volume cutoff, see (6.7) and (6.14), at each step of the iteration, so that the recursive construction allows us to prove locality of the renormalized potentials, see Theorem 6.2. The convergence stated in Theorem 2.2 is an easy byproduct of the whole analysis.

3. Bounds on the Badness Probability

To compute the partition function of the *constrained models*, see Sect. 4 below, we face the typical problem of the Griffiths' phase in disordered systems: anomalous values of the image variables, which do occur somewhere, might produce arbitrarily large regions of strong correlation [2]. In the present section we obtain some probability estimates on the multi–scale geometry of these regions, based on the hypotheses that such anomalous values have small probability.

In this section we denote the lattice $\mathbb{Z}^d$ by $\mathbb{L}$. In order to use a setup compatible with the one in [2], we use the distance $\mathrm{D}(x, y) := \sum_{i=1}^{d} |x_i - y_i|$ for all $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbb{L}$; on the other hand we recall that $\mathrm{d}(x, y) = \sup_{i=1,\dots,d} |x_i - y_i|$ as defined in Subsect. 2.1. Accordingly for $X \subset \mathcal{L}$ we set $\mathrm{Diam}(X) := \sup\{\mathrm{D}(x, y) : x, y \in X\}$. Moreover, given $X \subset \mathbb{L}$ and $m \geq 0$ we set $O_m(X) := \{y \in \mathbb{L} : \mathrm{D}(y, X) \leq m\}$ as the *m–neighborhood* of X w.r.t the metric D. If $x \in \mathbb{L}$ we write $O_m(x)$ instead of $O_m(\{x\})$. We also recall that $B_m(X)$, see Subsect. 2.1, is the m–neighborhood of X w.r.t. the metric d; of course $B_m(X) \supset O_m(X)$ for all $X \subset \mathbb{L}$.

We describe the strength of the disorder at the site x in terms of a binary variable $\omega_x \in \{0, 1\}$. We denote by $\omega \in \Omega := \{0, 1\}^{\mathbb{L}}$ the random field $\{\omega_x,\ x \in \mathbb{L}\}$; we consider Ω endowed with its Borel σ–algebra $\mathcal{A}$ and we let $\mathbb{Q}$, a probability in Ω, be the distribution of ω. We also introduce the family of σ–algebras on Ω defined by $\mathcal{A}_\Lambda := \sigma\{\omega_x,\ x \in \Lambda\}$. We measure the diluteness of the system via the parameter

$$p := \sup_{x\in\mathbb{L}} \mathbb{Q}(\omega_x = 1) \tag{3.1}$$

which, in our analysis, will be sufficiently small. We also assume that the correlations under $\mathbb{Q}$ are exponentially decaying; more precisely we assume there exist reals $b'' > 0$ and $B'' < \infty$ such that for each pair of local functions f, g such that $|\operatorname{supp}(f)| \wedge |\operatorname{supp}(g)| \exp\{-b''\mathrm{D}(\operatorname{supp}(f), \operatorname{supp}(g))\} \leq 1$ we have

$$|\mathbb{Q}(f; g)| \leq B'' \|f\|_\infty \|g\|_\infty |\operatorname{supp}(f)| \wedge |\operatorname{supp}(g)|\, e^{-b''\mathrm{D}(\operatorname{supp}(f),\operatorname{supp}(g))}. \tag{3.2}$$

We have a first classification of sites in *good* (where $\omega_x = 0$) and *bad* (where $\omega_x = 1$). We strengthen the notion of *steep scales* introduced in [2].

Definition 3.1. *We say that two strictly increasing sequences* $\Gamma = \{\Gamma_j\}_{j\geq 0}$ *and* $\gamma = \{\gamma_j\}_{j\geq 0}$ *are* **moderately steep scales** *iff they satisfy the following conditions:*

1. $\Gamma_0 = 0$, $\gamma_0 \geq 0$, $\Gamma_1 \geq 2$, *and* $\Gamma_j < \gamma_j/2$ *for any* $j \geq 1$.
2. *For* $j \geq 0$ *set* $\vartheta_j := \sum_{i=0}^{j} (\Gamma_i + \gamma_i)$ *and* $\lambda := \inf_{j\geq 0}(\Gamma_{j+1}/\vartheta_j)$*; then* $\lambda \geq 10$.

3. *We have* $\sum_{j=1}^{\infty} \frac{\Gamma_j}{\gamma_j} \leq \frac{1}{2}$, *where we understand* $\Gamma_0/\gamma_0 = 0$ *even in the case* γ_0.
4. *There exist reals* $a > 0$ *and* $\varepsilon \in (0, 1)$ *such that* $2 \cdot 3^d \vartheta_{k+1}^d \leq \exp\{a(1+\varepsilon)^k\}$ *for all* $k \geq 0$.
5. *For* a *and* ε *as above we have that* $\sum_{k=1}^{\infty} \vartheta_k^s \exp\{-a(1+\varepsilon)^k/q\} < \infty$ *for all* $s \geq 0$ *and* $q > 1$.

We remark that with respect to the definition of steep scales given in [2] we have added Conditions 4 and 5, and strengthened Item 2 to $\lambda \geq 10$. An explicit example of moderately steep scales is given in (4.19) below.

Definition 3.2. *We say that* $\mathcal{G} := \{\mathcal{G}_j\}_{j\geq 0}$, *where each* $\mathcal{G}_j$ *is a collection of finite subsets of* $\mathbb{L}$ *is a* **graded disintegration of** $\mathbb{L}$ *iff*

1. *for each* $g \in \bigcup_{j\geq 0} \mathcal{G}_j$ *there exists a unique* $j \geq 0$, *which is called the* **grade** *of* g, *such that* $g \in \mathcal{G}_j$;
2. *the collection* $\bigcup_{j\geq 0} \mathcal{G}_j$ *of finite subsets of* $\mathbb{L}$ *is a partition of the lattice* $\mathbb{L}$ *namely, it is a collection of not empty pairwise disjoint finite subsets of* $\mathbb{L}$ *such that*

$$\bigcup_{j\geq 0} \bigcup_{g\in\mathcal{G}_j} g = \mathbb{L}. \tag{3.3}$$

Given $\mathbb{G}_0 \subset \mathbb{L}$ *and* Γ, γ *steep scales, we say that a graded disintegration* $\mathcal{G}$ *is a* gentle disintegration *of* $\mathbb{L}$ *with respect to* $\mathbb{G}_0, \Gamma, \gamma$ *iff the following recursive conditions hold:*

3. $\mathcal{G}_0 = \big\{\{x\},\ x \in \mathbb{G}_0\big\}$;
4. *if* $g \in \mathcal{G}_j$ *then* $\mathrm{Diam}(g) \leq \Gamma_j$ *for any* $j \geq 1$;
5. *set* $\mathbb{G}_j := \bigcup_{g\in\mathcal{G}_j} g \subset \mathbb{L}$, $\mathbb{B}_0 := \mathbb{L} \setminus \mathbb{G}_0$ *and* $\mathbb{B}_j := \mathbb{B}_{j-1} \setminus \mathbb{G}_j$, *then for any* $g \in \mathcal{G}_j$ *we have* $\mathrm{D}\big(g, \mathbb{B}_{j-1} \setminus g\big) > \gamma_j$ *for any* $j \geq 1$;
6. $\forall x \in \mathbb{L}$ *we have* $k_x := \sup\big\{j \geq 1 : \exists g \in \mathcal{G}_j \text{ such that } \mathrm{d}(x, \mathcal{Q}(g)) \leq \vartheta_j\big\} < \infty$, *where we recall* $\mathcal{Q}(g)$ *has been defined in Subsect. 2.1.*

Sites in $\mathbb{G}_0$ *(resp.* $\mathbb{B}_0$*) are called* **good** *(resp.* **bad***); similarly we call* j*–gentle (resp.* j*–bad) the sites in* $\mathbb{G}_j$ *(resp.* $\mathbb{B}_j$*). Elements of* $\mathcal{G}_j$, *with* $j \geq 1$, *are called* j*–gentle atoms. Finally, we set* $\mathcal{G}_{\geq j} := \bigcup_{i\geq j} \mathcal{G}_i$.

The results of the present section are summarized in the following theorem.

Theorem 3.3. *Let the sequences* Γ, γ *be moderately steep scales in the sense of Definition 3.1. Assume also that (3.2) holds,* $p \leq \exp\{-a/(1-\varepsilon)\}$ *and the sequences* Γ, γ *are such that*

$$2 \cdot 9^d B'' \vartheta_{k+1}^{2d} \exp\{-b''\Gamma_{k+1}/20\} \leq \exp\left\{-\frac{a}{1-\varepsilon}(1+\varepsilon)^{k+1}\right\} \tag{3.4}$$

for any $k \geq 0$. *Set finally* $\mathbb{G}_0(\omega) := \{x \in \mathbb{L} : \omega_x = 0\}$. *Then there exists a set* $\bar{\Omega} \in \mathcal{A}$ *with* $\mathbb{Q}(\bar{\Omega}) = 1$ *such that*

1. *for each* $\omega \in \bar{\Omega}$ *there exists a gentle disintegration* $\mathcal{G} = \mathcal{G}(\omega)$ *in the sense of Definition 3.2;*

2. *for each* $x \in \mathbb{L}$ *and* $X \subset\subset \mathbb{L}$ *we have that* $\{\omega : \mathbb{G}_k(\omega) \ni x\} \in \mathcal{A}_{O_{\vartheta_k}(x)}$ *and also* $\{\omega : \mathcal{G}_k(\omega) \ni X\} \in \mathcal{A}_{O_{\vartheta_k}(X)}$.

Let us first describe an algorithm to construct the families $\mathcal{G}_k$ for $k \geq 1$; from this it will follow Item 2 in Theorem 3.3. Given a configuration $\omega \in \Omega$ and Γ, γ *moderately steep scales*, we define the following inductive procedure in a finite volume $\Lambda \subset\subset \mathbb{L}$ which constructs the *k–gentle* sites in Λ:

1. set $k = 1$;
2. set $i = 1$ and $V = \emptyset$;
3. if $(\mathbb{B}_{k-1} \cap \Lambda) \setminus V = \emptyset$ then goto 7;
4. pick a point $x \in (\mathbb{B}_{k-1} \cap \Lambda) \setminus V$, set $A = O_{\Gamma_k}(x) \cap \mathbb{B}_{k-1}$ and $V = V \cup A$;
5. if $\mathrm{Diam}(A) \leq \Gamma_k$ and $\mathrm{D}(A, \mathbb{B}_{k-1} \setminus A) > \gamma_k$ then $g_k^i = A$ and $i = i + 1$;
6. goto 3;
7. set $\mathcal{G}_k := \{g_k^m,\ m = 1, \ldots, i-1\}$, with the convention $\mathcal{G}_k = \emptyset$ if $i = 1$, $\mathbb{G}_k := \bigcup_{m=1}^{i-1} g_k^m$, and $\mathbb{B}_k := \mathbb{B}_{k-1} \setminus \mathbb{G}_k$;
8. set $k = k + 1$, if $\Gamma_k \leq \mathrm{Diam}(\Lambda)$ goto 2 else exit.

Let us briefly describe what the above algorithm does. At step k we have inductively constructed $\mathbb{B}_{k-1}$, the set of $(k-1)$–bad sites; we stress that sites in $\mathbb{L} \setminus \Lambda$ may belong to $\mathbb{B}_{k-1}$. Among the sites in $\mathbb{B}_{k-1} \cap \Lambda$ we are now looking for the k–gentle ones. The set V is used to keep track of the sites tested for k–gentleness. At Step 4 we pick a new site $x \in \mathbb{B}_{k-1} \cap \Lambda$ and test it, at Step 5, for k–gentleness w.r.t. $\mathbb{B}_{k-1}$, i.e. including also bad sites in $\mathbb{L} \setminus \Lambda$. Note that the families $\mathcal{G}_k$ for any $k \geq 1$ do not depend on the way in which x is chosen at Step 4 of the algorithm. Suppose, indeed, to choose $x \in (\mathbb{B}_{k-1} \cap \Lambda) \setminus V$ at Step 4 and to find that $A = O_{\Gamma_k}(x) \cap \mathbb{B}_{k-1}$ is a k–gentle atom. Consider $x' \in A$ such that $x' \neq x$ and set $A' := O_{\Gamma_k}(x') \cap \mathbb{B}_{k-1}$: since A satisfies the test for k–gentleness at Step 5 of the algorithm, we have $A \subset A'$. By changing the role of x and x' we get $A = A'$.

After a finite number of operations, the algorithm stops and outputs the family $\mathcal{G}_k(\Lambda)$ (note we wrote explicitly the dependence on Λ) with the following property. If $g \in \mathcal{G}_k(\Lambda)$ then $\mathrm{Diam}(g) \leq \Gamma_k$ and $\mathrm{D}(g, \mathbb{B}_{k-1}(\Lambda) \setminus g) > \gamma_k$. We call a set $g \in \mathcal{G}_k(\Lambda)$ an atom of *k–gentle* sites; note however that g is not necessarily connected.

We finally take an increasing sequence of sets $\Lambda_i \subset\subset \mathbb{L}$, invading $\mathbb{L}$ and we sequentially perform the above algorithm. This means the algorithm for Λ_i is performed independently of the outputs previously obtained, i.e., for Λ_j $j < i$. It is easy to show that if $g \in \mathcal{G}_k(\Lambda_i)$ then $g \in \mathcal{G}_k(\Lambda_{i+1})$; therefore $\mathcal{G}_k(\Lambda_i)$ is increasing in $i \geq 1$, so that we can define $\mathcal{G}_k := \lim_{i\to\infty} \mathcal{G}_k(\Lambda_i) = \bigcup_i \mathcal{G}_k(\Lambda_i)$ and $\mathbb{G}_k := \lim_{i\to\infty} \mathbb{G}_k(\Lambda_i) = \bigcup_{g\in\mathcal{G}_k} g$. Hence, $\mathbb{B}_k(\Lambda_i) = \mathbb{B}_{k-1}(\Lambda_i) \setminus \mathbb{G}_k(\Lambda_i) = \mathbb{L} \setminus \cup_{j=0}^{k-1} \mathbb{G}_j(\Lambda_i)$ is decreasing in $i \geq 1$, so that $\mathbb{B}_k := \lim_{i\to\infty} \mathbb{B}_k(\Lambda_i) = \bigcap_i \mathbb{B}_k(\Lambda_i)$. We also remark that, by construction, $\{\mathbb{B}_k,\ k \geq 0\}$ is a decreasing sequence. We say $x \in \mathbb{L}$ is *k–gentle* (resp. *k–bad*) iff $x \in \mathbb{G}_k$ (resp. $x \in \mathbb{B}_k$).

Note that it follows from the construction that it is possible to decide whether a site x at step k is k–gentle by looking only at the ω's inside a ball centered at x of radius ϑ_k (as defined in Item 2 of Definition 3.1).

Lemma 3.4. *Let* $\mathbb{G}_k$ *and* $\mathcal{G}_k$, $k = 0, 1, \ldots$, *as constructed above. Then Item 2 in Theorem 3.3 holds, i.e. for each* $x \in \mathbb{L}$,

$$\{\omega : x \in \mathbb{G}_k(\omega)\} \in \mathcal{A}_{O_{\vartheta_k}(x)} \tag{3.5}$$

and for each $X \subset \mathbb{L}$

$$\{\omega : \ X \in \mathcal{G}_k(\omega)\} \in \mathcal{A}_{O_{\vartheta_k}(X)}. \tag{3.6}$$

Proof. We first prove (3.5). We proceed by induction. For $k = 0$ (3.5) holds trivially. Let $O := O_{\Gamma_k}(x)$. From the algorithmic construction above we have

$$\begin{aligned}\{x \in \mathbb{G}_k\} = \{x \in \mathbb{B}_{k-1}\} \cap \{\mathrm{Diam}\,(O \cap \mathbb{B}_{k-1}) \le \Gamma_k\} \\ \cap \{\mathrm{D}\,(\mathbb{B}_{k-1} \cap O, \mathbb{B}_{k-1} \setminus O) > \gamma_k\}\,.\end{aligned} \tag{3.7}$$

Since ϑ_k is increasing, by the inductive hypotheses

$$\{x \in \mathbb{B}_{k-1}\} = \bigcap_{h=0}^{k-1} \{x \notin \mathbb{G}_h\} \in \bigvee_{h=0}^{k-1} \mathcal{A}_{O_{\vartheta_h}(x)} = \mathcal{A}_{O_{\vartheta_{k-1}}(x)}. \tag{3.8}$$

On the other hand

$$\{\mathrm{Diam}\,(O \cap \mathbb{B}_{k-1}) \le \Gamma_k\} = \bigcap_{\substack{y,z \in O \\ \mathrm{D}(y,z) > \Gamma_k}} (\{y \notin \mathbb{B}_{k-1}\} \cup \{z \notin \mathbb{B}_{k-1}\}) \in \bigvee_{y \in O} \mathcal{A}_{O_{\vartheta_{k-1}}(y)}, \tag{3.9}$$

where we used (3.8). Finally

$$\begin{aligned}\{\mathrm{D}\,(\mathbb{B}_{k-1} \cap O, \mathbb{B}_{k-1} \setminus O) > \gamma_k\} &= \{\nexists (y,z) \in (O \cap \mathbb{B}_{k-1}) \times (\mathbb{B}_{k-1} \setminus O) : \mathrm{D}(y,z) \le \gamma_k\} \\ &= \bigcap_{\substack{y \in O, z \in \mathbb{L} \setminus O: \\ \mathrm{D}(y,z) \le \gamma_k}} (\{y \notin \mathbb{B}_{k-1}\} \cup \{z \notin \mathbb{B}_{k-1}\})\,,\end{aligned}$$

hence

$$\{\mathrm{D}\,(\mathbb{B}_{k-1} \cap O, \mathbb{B}_{k-1} \setminus O) > \gamma_k\} \in \bigvee_{\substack{y \in \mathbb{L}: \\ \mathrm{D}(x,y) \le \Gamma_k + \gamma_k}} \mathcal{A}_{O_{\vartheta_{k-1}}(y)}, \tag{3.10}$$

where we used again (3.8). Recalling $\vartheta_k = \vartheta_{k-1} + \Gamma_k + \gamma_k$, Eq. (3.5) follows from (3.7)–(3.10).

Similar arguments can be used to prove (3.6). If $\mathrm{Diam}(X) > \Gamma_k$ then $X \notin \mathcal{G}_k$. We consider, now, the case $\mathrm{Diam}(X) \le \Gamma_k$; we have:

$$\begin{aligned}\{X \in \mathcal{G}_k\} &= \{x \in \mathbb{B}_{k-1},\ \forall x \in X\} \cap \{\mathrm{D}\,(X, \mathbb{B}_{k-1} \setminus X) > \gamma_k\} \\ &= \bigcap_{x \in X} \{x \in \mathbb{B}_{k-1}\} \cap \{\mathrm{D}\,(X, \mathbb{B}_{k-1} \setminus X) > \gamma_k\}\,.\end{aligned} \tag{3.11}$$

Now, from (3.8) we have that

$$\bigcap_{x \in X} \{x \in \mathbb{B}_{k-1}\} \in \mathcal{A}_{O_{\vartheta_{k-1}}(X)}. \tag{3.12}$$

Moreover,

$$\begin{aligned}\{\mathrm{D}\,(X, \mathbb{B}_{k-1} \setminus X) > \gamma_k\} &= \{\nexists (x,y) \in X \times (\mathbb{B}_{k-1} \setminus X) :\ \mathrm{D}(x,y) \le \gamma_k\} \\ &= \bigcap_{y \in \mathbb{L} \setminus X : \mathrm{D}(y,X) \le \gamma_k} \{y \notin \mathbb{B}_{k-1}\} \in \mathcal{A}_{O_{\vartheta_{k-1}+\gamma_k}(X)},\end{aligned} \tag{3.13}$$

and from (3.11)–(3.13) we finally get (3.6). □

Theorem 3.5. *Let the hypotheses of Theorem 3.3 be satisfied; recall a and ε have been defined in Item 4 of Definition 3.1. Then*

$$\sup_{x\in\mathbb{L}} \mathbb{Q}\,(\omega\ :\ x \in \mathbb{B}_k) \le \exp\Big\{-\frac{a}{1-\varepsilon}(1+\varepsilon)^k\Big\}. \tag{3.14}$$

Postponing the proof of the above bound, let us show how it implies, via a straightforward application of Borel–Cantelli lemma, Theorem 3.3.

Proof of Theorem 3.3. Proof of Item 1. For each $\omega \in \Omega$ let $\mathcal{G} \equiv \mathcal{G}(\omega) = \{\mathcal{G}_j(\omega)\}_{j\ge 0}$ be constructed by the algorithm described below Theorem 3.3. We have to show that $\mathcal{G}$ satisfies Items 1–6 in Definition 3.2 $\mathbb{Q}$–a.s.: Items 1, 3, 4 and 5 hold by construction. We prove first that there exists a set $\bar{\Omega} \subset \Omega$ of full $\mathbb{Q}$–measure such that Item 2 (of Definition 3.2) holds, namely such that $\bigcup_{j\ge 0}\mathcal{G}_j(\omega)$ is a partition of the lattice $\mathbb{L}$. Let $\mathbb{B}_\infty$ be the random subset of the lattice given by $\mathbb{B}_\infty := \lim_{n\to\infty}\mathbb{B}_n = \bigcap_{n=0}^{\infty}\mathbb{B}_n$. From Theorem 3.5 and the Borel–Cantelli lemma we get

$$0 = \mathbb{Q}\left(\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{\infty}\{x\in\mathbb{B}_k\}\right) = \mathbb{Q}\left(\bigcap_{n=0}^{\infty}\{x\in\mathbb{B}_n\}\right) = \mathbb{Q}\left(\{x\in\mathbb{B}_\infty\}\right), \tag{3.15}$$

where we have used that $\mathbb{B}_n,\ n\in\mathbb{N}$, is a decreasing family of subsets of the lattice. Whence, by taking a countable union, we get

$$0 = \mathbb{Q}\left(\bigcup_{x\in\mathbb{L}}\{x\in\mathbb{B}_\infty\}\right) = \mathbb{Q}\,(\mathbb{B}_\infty \neq \emptyset) = 1 - \mathbb{Q}\left(\mathbb{L} = \bigcup_{j=0}^{\infty}\mathbb{G}_j\right).$$

We prove, finally, that also Item 6 of Definition 3.2 is satisfied $\mathbb{Q}$–a.s.: it is enough to note that for $x \in \mathbb{L}$ we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{Q}\,(\omega:\ \exists g\in\mathcal{G}_k(\omega):\ \mathrm{d}(x,\mathcal{Q}(g))\le\vartheta_k)\\
&\quad\le \sum_{k=1}^{\infty}\mathbb{Q}\,(\omega:\ \exists y\in\mathbb{B}_{k-1}(\omega):\ \mathrm{d}(x,y)\le\vartheta_k+\Gamma_k)\\
&\quad\le \sum_{k=1}^{\infty}[2(\vartheta_k+\Gamma_k)+1]^d\sup_{y\in\mathbb{L}}\mathbb{Q}\,(\omega:\ y\in\mathbb{B}_{k-1}(\omega))\\
&\quad\le \sum_{k=1}^{\infty}[2(\vartheta_k+\Gamma_k)+1]^d\exp\Big\{-\frac{a}{1-\varepsilon}(1+\varepsilon)^{k-1}\Big\}\\
&\quad\le \sum_{k=1}^{\infty}[2(\vartheta_k+\Gamma_k)+1]^d\exp\Big\{-a(1+\varepsilon)^{k}\Big\}<\infty,
\end{aligned} \tag{3.16}$$

where we used the bound (3.14) and Item 5 in Definition 3.1. The proof is completed by applying again Borel–Cantelli.

Proof of Item 2 of the theorem. It has already been proven in Lemma 3.4 for the graded disintegration constructed via the algorithm described below Theorem 3.3. □

The key step in proving Theorem 3.5 is the following recursive estimate on the probability of the degree of badness.

Lemma 3.6. *Let the hypotheses of Theorem 3.3 be satisfied. Set* $p_k := \sup_{x\in\mathbb{L}} \mathbb{Q}\,(x \in \mathbb{B}_k)$*, for* $k = 0, 1, \ldots$*; set also* $A_k(x) := O_{\gamma_k+\Gamma_k}(x) \setminus O_{(\Gamma_k-1)/2}(x)$ *and* $|A_k| = |A_k(x)|$*. Then for each* $k = 0, 1, \ldots$ *we have*

$$p_{k+1} \le |A_{k+1}|\Big(p_k^2 + B''|O_{\vartheta_k}|\exp\{-b''\Gamma_{k+1}/20\}\Big). \tag{3.17}$$

Proof. Recalling the definition of the k–bad set $\mathbb{B}_k$ we have

$$\{x \in \mathbb{B}_{k+1}\} = \{x \in \mathbb{B}_k\} \cap \{x \notin \mathbb{G}_{k+1}\}\,. \tag{3.18}$$

On the other hand, by the construction of the $(k+1)$–gentle sites,

$$\{x \in \mathbb{B}_k\} \cap \{x \notin \mathbb{G}_{k+1}\} \subset \{x \in \mathbb{B}_k\} \cap \{\exists y \in A_{k+1}(x):\ y \in \mathbb{B}_k\} \tag{3.19}$$

indeed, given $\mathbb{B}_k$, if there were no k–bad site in the annulus $A_{k+1}(x)$ then x would have been $(k+1)$–gentle. From (3.18) and (3.19),

$$\begin{aligned}\mathbb{Q}\,(x \in \mathbb{B}_{k+1}) &\le \mathbb{Q}\Bigg(\bigcup_{y\in A_{k+1}(x)} \{x \in \mathbb{B}_k\} \cap \{y \in \mathbb{B}_k\}\Bigg)\\ &\le \sum_{y\in A_{k+1}(x)} \mathbb{Q}\big(\{x \in \mathbb{B}_k\} \cap \{y \in \mathbb{B}_k\}\big)\\ &= \sum_{y\in A_{k+1}(x)} \Big[\mathbb{Q}\,(\{x \in \mathbb{B}_k\})\,\mathbb{Q}\,(\{y \in \mathbb{B}_k\}) + \mathbb{Q}\,\big(\mathbb{1}_{\{x\in\mathbb{B}_k\}};\, \mathbb{1}_{\{y\in\mathbb{B}_k\}}\big)\Big].\end{aligned} \tag{3.20}$$

We note, now, that for $x \in \mathbb{L}$ and $y \in A_{k+1}(x)$ we have

$$\begin{aligned}\mathrm{D}(O_{\vartheta_k}(x), O_{\vartheta_k}(y)) &\ge \Big[\frac{\Gamma_{k+1}-1}{2}\Big] - 2\big[\vartheta_k\big] + 1 \ge \frac{\Gamma_{k+1}}{2} - 2\vartheta_k - \frac{1}{2}\\ &\ge \frac{\lambda-8}{4\lambda}\Gamma_{k+1} \ge \frac{1}{20}\Gamma_{k+1}\end{aligned} \tag{3.21}$$

recall we assumed $\lambda \ge 10$ in item 2 of Definition 3.1. By Lemma 3.4, (3.2), and (3.20), we finally get the bound (3.17). □

Proof of Theorem 3.5. The thesis follows by induction from $p_0 := p \le \exp\{-a/(1-\varepsilon)\}$, Lemma 3.6, item 4 in Definition 3.1, Eq. (3.4), $|A_{k+1}| \le 3^d\vartheta_{k+1}^d$, and $|O_{\vartheta_k}| \le 3^d\vartheta_{k+1}^d$. □

4. The Constrained Models

In dealing with the renormalization group transformation it is necessary to pack spins associated to different sites of the lattice so that a new variable, often called *block spin*, is obtained.

4.1. The block spin models. Recall the general setup introduced in Subsects. 2.2–2.4 for a spin model on lattice $\mathcal{L}^{(s)}$, with $s \geq 1$ integer, with potential $\Phi^{(s)}$, and Gibbs measure $\mu^{(s)}$. For u a positive multiple of s we consider the lattice $\mathcal{L}^{(u)}$ and associate to each site $i \in \mathcal{L}^{(u)}$ the *single site block spin configuration space*

$$\mathcal{S}_i^{(s),u} := \bigotimes_{x \in Q_{u/s}^{(s)}(i)} \mathcal{S}_x^{(s)} = \mathcal{S}^{(s)}_{Q_{u/s}^{(s)}(i)}. \tag{4.1}$$

We can then consider the *block spin configuration space* $\mathcal{S}_I^{(s),u} := \otimes_{i \in I} \mathcal{S}_i^{(s),u}$, for any $I \subset \mathcal{L}^{(u)}$, equipped with the product topology. As usual we let $\mathcal{S}_{\mathcal{L}^{(u)}}^{(s),u} =: \mathcal{S}^{(s),u}$ and denote its Borel σ–algebra by $\mathcal{F}^{(s),u}$. Moreover, for each $I \subset \mathcal{L}^{(u)}$ we set $\mathcal{F}_I^{(s),u} := \sigma\{\zeta_i \in \mathcal{S}_i^{(s),u},\ i \in I\} \subset \mathcal{S}^{(s),u}$.

As for the lattices we introduce operators which allow to pack spins and unpack block spins. With an abuse of notation we shall use the same symbol as in Subsect. 2.1. We define the *packing* operator $\mathcal{O}_s^u : \mathcal{S}^{(s)} \to \mathcal{S}^{(s),u}$ associating to each spin configuration $\sigma \in \mathcal{S}^{(s)}$ the unique block spin configuration $\mathcal{O}_s^u \sigma \in \mathcal{S}^{(s),u}$ such that $(\mathcal{O}_s^u \sigma)_i := \{\sigma_x,\ x \in Q_{u/s}^{(s)}(i)\}$ for all $i \in \mathcal{L}^{(u)}$. The *unpacking* operator $\mathcal{O}_u^s : \mathcal{S}^{(s),u} \to \mathcal{S}^{(s)}$ associates to each block spin configuration $\zeta \in \mathcal{S}^{(s),u}$ the unique spin configuration $\mathcal{O}_u^s \zeta \in \mathcal{S}^{(s)}$ such that $\zeta_i = \{(\mathcal{O}_u^s \zeta)_x,\ x \in Q_{u/s}^{(s)}(i)\}$ for all $i \in \mathcal{L}^{(u)}$. Note that in the case of infinite volume configurations the packing and the unpacking operators are the inverse of each other.

We remark also that the two operators allow the packing of the spin σ–algebra and the unpacking of the block spin one namely, for each $I \subset \mathcal{L}^{(u)}$ and $\Lambda \subset \mathcal{L}^{(s)}$ we have

$$\mathcal{O}_u^s\big(\mathcal{F}_I^{(s),u}\big) = \mathcal{F}^{(s)}_{\mathcal{O}_u^s I} \quad \text{and} \quad \mathcal{O}_s^u\big(\mathcal{F}_\Lambda^{(s)}\big) \subset \mathcal{F}^{(s),u}_{\mathcal{O}_s^u \Lambda}, \tag{4.2}$$

where in the last relation the equality between the two σ–algebras stands if and only if $\mathcal{O}_u^s \mathcal{O}_s^u \Lambda = \Lambda$.

To a block spin configuration we can naturally associate the potential $\Phi^{(s),u}$ defined as follows; for each $I \subset\subset \mathcal{L}^{(u)}$ the function $\Phi_I^{(s),u} : \mathcal{S}^{(s),u} \to \mathbb{R}$ is defined as

$$\Phi_I^{(s),u} := \sum_{\substack{X \subset \mathcal{L}^{(s)}: \\ \mathcal{O}_s^u X = I}} [\Phi_X^{(s)} \circ \mathcal{O}_u^s]. \tag{4.3}$$

We remark that $\Phi^{(s),u} \in \mathcal{F}_I^{(s),u}$. Given $I \subset\subset \mathcal{L}^{(u)}$, we consider the *block spin Hamiltonian* $H_I^{(s),u} : \mathcal{S}^{(s),u} \to \mathbb{R}$ associating to each block spin configuration $\zeta \in \mathcal{S}^{(s),u}$ the Hamiltonian

$$H_I^{(s),u}(\zeta) := \sum_{J \cap I \neq \emptyset} \Phi_J^{(s),u}(\zeta) \quad \text{and} \quad E_I^{(s),u}(\zeta) := \sum_{J \subset I} \Phi_J^{(s),u}(\zeta). \tag{4.4}$$

It is easy to show that, given $I \subset\subset \mathcal{L}^{(u)}$ and the block spin configuration $\zeta \in \mathcal{S}^{(s),u}$, the Hamiltonian $H_I^{(s),u}$ is the Hamiltonian of the unique spin configuration $\mathcal{O}_u^s \zeta \in \mathcal{S}^{(s)}$ obtained by unpacking ζ; indeed

$$H_I^{(s),u}(\zeta) = \sum_{J\cap I\neq\emptyset} \Phi_J^{(s),u}(\zeta) = \sum_{J\cap I\neq\emptyset} \sum_{\substack{X\subset\mathcal{L}^{(s)}:\\ \mathcal{O}_s^u X=J}} \Phi_X^{(s)}(\mathcal{O}_u^s\zeta)$$
$$= \sum_{\substack{X\subset\mathcal{L}^{(s)}:\\ X\cap\mathcal{O}_u^s I\neq\emptyset}} \Phi_X^{(s)}(\mathcal{O}_u^s\zeta) = H^{(s)}_{\mathcal{O}_u^s I}(\mathcal{O}_u^s\zeta). \tag{4.5}$$

We can finally define a Gibbs measure on the block spin configuration space $\mathcal{S}^{(s),u}$, with its σ–algebra $\mathcal{F}^{(s),u}$, by considering the measure $\mu^{(s),u} := \mu^{(s)} \circ \mathcal{O}_u^s$ which is Gibbsian w.r.t. the potential (4.3).

We note that it is possible to make block spins out of block spins, namely, we can consider $\mathcal{S}^{(s),u}$ as the starting configuration space, fix a multiple v of u, and construct the block spin configuration space $\mathcal{S}^{(s),u,v}$. Exploiting the fact that v is a multiple of s it is possible to construct the block spin space $\mathcal{S}^{(s),v}$; note that the spaces $\mathcal{S}^{(s),u,v}$ and $\mathcal{S}^{(s),v}$ are different because they are produced by grouping the original spins, living on scale s, in two different ways.

For the sake of clearness we list here the particular cases in which we will make use of the block spin setup introduced above. First of all, we fix the the renormalization scale ℓ and the rougher scale $\wp \equiv \wp(\ell) := d\ell$. On one hand we consider as original lattice model the lattice gas μ on $\mathcal{L}$ with configuration space $\mathcal{X}$ and algebra of the events $\mathcal{F}$, see Subsect. 2.5, and construct the block spin space $\mathcal{X}^{(1),\ell} \equiv \mathcal{X}^\ell$, its σ–algebra $\mathcal{F}^\ell$, and the Gibbs measure $\mu^\ell = \mu \circ \mathcal{O}_\ell$, with $\mathcal{O}_\ell : \mathcal{X}^\ell \to \mathcal{X}$ the unpacking operator. Then, on the rougher scale $\wp$, we construct the space $\mathcal{X}^{(1),\ell,\wp} \equiv \mathcal{X}^{\ell,\wp}$, its σ–algebra $\mathcal{F}^{\ell,\wp}$, and the Gibbs measure $\mu^{\ell,\wp} = \mu^\ell \circ \mathcal{O}^\ell_\wp$ with $\mathcal{O}^\ell_\wp : \mathcal{X}^{\ell,\wp} \to \mathcal{X}^\ell$ the unpacking operator. On the other hand, we consider as original lattice model the renormalized model $\nu^{(\ell)}$ on $\mathcal{L}^{(\ell)}$ with configuration space $\mathcal{M}^{(\ell)}$ and algebra of events $\mathcal{B}^{(\ell)}$, see Subsect. 2.6, and we construct the block spin space $\mathcal{M}^{(\ell),\wp}$, its σ–algebra $\mathcal{B}^{(\ell),\wp}$ and the measure $\nu^{(\ell),\wp} = \nu^{(\ell)} \circ \mathcal{O}^\ell_\wp$, with $\mathcal{O}^\ell_\wp : \mathcal{M}^{(\ell),\wp} \to \mathcal{M}^{(\ell)}$ the unpacking operator. The elements of $\mathcal{M}^{(\ell)}$ will be denoted by m, and by n those of $\mathcal{M}^{(\ell),\wp}$.

4.2. The constrained models. Let ℓ be the size of the BAT transformation, see Subsect. 2.6, and pick a configuration of renormalized variables and $m \in \mathcal{M}^{(\ell)}$. We define the *single site constrained configuration space*

$$\mathcal{X}^{(\ell)}_{m,i} := \left\{\zeta \in \mathcal{X}^\ell_i : M^{(\ell)}_i(\zeta) = m_i\right\} \subset \mathcal{X}^\ell_i \tag{4.6}$$

which will be equipped with the discrete topology. For $I \subset \mathcal{L}^{(\ell)}$ we consider the *constrained configuration space* $\mathcal{X}^{(\ell)}_{m,I} := \bigotimes_{i\in I} \mathcal{X}^{(\ell)}_{m,i} \subset \mathcal{X}^\ell_I$ equipped with the product topology; we remark that $\bigcup_{m\in\mathcal{M}^{(\ell)}} \mathcal{X}^{(\ell)}_{m,I} = \mathcal{X}^{(\ell)}_I$. As usual we let $\mathcal{X}^{(\ell)}_{m,\mathcal{L}^{(\ell)}} =: \mathcal{X}^{(\ell)}_m$ and denote by $\mathcal{F}^{(\ell)}_m$ the Borel σ–algebra of $\mathcal{X}^{(\ell)}_m$; for each $I \subset\subset \mathcal{L}^{(\ell)}$ we set $\mathcal{F}^{(\ell)}_{m,I} := \sigma\{\zeta_i \in \mathcal{X}^{(\ell)}_{m,i},\ i \in I\} \subset \mathcal{F}^{(\ell)}_m$. Finally, we consider the block spin potential $U^{(1),\ell} \equiv U^\ell$ constructed as in (4.3) starting from the lattice gas potential $U^{(1)} \equiv U = (z, U_{>1})$.

We consider, now, $\tau \in \mathcal{X}^\ell$ and emphasize that τ does not depend on the fixed m, in the sense that it is chosen arbitrarily in a set not depending on m. Let $I \subset\subset \mathcal{L}^{(\ell)}$, we define the probability measure for the constrained model on I with boundary condition τ as follows: for each $\zeta \in \mathcal{X}^{(\ell)}_{m,I}$,

Table 1. Notation for the object, constrained and image model; in the table $m \in \mathcal{M}^{(\ell)}$ is a given renormalized configuration, $\sigma \in \mathcal{X}$ and $\tau \in \mathcal{X}^{(\ell)}$ are fixed boundary conditions

	object model	constrained model	image model
lattice	$\Lambda \subset \mathcal{L} = \mathbb{Z}^d$	$I \subset \mathcal{L}^{(\ell)} = (\ell\mathbb{Z})^d$	$I \subset \mathcal{L}^{(\ell)} = (\ell\mathbb{Z})^d$
configuration space	$\mathcal{X}_\Lambda = \{0,1\}^\Lambda$	$\mathcal{X}^{(\ell)}_{m,I} = \bigotimes_{i\in I} \mathcal{X}^{(\ell)}_{m,i}$	$\mathcal{M}^{(\ell)} = \bigotimes_{i\in I} \mathcal{M}^{(\ell)}_i$
σ–algebra	$\mathcal{F}_\Lambda$	$\mathcal{F}^{(\ell)}_{m,I}$	$\mathcal{B}^{(\ell)}_I$
measure	μ^σ_Λ	$\mu^{(\ell),\tau}_{m,I}$	$\nu^{(\ell),\tau}_I$

$$\mu^{(\ell),\tau}_{m,I}(\zeta) := \frac{1}{Z^{(\ell)}_{m,I}(\tau)}\, e^{H^\ell_I(\zeta\tau_{I^c})},$$

where the Hamiltonian H^ℓ_I is defined as in (4.4) and the partition function $Z^{(\ell)}_{m,I}(\tau)$ is given by

$$Z^{(\ell)}_{m,I}(\tau) := \sum_{\zeta\in\mathcal{X}^{(\ell)}_{m,I}} e^{H^\ell_I(\zeta\tau_{I^c})}. \tag{4.7}$$

Note that the function $H^\ell_I : \mathcal{X}^\ell \to \mathbb{R}$ can be evaluated in $\zeta\tau_{I^c}$, indeed $\zeta \in \mathcal{X}^{(\ell)}_{m,I} \subset \mathcal{X}^{(\ell)}_I$ and $\tau \in \mathcal{X}^\ell$ imply $\zeta\tau_{I^c} \in \mathcal{X}^\ell$.

We remark that the function $Z^{(\ell)}_{m,I}(\cdot) \in \mathcal{F}^\ell_{I^c}$ can be looked at as the partition function of a not translationally invariant finite volume system which is the original lattice gas *constrained* to have fixed values $\rho|Q_\ell| + m_i\sqrt{|Q_\ell|\chi}$ of the total number of particles in each block $Q_\ell(i)$ for all $i \in I$. Its elementary variables are the original spin configurations in each block $Q_\ell(i)$ compatible with the assigned value m_i namely, the set $\mathcal{X}^{(\ell)}_{m,I}$ defined in (4.6). Finally we note that for each $\tau \in \mathcal{X}^\ell$ we have $m \mapsto Z^{(\ell)}_{m,I}(\tau) \in \mathcal{B}^{(\ell)}_I$.

The finite volume renormalized measure $\nu^{(\ell),\tau}_I$, introduced in Subsect. 2.6, which is a probability measure on $\mathcal{M}^{(\ell)}_I$, can be written in the Gibbsian form w.r.t. to the renormalized Hamiltonian given by $\log Z^{(\ell)}_{m,I}(\tau)$. Our aim will then be to compute the partition function $Z^{(\ell)}_{m,I}(\tau)$ for given $m \in \mathcal{M}^{(\ell)}$. More precisely we are interested in finding an expression for $\log Z^{(\ell)}_{m,I}(\tau)$ that allows to extract the renormalized potential with a procedure having sense in the thermodynamics limit.

4.3. On goodness and badness. As mentioned at the end of Subsect. 4.1 technical reasons, connected to the computation developed in Sect. 5 below, force the introduction of the rougher scale $\wp = d\ell$. We then pack the renormalized variables m_i lying inside cubes of $\mathcal{L}^{(\ell)}$ of side length d to form a renormalized block spin n_t, with $t \in \mathcal{L}^{(\wp)}$. More precisely we consider the block spin space $\mathcal{M}^{(\ell),\wp}$, its σ–algebra $\mathcal{B}^{(\ell),\wp}$ and the measure $\nu^{(\ell),\wp} = \nu^{(\ell)} \circ \mathcal{O}^\ell_\wp$, with $\mathcal{O}^\ell_\wp : \mathcal{M}^{(\ell),\wp} \to \mathcal{M}^{(\ell)}$ the unpacking operator.

We define, now, the good part of the lattice $\mathcal{L}^{(\wp)}$. We fix $\delta \in (0, 1/6)$ and $n \in \mathcal{M}^{(\ell),\wp}$; recall χ has been defined in (2.8), we set

$$\mathcal{L}^{(\wp)}_\delta(n) \equiv \mathcal{L}^{(\wp)}_\delta := \{t \in \mathcal{L}^{(\wp)} : |(\mathcal{O}^\ell_\wp n)_i| \le \ell^{d(1/6-\delta)}\chi^{-1/2} \text{ for all } i \in Q^{(\ell)}_{\wp/\ell}(t)\}. \tag{4.8}$$

We say that a site $t \in \mathcal{L}^{(\wp)}$ is *good* w.r.t $n \in \mathcal{M}^{(\ell),\wp}$ if $t \in \mathcal{L}_\delta^{(\wp)}(n)$; if t is not good we say it is *bad*. Loosely speaking a cube of side $d\ell$ of the original lattice is good if the empirical density in *all* its d^d sub–cubes of side ℓ differs from the infinite volume mean ρ less than $\ell^{-d(1/3+\delta)}$; this choice ensures the validity of the central limit theorem inside the good blocks, see [1, Theorem 4.5].

4.4. On the goodness of good sites. As we have already discussed in Subsect. 2.8 our strategy of proof consists in performing a cluster expansion, similar to the one used in [1], in the good region of the lattice and to use the sparseness of the bad sites to carry out the sum over the *bad* part of the lattice. In this subsection we deduce the property of the good blocks that will enable us to cluster expand the partition function of the constrained models in this region.

We recall that ℓ is the scale of the renormalization transformation. Let $i \in \mathcal{L}^{(\wp)}$ and $k \in \{1, \dots, d\}$; we denote by $P^{i,k}$ the family of all not empty subsets $I \subset \mathcal{L}^{(\wp)}$ such that for each $j \in I$ we have $j_k = i_k$ and $j_h \in \{i_h - \wp, i_h, i_h + \wp\}$ for all $h = 1, \dots, d$ and $h \neq k$. We set

$$I_\pm := \partial^{(\wp)} I \cap \{j \in \mathcal{L}^{(\wp)} : \ j_k = i_k \pm \wp\}$$

and, for $m \in \mathcal{M}^{(\ell)}$ and $\sigma \in \mathcal{X}^\ell$, $\sigma_\pm := \sigma_{\mathcal{O}_\wp^\ell I_\pm}$ and $\sigma_0 := \sigma_{\mathcal{O}_\wp^\ell (I_+ \cup I_-)^c}$. Recall $\delta > 0$ has been introduced in (4.8); given $J \subset \mathcal{L}^{(\ell)}$ we set $\mathcal{D}_\delta^{(\ell)}(J) := \{m \in \mathcal{M}^{(\ell)} : |m_j| \leq \ell^{d(1/6-\delta)} \chi^{-1/2}, \ j \in J\}$.

Theorem 4.1. *Let the lattice gas potential U satisfy Condition SM(ℓ_0, b, B). We have that there exists a real $C = C(\delta, \ell_0, b, B, \|U\|_0, r, d) < \infty$ such that for each ℓ multiple of ℓ_0 and $i \in \mathcal{L}^{(\wp)}$ we have*

$$\sup_{k=1,\dots,d} \ \sup_{I \in P^{i,k}} \ \sup_{m \in \mathcal{D}_\delta^{(\ell)}(\mathcal{O}_\wp^\ell I)} \ \sup_{\sigma,\zeta,\tau \in \mathcal{X}^\ell} \left| \frac{Z^{(\ell)}_{m,\mathcal{O}_\wp^\ell I}(\sigma_+\sigma_-\tau_0)\, Z^{(\ell)}_{m,\mathcal{O}_\wp^\ell I}(\zeta_+\zeta_-\tau_0)}{Z^{(\ell)}_{m,\mathcal{O}_\wp^\ell I}(\sigma_+\zeta_-\tau_0)\, Z^{(\ell)}_{m,\mathcal{O}_\wp^\ell I}(\zeta_+\sigma_-\tau_0)} - 1 \right| \leq \frac{C}{\ell}. \tag{4.9}$$

To prove the above theorem we shall use Lemma 4.2 below in which it is proven that the strong mixing condition holds uniformly in the activity. To state precisely such a property we introduce the notion of *lattice gas with not homogeneous activity*: consider the configuration space $\mathcal{X} := \{0, 1\}^{\mathcal{L}}$ of the lattice gas, the Borel σ–algebra $\mathcal{F}$, see Subsect. 2.5, and the family of local functions $U_{>1} = \{U_X \in \mathcal{F}_X, \ X \subset\subset \mathcal{L}, |X| > 1\}$. Let $\underline{z} := \{z_x \in [0, \infty), \ x \in \mathcal{L}\}$, the lattice gas potential with not homogeneous activity is the family of functions $U^{\underline{z}} := \{U_X \in \mathcal{F}_X, \ X \subset\subset \mathcal{L}\}$ with

$$U_X^{\underline{z}}(\eta) := \begin{cases} \eta_x \log z_x & \text{if there exists } x \in \mathcal{L} \text{ such that } X = \{x\} \\ U_X(\eta) & \text{if } |X| > 1 \end{cases}$$

for all $\eta \in \mathcal{X}$; we shall use the notation $U^{\underline{z}} := (\underline{z}, U_{>1})$. Given $\Lambda \subset\subset \mathcal{L}^{(\ell)}$ and $\tau \in \mathcal{X}$, the finite volume Gibbs measure with boundary condition τ associated with the lattice gas potential with not homogeneous activity $U^{\underline{z}}$ is denoted by $\mu^\tau_{\Lambda,\underline{z}}$ and the corresponding partition function by $Z^\tau_{\Lambda,\underline{z}}$. Namely, we have

$$Z^\tau_{\Lambda,\underline{z}} := \sum_{\eta \in \mathcal{X}_\Lambda} \exp\Big\{ \sum_{\substack{X \cap \Lambda \neq \emptyset: \\ |X| > 1}} U_X(\eta\tau_{\Lambda^c}) + \sum_{x \in \Lambda} \eta_x \log z_x \Big\}. \tag{4.10}$$

It is easy to see that by [13, 14], see Remark 2 in [1, p. 849], the following lemma, stating that the strong mixing condition (2.7) is satisfied uniformly in the activities, holds.

Lemma 4.2. *Let the lattice gas potential $U = (z, U_{>1})$ satisfy Condition SM(ℓ_0, b, B). Then there exist $\varepsilon > 0$, ℓ_0' multiple of ℓ_0, and two positive reals $b' = b'(\varepsilon, b, B, \ell_0)$ and $B' = B'(\varepsilon, b, B, \ell_0) < \infty$ such that for $\underline{z} = \{z_x \in [0, \infty) : x \in \mathcal{L}\}$ such that $|z_x - z| \le \varepsilon$, for all $x \in \mathcal{L}$, the lattice gas potential with not homogeneous activity $U^{\underline{z}} = (\underline{z}, U_{>1})$ satisfies SM(ℓ_0', b', B').*

Proof of Theorem 4.1. Let μ_z be the unique infinite volume Gibbs measure of the lattice gas with potential $U = (z, U_{>1})$ and set $\varrho : (0, +\infty) \ni z \longrightarrow \varrho(z) := \mu_z(\eta_0) \in (0, 1)$. Let $\varepsilon > 0$ be as in Lemma 4.2, by the continuity of ϱ we can choose $\varepsilon' > 0$ such that $\varrho^{-1}\left(\varrho(z) - 2\varepsilon', \varrho(z) + 2\varepsilon'\right) \subset [z - \varepsilon, z + \varepsilon]$. The thesis follows by Lemma 4.2 and [1, Prop. 5.1]. □

4.5. On the sparseness of bad sites. In this subsection we state precisely in which sense the bad sites in $\mathcal{L}^{(\wp)}$ are sparse. We define the map $\pi : \mathcal{M}^{(\ell),\wp} \to \{0, 1\}^{\mathcal{L}^{(\wp)}}$ by setting for each $n \in \mathcal{M}^{(\ell),\wp}$ and $t \in \mathcal{L}^{(\wp)}$,

$$\left(\pi(n)\right)_t := \begin{cases} 0 & \text{if } t \in \mathcal{L}_\delta^{(\wp)}(n) \\ 1 & \text{otherwise} \end{cases} . \tag{4.11}$$

As a first step we show that the probability that a site is bad is exponentially small in ℓ.

Theorem 4.3. *Let the lattice gas potential $U = (z, U_{>1})$ satisfy Condition SM(ℓ_0, b, B). Then there exists a real $C = C(\varepsilon, \ell_0, b, B) > 0$ such that for any positive integer ℓ we have that*

$$\sup_{t \in \mathcal{L}^{(\wp)}} \nu^{(\ell),\wp}((\pi(n))_t = 1) \le \exp\{-C\ell^{(1/3-2\delta)d}\}. \tag{4.12}$$

Proof. We have

$$\begin{aligned} \sup_{t\in\mathcal{L}^{(\wp)}} \nu^{(\ell),\wp}((\pi(n))_t = 1) &= \sup_{t\in\mathcal{L}^{(\wp)}} \nu^{(\ell)}\big(\exists i \in Q^{(\ell)}_{\wp/\ell}(t) : |m_i| > \ell^{d(1/6-\delta)}\chi^{-1/2}\big) \\ &\le d^d \sup_{i\in\mathcal{L}^{(\ell)}} \mu\big(|M_i^{(\ell)}| > \ell^{d(1/6-\delta)}\chi^{-1/2}\big). \end{aligned}$$

We pick $i \in \mathcal{L}^{(\ell)}$. To bound the right hand side of the above inequality, we recall (2.11), consider $L > \ell$ integer, set $\Delta_L(i) := \{x \in \mathcal{L} : \mathrm{d}(x, Q_\ell(i)) \le L\}$, and use the exponential Chebyshev inequality, with $h \ge 0$, as follows

$$\begin{aligned} &\mu\big(M_i^{(\ell)} > \ell^{d(1/6-\delta)}\chi^{-1/2}\big) = \mu\Big(\sum_{x\in Q_\ell(i)} (\eta_x - \rho) > \ell^{(2/3-\delta)d}\Big) \\ &\le e^{-h\ell^{(2/3-\delta)d}}\, \mu\Big(\exp\big\{h \sum_{x\in Q_\ell(i)} (\eta_x - \rho)\big\}\Big) \\ &= e^{-h\ell^{(2/3-\delta)d}} \int \mu(d\tau)\, \mu^\tau_{\Delta_L(i)}\Big(\exp\big\{h \sum_{x\in Q_\ell(i)} (\eta_x - \rho)\big\}\Big) \\ &= e^{-h\ell^{(2/3-\delta)d}} \int \mu(d\tau)\, \exp\big\{\log Z^\tau_{\Delta_L(i),\underline{z}(i,h)} - \log Z^\tau_{\Delta_L(i),\underline{z}(i,0)} - h\rho\ell^d\big\}, \end{aligned} \tag{4.13}$$

where we used the DLR equations (2.6) and, for $\tau \in \mathcal{X}$ and $\Lambda \subset\subset \mathcal{L}$, we have considered the partition function

$$Z^{\tau}_{\Lambda,\underline{z}(i,h)} = \sum_{\eta\in\mathcal{X}_\Lambda} \exp\Big\{H_\Lambda(\eta\tau_{\Lambda^c}) + h \sum_{x\in\Lambda\cap Q_\ell(i)} \eta_x\Big\} \tag{4.14}$$

of a lattice gas with not homogeneous activity $\underline{z}(i,h)$ such that $z_x(i,h) = ze^h$ for all $x \in Q_\ell(i)$ and $z_x(i,h) = z$ otherwise; recall z is the activity of the original lattice gas. Note that $Z^{\tau}_{\Lambda,\underline{z}(i,0)}$ coincides with the partition function Z^{τ}_{Λ} of the original lattice gas.

From the strong mixing condition $\mathrm{SM}(\ell_0, b, B)$ it follows that there exist two positive reals $C_1(\ell_0, b, B) < \infty$ and $C_2(\ell_0, b, B) > 0$ such that for any L multiple of ℓ_0 and $\tau \in \mathcal{X}$ we have

$$\begin{aligned}\bigg|\bigg(\frac{d\log Z^{\tau}_{\Delta_L(i),\underline{z}(i,h)}}{dh}\bigg)_{h=0} - \rho\ell^d\bigg| &= \bigg|\mu^{\tau}_{\Delta_L(i)}\Big(\sum_{x\in Q_\ell(i)}\eta_x\Big) - \mu\Big(\sum_{x\in Q_\ell(i)}\eta_x\Big)\bigg|\\ &\le C_1\,\ell^d\, e^{-C_2(L-\ell)}.\end{aligned} \tag{4.15}$$

By Lemma 4.2 there exist $\varepsilon > 0$, ℓ_0' multiple of ℓ_0, and the two reals $b' = b'(\varepsilon, b, B, \ell_0)$ and $B' = B'(\varepsilon, b, B, \ell_0)$ such that the perturbed lattice gas potential satisfies $\mathrm{SM}(\ell_0', b', B')$ for all $0 \le h \le \varepsilon$. Hence, if L is a multiple of ℓ_0' we have that there exists a real $0 < C_3 = C_3(\varepsilon, \ell_0, b, B) < \infty$ such that for any $h \in [0, \varepsilon]$ and $\tau \in \mathcal{X}$ the following bound holds

$$\bigg|\frac{d^2\log Z^{\tau}_{\Delta_L(i),\underline{z}(i,h)}}{dh^2}\bigg| = \bigg|\sum_{x,y\in Q_\ell(i)} \mu^{\tau}_{\Delta_L(i),\underline{z}(i,h)}(\eta_x;\eta_y)\bigg| \le 2C_3\,\ell^d, \tag{4.16}$$

where we recall $\mu^{\tau}_{\Lambda,\underline{z}}$, for $\Lambda \subset\subset \mathcal{L}$ and $\underline{z} \in [0,\infty)^{\mathcal{L}}$, is the finite volume Gibbs measure of the lattice gas with not homogeneous activity $\underline{z}$ and boundary condition $\tau \in \mathcal{X}$. By expanding the exponent on the right hand side of (4.13) and using (4.15) and (4.16) we get

$$\mu\big(M_i^{(\ell)} > \ell^{d(1/6-\delta)}\chi^{-1/2}\big) \le \exp\{-h\ell^{(2/3-\delta)d} + C_1\,\ell^d\,e^{-C_2(L-\ell)} + h^2C_3\,\ell^d\}. \tag{4.17}$$

Taking the limit $L \to \infty$ we finally get

$$\mu\big(M_i^{(\ell)} > \ell^{d(1/6-\delta)}\chi^{-1/2}\big) \le \exp\{-h(\ell^{(2/3-\delta)d} - hC_3\,\ell^d)\}. \tag{4.18}$$

The bound (4.12) follows by choosing $h = \ell^{-(1/3+\delta)d}/(2C_3)$; indeed the steps in (4.13) can be repeated to bound $\mu\big(M_i^{(\ell)} < -\ell^{d(1/6-\delta)}\chi^{-1/2}\big)$. □

In Theorem 4.4 below we shall state that the bad sites of $\mathcal{L}^{(\wp)}$ are sparse in the following sense. There exists a full measure subset of $\mathcal{M}^{(\ell),\wp}$, such that for each n in such a set there exists a gentle disintegration, see Definition 3.2, of the lattice $\mathcal{L}^{(\wp)}$ with respect to its good part $\mathcal{L}^{(\wp)}_{\delta}$ and two suitable moderately steep scales Γ, γ. The two sequences are chosen as in [2, Remark 2.3] namely, given $\beta \ge 9$ we set $\Gamma_0 = \gamma_0 := 0$,

$$\Gamma_k := e^{(\beta+1)(3/2)^k} \qquad \text{and} \qquad \gamma_k := \frac{1}{8}e^{\beta(3/2)^{k+1}} \qquad \text{for } k \ge 1. \tag{4.19}$$

Those sequences are steep scales namely, they satisfy Items 1–3 in Definition 3.1. Moreover, see the remark below Theorem 2.5 in [2], we choose β large enough so that the supplementary conditions on the steep scales in the hypotheses of [2, Theorem 2.5] are satisfied with $\alpha = 1$. It is easy to prove that for

$$\varepsilon \in (1/2, 1) \quad \text{and} \quad a \geq 9d\beta/2 \tag{4.20}$$

the steep scales Γ, γ are moderate namely, they also fulfill Items 4–5 in Definition 3.1. The conditions above on β and a are met for ℓ integer large enough if we set

$$a \equiv a_\delta(\ell) := \frac{9}{2} d\big[\beta \vee \ell^{(1/3-2\delta)d/2}\big], \tag{4.21}$$

where $\delta \in (0, 1/6)$ is the real number which has been picked up before (4.8) to define the good part of the lattice $\mathcal{L}^{(\wp)}$.

In order to prove Theorem 2.2, we had to choose the parameter a diverging with the renormalization scale ℓ. In fact we shall need that the probability that a site of $\mathcal{L}^{(\wp)}$ belongs to a k–gentle atom vanishes fast enough as $\ell \to \infty$. On the other hand the existence of the gentle disintegration of $\mathcal{L}^{(\wp)}$ is proven on the basis of Theorem 3.3 whose hypotheses are satisfied if the probability p for a site to be bad is smaller than $\exp\{-a/(1-\varepsilon)\}$. In our application this probability is estimated with the stretched exponential in (4.12); to ensure that for ℓ large enough p is smaller than $\exp\{-a/(1-\varepsilon)\}$ the function $a_\delta(\ell)$ must diverge sufficiently slow. The choice (4.21) meets both the above requirements.

Theorem 4.4. *Let the lattice gas potential U satisfy Condition SM(ℓ_0, b, B). Consider the two moderately steep scales Γ, γ defined in (4.19).*

Then for each ℓ large enough multiple of ℓ_0 there exists a $\mathcal{B}^{(\ell),\wp}$–measurable subset $\bar{\mathcal{M}}^{(\ell),\wp} \subset \mathcal{M}^{(\ell),\wp}$ with $\nu^{(\ell),\wp}(\bar{\mathcal{M}}^{(\ell),\wp}) = 1$ such that

1. *For each $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ there exists a gentle disintegration $\mathcal{G}(n)$, see Definition 3.2, of $\mathcal{L}^{(\wp)}$ with respect to $\mathbb{G}_0(n) := \mathcal{L}_\delta^{(\wp)}(n)$, Γ, and γ.*
2. *For each $t \in \mathcal{L}^{(\wp)}$ and $X \subset\subset \mathcal{L}^{(\wp)}$ we have that $\{n : \mathbb{G}_k(n) \ni t\} \in \mathcal{B}^{(\ell),\wp}_{B^{(\wp)}_{\vartheta_k}(t)}$ and also $\{n : \mathcal{G}_k(n) \ni X\} \in \mathcal{B}^{(\ell),\wp}_{B^{(\wp)}_{\vartheta_k}(X)}$.*

Proof. We use the setup of Sect. 3 with $\mathbb{L} = \wp^{-1}\mathcal{L}^{(\wp)}$. Recall the map $\pi : \mathcal{M}^{(\ell),\wp} \to \{0, 1\}^{\mathcal{L}^{(\wp)}}$ has been defined in (4.11). Note that for each $x, y \in \mathbb{L}$ we have

$$\mathrm{D}(x, y) = \sum_{i=1}^{d} |x_i - y_i| \leq d \sup_{i\in\{1,\dots,d\}} |x_i - y_i| = (d/\wp)\mathrm{d}_\wp(\wp x, \wp y).$$

From Condition SM(ℓ_0, b, B) it follows that the measure $\mathbb{Q} := \nu^{(\ell),\wp} \circ \pi^{-1}$ on the set $\Omega := \{0, 1\}^{\mathcal{L}^{(\wp)}}$, endowed with its Borel σ–algebra $\mathcal{A}$, satisfies the bound (3.2) with constants $b'' = \wp b/d$ and $B'' = \wp^d B$. By taking ℓ large enough the scales Γ, γ in (4.19) satisfy (3.4). Moreover by Theorem 4.3,

$$p := \sup_{t\in\mathcal{L}^{(\wp)}} \mathbb{Q}\left(\{\omega : \omega_t = 1\}\right) = \sup_{t\in\mathcal{L}^{(\wp)}} \nu^{(\ell),\wp}\left(\{n : (\pi(n))_t = 1\}\right) \leq e^{-C\ell^{(1/3-2\delta)d}}.$$

We can therefore apply Theorem 3.3, we set $\bar{\mathcal{M}}^{(\ell),\wp} := \pi^{-1}(\bar{\Omega})$. The thesis follows by noticing that for each $X \subset \mathcal{L}^{(\wp)}$ we have $B_s^{(\wp)} \supset \wp O_s(\wp^{-1}X)$ for all $s > 0$. □

5. Cluster Expansion in the Good Part of the Lattice

In this section we start to compute the renormalized potentials; our main technique, as in [1], will be the scale adapted cluster expansion.

Let ℓ be the renormalization scale, recall $\wp = d\ell$; for $m \in \mathcal{M}^{(\ell)}$ the set $\mathcal{L}_\delta^{(\wp)}(\mathcal{O}_\ell^\wp m) \equiv \mathcal{L}_\delta^{(\wp)} \subset \mathcal{L}^{(\wp)}$ has been defined in (4.8). Pick $\Lambda \subset\subset \mathcal{L}^{(\wp)}$, a configuration of the renormalized variables $m \in \mathcal{M}^{(\ell)}$, and a boundary condition $\tau \in \mathcal{X}^\ell$; set $J := \mathcal{O}_\wp^\ell \Lambda$, $\Lambda_\delta := \Lambda \cap \mathcal{L}_\delta^{(\wp)}$ and $J_\delta := \mathcal{O}_\wp^\ell \Lambda_\delta$. We write

$$Z_{m,J}^{(\ell)}(\tau) = \sum_{\eta \in \mathcal{X}_{m,J}^{(\ell)}} \exp\{H_J^\ell(\eta\tau_{J^c})\} = \sum_{\sigma \in \mathcal{X}_{m,J\setminus J_\delta}^{(\ell)}} \sum_{\eta \in \mathcal{X}_{m,J_\delta}^{(\ell)}} \exp\{H_J^\ell(\sigma\eta\tau_{J^c})\}.$$

In this section we fix $\sigma \in \mathcal{X}_{m,J\setminus J_\delta}^{(\ell)}$ and compute the partition function associated to the good part J_δ of the set J namely, we compute

$$Z_{m,J_\delta}^{(\ell)}(\sigma\tau_{J^c}) = \sum_{\eta \in \mathcal{X}_{m,J_\delta}^{(\ell)}} \exp\{H_{J_\delta}^\ell(\eta\sigma\tau_{J^c})\}. \tag{5.1}$$

We rewrite this problem on the scale $\wp$, that is we apply the procedure described in Subsect. 4.1 to the constrained models introduced in Subsect. 4.2 on the scale ℓ to group the block spin variables on the scale $\wp$. We fix a configuration $n \in \mathcal{M}^{(\ell),\wp}$, the corresponding renormalized configuration is $m \equiv m(n) := \mathcal{O}_\wp^\ell n$. We recall the notion of constrained model defined on the configuration space $\mathcal{X}_m^{(\ell)}$ and, via the procedure discussed in Subsect. 4.1, we construct the configuration space $\mathcal{X}_m^{(\ell),\wp}$ and its σ–algebra $\mathcal{F}_m^{(\ell),\wp}$; for $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ we set $\mathcal{F}_{m,\Lambda}^{(\ell),\wp} := \{\zeta_i \in \mathcal{X}_{m,i}^{(\ell),\wp},\ i \in \Lambda\} \subset \mathcal{F}_m^{(\ell),\wp}$.

Finally we consider the potential $U^{(1),\ell,\wp} \equiv U^{\ell,\wp}$, obtained by applying the procedure in Subsect. 4.1 to the original lattice gas potential introduced in Subsect. 2.5, and, given $\Lambda \subset\subset \mathcal{L}^{(\wp)}$, we consider the Hamiltonian $H_\Lambda^{\ell,\wp}$ and the self–interaction $E_\Lambda^{\ell,\wp}$; we obviously have that for each $\zeta \in \mathcal{X}^{\ell,\wp}$,

$$H_\Lambda^{\ell,\wp}(\zeta) = H^\ell_{\mathcal{O}_\wp^\ell \Lambda}(\mathcal{O}_\wp^\ell \zeta) = H_{\mathcal{O}_\ell \mathcal{O}_\wp^\ell \Lambda}(\mathcal{O}_\ell \mathcal{O}_\wp^\ell \zeta).$$

For $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ we can then write the finite volume Gibbs measure with boundary condition $\xi \in \mathcal{X}^{\ell,\wp}$ as

$$\mu_{m,\Lambda}^{(\ell),\wp,\xi}(\zeta) := \frac{1}{Z_{m,\Lambda}^{(\ell),\wp}(\xi)} e^{H_\Lambda^{\ell,\wp}(\zeta\xi_{\Lambda^c})}, \qquad \zeta \in \mathcal{X}_{m,\Lambda}^{(\ell),\wp}.$$

The partition function above is given by

$$Z_{m,\Lambda}^{(\ell),\wp}(\xi) := \sum_{\zeta \in \mathcal{X}_{m,\Lambda}^{(\ell),\wp}} e^{H_\Lambda^{\ell,\wp}(\zeta\xi_{\Lambda^c})}. \tag{5.2}$$

Note that the boundary condition $\xi \in \mathcal{X}^{(\ell),\wp}$ is chosen independently of the renormalized configuration m. We have that $Z_{m,\Lambda}^{(\ell),\wp}(\cdot) \in \mathcal{F}_{\Lambda^c}^{\ell,\wp}$ and $m \mapsto Z_{m,\Lambda}^{(\ell),\wp}(\xi) \in \mathcal{B}_\Lambda^{(\ell),\wp}$.

It is easy to show that $Z_{m,J_\delta}^{(\ell)}(\sigma\tau_{J^c}) = Z_{m,\Lambda_\delta}^{(\ell),\wp}(\mathcal{O}_\ell^\wp(\sigma\tau_{J^c}))$. In the following theorem we shall denote by $\xi := \mathcal{O}_\ell^\wp\big(\sigma\tau_{J^c}\big)$ the block spin configuration outside $\Lambda_\delta = \Lambda \cap \mathcal{L}_\delta^{(\wp)}$.

Theorem 5.1. *Let the lattice gas potential U satisfy Condition SM(ℓ_0, b, B). Then for each ℓ a large enough multiple of ℓ_0, $n \in \mathcal{M}^{(\ell),\wp}$, and $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ there exist a family of local functions $\{V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) : \mathcal{X}^{\ell,\wp} \to \mathbb{R},\ X \subset\subset \mathcal{L}^{(\wp)}\}$, a real $K_\Lambda^{(\wp)}$, and an integer κ such that*

1. *for any $\xi \in \mathcal{X}^{\ell,\wp}$ we have the absolutely convergent expansion*

$$\log Z^{(\ell),\wp}_{\mathcal{O}^\ell_\wp n,\Lambda_\delta}(\xi) = K_\Lambda^{(\wp)} - \frac{1}{2}\sum_{i\in\mathcal{O}^\ell_\wp\Lambda}(\mathcal{O}^\ell_\wp n)_i^2 + \sum_{\substack{X\subset\subset\mathcal{L}^{(\wp)}:\\ X\cap\Lambda\neq\emptyset}} V_{X,\Lambda}^{(\ell),\wp}(\xi, n), \tag{5.3}$$

   where $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n)$ is constant if $X\cap\Lambda_\delta = \emptyset$; moreover, $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) = 0$ if $X\cap\Lambda_\delta = \emptyset$ and $\operatorname{diam}_\wp(X) > \kappa$.

For any $X \subset\subset \mathcal{L}^{(\wp)}$

2. *we have that $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) \in \mathcal{F}^{\ell,\wp}_{X\cap\Lambda_\delta^c}$;*
3. *if X is not $\wp$–connected then $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) = 0$;*
4. *if $X \cap \Lambda_\delta \neq \emptyset$ we have that $X \cap \left(\overline{\Lambda_\delta}^{(\wp),\kappa}\right)^c \neq \emptyset$ implies $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) = 0$.*

Moreover

5. *there exist reals $\alpha_1 > 0$ and $A_1 < \infty$ depending on ℓ_0, b, B, $\|U\|$, r, d, and δ such that we have*

$$\sup_{x\in\mathcal{L}^{(\wp)}} \sum_{\substack{X\subset\subset\mathcal{L}^{(\wp)}:\\ X\ni x}} e^{\alpha_\ell \mathbb{T}_\wp(X)} \sup_{\Lambda\subset\subset\mathcal{L}^{(\wp)}} \|V_{X,\Lambda}^{(\ell),\wp}(\cdot, n)\|_\infty \le A_\ell, \tag{5.4}$$

   where we have set $\alpha_\ell := \alpha_1 \log(e\ell)$ and $A_\ell := A_1\, \ell^{(\kappa+1)^d\alpha_1+d}$;
6. *we have that*

$$\lim_{\ell\to\infty} \sup_{\Lambda\subset\subset\mathcal{L}^{(\wp)}} \sup_{X\subset\Lambda} \sup_{\substack{n\in\mathcal{M}^{(\ell),\wp}:\\ \mathcal{L}^{(\wp)}_\delta(n)\supset X}} \|V_{X,\Lambda}^{(\ell),\wp}(\cdot, n)\|_\infty = 0, \tag{5.5}$$

   where the limit is taken along a sequence of multiples of ℓ_0;
7. *for any $\Lambda, \Lambda' \subset\subset \mathcal{L}^{(\wp)}$ and $X \subset\subset \mathcal{L}^{(\wp)}$ if $X \cap \Lambda = X \cap \Lambda'$ then $V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) = V_{X,\Lambda'}^{(\ell),\wp}(\cdot, n)$;*
8. *let $X, \Lambda \subset\subset \mathcal{L}^{(\wp)}$ and $n, n' \in \mathcal{M}^{(\ell),\wp}$ such that $n_X = n'_X$, then*

$$V_{X,\Lambda}^{(\ell),\wp}(\cdot, n) = V_{X,\Lambda}^{(\ell),\wp}(\cdot, n'). \tag{5.6}$$

We have a situation very similar to the one in [1] where we considered the case of a torus; the sole difference is that, now, Λ_δ is an arbitrary finite subset of $\mathcal{L}^{(\wp)}$, hence its boundary can be geometrically complicated. To simplify the exposition, like in [1], we will treat explicitly only the two–dimensional case. The general d–dimensional case can be treated analogously, following the methods of [28]. We mention that a similar expansion has been used in [4] to study coupled maps.

As in [1] we will transform the constrained system, whose partition function is $Z^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)$, into a small activity polymer system. More precisely, we shall prove the following formula

$$Z^{(\ell),\wp}_{m,\Lambda_\delta}(\xi) = \bar{Z}^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)\,\Xi^{(\ell),\wp}_{m,\Lambda_\delta}(\xi), \tag{5.7}$$

where $\bar{Z}^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)$ is a product of partition functions on suitable finite volumes; the dependence on m of the single factors is local. Moreover, the *reference system* around which we perform the perturbative expansion is described by the partition function $\bar{Z}^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)$. On the other hand $\Xi^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)$ is the partition function of a gas of polymers, see (5.43) below.

The expression (5.7) is well suited to compute the renormalized potentials; in order to get good estimates on them we need that the polymer system described by $\Xi^{(\ell),\wp}_{m,\Lambda_\delta}(\xi)$ is in the small activity region thanks to the uniform bound in Theorem 4.1. In other words, the bound (4.9) implies that the finite size condition of [28] is satisfied on Λ_δ. More precisely, recalling the notation introduced in Subsect. 4.4, there exists a real $C < \infty$ depending on ℓ_0, b, B, $\|U\|_0$, r, d, and δ such that for ℓ multiple of ℓ_0, $m \in \mathcal{M}^{(\ell)}$, and $i \in \mathcal{L}^{(\wp)}$ we have

$$\sup_{k=1,\dots,d}\ \sup_{I\in P^{i,k}}\ \sup_{\sigma,\zeta,\tau\in\mathcal{X}^{\ell,\wp}} \left| \frac{Z^{(\ell),\wp}_{m,I\cap\mathcal{L}^{(\wp)}_\delta}(\sigma_+\sigma_-\tau_0)\, Z^{(\ell),\wp}_{m,I\cap\mathcal{L}^{(\wp)}_\delta}(\zeta_+\zeta_-\tau_0)}{Z^{(\ell),\wp}_{m,I\cap\mathcal{L}^{(\wp)}_\delta}(\sigma_+\zeta_-\tau_0)\, Z^{(\ell),\wp}_{m,I\cap\mathcal{L}^{(\wp)}_\delta}(\zeta_+\sigma_-\tau_0)} - 1 \right| \le \frac{C}{\ell}. \tag{5.8}$$

We start, now, the computation yielding the expansion (5.7). We pick $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ and $n \in \mathcal{M}^{(\ell),\wp}$; to simplify the notation we set $m = \mathcal{O}^{\ell}_{\wp} n$ and $\Delta := \Lambda_\delta = \Lambda \cap \mathcal{L}^{(\wp)}_\delta(n)$. Recall $e^{(\wp)}_1 = (\wp, 0)$ and $e^{(\wp)}_2 = (0, \wp)$; we partition $\mathcal{L}^{(\wp)}$ into the four sub–lattices $\mathcal{A} := \mathcal{L}^{(2\wp)}$, $\mathcal{B} := \mathcal{L}^{(2\wp)} + e^{(\wp)}_2$, $\mathcal{C} := \mathcal{L}^{(2\wp)} + e^{(\wp)}_1 + e^{(\wp)}_2$, and $\mathcal{D} := \mathcal{L}^{(2\wp)} + e^{(\wp)}_1$. We label the points in those sub–lattices by $k \in \mathcal{L}^{(2\wp)}$ as follows: $A_k := k \in \mathcal{A}$, $B_k := k + e^{(\wp)}_2 \in \mathcal{B}$, $C_k := k + e^{(\wp)}_1 + e^{(\wp)}_2 = B_k + e^{(\wp)}_1 \in \mathcal{C}$, $D_k := k + e^{(\wp)}_1 = C_k - e^{(\wp)}_2 \in \mathcal{D}$. It is useful, here and in the sequel, to think of $e^{(\wp)}_1$ as horizontal and as $e^{(\wp)}_2$ as vertical.

Recalling definition (4.4) for $\xi \in \mathcal{X}^{\ell,\wp}$, $x \in \mathcal{L}^{(\wp)}$ we define the function $E^{\ell,\wp}_{x;\Delta}(\cdot|\xi) : \mathcal{X}^{\ell,\wp} \to \overline{\mathbb{R}}$ as

$$E^{\ell,\wp}_{x;\Delta}(\eta|\xi) := \begin{cases} E^{\ell,\wp}_{\{x\}}(\eta) & \text{if } x \in \Delta \\ 0 & \text{if } x \notin \Delta \text{ and } \eta_x = \xi_x \\ -\infty & \text{if } x \notin \Delta \text{ and } \eta_x \neq \xi_x \end{cases}, \tag{5.9}$$

where we recall that by $E^{\ell,\wp}_{\{x\}}$ we mean $E^{(1),\ell,\wp}_{\{x\}}$, see the discussion below (4.5). We shall understand, below, $\exp\{-\infty\} = 0$. We have that $E^{\ell,\wp}_{x;\Delta}(\cdot|\xi) \in \mathcal{F}^{\ell,\wp}_{\{x\}}$; we notice here that in the following we will sometimes misuse the notation and write $E^{\ell,\wp}_{x;\Delta}(\eta_x|\xi)$ instead of $E^{\ell,\wp}_{x;\Delta}(\eta|\xi)$. We define the interaction $\widetilde{W}^{\ell,\wp}_{X_1,X_2} : \mathcal{X}^{\ell,\wp} \to \mathbb{R}$ between two disjoint sets $X_1, X_2 \subset\subset \mathcal{L}^{(\wp)}$ by setting

$$\widetilde{W}^{\ell,\wp}_{X_1,X_2} := E^{\ell,\wp}_{X_1\cup X_2} - E^{\ell,\wp}_{X_1} - E^{\ell,\wp}_{X_2}. \tag{5.10}$$

Notice that $\widetilde{W}^{\ell,\wp}_{X_1,X_2} \in \mathcal{F}^{\ell,\wp}_{X_1\cup X_2}$. For $x \in \mathcal{L}^{(\wp)}$ we define the function $W^{\ell,\wp}_{x;\Delta} : \mathcal{X}^{\ell,\wp} \to \mathbb{R}$ by setting

$$W^{\ell,\wp}_{x;\Delta} := \begin{cases} 0 & \text{if } x \notin \Delta \\ \widetilde{W}^{\ell,\wp}_{\{x\},\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} & \text{if } x \in \Delta\cap\mathcal{D} \\ \widetilde{W}^{\ell,\wp}_{\{x\},\mathcal{A}\cup\mathcal{B}\cup(\mathcal{D}\cap\Delta^{c})} & \text{if } x \in \Delta\cap\mathcal{C} \\ \widetilde{W}^{\ell,\wp}_{\{x\},\mathcal{A}\cup[(\mathcal{C}\cup\mathcal{D})\cap\Delta^{c}]} & \text{if } x \in \Delta\cap\mathcal{B} \\ \widetilde{W}^{\ell,\wp}_{\{x\},(\mathcal{B}\cup\mathcal{C}\cup\mathcal{D})\cap\Delta^{c}} & \text{if } x \in \Delta\cap\mathcal{A} \end{cases} . \tag{5.11}$$

By using definitions (5.9), (5.11), and choosing ℓ large enough such that $\wp = d\ell > r$ (recall r is the range of the original interaction, so that the block spin interaction has range one), we have that for $\eta, \xi \in \mathcal{X}^{\ell,\wp}$, such that $\eta_{\Delta^c} = \xi_{\Delta^c}$,

$$\begin{aligned} H^{\ell,\wp}_{\Delta}(\eta) = \sum_{k\in\mathcal{L}^{(2\wp)}} \big[& E^{\ell,\wp}_{A_k}(\eta|\xi) + W^{\ell,\wp}_{A_k;\Delta}(\eta) + E^{\ell,\wp}_{B_k}(\eta|\xi) + W^{\ell,\wp}_{B_k;\Delta}(\eta) \\ & + E^{\ell,\wp}_{C_k}(\eta|\xi) + W^{\ell,\wp}_{C_k;\Delta}(\eta) + E^{\ell,\wp}_{D_k}(\eta|\xi) + W^{\ell,\wp}_{D_k;\Delta}(\eta) \big]. \end{aligned} \tag{5.12}$$

For $V \subset \mathcal{L}^{(\wp)}$ we introduce the set

$$\mathcal{Y}^{(\ell),\wp}_{\Delta,m,V} := \bigotimes_{x\in\Delta\cap V} \mathcal{X}^{(\ell),\wp}_{m,\{x\}} \otimes \bigotimes_{x\in\Delta^c\cap V} \mathcal{X}^{\ell,\wp}_{\{x\}} \tag{5.13}$$

as usual if $V = \mathcal{L}^{(\wp)}$ we drop it from the notation. Hence, we have that for $\xi \in \mathcal{X}^{\ell,\wp}$ the partition function in Δ can be written in the following way:

$$\begin{aligned} Z^{(\ell),\wp}_{m,\Delta}(\xi) &= \sum_{\eta\in\mathcal{X}^{(\ell),\wp}_{m,\Delta}} \exp\left\{H^{\ell,\wp}_{\Delta}(\eta\xi_{\Delta^c})\right\} \\ &= \sum_{\alpha\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{A}}} \Big(\prod_{k\in\mathcal{L}^{(2\wp)}} \exp\left\{E^{\ell,\wp}_{A_k;\Delta}(\alpha_{A_k}|\xi) + W^{\ell,\wp}_{A_k;\Delta}(\alpha\xi_{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}})\right\}\Big) \\ &\quad\times \sum_{\beta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{B}}} \Big(\prod_{k\in\mathcal{L}^{(2\wp)}} \exp\left\{E^{\ell,\wp}_{B_k;\Delta}(\beta_{B_k}|\xi) + W^{\ell,\wp}_{B_k;\Delta}(\alpha\beta\xi_{\mathcal{C}\cup\mathcal{D}})\right\}\Big) \\ &\quad\times \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \Big(\prod_{k\in\mathcal{L}^{(2\wp)}} \exp\left\{E^{\ell,\wp}_{C_k;\Delta}(\gamma_{C_k}|\xi) + W^{\ell,\wp}_{C_k;\Delta}(\alpha\beta\gamma\xi_{\mathcal{D}})\right\}\Big) \\ &\quad\times \sum_{\delta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{D}}} \Big(\prod_{k\in\mathcal{L}^{(2\wp)}} \exp\left\{E^{\ell,\wp}_{D_k;\Delta}(\delta_{D_k}|\xi) + W^{\ell,\wp}_{D_k;\Delta}(\alpha\beta\gamma\delta)\right\}\Big). \end{aligned} \tag{5.14}$$

Notice that although the sum defining the partition function is extended to the volume Δ, it is convenient, for practical reasons, to consider the sums extended to the whole lattice $\mathcal{L}^{(\wp)}$. This has been realized in the last step of (5.14) via the definition (5.9) of the function $E^{(\ell),\wp}_{x;\Delta}$.

In order to get (5.7) we perform a sequence of decimations; we fix ξ and sum over δ, γ, β, and, finally, α following this prescribed order. At each decimation step, we perform three operations, called *unfolding*, *splitting* and *gluing*, which will show that the system of variables corresponding to the sub–lattice involved in the decimation is weakly coupled. This weak coupling is a consequence of the factorization properties of the partition functions on suitable finite volumes which follow from (5.8).

We pick a reference configuration $\bar\eta \in \mathcal{X}^{\ell,\wp}$ and let $\bar\xi := \bar\eta_\Delta \xi_{\Delta^c}$. By computing the last sum for $\delta \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{D}}$ in (5.14) and recalling $\wp > r$, we get

$$\sum_{\delta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{D}}} \prod_{k\in\mathcal{L}^{(2\wp)}} \exp\left\{E^{\ell,\wp}_{D_k;\Delta}(\delta_{D_k}|\xi) + W^{\ell,\wp}_{D_k;\Delta}(\alpha\beta\gamma\delta)\right\} = \prod_{k\in\mathcal{L}^{(2\wp)}} Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\beta\gamma\bar\xi_{\mathcal{D}}), \tag{5.15}$$

where from now on we understand $Z^{(\ell),\wp}_{m,\emptyset} = 1$. We also note that $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}$ depends only on the block spin configuration in the boundary of $\{D_k\}\cap\Delta$ namely, $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta} \in \mathcal{F}^{(\ell),\wp}_{\partial^{(\wp)}[\{D_k\}\cap\Delta]}$, in particular it does not depend on $\bar\xi_{\mathcal{D}}$. Finally we note that by definition (5.9), when (5.15) is plugged into (5.14), the function $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\cdot)$ will be evaluated in the configuration $(\alpha\beta\gamma)_\Delta\bar\xi_{\Delta^c\cup\mathcal{D}}$.

Given $D_k \in \mathcal{D}$ we denote by $(\beta\gamma)_{\mathrm{u}}$, resp. $(\beta\gamma)_{\mathrm{d}}$, the restriction of the configuration $\beta\gamma$ to the half–space above, resp. below, D_k. We now unfold the partition function $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}$ in the $e_2^{(\wp)}$ direction namely, we write

$$\begin{aligned} Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\beta\gamma\bar\xi_{\mathcal{D}}) &= Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\beta\gamma)_{\mathrm{d}}\bar\xi_{\mathcal{D}}) \\ &= \frac{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar\xi_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar\xi_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar\xi_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar\xi_{\mathcal{D}})}{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\bar\xi_{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}})} \\ &\quad \times\left[1+\Phi_{D_k}(\alpha\beta\gamma,\bar\xi)\right], \end{aligned} \tag{5.16}$$

where, recall $\bar\xi = \bar\eta_\Delta\xi_{\Delta^c}$, we have defined the function $\Phi_{D_k} : \mathcal{X}^{\ell,\wp}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \times \mathcal{X}^{\ell,\wp} \to \mathbb{R}$ as follows

$$\Phi_{D_k}(\alpha\beta\gamma,\xi) := \frac{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\beta\gamma)_{\mathrm{d}}\bar\xi_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\bar\xi_{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}})}{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar\xi_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar\xi_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar\xi_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar\xi_{\mathcal{D}})} - 1, \tag{5.17}$$

which can be considered as an effective interaction potential among the α,β,γ–variables due to the decimation on δ. To simplify the notation we do not make explicit the parametric dependence of Φ_{D_k} on $\wp$, Δ, and m. From the measurability properties of the partition function $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}$ we get

$$\Phi_{D_k}(\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{D_k\}\cap(\mathcal{A}\cup\mathcal{B}\cup\mathcal{C})} = \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{D_k\}} \;\text{ and }\; \Phi_{D_k}(\alpha\beta\gamma,\cdot) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{D_k\}\cap\Delta^c} \tag{5.18}$$

for all $\alpha \in \mathcal{X}^{\ell,\wp}_{\mathcal{A}}$, $\beta \in \mathcal{X}^{\ell,\wp}_{\mathcal{B}}$, $\gamma \in \mathcal{X}^{\ell,\wp}_{\mathcal{C}}$, and $\xi \in \mathcal{X}^{\ell,\wp}$, where we recall the definition of boundary given in Subsect. 2.1. The bound (5.8) implies that Φ_{D_k}, as well as similar effective interactions that will be defined later on, is uniformly small. We note, finally, that $\Phi_{D_k} = 0$ if $\{D_k\} \cap \Delta = \emptyset$.

We next split the product of the numerator in (5.16) in the $e_2^{(\wp)}$ direction namely, we write

$$\prod_{k\in\mathcal{L}^{(2\wp)}} Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar{\xi}_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar{\xi}_{\mathcal{D}})$$
$$= \prod_{k\in\mathcal{L}^{(2\wp)}} Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar{\xi}_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k+e_2^{(2\wp)}\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar{\xi}_{\mathcal{D}}). \tag{5.19}$$

By (5.15), (5.16) and (5.19) we have that

$$\sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \prod_{k\in\mathcal{L}^{(2\wp)}} e^{E^{\ell,\wp}_{C_k;\Delta}(\gamma_{C_k}|\xi)+W^{\ell,\wp}_{C_k;\Delta}(\alpha\beta\gamma\xi_{\mathcal{D}})} \sum_{\delta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{D}}} \prod_{k\in\mathcal{L}^{(2\wp)}} e^{E^{\ell,\wp}_{D_k;\Delta}(\delta_{D_k}|\xi)+W^{\ell,\wp}_{D_k;\Delta}(\alpha\beta\gamma\delta)}$$
$$= \prod_{k\in\mathcal{L}^{(\wp)}} Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\bar{\xi}_{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}})^{-1}$$
$$\times \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \prod_{k\in\mathcal{L}^{(2\wp)}} \Big[e^{E^{\ell,\wp}_{C_k;\Delta}(\gamma_{C_k}|\xi)+W^{\ell,\wp}_{C_k;\Delta}(\alpha\beta\gamma\xi_{\mathcal{D}})} (1+\Phi_{D_k}(\alpha\beta\gamma,\bar{\xi}))$$
$$\times Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar{\xi}_{\mathcal{D}}) Z^{(\ell),\wp}_{m,\{D_k+e_2^{(2\wp)}\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar{\xi}_{\mathcal{D}})\Big]$$
$$= \Bigg[\prod_{k\in\mathcal{L}^{(\wp)}} \frac{Z^{(\ell),\wp}_{m,\widetilde{C}_k\cap\Delta}((\alpha\beta)_{\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})}\bar{\xi}_{[\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})]^{\mathrm{c}}})}{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha\bar{\xi}_{\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}})}\Bigg]$$
$$\times \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \nu_{\mathcal{C}}(\gamma|\alpha\beta,\bar{\xi}) \prod_{k\in\mathcal{L}^{(\wp)}} (1+\Phi_{D_k}(\alpha\beta\gamma,\bar{\xi})), \tag{5.20}$$

where we have defined $\widetilde{C}_k := \{D_k + e_2^{(2\wp)}, C_k, D_k\} \subset \mathcal{L}^{(\wp)}$, see Fig. 2 below, and introduced the product measure

$$\nu_{\mathcal{C}}(\gamma|\alpha\beta,\bar{\xi}) := \prod_{k\in\mathcal{L}^{(2\wp)}} \nu_{C_k}(\gamma_{C_k}|\alpha\beta,\bar{\xi}) \tag{5.21}$$

with

$$\nu_{C_k}(\gamma_{C_k}|\alpha\beta,\bar{\xi}_{\mathcal{D}}) := \exp\big\{E^{\ell,\wp}_{C_k;\Delta}(\gamma_{C_k}|\xi) + W^{\ell,\wp}_{C_k;\Delta}(\alpha\beta\gamma\xi_{\mathcal{D}})\big\}$$
$$\times \frac{Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar{\xi}_{\mathcal{D}})\; Z^{(\ell),\wp}_{m,\{D_k+e_2^{(2\wp)}\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar{\xi}_{\mathcal{D}})}{Z^{(\ell),\wp}_{m,\widetilde{C}_k\cap\Delta}((\alpha\beta)_{\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})}\bar{\xi}_{[\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})]^{\mathrm{c}}})}. \tag{5.22}$$

To simplify the notation we do not make explicit the parametric dependence of ν_{C_k} on $\wp$, Δ, and m. The definition above is well posed because the right-hand side depends on the configuration γ only through its restriction to C_k. Recalling $\wp > r$ we have that $Z^{(\ell),\wp}_{m,\widetilde{C}_k\cap\Delta} \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}[\widetilde{C}_k\cap\Delta]}$, moreover by using definitions (5.22), (5.9), (5.11), and the properties of measurability of the partition function $Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}$ we get

$$\nu_{C_k}(\gamma_{C_k}|\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})} \qquad \text{and} \qquad \nu_{C_k}(\gamma_{C_k}|\alpha\beta,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{C_k\}}^{(\wp),2}\cap\Delta^{\mathrm{c}}} \tag{5.23}$$

for all $\alpha \in \mathcal{X}^{\ell,\wp}_{\mathcal{A}}$, $\beta \in \mathcal{X}^{\ell,\wp}_{\mathcal{B}}$, $\gamma \in \mathcal{X}^{\ell,\wp}_{\mathcal{C}}$, and $\xi \in \mathcal{X}^{\ell,\wp}_{m}$. Moreover, we remark that ν_{C_k} is a probability measure on $\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\{C_k\}}$ since the gluing identity

$$\begin{aligned} &Z^{(\ell),\wp}_{m,\widetilde{C}_k\cap\Delta}((\alpha\beta)_{\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})}\bar{\xi}_{[\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})]^{\mathrm{c}}}) \\ &\quad = \sum_{\gamma_{C_k}\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,C_k}} \exp\left\{E^{\ell,\wp}_{C_k;\Delta}(\gamma_{C_k}|\xi) + W^{\ell,\wp}_{C_k;\Delta}(\alpha\beta\gamma\xi_{\mathcal{D}})\right\} \\ &\qquad \times\ Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{u}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{d}}\bar{\xi}_{\mathcal{D}})\, Z^{(\ell),\wp}_{m,\{D_k+e_2^{(2\wp)}\}\cap\Delta}(\alpha(\beta\gamma)_{\mathrm{d}}(\bar{\xi}_{\mathcal{B}\cup\mathcal{C}})_{\mathrm{u}}\bar{\xi}_{\mathcal{D}}) \end{aligned} \tag{5.24}$$

holds. We finally remark that $\nu_{C_k}(\gamma_{C_k}|\alpha\beta,\bar{\xi}_{\mathcal{D}}) = \mathbb{1}_{\{\gamma_{C_k}=\bar{\xi}_{C_k}\}}$ whenever $C_k \notin \Delta$.

By following the procedure of [1], with the modifications illustrated above, we straightforwardly get (5.7) with

$$\bar{Z}^{(\ell),\wp}_{m,\Delta}(\xi) := \prod_{k\in\mathcal{L}^{(2\wp)}} \frac{Z^{(\ell),\wp}_{m,\widetilde{A}_k\cap\Delta}(\bar{\xi})\; Z^{(\ell),\wp}_{m,\{D_k\}\cap\Delta}(\bar{\xi})}{Z^{(\ell),\wp}_{m,F_k\cap\Delta}(\bar{\xi})\; Z^{(\ell),\wp}_{m,\widetilde{C}_k\cap\Delta}(\bar{\xi})}, \tag{5.25}$$

where $F_k := \{C_k - e_1^{(2\wp)}, B_k, C_k\}$ and $\widetilde{A}_k := \{A_k\} \cup \partial^{(2\wp)}\{A_k\}$, see Fig. 2, and

$$\begin{aligned} \Xi^{(\ell),\wp}_{m,\Delta}(\xi) := &\sum_{\alpha\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{A}}} \prod_{k\in\mathcal{L}^{(2\wp)}} \nu_{A_k}(\alpha_{A_k}|\bar{\xi})\,\big(1+\Psi_{D_k}(\alpha,\bar{\xi})\big)\big(1+\Psi_{A_k}(\alpha,\bar{\xi})\big) \\ &\times\big(1+\Phi_{B_k}(\alpha,\bar{\xi})\big) \\ &\times \sum_{\beta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{B}}} \prod_{k\in\mathcal{L}^{(2\wp)}} \nu_{B_k}(\beta_{B_k}|\alpha,\bar{\xi})\,\big(1+\Phi_{C_k}(\alpha\beta,\bar{\xi})\big) \\ &\times \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \prod_{k\in\mathcal{L}^{(2\wp)}} \nu_{C_k}(\gamma_{C_k}|\alpha\beta,\bar{\xi})\,\big(1+\Phi_{D_k}(\alpha\beta\gamma,\bar{\xi})\big), \end{aligned} \tag{5.26}$$

where the Ψ's and Φ's are error terms similar to the one explicitly defined in (5.17), and each ν_x is a probability measure on $\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\{x\}}$, for $x \in \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$, similar to the one in (5.22). All these functions can be defined as in [1], we do not enter here into these details, we just recall their measurability properties. For each $\alpha \in \mathcal{X}^{\ell,\wp}_{\mathcal{A}}$, $\beta \in \mathcal{X}^{\ell,\wp}_{\mathcal{B}}$, $\gamma \in \mathcal{X}^{\ell,\wp}_{\mathcal{C}}$, and $\xi \in \mathcal{X}^{\ell,\wp}$ we have

$$\begin{aligned} &\Phi_{C_k}(\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{C_k\}\cap(\mathcal{A}\cup\mathcal{B})}, \quad \Phi_{C_k}(\alpha\beta,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{C_k\}}^{(\wp),2}\cap\Delta^{\mathrm{c}}}, \quad \Phi_{B_k}(\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{B_k\}\cap\mathcal{A}}, \\ &\Phi_{B_k}(\alpha,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{B_k\}}^{(\wp),2}\cap\Delta^{\mathrm{c}}}, \quad \Psi_{D_k}(\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{D_k\}\cap\mathcal{A}}, \quad \Psi_{D_k}(\alpha,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{D_k\}}^{(\wp)}\cap\Delta^{\mathrm{c}}}, \\ &\Psi_{A_k}(\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{A_k}, \quad \Psi_{A_k}(\alpha,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{A_k\}}^{(\wp),2}\cap\Delta^{\mathrm{c}}}, \end{aligned} \tag{5.27}$$

and

$$\nu_{B_k}(\beta_{B_k}|\cdot,\xi) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}\{B_k\}\cap\mathcal{A}},\ \nu_{B_k}(\beta_{B_k}|\alpha,\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{B_k\}}^{(\wp),2}\cap\Delta^c} \text{ and } \nu_{A_k}(\alpha_{A_k}|\cdot) \in \mathcal{F}^{\ell,\wp}_{\overline{\{A_k\}}^{(\wp),2}\cap\Delta^c}. \tag{5.28}$$

We also set

$$\nu_{\mathcal{B}}(\beta|\xi) := \prod_{B\in\mathcal{B}} \nu_B(\beta_B|\alpha,\xi) \quad \text{and} \quad \nu_{\mathcal{A}}(\alpha|\xi) := \prod_{A\in\mathcal{A}} \nu_A(\alpha_A|\xi) \tag{5.29}$$

for all $\alpha \in \mathcal{X}^{\ell,\wp}_{\mathcal{A}}$, $\beta \in \mathcal{X}^{\ell,\wp}_{\mathcal{B}}$, and $\xi \in \mathcal{X}^{\ell,\wp}$.

We next rewrite the functions $\bar{Z}^{(\ell),\wp}_{m,\Delta}$ and $\Xi^{(\ell),\wp}_{m,\Delta}$ having in mind that our goal is the definition of the family $\{V^{(\ell),\wp}_{X,\Lambda},\ X \subset\subset \mathcal{L}^{(\wp)}\}$ whose existence has been stated in the theorem. We first define the collection of subsets of the lattice $\mathcal{L}^{(\wp)}$

$$\mathcal{G} := \bigcup_{k\in\mathcal{L}^{(2\wp)}} \{\{D_k\}, \widetilde{C}_k,\ F_k,\ \widetilde{A}_k\} \tag{5.30}$$

and for all $k \in \mathcal{L}^{(2\wp)}$ we set

$$g(\{D_k\}) = +1, \quad g(\widetilde{A}_k) = +1, \quad g(F_k) = -1, \quad \text{and} \quad g(\widetilde{C}_k) = -1. \tag{5.31}$$

From (5.25) we then have

$$\log \bar{Z}^{(\ell),\wp}_{m,\Delta}(\xi) = \sum_{G\in\mathcal{G}} g(G) \log Z^{(\ell),\wp}_{m,G\cap\Delta}(\bar{\xi}). \tag{5.32}$$

Recalling that we always understand $Z^{(\ell),\wp}_{m,\emptyset} = 1$ and that Δ is a finite subset of the lattice $\mathcal{L}^{(\wp)}$, we have that the sum in (5.32) has indeed a finite number of terms. We prove, now, that

$$\sum_{i\in\mathcal{O}^{\ell}_{\wp}Y} \frac{1}{2}m_i^2 = \sum_{G\in\mathcal{G}} g(G) \sum_{i\in\mathcal{O}^{\ell}_{\wp}(G\cap Y)} \frac{1}{2}m_i^2 \tag{5.33}$$

for all $Y \subset \mathcal{L}^{(\wp)}$. Indeed, we first remark that

$$\sum_{G\in\mathcal{G}} g(G) \sum_{i\in\mathcal{O}^{\ell}_{\wp}(G\cap Y)} \frac{1}{2}m_i^2 = \sum_{i\in\mathcal{O}^{\ell}_{\wp}Y} \frac{1}{2}m_i^2 \sum_{G\in\mathcal{G}:\,\mathcal{O}^{\ell}_{\wp}G\ni i} g(G). \tag{5.34}$$

The identity (5.33) follows from (5.34) once we prove that $\sum_{G\in\mathcal{G}:\,\mathcal{O}^{\ell}_{\wp}G\ni i} g(G) = +1$ for each $i \in \mathcal{L}^{(\ell)}$. Pick $i \in \mathcal{L}^{(\ell)}$ and suppose there exists $k' \in \mathcal{L}^{(2\wp)}$ such that $i \in \mathcal{O}^{\ell}_{\wp}\{D_{k'}\}$. The only G's of $\mathcal{G}$ such that $\mathcal{O}^{\ell}_{\wp}G \ni i$ are $\widetilde{A}_{k'}$, $\widetilde{A}_{k'+e_1^{(2\wp)}}$, $\widetilde{C}_{k'}$, $\widetilde{C}_{k'-e_2^{(2\wp)}}$, and $\{D_{k'}\}$, see Fig. 2. Then

$$\sum_{G\in\mathcal{G}:\,\mathcal{O}^{\ell}_{\wp}G\ni i} g(G) = g(\widetilde{A}_{k'}) + g(\widetilde{A}_{k'+e_1^{(2\wp)}}) + g(\widetilde{C}_{k'}) + g(\widetilde{C}_{k'-e_2^{(2\wp)}}) + g(\{D_{k'}\}) = +1,$$

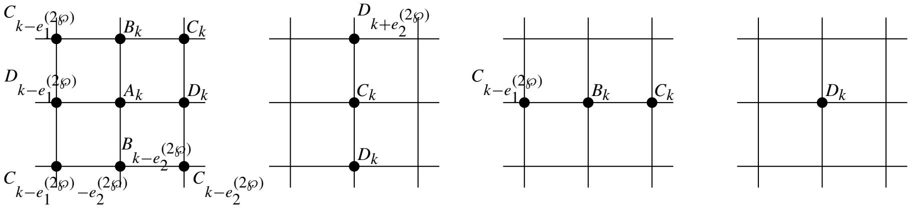

Fig. 2. From the left to the right the sets $\widetilde{A}_k, \widetilde{C}_k, F_k, \{D_k\} \subset \mathcal{L}^{(\wp)}$ are depicted for some $k \in \mathcal{L}^{(2\wp)}$. Solid circles denote the sites belonging to those subsets; intersections of lines represent sites in $\mathcal{L}^{(\wp)}$

where in the last equality we have used (5.31). The other three cases, where $i \in \mathcal{O}^\ell_\wp\{A_{k'}\}$, $i \in \mathcal{O}^\ell_\wp\{B_{k'}\}$, or $i \in \mathcal{O}^\ell_\wp\{C_{k'}\}$ for a suitable $k' \in \mathcal{L}^{(2\wp)}$, can be treated similarly.

We recall that μ^τ_X, for $X \subset\subset \mathcal{L}$, is the finite volume (grancanonical) Gibbs measure of the original lattice gas, see Subsect. 2.5, χ is the infinite volume compressibility defined in (2.8), and that for $i \in \mathcal{L}^{(\ell)}$ the function $M^{(\ell)}_i$ is defined in (2.11). Then for $G \in \mathcal{G}$ we define the $|\mathcal{O}^\ell_\wp(G \cap \Lambda)| \times |\mathcal{O}^\ell_\wp(G \cap \Lambda)|$ covariance matrix

$$\left(\mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}\right)_{i,j} := 2\pi\chi\ell^d \mu^{\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar{\eta}}_{\mathcal{O}_\wp(G\cap\Lambda)}(M^{(\ell)}_i, M^{(\ell)}_j) \tag{5.35}$$

for $i, j \in \mathcal{O}^\ell_\wp(G \cap \Lambda)$, with $\bar{\eta}$ the reference in $\mathcal{X}^{(\ell),\wp}$ chosen before (5.15). We understand $\mathbb{V}^{(\ell),\bar{\eta}}_\emptyset$ is equal to the 1×1 matrix with its sole element equal to 1. We let, as in Subsect. 2.5, $Z_X(\tau)$, with $X \subset\subset \mathcal{L}$ and $\tau \in \mathcal{X}$, be the (grancanonical) partition function of the original lattice gas model, on X with boundary condition τ. Then we define the real

$$K^{(\wp)}_\Lambda := \sum_{G\in\mathcal{G}} g(G) \log\left[Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar{\eta})/\sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}\right]. \tag{5.36}$$

By using (5.32), (5.33) for $Y = \Lambda$, and (5.36) we rewrite $\log \bar{Z}^{(\ell),\wp}_{m,\Delta}(\xi)$ as follows

$$\begin{aligned}\log \bar{Z}^{(\ell),\wp}_{m,\Delta}(\xi) = K^{(\wp)}_\Lambda - \sum_{i\in\mathcal{O}^\ell_\wp\Lambda} \frac{1}{2}m_i^2 \\ + \sum_{G\in\mathcal{G}} g(G)\Big[\log \frac{Z^{(\ell),\wp}_{m,G\cap\Delta}(\bar{\xi})\sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}}{Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar{\eta})} + \sum_{i\in\mathcal{O}^\ell_\wp(G\cap\Lambda)} \frac{1}{2}m_i^2\Big].\end{aligned} \tag{5.37}$$

Consider, now, the function $\Xi^{(\ell),\wp}_{m,\Delta}$ defined in (5.26); we show that it can be rewritten as the partition function of a gas of polymers. We first associate to each error term $\Phi_{D_k}, \Phi_{C_k}, \ldots, \Psi_{D_k}$ appearing in (5.26) a subset of the lattice that will be called *bond*. More precisely, for the Φ error terms we set

$$\begin{aligned}&e(\Phi_{D_k}) := \partial^{(\wp)}\{D_k\} \cup (\partial^{(\wp),2}\{D_k\} \cap \mathcal{A}), e(\Phi_{C_k}) := \partial^{(\wp)}\{C_k\} \cap (\mathcal{A} \cup \mathcal{B}), \\ &e(\Phi_{B_k}) := \partial^{(\wp)}\{B_k\} \cap \mathcal{A}.\end{aligned} \tag{5.38}$$

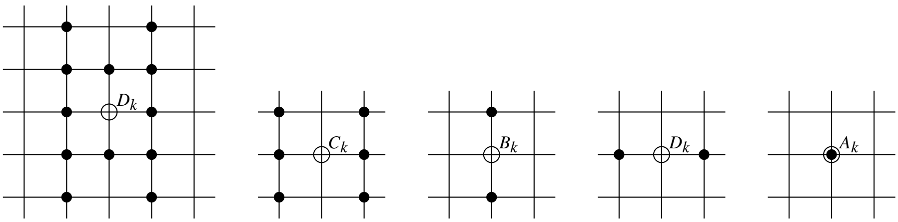

Fig. 3. From the left to the right the bonds $e(\Phi_{D_k})$, $e(\Phi_{C_k})$, $e(\Phi_{B_k})$, $e(\Psi_{D_k})$, and $e(\Psi_{A_k})$ are depicted for some $k \in \mathcal{L}^{(2\wp)}$. Solid circles denote the sites belonging to the bonds; open circles denote the site labelling a bond; intersections of lines represent sites in $\mathcal{L}^{(\wp)}$

For the Ψ error terms we set

$$e(\Psi_{D_k}) := \partial^{(\wp)}\{D_k\} \cap \mathcal{A} \quad \text{and} \quad e(\Psi_{A_k}) := \{A_k\}, \tag{5.39}$$

see Fig. 3. Moreover, in this section we denote by

$$\mathcal{E} := \bigcup_{k \in \mathcal{L}^{(2\wp)}} \{e(\Phi_{D_k}), e(\Phi_{C_k}), e(\Phi_{B_k}), e(\Psi_{D_k}), e(\Psi_{A_k})\} \tag{5.40}$$

the collection of all the bonds. For each $e \in \mathcal{E}$ we denote by $\Theta_e : \mathcal{X}^{\ell,\wp}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}} \times \mathcal{X}^{\ell,\wp} \to \mathbb{R}$ the error term with which the bond e is associated and we call it *weight* of the bond; for instance if $e = e(\Phi_{D_k})$ then $\Theta_e = \Phi_{D_k}$. We notice that by expanding the products in (5.26) we get also addends with a single error term which must be averaged against the measures ν's; the bond have been defined so that the infinite volume average in (5.26) can be replaced by the average restricted to the bond itself. More precisely, consider the bond $e \in \mathcal{E}$ and the corresponding error term Θ_e, by using (5.18), (5.27), (5.23), and (5.28) we have that for each $\xi \in \mathcal{X}^{\ell,\wp}$,

$$\begin{aligned}
&\sum_{\alpha \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{A}}} \sum_{\beta \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{B}}} \sum_{\gamma \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \nu_{\mathcal{A}}(\alpha|\xi)\nu_{\mathcal{B}}(\beta|\alpha,\xi)\nu_{\mathcal{C}}(\gamma|\alpha\beta,\xi)\,\Theta_e(\alpha\beta\gamma,\xi)\\
&\quad = \sum_{\alpha \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,e\cap\mathcal{A}}} \sum_{\beta \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,e\cap\mathcal{B}}} \sum_{\gamma \in \mathcal{Y}^{(\ell),\wp}_{\Delta,m,e\cap\mathcal{C}}} \prod_{A \in e\cap\mathcal{A}} \nu_A(\alpha_A|\xi) \prod_{B \in e\cap\mathcal{B}} \nu_B(\beta_B|\alpha,\xi)\\
&\qquad \times \prod_{C \in e\cap\mathcal{C}} \nu_C(\gamma_C|\alpha\beta,\xi)\,\Theta_e(\alpha\beta\gamma,\xi).
\end{aligned} \tag{5.41}$$

Consider, now, a collection $\{e_1, \ldots, e_k\}$ of pairwise different elements of $\mathcal{E}$, we say that such a collection is a polymer if and only if for each $i, i' \in \{1, \ldots, k\}$ there exists $i_1, \ldots, i_s \in \{1, \ldots, k\}$ such that $e_i = e_{i_1}$, $e_{i_1} \cap e_{i_2} \neq \emptyset, \cdots, e_{i_{s-1}} \cap e_{i_s} \neq \emptyset$, $e_{i_s} = e_{i'}$. We denote by $\mathcal{R}$ the collection of all polymers and for each $R \in \mathcal{R}$ we set

$$\widetilde{R} := \bigcup_{e \in R} e \subset \mathcal{L}^{(\wp)}. \tag{5.42}$$

By expanding the products in (5.26) and by standard polymerization, for $\xi \in \mathcal{X}^{\ell,\wp}$ we have that

$$\Xi^{(\ell),\wp}_{m,\Delta}(\xi) = 1 + \sum_{k\geq 1} \sum_{\substack{R_1,\dots,R_k\in\mathcal{R}:\\ \widetilde{R}_i\cap\widetilde{R}_j=\emptyset, i\neq j}} \prod_{j=1}^{k} \zeta^{(\ell),\wp}_{m,R_j,\Delta}(\xi), \tag{5.43}$$

where the *activity* $\zeta^{(\ell),\wp}_{m,R,\Delta}$ associated with a polymer $R \in \mathcal{R}$ is given by

$$\begin{aligned}\zeta^{(\ell),\wp}_{m,R,\Delta}(\xi) &:= \sum_{\alpha\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{A}}} \sum_{\beta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{B}}} \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\mathcal{C}}} \nu_{\mathcal{A}}(\alpha|\xi)\nu_{\mathcal{B}}(\beta|\alpha,\xi)\nu_{\mathcal{C}}(\gamma|\alpha\beta,\xi)\prod_{e\in R}\Theta_e(\alpha\beta\gamma,\bar{\xi})\\ &= \sum_{\alpha\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\widetilde{R}\cap\mathcal{A}}} \sum_{\beta\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\widetilde{R}\cap\mathcal{B}}} \sum_{\gamma\in\mathcal{Y}^{(\ell),\wp}_{\Delta,m,\widetilde{R}\cap\mathcal{C}}} \prod_{A\in\widetilde{R}\cap\mathcal{A}} \nu_A(\alpha_A|\xi)\\ &\quad\times \prod_{B\in\widetilde{R}\cap\mathcal{B}} \nu_B(\beta_B|\alpha,\xi) \prod_{C\in\widetilde{R}\cap\mathcal{C}} \nu_C(\gamma_C|\alpha\beta,\xi)\prod_{e\in R}\Theta_e(\alpha\beta\gamma,\xi),\end{aligned} \tag{5.44}$$

where the last equality holds by the same arguments used to prove (5.41). We remark that the sum in (5.43) is restricted to a finite number of "non–intersecting" polymers, indeed the error term Θ_e associated to a bond sufficiently far from Δ is equal to zero. This can be easily checked in the case of Φ_{D_k}: by using definition (5.17) and recalling $Z^{(\ell),\wp}_{m,\emptyset} = 1$, we have that $\{D_k\}\cap\Delta = \emptyset$ implies $\Phi_{D_k} = 0$. By looking at the definitions of the error terms Θ_e, those given in [1] and suitably modified as we did in (5.17), it is easy to check that for each $e \in \mathcal{E}$,

$$\overline{e}^{(\wp)} \subset \Delta^{\mathrm{c}} \implies \Theta_e = 0. \tag{5.45}$$

Finally, we note that the activity $\zeta^{(\ell),\wp}_{m,R,\Delta}(\xi)$ of a polymer $R \in \mathcal{R}$ is a local function of ξ, indeed by using (5.44), (5.28), and (5.27) we have that

$$\zeta^{(\ell),\wp}_{m,R,\Delta} \in \mathcal{F}^{\ell,\wp}_{(\widetilde{R}\cup\partial^{(\wp),2}\widetilde{R})\cap\Delta^{\mathrm{c}}}. \tag{5.46}$$

Let us consider a collection of polymers $\{R_1,\dots,R_k\}$; we say that it is a *cluster of polymers* if and only if for each $i,i' \in \{1,\dots,k\}$ there exists $i_1,\dots,i_s \in \{1,\dots,k\}$ such that $R_i = R_{i_1}$, $\widetilde{R}_{i_1}\cap\widetilde{R}_{i_2} \neq \emptyset, \cdots, \widetilde{R}_{i_{s-1}}\cap\widetilde{R}_{i_s} \neq \emptyset$, $R_{i_s} = R_{i'}$. We denote by $\underline{\mathcal{R}}$ the collection of all clusters of polymers and for each $\underline{R} \in \underline{\mathcal{R}}$ we set

$$\widetilde{\underline{R}} := \bigcup_{R\in\underline{R}} \widetilde{R} \subset \mathcal{L}^{(\wp)}. \tag{5.47}$$

We finally introduce some combinatorial factors as follows: let $F(R_1,\dots,R_k)$ be the collection of connected subgraphs with vertex set $\{1,\dots,k\}$ of the graph with vertices $\{1,\dots,k\}$ and edges $\{i,j\}$ corresponding to pairs R_i, R_j such that $\widetilde{R}_i\cap\widetilde{R}_j \neq \emptyset$, then

$$\varphi_T(R_1,\dots,R_k) := \frac{1}{k!}\sum_{f\in F(R_1,\dots,R_k)} (-1)^{\#\text{ edges in } f}; \tag{5.48}$$

we set the sum equal to zero if F is empty and one if $k = 1$. Then, by a standard cluster expansion, see for instance [22,24], under suitable small activity conditions that we shall specify later on, the polymer gas partition function (5.43) can be written as follows

$$\log \Xi^{(\ell),\wp}_{m,\Delta}(\xi) = \sum_{\underline{R}\in\underline{\widetilde{R}}} \varphi_T(\underline{R})\zeta^{(\ell),\wp}_{m,\underline{R},\Delta}(\xi), \tag{5.49}$$

where for each $\underline{R} \in \underline{\widetilde{R}}$ we have set

$$\zeta^{(\ell),\wp}_{m,\underline{R},\Delta} := \prod_{R\in\underline{R}} \zeta^{(\ell),\wp}_{m,R,\Delta} \in \mathcal{F}^{\ell,\wp}_{(\underline{\widetilde{R}}\cup\partial^{(\wp),2}\underline{\widetilde{R}})\cap\Delta^{c}}. \tag{5.50}$$

As remarked above, see (5.45), the activity of polymers containing at least a bond e such that $\overline{e}^{(\wp)} \subset \Delta^{c}$ is equal to zero, so that only polymers with support close to Δ have non–zero activity. Nevertheless, the sum on the right–hand side of (5.49) is infinite due to the fact that in a cluster of polymers a given polymer can be repeated arbitrarily many times. We next prove that for ℓ a large enough multiple of ℓ_0 the series is absolutely convergent. We shall use the technique developed in [8] to get a uniform estimate of the sum of the activity of all the polymers whose support contains a given site $x \in \mathcal{L}^{(\wp)}$; such an estimate will be then used as the input of the abstract theory developed in [24] to estimate the sum (5.49) which is extended to the clusters of polymers whose support intersects Δ.

Let $e \in \mathcal{E}$, consider the corresponding error term Θ_e. By looking at the definition of Φ_{D_k} given in (5.17) and at the similar expressions in [1] for the other error terms we have that (5.8) implies

$$\sup_{\alpha\in\mathcal{X}^{\ell,\wp}_{\mathcal{A}}} \sup_{\beta\in\mathcal{X}^{\ell,\wp}_{\mathcal{B}}} \sup_{\gamma\in\mathcal{X}^{\ell,\wp}_{\mathcal{C}}} \sup_{\xi\in\mathcal{X}^{\ell,\wp}} |\Theta_e(\alpha\beta\gamma,\xi)| \le \frac{C}{\ell} \tag{5.51}$$

for ℓ a multiple of ℓ_0. Consider, now, a polymer and its activity $\zeta^{(\ell),\wp}_{m,R,\Delta}$ defined in (5.44); from (5.51) we have the bound

$$\|\zeta^{(\ell),\wp}_{m,R,\Delta}\|_\infty \le \prod_{e\in R} \frac{C}{\ell} \le \varepsilon^{2|\widetilde{R}|}, \tag{5.52}$$

where we have set $\varepsilon = \varepsilon(\ell) := (C/\ell)^{1/(2\kappa')}$, with $\kappa' = \kappa'(d)$ the maximum cardinality of the bonds (equal to 12 in dimension two, see Fig. 3), and we are considering ℓ so large that $\varepsilon(\ell) < 1$. We remark that for the current purpose it would have been sufficient to define $\varepsilon(\ell) = (C/\ell)^{1/\kappa'}$; the extra factor 2 will be used in the proof of Item 5.

For each polymer $R \in \mathcal{R}$ we set, now, $\bar{\zeta}_R = \bar{\zeta}_R(\ell) := [\varepsilon(\ell)]^{|\widetilde{R}|}$ and we prove that for ℓ large enough

$$\sup_{x\in\mathcal{L}^{(\wp)}} \sum_{R\in\mathcal{R}:\,\widetilde{R}\ni x} \bar{\zeta}_R(\ell)\, e^{|\widetilde{R}|} \le 1. \tag{5.53}$$

Indeed, from (5.38)–(5.40) we have that there exist a real $\kappa'' = \kappa''(d)$ such that $|\{e \in \mathcal{E} : e \ni x\}| \le \kappa''$ for all $x \in \mathcal{L}^{(\wp)}$. Moreover by choosing ℓ large enough we have that

$\exp\{\kappa''\} \leq [e\varepsilon(2 - e\varepsilon)]^{-1}$. We can now perform the estimate in [8, Appendix B], by replacing there $\zeta(R)$ with $\bar{\zeta}_R$, σ with $e\varepsilon$, and φ_e with 1, to obtain

$$\sup_{x \in \mathcal{L}^{(\wp)}} \sum_{R \in \mathcal{R}\colon \widetilde{R} \ni x} \bar{\zeta}_R \, e^{|\widetilde{R}|} \leq e\varepsilon\kappa'' \Big[1 + \frac{e^{\kappa''} - 1}{1 + (e\varepsilon)^2 e^{\kappa''} - 2e\varepsilon e^{\kappa''}} \Big]. \tag{5.54}$$

The bound (5.53) now follows trivially for ℓ large enough.

We are now ready to apply the abstract theory developed in [24]. Given a polymer $S \in \mathcal{R}$, by using (5.53), we have that

$$\sum_{\substack{R \in \mathcal{R}: \\ \widetilde{R} \cap \widetilde{S} \neq \emptyset}} \bar{\zeta}_R \, e^{|\widetilde{R}|} \leq \sum_{x \in \widetilde{S}} \sum_{\substack{R \in \mathcal{R}: \\ \widetilde{R} \ni x}} \bar{\zeta}_R \, e^{|\widetilde{R}|} \leq |\widetilde{S}| \implies \sum_{\substack{\underline{R} \in \underline{\mathcal{R}}: \\ \widetilde{\underline{R}} \cap \widetilde{S} \neq \emptyset}} \varphi_T(\underline{R}) \prod_{R \in \underline{R}} \bar{\zeta}_R \leq |\widetilde{S}|, \tag{5.55}$$

where the last bound is a direct consequence of the theorem in [24] whenever we choose there $a(R) = |\widetilde{R}|$. The absolute convergence of (5.49) for ℓ large enough follows easily from (5.55) once we recall that the activity of a cluster of polymer $\underline{R}$ such that $\widetilde{\underline{R}} \cap \Delta = \emptyset$ is equal to zero and we note that for ℓ large enough,

$$\|\zeta^{(\ell),\wp}_{m,R,\Delta}\|_\infty \leq (\bar{\zeta}_R)^2 \leq \bar{\zeta}_R, \tag{5.56}$$

where the first inequality is just a rewriting of (5.52).

Proof of Theorem 5.1. Item 1. First of all we recall $m = \mathcal{O}^\ell_\wp n$ and define the family $V^{(\ell),\wp}_{X,\Lambda}$ in the following way: for any $\xi \in \mathcal{X}^{\ell,\wp}_m$ and $X \subset\subset \mathcal{L}^{(\wp)}$ we set

$$\begin{aligned} V^{(\ell),\wp}_{X,\Lambda}(\xi, n) := &\sum_{\substack{G \in \mathcal{G}: \\ \overline{G}^{(\wp)} = X}} g(G) \Big[\log \frac{Z^{(\ell),\wp}_{m,G\cap\Delta}(\bar{\xi}) \sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}}{Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell \mathcal{O}^\ell_\wp \bar{\eta})} + \sum_{i \in \mathcal{O}^\ell_\wp(G\cap\Lambda)} \frac{1}{2} m_i^2 \Big] \\ &+ \sum_{\substack{\underline{R} \in \underline{\widetilde{R}}: \\ \widetilde{\underline{R}} \cup \partial^{(\wp),2} \widetilde{\underline{R}} = X}} \varphi_T(\underline{R}) \zeta^{(\ell),\wp}_{m,\underline{R},\Delta}(\xi). \end{aligned} \tag{5.57}$$

We prove, now, that for any $X \subset\subset \mathcal{L}^{(\wp)}$,

$$X \subset \Lambda^c \implies V^{(\ell),\wp}_{X,\Lambda} = 0. \tag{5.58}$$

Indeed, let $X \subset \Lambda^c$. Since

$$\overline{G}^{(\wp)} = X \subset \Lambda^c \implies G \subset \Lambda^c \implies G \cap \Delta = \emptyset = G \cap \Lambda$$

we have that the first sum in (5.57) is zero. Moreover, since $\widetilde{\underline{R}} \cup \partial^{(\wp),2} \widetilde{\underline{R}} = X \subset \Lambda^c \subset \Delta^c$, definitions (5.44), (5.50), and (5.45) imply that the second sum in (5.57) is zero as well. The expansion (5.3) finally follows from (5.7), (5.37), (5.49), and (5.57).

Suppose, now, that $X \cap \Delta = \emptyset$, by the same arguments used above we can easily prove that

$$V^{(\ell),\wp}_{X,\Lambda}(\xi, n) = \sum_{\substack{G \in \mathcal{G}: \\ \overline{G}^{(\wp)} = X}} g(G) \Big[\log \frac{\sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}}{Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell \mathcal{O}^\ell_\wp \bar{\eta})} + \sum_{i \in \mathcal{O}^\ell_\wp(G\cap\Lambda)} \frac{1}{2} m_i^2 \Big].$$

Hence, $V^{(\ell),\wp}_{X,\Lambda}(\cdot, n)$ is constant. Finally, we note that if we also have $\mathrm{diam}_\wp(X) > 5$, then $V^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = 0$ since there exists no $G \in \mathcal{G}$ such that $\overline{G}^{(\wp)} = X$. The proof of the item is completed by choosing κ large enough; in dimension two $\kappa \geq 5$ does the job.
Item 2. The statement follows from (5.57), the measurability property (5.50), and the following remarks: since $\wp > r$, where r is the range of the original lattice gas interaction, $Z^{(\ell),\wp}_{m,Y}(\cdot) \in \mathcal{F}^{\ell,\wp}_{\partial^{(\wp)}Y}$ for all $Y \subset \mathcal{L}^{(\wp)}$; $\partial^{(\wp)}[G \cap \Delta] \subset X$ whenever $\overline{G}^{(\wp)} = X$; $\bar{\bar{\xi}}_\Delta = \bar{\eta}_\Delta$.
Item 3. The statement is true by construction.
Item 4. The statement trivially follows from (5.57), (5.45), (5.40), and (5.30) by choosing κ large enough. In dimension two it is enough when $\kappa \geq 8$.
Item 5. We first recall that U is the potential of the original lattice gas model, r its range (see Subsect. 2.5), ℓ_0, b, and B the strong mixing constants (see Condition $\mathrm{SM}^{(s)}(\ell_0, b, B)$ in Subsect. 2.4). Pick $x \in \mathcal{L}^{(\wp)}$, and let $\alpha_1 > 0$ be chosen later; by using (5.57), the triangular inequality, and the fact that $|g(G)| = 1$ for all $G \in \mathcal{G}$, we have

$$
\begin{aligned}
&\sum_{X \ni x} e^{\alpha_\ell \mathbb{T}_\wp(X)} \sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \| V^{(\ell),\wp}_{X,\Lambda}(\cdot, n) \|_\infty \\
&\quad \leq \sum_{X \ni x} e^{\alpha_\ell \mathbb{T}_\wp(X)} \sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \sup_{\xi \in \mathcal{X}^{\ell,\wp}} \sum_{\substack{G \in \mathcal{G}: \\ \overline{G}^{(\wp)} = X}} \left| \log \frac{Z^{(\ell),\wp}_{m,G\cap\Delta}(\bar{\xi}) \sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}}{Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell \mathcal{O}^\ell_\wp \bar{\eta})} + \sum_{i \in \mathcal{O}^\ell_\wp(G\cap\Lambda)} \frac{1}{2} m_i^2 \right| \\
&\qquad + \sum_{X \ni x} e^{\alpha_\ell \mathbb{T}_\wp(X)} \sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \left\| \sum_{\underline{R} \in \widetilde{\underline{R}}:\, \widetilde{\underline{R}} \cup \partial^{(\wp),2} \widetilde{\underline{R}} = X} \varphi_T(\underline{R}) \zeta^{(\ell),\wp}_{m,\underline{R},\Delta}(\cdot) \right\|_\infty,
\end{aligned}
\tag{5.59}
$$

where $\mathbb{T}_\wp(X)$ has been defined in (2.1), α_ℓ is as in the hypothesis of the theorem, and $\Delta = \Lambda \cap \mathcal{L}^{(\wp)}_\delta$.

We now bound the first sum in the right–hand side of (5.59). By (5.30) we have that the terms corresponding to $X \subset \mathcal{L}^{(\wp)}$ such that $\mathrm{diam}_\wp(X) > \kappa$, where κ is as in the proof of Item 1, are equal to zero. Consider, now, $X \subset\subset \mathcal{L}^{(\wp)}$ such that $\mathrm{diam}(X) \leq \kappa$; we have that

$$
e^{\alpha_\ell \mathbb{T}_\wp(X)} = (e\ell)^{\alpha_1 \mathbb{T}_\wp(X)} \leq (e\ell)^{(\kappa+1)^d \alpha_1}.
$$

Moreover, since for each $G \in \mathcal{G}$ one has $\mathrm{diam}_\wp(G) \leq \kappa - 2$, there exists a real C' depending on ℓ_0, b, B, $\|U\|_0$, r, and the dimension of the space d such that

$$
\sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \sup_{\xi \in \mathcal{X}^{(\ell),\wp}} \sum_{\substack{G \in \mathcal{G}: \\ \overline{G}^{(\wp)} = X}} \left| \log \frac{Z^{(\ell),\wp}_{m,G\cap\Delta}(\bar{\xi}) \sqrt{\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}}}{Z_{\mathcal{O}_\wp(G\cap\Lambda)}(\mathcal{O}_\ell \mathcal{O}^\ell_\wp \bar{\eta})} + \sum_{i \in \mathcal{O}^\ell_\wp(G\cap\Lambda)} \frac{1}{2} m_i^2 \right| \leq C' \ell^d.
$$

Indeed, the bound is easy for the logarithm of the partition functions, follows from (2.10) for the $m_i^2/2$ contribution, and follows from the strong mixing condition $\mathrm{SM}(\ell_0, b, B)$ and the result in [1, Sect. 4] for the $\det \mathbb{V}^{(\ell),\bar{\eta}}_{G\cap\Lambda}$ terms. We can therefore conclude that the first sum in the right–hand side of (5.59) is bounded by

$$|\{X \subset\subset \mathcal{L}^{(\wp)} : X \ni 0, \operatorname{diam}_{\wp}(X) \le \kappa\}| \times (e\ell)^{(\kappa+1)^d \alpha_1} \times C'\ell^d =: C''\ell^{(\kappa+1)^d\alpha_1+d}, \tag{5.60}$$

where 0 denotes the origin of the lattice $\mathcal{L}^{(\wp)}$.

We bound, now, the second sum in the right–hand side of (5.59). Recall $\bar{\zeta}$ has been defined above (5.53) and choose ℓ so large that $\varepsilon(\ell) < 1$. Let $X \subset\subset \mathcal{L}^{(\wp)}$, we claim that for each cluster of polymers $\underline{R}$ such that $\underline{\widetilde{R}} \cup \partial^{(\wp),2}\underline{\widetilde{R}} = X$ we have that

$$\prod_{R\in\underline{\mathcal{R}}} \bar{\zeta}_R = \prod_{R\in\underline{\mathcal{R}}} \varepsilon^{|\widetilde{R}|} \le \varepsilon^{|\underline{\widetilde{R}}|} \le \varepsilon^{|X|/5^d} \le \varepsilon^{\mathbb{T}_{\wp}(X)/5^d} = e^{(\mathbb{T}_{\wp}(X)/5^d)\log\varepsilon}, \tag{5.61}$$

where we have used $\mathbb{T}_{\wp}(X) = |X| - 1$. We choose $\alpha_1 < 1/(2\cdot 5^d\kappa')$, recall κ' has been defined below (5.52). By taking ℓ large enough we have

$$e^{\alpha_\ell \mathbb{T}_{\wp}(X)} \prod_{R\in\underline{R}} \bar{\zeta}_R \le 1 \tag{5.62}$$

for any $X \subset\subset \mathcal{L}^{(\wp)}$ and $\underline{R}$ such that $\underline{\widetilde{R}} \cup \partial^{(\wp),2}\underline{\widetilde{R}} = X$. Therefore, recalling (5.52), the second term on the r.h.s. of (5.59) can be bounded by

$$\begin{aligned}
&\sum_{X\ni x} e^{\alpha_\ell \mathbb{T}_{\wp}(X)} \sum_{\substack{\underline{R}\in\underline{\widetilde{R}}:\\ \underline{\widetilde{R}}\cup\partial^{(\wp),2}\underline{\widetilde{R}}=X}} |\varphi_T(\underline{R})| \prod_{R\in\underline{R}} (\bar{\zeta}_R)^2\\
&\le \sum_{X\ni x} \sum_{\substack{\underline{R}\in\underline{\widetilde{R}}:\\ \underline{\widetilde{R}}\cup\partial^{(\wp),2}\underline{\widetilde{R}}=X}} |\varphi_T(\underline{R})| \prod_{R\in\underline{R}} \bar{\zeta}_R \le \sum_{\substack{e\in\mathcal{E}:\\ \bar{e}^{(\wp),2}\ni x}} \sum_{\substack{\underline{R}\in\underline{\widetilde{R}}:\\ \underline{\widetilde{R}}\cap e\neq\emptyset}} |\varphi_T(\underline{R})| \prod_{R\in\underline{R}} \bar{\zeta}_R\\
&\le \sum_{\substack{e\in\mathcal{E}:\\ \bar{e}^{(\wp),2}\ni x}} |e| =: \kappa'''(d),
\end{aligned} \tag{5.63}$$

where we used (5.55). The bound (5.4) follows from (5.59), (5.60), and (5.63) by setting $A_1 := C'' + \kappa'''$.

Item 6. Pick $\Lambda \subset\subset \mathcal{L}^{(\wp)}$, $X \subset \Lambda$, $n \in \mathcal{M}^{(\ell),\wp}$ such that $\mathcal{L}_\delta^{(\wp)}(n) \supset X$, and set $m = m(n) = \mathcal{O}_\wp^\ell n$; then $\Delta := \Lambda \cap \mathcal{L}_\delta^{(\wp)} \supset X$. Since $\wp > r$, where r is the range of the original lattice gas interaction, $Z_{m,Y}^{(\ell),\wp}(\cdot) \in \mathcal{F}_{\partial^{(\wp)}Y}^{\ell,\wp}$ for all $Y \subset \mathcal{L}^{(\wp)}$; then for each $G \in \mathcal{G}$ such that $\overline{G}^{(\wp)} = X$ we have $Z_{m,G}^{(\ell),\wp} \in \mathcal{F}_X^{\ell,\wp}$. Recall, now, that $\bar{\eta}$ is the reference configuration picked up in $\mathcal{X}^{\ell,\wp}$ before (5.15) and that for each $\xi \in \mathcal{X}^{\ell,\wp}$ we set $\bar{\xi} := \bar{\eta}_\Delta\xi_{\Delta^c}$. Hence, for $G \in \mathcal{G}$,

$$\overline{G}^{(\wp)} = X \subset \Delta \implies \bar{\xi}_X = \bar{\eta}_X \implies Z_{m,G}^{(\ell),\wp}(\bar{\xi}) = Z_{m,G}^{(\ell),\wp}(\bar{\eta}). \tag{5.64}$$

By using (5.57), (5.64), and the triangular inequality we have that

$$\begin{aligned}
\|V_{X,\Lambda}^{(\ell),\wp}(\cdot,n)\|_\infty \le &\sum_{\substack{G\in\mathcal{G}:\\ \overline{G}^{(\wp)}=X}} \Big|\log\frac{Z_{m,G}^{(\ell),\wp}(\bar{\eta})\sqrt{\det\mathbb{V}_G^{(\ell),\bar{\eta}}}}{Z_{\mathcal{O}_\wp G}(\mathcal{O}_\ell\mathcal{O}_\wp^\ell\bar{\eta})} + \sum_{i\in\mathcal{O}_\wp^\ell G}\frac{1}{2}m_i^2\Big|\\
&+\Big\|\sum_{\underline{R}\in\underline{\widetilde{R}}:\,\underline{\widetilde{R}}\cup\partial^{(\wp),2}\underline{\widetilde{R}}=X} \varphi_T(\underline{R})\zeta_{m,\underline{R},\Delta}^{(\ell),\wp}(\cdot)\Big\|_\infty.
\end{aligned} \tag{5.65}$$

The estimate (5.63) provides immediately an upper bound to the second term on the right–hand side of (5.65) vanishing as $\ell \to \infty$. We consider, now, the first term on the right–hand side of (5.65): we first notice that (5.2), (4.5), and (2.11) imply

$$Z^{(\ell),\wp}_{m,G}(\bar\eta) = \sum_{\zeta\in\mathcal{X}^{(\ell),\wp}_{m,G}} e^{H^{\ell,\wp}_G(\zeta\,\bar\eta_{G^c})} = \sum_{\substack{\sigma\in\mathcal{X}_{\mathcal{O}_\wp G}:\\ M^{(\ell)}_i(\sigma_{Q_\ell(i)})=m_i,\, i\in\mathcal{O}^\ell_\wp G}} e^{H_G(\sigma(\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta)_{G^c})}.$$

Hence, we have

$$\frac{Z^{(\ell),\wp}_{m,G}(\bar\eta)}{Z_{\mathcal{O}_\wp G}(\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta)} = \mu^{\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta}_{\mathcal{O}_\wp G,z}\big(\{M^{(\ell)}_i = m_i,\ i\in\mathcal{O}^\ell_\wp G\}\big), \tag{5.66}$$

where we recall the notation for the Gibbs measure associated with the original lattice gas potential U, see Subsect. 2.5, with activity z. Recalling that by hypothesis U satisfies the strong mixing condition SM(ℓ_0, b, B), from Lemma 4.2 we have that there exists ℓ'_0, a multiple of ℓ_0, b', and B' positive reals, such that U satisfies SM(ℓ'_0, b', B') uniformly w.r.t. the activity in a neighbor of z small enough. We can then apply the local central limit theorem [1, Theorem 4.5] and (5.66) to write

$$\begin{aligned}
&\Big|\log\frac{Z^{(\ell),\wp}_{m,G}(\bar\eta)\sqrt{\det\mathbb{V}^{(\ell),\bar\eta}_G}}{Z_{\mathcal{O}_\wp G}(\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta)} + \sum_{i\in\mathcal{O}^\ell_\wp G}\frac{1}{2}m_i^2\Big|\\
&\quad= \Big|\log\Big[\mu^{\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta}_{\mathcal{O}_\wp G,z}\big(\{M^{(\ell)}_i = m_i,\ i\in\mathcal{O}^\ell_\wp G\}\big)\sqrt{\det\mathbb{V}^{(\ell),\bar\eta}_G}\Big] + \sum_{i\in\mathcal{O}^\ell_\wp G}\frac{1}{2}m_i^2\Big|\\
&\quad\le \Big|\frac{1}{2}\sum_{i,j\in\mathcal{O}^\ell_\wp G} m_i\Big(\delta_{ij} - 2\pi\chi\ell^d(\mathbb{V}^{(\ell),\bar\eta}_G)^{-1}_{ij}\Big)m_j\Big| + |\log(1+R^{\mathcal{O}_\ell\mathcal{O}^\ell_\wp\bar\eta}_{\mathcal{O}_\wp G}(m))|,
\end{aligned} \tag{5.67}$$

where there exist two positive reals δ' and C_1 depending on G, $\|U\|_0$, and δ, such that

$$\sup_{\sigma\in\mathcal{X}}\ \sup_{m\in\mathcal{M}^{(\ell)}:\,\mathcal{L}^{(\wp)}_\delta(\mathcal{O}^\wp_\ell m)\supset G} |R^\sigma_{\mathcal{O}_\wp G}(m)| \le \frac{C_1}{\ell^{\delta' d}}.$$

Moreover, by using the strong mixing condition it is not difficult to show, see results in [1, Subsect. 5.2], that there exists a positive real C_2 depending on $\|U\|_0$, such that

$$\Big|\delta_{ij} - 2\pi\chi\ell^d(\mathbb{V}^{(\ell),\bar\eta}_G)^{-1}_{ij}\Big| \le \frac{C_2}{\ell}.$$

By using (5.65), (5.67), (5.63), and the two above estimates we get

$$\begin{aligned}
&\sup_{\Lambda\subset\subset\mathcal{L}^{(\wp)}}\ \sup_{X\subset\Lambda}\ \sup_{\substack{n\in\mathcal{M}^{(\ell),\wp}:\\ \mathcal{L}^{(\wp)}_\delta(n)\supset X}} \|V^{(\ell),\wp}_{X,\Lambda}(\cdot,n)\|_\infty\\
&\quad\le \sup_{X\subset\subset\mathcal{L}^{(\wp)}}\ \sup_{\substack{n\in\mathcal{M}^{(\ell),\wp}:\\ \mathcal{L}^{(\wp)}_\delta(n)\supset X}}\ \sum_{\substack{G\in\mathcal{G}:\\ \overline{G}^{(\wp)}=X}} \Big(\frac{C_2}{2\ell}|G|^2\sup_{i\in\mathcal{O}^\ell_\wp G} m_i^2 + \frac{C_1}{\ell^{\delta' d}}\Big) + \kappa''' e^{-\alpha_\ell}\\
&\quad\le 2^{\kappa^d}\Big(\frac{C_2}{2\ell}|G|^2\ell^{1/3-2\delta} + \frac{C_1}{\ell^{\delta' d}}\Big) + \kappa''' e^{-\alpha_\ell}.
\end{aligned}$$

By taking the limit $\ell \to \infty$ we complete the proof of (5.5).
Item 7. Suppose $X \cap \Lambda = X \cap \Lambda'$, then we have $X \cap \Delta = X \cap \Delta'$, where $\Delta' := \Lambda' \cap \mathcal{L}_\delta^{(\wp)}(n)$. The thesis follows from (5.57) and the explicit expression (5.44) of the activity. The key point is that the sums in (5.57) are extended to subsets of the lattice inside X and to a cluster of polymers $\underline{R}$ such that $\widetilde{\underline{R}} \subset X$, and the intersection of Λ and Λ' with X is the same.
Item 8. It follows directly from (5.57) and (5.44). □

6. Construction of the Renormalized Potential and Convergence

In this section we construct the renormalized potential and prove the main Theorems 2.1 and 2.2.

6.1. Cluster expansion in the bad part of the lattice. In this subsection we apply the framework in [2] to develop a multi–scale cluster expansion for the *constrained model* in the *bad* part of the lattice on the basis of the uniformly convergent cluster expansion in the good part of the lattice proven in Theorem 5.1. Recall that in Sect. 4.1 we have introduced $\wp = d\ell$, with ℓ the renormalization scale and d the dimension of the lattice, on which we have defined the notion of goodness.

We are now ready to evoke [2, Thm. 2.5]. Let $\bar{\mathcal{M}}^{(\ell),\wp} \subset \mathcal{M}^{(\ell),\wp}$ be the set of full $\nu^{(\ell),\wp}$–measure in Theorem 4.4. For each $x \in \mathcal{L}^{(\wp)}$ and $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ we let $k_x(n) < \infty$ be the integer such that Item 6 in Definition 3.2 holds true and set

$$\varrho := \Big[\frac{1}{1-q}\Big(1 + \frac{1}{\alpha_\ell}\log A_\ell\Big)\Big] \vee 0 \qquad \text{and} \qquad r_x^{(\wp)}(n) := \big[\Gamma_{k_x(n)} + 2\vartheta_{k_x(n)}\big] \vee \varrho, \tag{6.1}$$

where $q := 2^{-5}3^{-2}$ and α_ℓ and A_ℓ are as in Theorem 5.1.

Theorem 6.1. *Let the lattice gas potential U satisfy Condition SM(ℓ_0, b, B). Let Γ, γ be the two moderate steep scales in (4.19), and $\bar{\mathcal{M}}^{(\ell),\wp} \subset \mathcal{M}^{(\ell),\wp}$ be the set of full $\nu^{(\ell),\wp}$–measure in Theorem 4.4. Then for each ℓ a large enough multiple of ℓ_0, each $n \in \bar{\mathcal{M}}^{(\ell),\wp}$, and each $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ there exist two families of functions $\{\Psi_{X,\Lambda}^{(\ell),\wp}(\cdot, n) : \mathcal{X}^{\ell,\wp} \to \mathbb{R},\ X \subset\subset \mathcal{L}^{(\wp)}\}$ and $\{\Phi_{X,\Lambda}^{(\ell),\wp}(\cdot, n) : \mathcal{X}^{\ell,\wp} \to \mathbb{R},\ X \subset\subset \mathcal{L}^{(\wp)}\}$ such that*

1. *For each $\xi \in \mathcal{X}^{\ell,\wp}$ we have the expansion*

$$\log Z^{(\ell),\wp}_{\mathcal{O}^\ell_\wp n,\Lambda}(\xi) = K_\Lambda^{(\wp)} - \frac{1}{2}\sum_{i\in\mathcal{O}^\ell_\wp\Lambda}(\mathcal{O}^\ell_\wp n)_i^2 + \sum_{X\cap\Lambda\neq\emptyset}\Big[\Psi^{(\ell),\wp}_{X,\Lambda}(\xi, n) + \Phi^{(\ell),\wp}_{X,\Lambda}(\xi, n)\Big], \tag{6.2}$$

   where $K_\Lambda^{(\wp)}$ is as in Theorem 5.1.
2. *For each $X \subset\subset \mathcal{L}^{(\wp)}$ we have $\Psi_{X,\Lambda}^{(\ell),\wp}(\cdot, n), \Phi_{X,\Lambda}^{(\ell),\wp}(\cdot, n) \in \mathcal{F}^{\ell,\wp}_{X\cap\Lambda^c}$.*

Moreover, for each $n \in \bar{\mathcal{M}}^{(\ell),\wp}$

3. *For each $\Lambda, \Lambda' \subset\subset \mathcal{L}^{(\wp)}$, and each $X \subset\subset \mathcal{L}^{(\wp)}$ we have*
$$X \cap \Lambda = X \cap \Lambda' \implies \Psi^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = \Psi^{(\ell),\wp}_{X,\Lambda'}(\cdot, n) \text{ and } \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = \Phi^{(\ell),\wp}_{X,\Lambda'}(\cdot, n).$$

4. *Let $x \in \mathcal{L}^{(\wp)}$, for any $X \subset\subset \mathcal{L}^{(\wp)}$ if $X \ni x$ and $\mathrm{diam}_\wp(X) > r^{(\wp)}_x(n)$, then for each $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ we have $\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = 0$. In particular, for each $x \in \mathcal{L}^{(\wp)}$ there exists a positive real $c^{(\wp)}_x(n) < \infty$ such that*
$$\sum_{X \ni x} \sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \| \Psi^{(\ell),\wp}_{X,\Lambda}(\cdot, n)\|_\infty \le c^{(\wp)}_x(n). \tag{6.3}$$

5. *We have*
$$\sup_{x \in \mathcal{L}^{(\wp)}} \sum_{X \ni x} e^{q\alpha_\ell \,\mathrm{diam}_\wp(X)/d} \sup_{\Lambda \subset\subset \mathcal{L}^{(\wp)}} \| \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot, n)\|_\infty$$
$$\le e^{-\alpha_\ell/d} + e^{-q\alpha_\ell\gamma_1/d} \Big(\frac{1 + e^{-q\alpha_\ell/(2d^2)}}{1 - e^{-q\alpha_\ell/(2d^2)}} \Big)^d. \tag{6.4}$$

Proof of Theorem 6.1. By Theorem 4.4 for each $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ there exists a gentle disintegration $\mathcal{G}(n)$ of $\mathcal{L}^{(\wp)}$ with respect to $\mathbb{G}_0(n) := \mathcal{L}^{(\wp)}_\delta(n)$, Γ, and γ. Moreover, Theorem 5.1 and (4.19) ensure that for ℓ large enough [2, Condition 2.1] is fulfilled with A and α given respectively by A_ℓ and α_ℓ/d. Note that the factor $1/d$ is due to the fact that here we are using, as distance on the lattice $\mathcal{L}^{(\wp)}$, the supremum of the coordinates, while in [2] we used their sum. Moreover, we note that Items 1–4 in the hypotheses of [2, Theorem 2.5] are satisfied by the scales Γ, γ in (4.19).

Items 1–5 are, then, a simple restatement of results in [2, Theorem 2.5] once we define the real
$$c^{(\wp)}_x(n) := A_\ell + k_x(n)(\Gamma_{k_x(n)} + 1 + 2\vartheta_{k_x(n)})^{2d}$$
$$\times \big[\wp^d(\log 2 + \|U\|_0) + k_x(n)(1 \vee A_\ell)(8^d + 1)\big] \tag{6.5}$$
for all $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ and $x \in \mathcal{L}^{(\wp)}$. □

6.2. Locality of the renormalized potential. To prove the Gibbsianity of the renormalized measure we need to introduce functions of the renormalized variable n which will play the role of potentials. In the subsection we state and prove a locality property of the finite volume potentials.

Theorem 6.2. *Assume the hypotheses of Theorem 6.1 are satisfied. Let also $X, \Lambda \subset\subset \mathcal{L}^{(\wp)}$, $n, n' \in \bar{\mathcal{M}}^{(\ell),\wp}$ such that $n_X = n'_X$. Then*
$$\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = \Psi^{(\ell),\wp}_{X,\Lambda}(\cdot, n') \quad \text{and} \quad \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot, n) = \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot, n'). \tag{6.6}$$

The proof of Theorem 6.2 needs to some extent the details of the recursive construction of $\Psi^{(\ell),\wp}_{X,\Lambda}$ and $\Phi^{(\ell),\wp}_{X,\Lambda}$ provided in [2] to which we refer for more details; we outline here the main idea beneath the computation.

We pick $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ and recall the notion of gentle disintegration given in Definition 3.2; for $j \ge 1$ we say $G, G' \subset \mathcal{G}_{\ge j}(n)$ are *j–connected* iff $G \cap G' \cap \mathcal{G}_j(n) \ne \emptyset$.

A system $G_1, \ldots, G_k$ with $G_h \subset \mathcal{G}_{\geq j}(n)$ is said to be *j–connected* iff for each $h, h' \in \{1, \ldots, k\}$ there exists $h_1, \ldots, h_{k'} \in \{1, \ldots, k\}$ such that $G_h = G_{h_1}$, $G_{h_{k'}} = G_{h'}$ and G_{h_i} is j–connected to $G_{h_{i+1}}$ for all $i = 1, \ldots, k' - 1$.

A j–polymer is a collection $\{(G_1, s_1), \cdots, (G_k, s_k)\}$, with $G_h \subset \mathcal{G}_{\geq j}(n)$ and $s_h \geq 0$ integers for $h = 1, \ldots, k$, such that the system $G_1, \ldots, G_k$ is j–connected. We denote by $\mathcal{R}_j(n)$ the collection of all the j–polymers. Given a j–polymer $R = \{(G_1, s_1), \ldots, (G_k, s_k)\}$ and $i \geq j$ we set $R\restriction_i := \bigcup_{h=1}^{k} G_h \cap \mathcal{G}_i(n) \subset \mathcal{G}_i(n)$ and $R\restriction_{\geq i} := \bigcup_{i' \geq i} R\restriction_{i'} \subset \mathcal{G}_{\geq i}(n)$. We also introduce the support of the polymer

$$\operatorname{supp} R := \bigcup_{h=1}^{k} Y_{s_h}(G_h) \subset \mathcal{L}^{(\wp)} \quad \text{with} \quad Y_s(G_h) := \left\{ x \in \mathbb{L} : \mathrm{d}_{\wp}(x, \mathcal{Q}^{(\wp)}(\widehat{G_h})) \leq \vartheta_j + s \right\} \tag{6.7}$$

for all non–negative integer s and $h = 1, \ldots, k$, where we have $\widehat{G} := \bigcup_{g \in G} g$ for all $G \subset \mathcal{G}_{\geq 1}(n)$ and we recall $\mathcal{Q}^{(\wp)}(\Delta)$ is, for all $\Delta \subset\subset \mathcal{L}^{(\wp)}$, the smallest parallelepiped with faces parallel to the coordinate directions and containing Δ. Moreover for each $s \geq 0$, $h = 1, \ldots, k$, we set $y_s(G_h) := Y_s(G_h) \setminus Y_{s-1}(G_h)$, where we understand $Y_{-1}(G_h) = \emptyset$. We note that the set $Y_s(G_h)$ will realize, see (6.14), the volume cutoff mentioned at the end of Subsect. 2.8.

Given two j–polymers $R, S \in \mathcal{R}_j(n)$ we say they are *j–compatible* iff $R\restriction_j \cap S\restriction_j = \emptyset$. Conversely we say that R, S are *j–incompatible* iff they are not *j–compatible*. We say that a collection $\underline{R} = \{R_1, \ldots, R_k\}$, where $R_h \in \mathcal{R}_j(n)$, $h = 1, \ldots, k$, of j–polymers forms a *cluster of j–polymers* iff it is not decomposable into two non empty subsets $\underline{R} = \underline{R}_1 \cup \underline{R}_2$ such that every pair $R_1 \in \underline{R}_1$, $R_2 \in \underline{R}_2$ is j–compatible. We denote by $\underline{\mathcal{R}}_j(n)$ the collection of all the clusters of j–polymers. For $i \geq j$, $\underline{R} \in \underline{\mathcal{R}}_j(n)$ we set $\underline{R}\restriction_i := \bigcup_{R \in \underline{R}} R\restriction_i$, $\underline{R}\restriction_{\geq i} := \bigcup_{i' \geq i} \underline{R}\restriction_{i'}$; we set $\operatorname{supp} \underline{R} := \bigcup_{R \in \underline{R}} \operatorname{supp} R$. We note that $\operatorname{supp} \underline{R}$ is a $\wp$–connected subset of $\mathcal{L}^{(\wp)}$.

For any $\Lambda \subset\subset \mathcal{L}^{(\wp)}$, $G \subset\subset \mathcal{G}_{\geq 1}(n)$, and $s \geq 0$ we define the two collections of subsets of the lattice

$$\begin{aligned} \Upsilon_\Lambda &:= \{Y \subset\subset \mathcal{L}^{(\wp)} : Y \cap \Lambda \neq \emptyset \text{ and } Y \cap \left(\overline{\Lambda}^{(\wp),\kappa}\right)^{\mathrm{c}} = \emptyset\}, \\ \Upsilon_\Lambda(G, s)(n) &:= \{Y \in \Upsilon_{\Lambda \cap \mathbb{G}_0(n)} : \xi(Y)(n) = G,\ Y \subset Y_s(G),\ Y \cap y_s(G) \neq \emptyset\}, \end{aligned} \tag{6.8}$$

where for each $Y \subset \mathcal{L}^{(\wp)}$ we have set

$$\xi(Y)(n) := \{g \in \mathcal{G}_{\geq 1}(n) : g \cap Y \neq \emptyset\} \subset \mathcal{G}_{\geq 1}(n), \tag{6.9}$$

and κ has been introduced in Theorem 5.1. Recalling Theorem 5.1, for $i \geq 1$, $g \in \mathcal{G}_i(n)$, $G \subset\subset \mathcal{G}_{\geq i}(n)$ such that $G \cap \mathcal{G}_i(n) \neq \emptyset$, and $s \geq 0$, we define the 0–order effective potential

$$\begin{aligned} \Psi_{g,\Lambda}^{(i,0)}(\cdot, n) &:= \sum_{Y \in \Upsilon_\Lambda(g,0)(n)} V_{Y,\Lambda}^{(\ell),\wp}(\cdot, n), \\ \Phi_{G,s,\Lambda}^{(i,0)}(\cdot, n) &:= \mathbb{1}_{\{(|G|,s) \neq (1,0)\}} \sum_{Y \in \Upsilon_\Lambda(G,s)(n)} V_{Y,\Lambda}^{(\ell),\wp}(\cdot, n). \end{aligned} \tag{6.10}$$

We next define by recursion on j the j–order effective potentials: as recursive hypotheses we assume that there exist the families

$$\{\Psi^{(i,k)}_{g,\Lambda}(\cdot,n) : \mathcal{X}^{\ell,\wp} \to \mathbb{R},\ \Lambda \subset\subset \mathbb{L}\} \qquad \text{and} \qquad \{\Phi^{(i,k)}_{G,s,\Lambda}(\cdot,n) : \mathcal{X}^{\ell,\wp} \to \mathbb{R},\ \Lambda \subset\subset \mathbb{L}\}$$

for any $k = 0, \dots, j-1$, any $i \geq k+1$, any $g \in \mathcal{G}_i(n)$, any $G \subset\subset \mathcal{G}_{\geq i}(n)$, such that $G \cap \mathcal{G}_i(n) \neq \emptyset$, and any $s \geq 0$. We integrate on the scale j and define the j–order effective potentials $\Psi^{(i,j)}_{g,\Lambda}$ and $\Phi^{(i,j)}_{G,s,\Lambda}$ for $i \geq j+1$, any $g \in \mathcal{G}_i(n)$, any $G \subset\subset \mathcal{G}_{\geq i}(n)$, such that $G \cap \mathcal{G}_i(n) \neq \emptyset$, and $s \geq 0$.

Given $g \in \mathcal{G}_j(n)$, $G \subset\subset \mathcal{G}_{\geq j}(n)$ such that $G \cap \mathcal{G}_j(n) \neq \emptyset$, and $s \geq 0$ we sum all the lower order contributions, obtained by performing the k–order cluster expansion with $k = 1, \cdots, j-1$, to the effective potentials associated with such a vertex, namely, we define

$$\Psi^{(j)}_{g,\Lambda}(\cdot,n) := \sum_{k=0}^{j-1} \Psi^{(j,k)}_{g,\Lambda}(\cdot,n) \qquad \text{and} \qquad \Phi^{(j)}_{G,s,\Lambda}(\cdot,n) := \sum_{k=0}^{j-1} \Phi^{(j,k)}_{G,s,\Lambda}(\cdot,n). \tag{6.11}$$

For each vertex $g \in \mathcal{G}_j(n)$ and block spin configuration $\xi \in \mathcal{X}^{\ell,\wp}$ we define the partition function

$$Z^{(j)}_{g,\Lambda}(\xi,n) := \sum_{\zeta \in \mathcal{X}^{(\ell),\wp}_{\mathcal{O}^{\ell}_{\wp}n,g}} \exp\Big\{ \sum_{\substack{Y \subset\subset \mathcal{L}^{(\wp)}:\, Y\cap\Lambda \neq \emptyset \\ Y\cap\Lambda \subset g\cap\Lambda}} U^{\ell,\wp}_Y(\zeta\xi_{g^c}) + \Psi^{(j)}_{g,\Lambda}(\zeta\xi_{g^c},n)\Big\}, \tag{6.12}$$

where $U^{\ell,\wp}_Y$ are the original lattice gas potentials rewritten, see the discussion before Theorem 5.1, for the block spin variable in $\mathcal{X}^{(1),\ell,\wp} \equiv \mathcal{X}^{\ell,\wp}$, and the probability measure $\nu^{(j)}_{g,\Lambda,n,\xi}$ on $\mathcal{X}^{(\ell),\wp}_{\mathcal{O}^{\ell}_{\wp}n,g}$ by setting

$$\nu^{(j)}_{g,\Lambda,n,\xi}(\zeta) := \frac{\delta_\xi(\zeta_{g\cap\Lambda^c})}{Z^{(j)}_{g,\Lambda}(\xi,n)} \exp\Big\{ \sum_{\substack{Y \subset\subset \mathcal{L}^{(\wp)}:\, Y\cap\Lambda \neq \emptyset \\ Y\cap\Lambda \subset g\cap\Lambda}} U^{\ell,\wp}_Y(\zeta\xi_{g^c}) + \Psi^{(j)}_{g,\Lambda}(\zeta\xi_{g^c},n)\Big\} \tag{6.13}$$

for any $\zeta \in \mathcal{X}^{(\ell),\wp}_{\mathcal{O}^{\ell}_{\wp}n,g}$.

We consider, now, a bond $G \subset\subset \mathcal{G}_{\geq j+1}(n)$, such that $G \cap \mathcal{G}_j(n) \neq \emptyset$, and $s \geq 0$; our aim is the definition of the j–order effective potential associated to such a bond and due to the integration over the j–gentle sites. We set

$$\begin{aligned} &\underline{\mathcal{R}}_j(G,s)(n) \\ &\quad := \Big\{ \underline{R} \in \underline{\mathcal{R}}_j(n) :\ \underline{R}\restriction_{\geq j+1} = G\,,\ \operatorname{supp}\underline{R} \subset Y_s(G),\ \operatorname{supp}\underline{R} \cap y_s(G) \neq \emptyset \Big\}. \end{aligned} \tag{6.14}$$

We define, now, the activity of a cluster of polymers $\underline{R} \in \underline{\mathcal{R}}_j(G,s)(n)$, whose set of vertices of gentleness order greater or equal to $j+1$ is given exactly by G, by setting

$$\zeta_{\underline{R},\Lambda}(\cdot,n) := \prod_{R \in \underline{R}} \zeta_{R,\Lambda}(\cdot,n), \tag{6.15}$$

where for $\xi \in \mathcal{X}^{\ell,\wp}$ we have set

$$\zeta_{R,\Lambda}(\xi,n) := \sum_{\zeta\in\mathcal{X}^{(\ell),\wp}_{\mathcal{O}^{\ell}_{\wp n,\widehat{R\restriction_j}}}} \prod_{g\in R\restriction_j} \nu^{(j)}_{g,\Lambda,n,\xi}(\zeta_g) \prod_{h=1}^{k} \left[\exp\left\{\Phi^{(j)}_{G_h,s_h,\Lambda}(\zeta\xi_{(\widehat{R\restriction_j})^c})\right\} - 1\right] \tag{6.16}$$

for all $R = \{(G_1,s_1),\dots(G_k,s_k)\} \in \underline{R}$.

We are now ready to define the j–order effective potentials. Let $i \geq j+1$, $g \in \mathcal{G}_i(n)$, $G \subset\subset \mathcal{G}_{\geq i}(n)$ such that $G \cap \mathcal{G}_i(n) \neq \emptyset$ and $s \geq 0$; then we set

$$\begin{aligned}\Psi^{(i,j)}_{g,\Lambda}(\cdot,n) &:= \sum_{\underline{R}\in\underline{\mathcal{R}}_j(g,0)(n)} \varphi_T\left(\underline{R}\right)\,\zeta_{\underline{R},\Lambda}(\cdot . n),\\ \Phi^{(i,j)}_{G,s,\Lambda}(\cdot,n) &:= \mathbb{1}_{\{(|G|,s)\neq(1,0)\}} \sum_{\underline{R}\in\underline{\mathcal{R}}_j(G,s)(n)} \varphi_T\left(\underline{R}\right)\,\zeta_{\underline{R},\Lambda}(\cdot,n).\end{aligned} \tag{6.17}$$

In [2, Sect. 4] it is proven that the j–order effective potentials depend only on those block spins associated to sites of order greater than j lying inside the vertices which label the function; more precisely

$$\Psi^{(i,j)}_{g,\Lambda}(\cdot,n) \in \mathcal{F}^{\ell,\wp}_{(Y_0(g)\cap\Lambda^c)\cup g} \quad \text{and} \quad \Phi^{(i,j)}_{G,s,\Lambda}(\cdot,n) \in \mathcal{F}^{\ell,\wp}_{(Y_s(G)\cap\Lambda^c)\cup\widehat{G}}, \tag{6.18}$$

where we recall $\widehat{G} := \bigcup_{g\in G} g$.

We can finally define the functions $\Psi^{(\ell),\wp}_{X,\Lambda}$ and $\Phi^{(\ell),\wp}_{X,\Lambda}$ whose existence has been stated in Theorem 6.1. More precisely for each $X, \Lambda \subset\subset \mathcal{L}^{(\wp)}$ and $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ we define

$$\begin{aligned}\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot,n) &= \mathbb{1}_{\{\mathrm{diam}_\wp(X)\leq\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n)=\emptyset\}} V^{(\ell),\wp}_{X,\Lambda}(\cdot,n)\\ &\quad + \sum_{j\geq1}\sum_{g\in\mathcal{G}_j(X)(n)} \log Z^{(j)}_{g,\Lambda}(\cdot,n),\\ \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot,n) &= \mathbb{1}_{\{\mathrm{diam}_\wp(X)>\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n)=\emptyset\}} V^{(\ell),\wp}_{X,\Lambda}(\cdot,n)\\ &\quad + \sum_{j\geq1}\sum_{\underline{R}\in\underline{\mathcal{R}}_j(X)(n)} \varphi_T\left(\underline{R}\right)\zeta_{\underline{R},\Lambda}(\cdot,n),\end{aligned} \tag{6.19}$$

where we have introduced the two sets

$$\begin{aligned}\mathcal{G}_j(X)\,(n) &:= \left\{g \in \mathcal{G}_j(n) :\ Y_0(g) = X\right\},\\ \underline{\mathcal{R}}_j(X)\,(n) &:= \left\{\underline{R} \in \underline{\mathcal{R}}_j(n) :\ \underline{R}\restriction_{\geq j+1} = \emptyset,\ \mathrm{supp}\,\underline{R} = X\right\}.\end{aligned} \tag{6.20}$$

We remark that the sums over j in (6.19) are extended to a finite number of terms, indeed for j such that $\vartheta_j > \mathrm{diam}_\wp(X)$ the sets $\mathcal{G}_j(X)(n)$ and $\underline{\mathcal{R}}_j(X)(n)$ are empty for all $n \in \bar{\mathcal{M}}^{(\ell),\wp}$. For each $X \subset\subset \mathcal{L}^{(\wp)}$ and $j \geq 1$ we finally set

$$\widetilde{\mathcal{G}}_j(X)(n) := \left\{g \in \mathcal{G}_j(n) :\ Y_0(g) \subset X\right\} \tag{6.21}$$

and $\widetilde{\mathcal{G}}_{\geq j}(X)(n) := \bigcup_{i\geq j} \widetilde{\mathcal{G}}_i(X)(n)$.

Lemma 6.3. *Let $X, \Lambda \subset\subset \mathcal{L}^{(\wp)}$, $n, n' \in \bar{\mathcal{M}}^{(\ell),\wp}$ such that $n_X = n'_X$, then*

1. *for each $j \geq 1$ we have $\widetilde{\mathcal{G}}_j(X)(n) = \widetilde{\mathcal{G}}_j(X)(n')$ and $\widetilde{\mathcal{G}}_{\geq j}(X)(n) = \widetilde{\mathcal{G}}_{\geq j}(X)(n')$;*
2. *for each $j \geq 1$ we have $\mathcal{G}_j(X)(n) = \mathcal{G}_j(X)(n')$ and $\underline{\mathcal{R}}_j(X)(n) = \underline{\mathcal{R}}_j(X)(n')$;*
3. *we have $\Upsilon_\Lambda(G,s)(n) = \Upsilon_\Lambda(G,s)(n')$ for any $G \subset \widetilde{\mathcal{G}}_{\geq 1}(X)(n) = \widetilde{\mathcal{G}}_{\geq 1}(X)(n')$ and $s \geq 0$ such that $Y_s(G) \subset X$;*
4. *for each $j \geq 1$ we have that $\underline{\mathcal{R}}_j(G,s)(n) = \underline{\mathcal{R}}_j(G,s)(n')$ for any $G \subset \widetilde{\mathcal{G}}_{\geq j+1}(X)(n) = \widetilde{\mathcal{G}}_{\geq j+1}(X)(n')$ and $s \geq 0$ such that $Y_s(G) \subset X$;*
5. *for each $j \geq 0$, $i \geq j+1$, $g \in \widetilde{\mathcal{G}}_i(X)(n) = \widetilde{\mathcal{G}}_i(X)(n')$, we have $\Psi^{(i,j)}_{g,\Lambda}(\cdot, n) = \Psi^{(i,j)}_{g,\Lambda}(\cdot, n')$;*
6. *for each $j \geq 0$, $i \geq j+1$, $G \subset \widetilde{\mathcal{G}}_{\geq i}(X)(n) = \widetilde{\mathcal{G}}_{\geq i}(X)(n')$ and $s \geq 0$ such that $G \cap \widetilde{\mathcal{G}}_i(X)(n) = G \cap \widetilde{\mathcal{G}}_i(X)(n') \neq \emptyset$ and $Y_s(G) \subset X$, we have $\Phi^{(i,j)}_{G,s,\Lambda}(\cdot, n) = \Phi^{(i,j)}_{G,s,\Lambda}(\cdot, n')$;*
7. *for each $j \geq 1$, $g \in \mathcal{G}_j(X)(n) = \mathcal{G}_j(X)(n')$, we have $Z^{(j)}_{g,\Lambda}(\cdot, n) = Z^{(j)}_{g,\Lambda}(\cdot, n')$;*
8. *for each $j \geq 1$, $\underline{R} \in \underline{\mathcal{R}}_j(X)(n) = \underline{\mathcal{R}}_j(X)(n')$, we have $\zeta_{\underline{R},\Lambda}(\cdot, n) = \zeta_{\underline{R},\Lambda}(\cdot, n')$.*

Proof of Lemma 6.3. We first prove Items 1–4 separately, then 5 and 6 by induction. Items 7 and 8 will be a byproduct of the proof of 5 and 6.

Item 1. Let $g \in \widetilde{\mathcal{G}}_j(X)(n)$; since $X \supset Y_0(g) \supset B^{(\wp)}_{\vartheta_j}(g)$, Item 2 in Theorem 4.4 and $n_X = n'_X$ imply $g \in \mathcal{G}_j(n')$. Now, $g \in \widetilde{\mathcal{G}}_j(X)(n')$ follows from $g \in \mathcal{G}_j(n')$ and the geometrical property $Y_0(g) \subset X$. Hence $\widetilde{\mathcal{G}}_j(X)(n) \subset \widetilde{\mathcal{G}}_j(X)(n')$ and, by interchanging the role of n and n', we get the equality. The second equality follows immediately from the first one.

Item 2. The proof of the first equality is similar to the proof of Item 1. *Proof of the second equality.* Let $\underline{R} \in \underline{\mathcal{R}}_j(X)(n)$ and $G = \{g_1, \ldots, g_k\} := \underline{R} \restriction_j$ be the collection of all the vertices the cluster of polymers $\underline{R}$ is built of. By definition $g_h \in \mathcal{G}_j(n)$ for any $h = 1, \ldots, k$. The definition of support of a polymer and $\text{supp}\,\underline{R} = X$ imply that $Y_0(g_h) \subset X$ for any $h = 1, \ldots, k$. Hence, $n_X = n'_X$ and Item 2 in Theorem 4.4 imply $g_h \in \mathcal{G}_j(n')$ for any $h = 1, \ldots, k$, which yields $\underline{R} \in \underline{\mathcal{R}}_j(X)(n')$. Hence $\underline{\mathcal{R}}_j(X)(n) \subset \underline{\mathcal{R}}_j(X)(n')$ and, by interchanging the role of n and n', we get the equality.

Item 3. Recall (6.8), let $Y \in \Upsilon_\Lambda(G,s)(n)$. Then we have

$$Y \subset Y_s(G) \quad \text{and} \quad Y \cap y_s(G) \neq \emptyset.$$

Moreover, $X \supset Y_s(G) \supset Y$, $n_X = n'_X$, and $Y \in \Upsilon_{\Lambda \cap \mathcal{L}^{(\wp)}_\delta(n)}$ imply $Y \in \Upsilon_{\Lambda \cap \mathcal{L}^{(\wp)}_\delta(n')}$. Finally, $\xi(Y)(n) = G$ and $n_Y = n'_Y$ imply $\xi(Y)(n') = G$. All the properties ensuring $Y \in \Upsilon(G,s)(n')$ have been verified, hence we have $\Upsilon(G,s)(n) \subset \Upsilon(G,s)(n')$ and, by interchanging the role of n and n', we get the equality.

Item 4. Let $\underline{R} \in \underline{\mathcal{R}}_j(G,s)(n)$, $F = \{f_1, \ldots, f_k\} := \underline{R} \restriction_{\geq j}$ be the collection of all the vertices the cluster of polymers $\underline{R}$ is built of (note $G \subset F$) and $I = \{i_1, \ldots, i_k\}$ the collection of integral numbers such that $f_h \in \mathcal{G}_{i_h}(n)$ for any $h = 1, \ldots, k$, namely, i_h is the grade of f_h. Remark that for each $h = 1, \ldots, k$ either $i_h = j$ or $f_h \in G$. We next prove that $f_h \in \widetilde{\mathcal{G}}_{i_h}(X)(n)$, for $h = 1, \ldots, k$, by showing that $Y_0(f_h) \subset X$. Indeed, in the case $f_h \in G$, we have that $G \subset \widetilde{\mathcal{G}}_{\geq j+1}(X)(n)$ implies $Y_0(f_h) \subset X$; on the other hand, if $i_h = j$, then, recall the definition (6.7) of support of a polymer, $X \supset Y_s(G) \supset \text{supp}\,\underline{R} \supset Y_0(f_h)$. Now, from Item 1 we get $f_h \in \widetilde{\mathcal{G}}_{i_h}(X)(n')$ for any $h = 1, \ldots, k$, which implies $\underline{R} \in \underline{\mathcal{R}}_j(n')$. We remark, finally, that $\underline{R} \in \underline{\mathcal{R}}_j(G,s)(n) \Longrightarrow$

$\underline{R}\upharpoonright_{\geq j+1} = G$, $\text{supp}\,\underline{R} \subset Y_s(G)$ and $\text{supp}\,\underline{R} \cap y_s(G) \neq \emptyset$. Hence, $\underline{R} \in \underline{\mathcal{R}}_j(G,s)(n')$. We then have $\underline{\mathcal{R}}_j(G,s)(n) \subset \underline{\mathcal{R}}_j(G,s)(n')$ and, by interchanging the role of n and n', we get the equality.

Items 5–6. We proceed by induction on j. Let $j = 0$. For each $i \geq 1$ and $g \in \widetilde{\mathcal{G}}_i(X)(n) = \widetilde{\mathcal{G}}_i(X)(n')$, by using (6.10), Item 3, $Y \in \Upsilon_\Lambda(g,0)(n) = \Upsilon_\Lambda(g,0)(n')$, and Item 8 in Theorem 5.1, we have that $\Psi^{(i,0)}_{g,\Lambda}(\cdot,n) = \Psi^{(i,0)}_{g,\Lambda}(\cdot,n')$. The statement in Item 6 for $j = 0$ is proven similarly.

Now, we fix the integer j and suppose that the statements in Items 5 and 6 are verified for all $k = 0,\ldots,j-1$, $i \geq k+1$. From the inductive hypotheses and (6.11) we have that the equality

$$\Psi^{(j)}_{g,\Lambda}(\cdot,n) = \Psi^{(j)}_{g,\Lambda}(\cdot,n') \tag{6.22}$$

holds for all $g \in \widetilde{\mathcal{G}}_j(X)(n) = \widetilde{\mathcal{G}}_j(X)(n')$. Hence, recalling (6.12) and (6.13), we have

$$Z^{(j)}_{g,\Lambda}(\cdot,n) = Z^{(j)}_{g,\Lambda}(\cdot,n') \quad \text{and} \quad \nu^{(j)}_{g,\Lambda,n,\xi} = \nu^{(j)}_{g,\Lambda,n',\xi} \tag{6.23}$$

for any $g \in \widetilde{\mathcal{G}}_j(X)(n) = \widetilde{\mathcal{G}}_j(X)(n')$ and $\xi \in \mathcal{X}^{\ell,\wp}$.

Analogously, from the inductive hypotheses and (6.11) we have

$$\Phi^{(j)}_{G,s,\Lambda}(\cdot,n) = \Phi^{(j)}_{G,s,\Lambda}(\cdot,n') \tag{6.24}$$

for any $G \subset \widetilde{\mathcal{G}}_{\geq j}(X)(n) = \widetilde{\mathcal{G}}_{\geq j}(X)(n')$ and $s \geq 0$ such that $G \cap \widetilde{\mathcal{G}}_j(X)(n) = G \cap \widetilde{\mathcal{G}}_j(X)(n') \neq \emptyset$ and $Y_s(G) \subset X$.

Consider, now, $i \geq j+1$ and $G \subset \widetilde{\mathcal{G}}_{\geq i}(X)(n) = \widetilde{\mathcal{G}}_{\geq i}(X)(n')$, such that $G \cap \widetilde{\mathcal{G}}_j(X)(n) = G \cap \widetilde{\mathcal{G}}_i(X)(n') \neq \emptyset$, and $s \geq 0$ such that $Y_s(G) \subset X$. Since $G \subset \widetilde{\mathcal{G}}_{\geq i}(X)(n) = \widetilde{\mathcal{G}}_{\geq i}(X)(n')$, then $g \in \widetilde{\mathcal{G}}_{\geq i}(X)(n) = \widetilde{\mathcal{G}}_{\geq i}(X)(n')$ for all $g \in G$. Let $\underline{R} \in \underline{\mathcal{R}}_j(G,s)(n) = \underline{\mathcal{R}}_j(G,s)(n')$, for all $g \in \underline{R}\upharpoonright_j$ we have that $Y_0(g) \subset \text{supp}\,\underline{R} \subset X$, i.e., $g \in \widetilde{\mathcal{G}}_j(X)(n) = \widetilde{\mathcal{G}}_j(X)(n')$. Consider, now, $R = \{(G_1,s_1),\ldots,(G_h,s_h)\} \in \underline{R}$; from definition (6.14) we have that $G_l \cap \widetilde{\mathcal{G}}_j(X)(n) = G_l \cap \widetilde{\mathcal{G}}_j(X)(n') \neq \emptyset$ and $Y_{s_l}(G_l) \subset \text{supp}\,\underline{R} \subset Y_s(G) \subset X$ for all $l = 1,\ldots,h$. Moreover, recalling that for all $l = 1,\ldots,h$ each $g \in G_l$ is either an element of $\underline{R}\upharpoonright j$ or an element of G, we have that $Y_0(g) \subset X$ and, hence, $G_l \subset \widetilde{\mathcal{G}}_{\geq j}(X)(n) = \widetilde{\mathcal{G}}_{\geq j}(X)(n')$. Then by using (6.23), (6.24), (6.15), and (6.16) we have that

$$\zeta_{\underline{R},\Lambda}(\cdot,n) = \zeta_{\underline{R},\Lambda}(\cdot,n') \tag{6.25}$$

The inductive proof is completed easily by using (6.17), (6.14), Item 4 above and (6.25). □

Proof of Theorem 6.2. We focus on the first of the two identities (6.6); the proof of the second can be achieved analogously. We recall (6.19) and notice that $n_X = n'_X$ implies

$$\mathbb{I}_{\{\text{diam}_\wp(X)\leq\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n)=\emptyset\}} = \mathbb{I}_{\{\text{diam}_\wp(X)\leq\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n')=\emptyset\}}.$$

Then from Item 8 in Theorem 5.1 we get

$$\begin{aligned} &\mathbb{I}_{\{\text{diam}_\wp(X)\leq\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n)=\emptyset\}} V^{(\ell),\wp}_{X,\Lambda}(\cdot,n) \\ &\quad = \mathbb{I}_{\{\text{diam}_\wp(X)\leq\varrho, X\cap\Lambda\neq\emptyset, \xi(X)(n')=\emptyset\}} V^{(\ell),\wp}_{X,\Lambda}(\cdot,n'). \end{aligned} \tag{6.26}$$

The first of the identities (6.6) finally follows from definition (6.19), the equality (6.26), and Items 2 and 7 of Lemma 6.3. □

6.3. Proof of Gibbsianity and convergence. We notice that for $X \subset\subset \mathcal{L}^{(\wp)}$ and $n \in \bar{\mathcal{M}}^{(\ell),\wp}$ Item 2 of Theorem 6.1 implies that $\Psi^{(\ell),\wp}_{X,X}(\cdot,n)$ and $\Phi^{(\ell),\wp}_{X,X}(\cdot,n)$ are constant functions namely, they do not depend on the first argument. In the sequel we shall write $\Psi^{(\ell),\wp}_{X,X}(n)$ and $\Phi^{(\ell),\wp}_{X,X}(n)$ respectively for $\Psi^{(\ell),\wp}_{X,X}(\cdot,n)$ and $\Phi^{(\ell),\wp}_{X,X}(\cdot,n)$.

We suppose, now, that the hypotheses of Theorem 6.1 are satisfied, we pick $\widetilde{n} \in \pi^{-1}(0)$ once and for all, recall the map π has been defined in (4.11), and for each $X \subset\subset \mathcal{L}^{(\wp)}$ we define the functions $\psi^{(\ell),\wp}_X : \mathcal{M}^{(\ell),\wp} \to \mathbb{R}$ and $\phi^{(\ell),\wp}_X : \mathcal{M}^{(\ell),\wp} \to \mathbb{R}$ as follows:

$$\psi^{(\ell),\wp}_X(n) := \Psi^{(\ell),\wp}_{X,X}(n_X\widetilde{n}_{X^c}) \quad \text{and} \quad \phi^{(\ell),\wp}_X(n) := \Phi^{(\ell),\wp}_{X,X}(n_X\widetilde{n}_{X^c}). \tag{6.27}$$

We note that, by definition, the functions $\psi^{(\ell),\wp}_X$ and $\phi^{(\ell),\wp}_X$ are local, that is $\psi^{(\ell),\wp}_X$, $\phi^{(\ell),\wp}_X \in \mathcal{B}^{(\ell),\wp}_X$, where we recall the σ–algebra $\mathcal{B}^{(\ell),\wp}$ has been introduced at the beginning of Subsect. 4.3.

Let $X, \Lambda \subset\subset \mathcal{L}^{(\wp)}$ be such that $\Lambda \supset X$ and $n \in \bar{\mathcal{M}}^{(\ell),\wp}$. The functions $\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot,n)$ and $\Phi^{(\ell),\wp}_{X,\Lambda}(\cdot,n)$ are constant, namely, $\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot,n), \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot,n) \in \mathcal{F}^{\ell,\wp}_{\emptyset}$, and moreover from Theorem 6.2 and item 3 in Theorem 6.1 we get

$$\Psi^{(\ell),\wp}_{X,\Lambda}(\cdot,n) = \psi^{(\ell),\wp}_X(n) \quad \text{and} \quad \Phi^{(\ell),\wp}_{X,\Lambda}(\cdot,n) = \phi^{(\ell),\wp}_X(n). \tag{6.28}$$

Proof of Theorem 2.1. To get the renormalized potentials of Theorem 2.1 we next pull the $\Psi^{(\ell),\wp}$ and $\Phi^{(\ell),\wp}$ back to the scale ℓ. We define the family $\{\psi^{(\ell)}_I, \phi^{(\ell)}_I : \mathcal{M}^{(\ell)} \to \mathbb{R},\ I \subset\subset \mathcal{L}^{(\ell)}\}$ as follows: for each $m \in \mathcal{M}^{(\ell)}$ we set

$$\psi^{(\ell)}_I(m) := \begin{cases} -m_i^2/2 & \text{if } I = \{i\} \text{ with } i \in \mathcal{L}^{(\ell)} \\ \psi^{(\ell),\wp}_X(\mathcal{O}^{\wp}_{\ell} m) & \text{if } |I| \geq 2 \text{ and } \exists X \subset \mathcal{L}^{(\wp)} : \mathcal{O}^{\ell}_{\wp} X = I\ ; \\ 0 & \text{otherwise} \end{cases} \tag{6.29}$$

note that by construction, see (6.19) and (6.27), if $|X| \leq 1$ then $\psi^{(\ell),\wp}_X = 0$, and

$$\phi^{(\ell)}_I(m) := \begin{cases} \phi^{(\ell),\wp}_X(\mathcal{O}^{\wp}_{\ell} m) & \text{if } \exists X \subset \mathcal{L}^{(\wp)} : \mathcal{O}^{\ell}_{\wp} X = I \\ 0 & \text{otherwise} \end{cases}. \tag{6.30}$$

Equivalently, for all $I \subset\subset \mathcal{L}^{(\ell)}$ such that $|I| > 2$, we can write

$$\psi^{(\ell)}_I = \sum_{\substack{X \subset \mathcal{L}^{(\wp)}: \\ \mathcal{O}^{\ell}_{\wp} X = I}} \psi^{(\ell),\wp}_X \circ \mathcal{O}^{\wp}_{\ell} \quad \text{and} \quad \phi^{(\ell)}_I = \sum_{\substack{X \subset \mathcal{L}^{(\wp)}: \\ \mathcal{O}^{\ell}_{\wp} X = I}} \phi^{(\ell),\wp}_X \circ \mathcal{O}^{\wp}_{\ell}; \tag{6.31}$$

we note, indeed, that for each $I \subset\subset \mathcal{L}^{(\ell)}$ there exists at most one $X \subset \mathcal{L}^{(\wp)}$ such that $\mathcal{O}^{\ell}_{\wp} X = I$.

Item 1. Since for each $X \subset\subset \mathcal{L}^{(\wp)}$ we have $\psi^{(\ell),\wp}_X, \phi^{(\ell),\wp}_X \in \mathcal{B}^{(\ell),\wp}_X$, the thesis follows from definition (6.31) and (4.2).

Item 2. We note that if we let $X \subset \mathcal{L}^{(\wp)}$ and $I := \mathcal{O}^{\ell}_{\wp} X \subset \mathcal{L}^{(\ell)}$, we have that I is ℓ–connected if and only if X is $\wp$–connected. Then the thesis follows immediately from definitions (6.31), (6.19), (6.20), and Item 3 in Theorem 5.1.

Item 3. Since the original lattice gas potential and the algorithmic construction of the gentle atoms in Sect. 3 are translationally invariant, the statement follows.
Item 4. Let $\bar{\mathcal{M}}^{(\ell),\wp} \subset \mathcal{M}^{(\ell),\wp}$ as in Theorem 4.4. We set $\bar{\mathcal{M}}^{(\ell)} := \mathcal{O}^{\ell}_{\wp}\bar{\mathcal{M}}^{(\ell),\wp}$, with $\mathcal{O}^{\ell}_{\wp}$ the unpacking operator. Recalling the definition of $\nu^{(\ell),\wp}$ given at the end of Subsect. 4.1, we have

$$1 = \nu^{(\ell),\wp}(\bar{\mathcal{M}}^{(\ell),\wp}) = \nu^{(\ell)}(\mathcal{O}^{\ell}_{\wp}\bar{\mathcal{M}}^{(\ell),\wp}) = \nu^{(\ell)}(\bar{\mathcal{M}}^{(\ell)}). \tag{6.32}$$

We recall (6.29), (6.27), and that $\widetilde{n}$ has been picked up above; for $m \in \bar{\mathcal{M}}^{(\ell)}$ and $I \subset\subset \mathcal{L}^{(\ell)}$ such that $|I| \geq 2$, we have that if there exists $X \subset\subset \mathcal{L}^{(\ell),\wp}$ such that $\mathcal{O}^{\ell}_{\wp}X = I$ we have

$$\psi^{(\ell)}_I(m) = \psi^{(\ell),\wp}_X(\mathcal{O}^{\wp}_{\ell}m) = \Psi^{(\ell),\wp}_{X,X}((\mathcal{O}^{\wp}_{\ell}m)_X\widetilde{n}_{X^c}) = \Psi^{(\ell),\wp}_{X,X}(\mathcal{O}^{\wp}_{\ell}m), \tag{6.33}$$

where the last equality follows from Theorem 6.2.

Recall (6.1), pick $m \in \bar{\mathcal{M}}^{(\ell)}$ and $i \in \mathcal{L}^{(\ell)}$, set $r^{(\ell)}_i(m) := 2dr^{(\wp)}_{x(i)}(\mathcal{O}^{\wp}_{\ell}m)$, where $x(i) \in \mathcal{L}^{(\wp)}$ is such that $\{x(i)\} = \mathcal{O}^{\wp}_{\ell}\{i\}$. Consider $I \subset\subset \mathcal{L}^{(\ell)}$ such that $I \ni i$ and $\mathrm{diam}_\ell(I) > r^{(\ell)}_i(m)$; from definition (6.1) and $\mathrm{diam}_\ell(I) > r^{(\ell)}_i(m) \geq d$ we have that $|I| \geq 2$. If any $X \subset\subset \mathcal{L}^{(\ell),\wp}$ such that $\mathcal{O}^{\ell}_{\wp}X = I$ does not exist we have $\psi^{(\ell)}_I(m) = 0$. On the other hand if $X \subset\subset \mathcal{L}^{(\ell),\wp}$ exists such that $\mathcal{O}^{\ell}_{\wp}X = I$, from (6.33) we have that $\psi^{(\ell)}_I(m) = 0$. Indeed Item 4 of Theorem 6.1 implies that $\Psi^{(\ell),\wp}_{X,X}(\mathcal{O}^{\wp}_{\ell}m) = 0$ once we note that $\mathcal{O}^{\wp}_{\ell}m \in \bar{\mathcal{M}}^{(\ell),\wp}$, $X \ni x(i)$, and

$$\mathrm{diam}_\wp(X) \geq \frac{1}{2d}\,\mathrm{diam}_\ell(I) > \frac{1}{2d}r^{(\ell)}_i(m) = r^{(\wp)}_{x(i)}(\mathcal{O}^{\wp}_{\ell}m).$$

Consider, now, $m \in \bar{\mathcal{M}}^{(\ell)}$, $i \in \mathcal{L}^{(\ell)}$, and $x(i)$ as above. By using (6.29), (6.33), and Item 3 in Theorem 6.1, we then have that

$$\sum_{I\ni i}\left|\psi^{(\ell)}_I(m)\right| = \frac{1}{2}m_i^2 + \sum_{\substack{I\subset\subset\mathcal{L}^{(\ell)}:\\ |I|\geq 2,\, I\ni i}}\left|\psi^{(\ell)}_I(m)\right| = \frac{1}{2}m_i^2 + \sum_{\substack{X\subset\subset\mathcal{L}^{(\wp)}:\\ X\ni x(i)}}\left|\Psi^{(\ell),\wp}_{X,X}(\mathcal{O}^{\wp}_{\ell}m)\right|.$$

The statement (2.12) follows from Item 4 in Theorem 6.1 by setting

$$c^{(\ell)}_i(m) := \frac{1}{2}m_i^2 + c^{(\wp)}_{x(i)}(\mathcal{O}^{\wp}_{\ell}m)$$

for all $m \in \bar{\mathcal{M}}^{(\ell)}$.
Item 5. By recalling definition (6.29) and by using the Minkowski inequality we have that

$$\begin{aligned}\sup_{i\in\mathcal{L}^{(\ell)}}\left[\nu^{(\ell)}\Big(\Big|\sum_{I\ni i}\psi^{(\ell)}_I\Big|^q\Big)\right]^{1/q} &= \sup_{i\in\mathcal{L}^{(\ell)}}\left\{\left[\nu^{(\ell)}\Big(\Big|-\frac{1}{2}m_i^2\Big|^q\Big)\right]^{1/q}\right.\\ &\quad\left.+\left[\nu^{(\ell)}\Big(\Big|\sum_{\substack{I\ni i:\\ |I|\geq 2}}\psi^{(\ell)}_I\Big|^q\Big)\right]^{1/q}\right\}\\ &\leq \frac{1}{2\chi}\ell^d + \sup_{i\in\mathcal{L}^{(\ell)}}\left[\nu^{(\ell)}\Big(\Big|\sum_{\substack{I\ni i:\\ |I|\geq 2}}\psi^{(\ell)}_I\Big|^q\Big)\right]^{1/q}\end{aligned} \tag{6.34}$$

with χ the infinite volume compressibility defined in (2.8).

To bound the second term of the right-hand side of the above inequality we use (6.29), (6.32), (6.33), (6.19), and the Minkowski inequality. We have

$$\sup_{i\in\mathcal{L}^{(\ell)}}\Big[\nu^{(\ell)}\Big(\Big|\sum_{\substack{I\ni i:\\|I|\geq 2}}\psi_I^{(\ell)}\Big|^q\Big)\Big]^{1/q}\leq\sup_{x\in\mathcal{L}^{(\wp)}}\Big[\nu^{(\ell)}\Big(\Big|\sum_{X\ni x}[\Psi_{X,X}^{(\ell),\wp}\circ\mathcal{O}_\ell^\wp]\Big|^q\Big)\Big]^{1/q}$$
$$\leq\sup_{x\in\mathcal{L}^{(\wp)}}\Big\{\Big[\int_{\bar{\mathcal{M}}^{(\ell)}}\nu^{(\ell)}(dm)\Big(\sum_{X\ni x}\mathbb{1}_{\{\mathrm{diam}_\wp(X)\leq\varrho,\xi(X)(\mathcal{O}_\ell^\wp m)=\emptyset\}}\|V_{X,X}^{(\ell),\wp}(\cdot,\mathcal{O}_\ell^\wp m)\|_\infty\Big)^q\Big]^{1/q}$$
$$+\Big[\int_{\bar{\mathcal{M}}^{(\ell)}}\nu^{(\ell)}(dm)\Big(\sum_{X\ni x}\sum_{j\geq 1}\sum_{g\in\mathcal{G}_j(X)(\mathcal{O}_\ell^\wp m)}\|\log Z_{g,X}^{(j)}(\cdot,\mathcal{O}_\ell^\wp m)\|_\infty\Big)^q\Big]^{1/q}\Big\}. \tag{6.35}$$

By using (5.4) and the Minkowski inequality we have

$$\sup_{i\in\mathcal{L}^{(\ell)}}\nu^{(\ell)}\Big(\Big|\sum_{\substack{I\ni i:\\|I|\geq 2}}\psi_I^{(\ell)}\Big|^q\Big)^{1/q}\leq A_\ell$$
$$+\sup_{x\in\mathcal{L}^{(\wp)}}\sum_{j\geq 1}\Big[\int_{\bar{\mathcal{M}}^{(\ell)}}\nu^{(\ell)}(dm)\Big(\sum_{X\ni x}\sum_{g\in\mathcal{G}_j(X)(\mathcal{O}_\ell^\wp m)}\|\log Z_{g,X}^{(j)}(\cdot,\mathcal{O}_\ell^\wp m)\|_\infty\Big)^q\Big]^{1/q}. \tag{6.36}$$

To bound the second term on the right–hand side of (6.36) we recall (6.20) and note that the sum over $g\in\mathcal{G}_j(X)(\mathcal{O}_\ell^\wp m)$ is zero if there exists no $g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m)$ such that $Y_0(g)\ni x$. Hence this term is estimated from above by

$$\sup_{x\in\mathcal{L}^{(\wp)}}\sum_{j\geq 1}\Big[\int_{\bar{\mathcal{M}}^{(\ell)}}\nu^{(\ell)}(dm)\,\mathbb{1}_{\{\exists g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m):\,Y_0(g)\ni x\}}$$
$$\times\Big(\sum_{X\ni x}\sum_{g\in\mathcal{G}_j(X)(\mathcal{O}_\ell^\wp m)}\|\log Z_{g,X}^{(j)}(\cdot,\mathcal{O}_\ell^\wp m)\|_\infty\Big)^q\Big]^{1/q}. \tag{6.37}$$

Let $j\geq 1$, definition (6.7) and the bound on the diameter of a j–gentle atom, see Item 4 in Definition 3.2, imply that for all $m\in\bar{\mathcal{M}}^{(\ell)}$ and $g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m)$ we have that $\mathrm{diam}_\wp Y_0(g)\leq\Gamma_j+2\vartheta_j$. The sum over X in (6.37) is then extended only to parallelepipedal subsets of the lattice $\mathcal{L}^{(\wp)}$ whose diameter is smaller than $\Gamma_j+2\vartheta_j$; this implies that this sum has at most $(\Gamma_j+2\vartheta_j)^{2d}$ terms. Moreover given X, there exists at most one $g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m)$ such that $Y_0(g)=X$. These remarks and the inequality in Item 3 of Theorem 3.2 of [2] imply the expression in (6.37) is bounded from above by

$$c_1\ell^d\sup_{x\in\mathcal{L}^{(\wp)}}\sum_{j\geq 1}\vartheta_j^{4d}\big[\nu^{(\ell)}(\{\exists g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m):\,Y_0(g)\ni x\})\big]^{1/q} \tag{6.38}$$

with c_1 a suitable real depending on the norm $\|U\|_0$ of the interaction. By the same estimate as in (3.16), we have

$$c_1\ell^d\sup_{x\in\mathcal{L}^{(\wp)}}\sum_{j\geq 1}\vartheta_j^{4d}\big[\nu^{(\ell)}(\{\exists g\in\mathcal{G}_j(\mathcal{O}_\ell^\wp m):\,Y_0(g)\ni x\})\big]^{1/q}$$
$$\leq c_2\ell^d\sum_{j\geq 1}\vartheta_j^{4d+d/q}e^{-a_\delta(\ell)(1+\varepsilon)^j/q}, \tag{6.39}$$

where ε and $a_\delta(\ell)$ are as in (4.20) and (4.21), and c_2 is a positive real depending on $\|U\|_0$. The thesis now follows from (6.34)–(6.39) and Item 5 in Definition 3.1.
Item 6. We recall that $\wp = d\ell$, $\alpha_1 > 0$ has been chosen below (5.61), α_ℓ have been introduced in Item 5 of Theorem 5.1, and q has been defined below (6.1). We set $\alpha' := q\alpha_1/(2d^2)$ and recall definitions (6.31) and (6.27). We have

$$\begin{aligned}
&\sup_{m\in\mathcal{M}^{(\ell)}} \sup_{i\in\mathcal{L}^{(\ell)}} \sum_{I\ni i} e^{\alpha' \operatorname{diam}_\ell(I)} |\phi_I^{(\ell)}(m)| \\
&\quad\le \sup_{m\in\mathcal{M}^{(\ell)}} \sup_{i\in\mathcal{L}^{(\ell)}} \sum_{I\ni i} e^{q\alpha_\ell \operatorname{diam}_\ell(I)/(2d^2)} |\phi_I^{(\ell)}(m)| \\
&\quad\le \sup_{m\in\mathcal{M}^{(\ell)}} \sup_{i\in\mathcal{L}^{(\ell)}} \sum_{I\ni i} e^{q\alpha_\ell \operatorname{diam}_\ell(I)/(2d^2)} \sum_{\substack{X\subset\mathcal{L}^{(\wp)}:\\ \mathcal{O}^\ell_\wp X=I}} |\Phi^{(\ell),\wp}_{X,X}((\mathcal{O}^\wp_\ell m)_X \widetilde{n}_{X^c})| \\
&\quad\le \sup_{m\in\mathcal{M}^{(\ell)}} \sup_{i\in\mathcal{L}^{(\ell)}} \sum_{\substack{X\subset\subset\mathcal{L}^{(\wp)}:\\ \mathcal{O}^\ell_\wp X\ni i}} e^{q\alpha_\ell \operatorname{diam}_\wp(X)/d} |\Phi^{(\ell),\wp}_{X,X}((\mathcal{O}^\wp_\ell m)_X \widetilde{n}_{X^c})| \\
&\quad\le e^{-\alpha_\ell/d} + e^{-q\alpha_\ell\gamma_1/d}\Big(\frac{1+e^{-q\alpha_\ell/(2d^2)}}{1-e^{-q\alpha_\ell/(2d^2)}}\Big)^d,
\end{aligned} \tag{6.40}$$

where we have used (6.4) and $\operatorname{diam}_\ell(\mathcal{O}^\ell_\wp X)/d \le 2\operatorname{diam}_\wp(X)$ for any $X\subset\mathcal{L}^{(\wp)}$.
Item 7. We follow an argument analogous to that in [1, Sect. 5.3]. Let $m'\in\mathcal{M}^{(\ell)}$ and $J\subset\subset\mathcal{L}^{(\ell)}$; we define the following probability kernel on $\mathcal{L}^{(\ell)}_J$:

$$q_J(m',m) := \frac{\exp\Big\{\sum_{I\cap J\neq\emptyset}\big[\psi_I^{(\ell)}(mm'_{J^c})+\phi_I^{(\ell)}(mm'_{J^c})\big]\Big\}}{\sum_{m\in\mathcal{M}^{(\ell)}_J}\exp\Big\{\sum_{I\cap J\neq\emptyset}\big[\psi_I^{(\ell)}(mm'_{J^c})+\phi_I^{(\ell)}(mm'_{J^c})\big]\Big\}}, \tag{6.41}$$

where the functions $\psi_I^{(\ell)}$ and $\phi_I^{(\ell)}$ have been defined in (6.29) and (6.30). Note that, given $m\in\mathcal{M}^{(\ell)}_J$, we have $q_J(\cdot,m)\in\mathcal{B}^{(\ell)}_{J^c}$.

Pick $J\subset\subset\mathcal{L}^{(\ell)}$, $f\in\mathcal{B}^{(\ell)}_J$, recall $M^{(\ell)}$ has been defined in (2.11), by definition of the renormalized measure $\nu^{(\ell)}$ and by standard measure theory we have $\mu(f(M^{(\ell)})) = \nu^{(\ell)}(f)$ and

$$\int_{\mathcal{M}^{(\ell)}} \nu^{(\ell)}(dm') \sum_{m\in\mathcal{M}^{(\ell)}_J} q_J(m',m)\, f(m) = \int_{\mathcal{X}} \mu(d\eta) \sum_{m\in\mathcal{M}^{(\ell)}_J} q_J(M^{(\ell)}(\eta),m)\, f(m). \tag{6.42}$$

Equations (2.15) will thus follow from

$$\mu(f(M^{(\ell)})) = \int_{\mathcal{X}} \mu(d\eta) \sum_{m\in\mathcal{M}^{(\ell)}_J} q_J(M^{(\ell)}(\eta),m)\, f(m). \tag{6.43}$$

For $X \subset \mathcal{L}^{(\ell)}$, let us introduce the family of σ–algebras $\mathcal{E}^{(\ell)}_X := \sigma\{M^{(\ell)}_i(\cdot)\,,\, i \in X\} \subset \mathcal{F}_{\mathcal{O}_\ell X}$ on the configuration space $\mathcal{X}$. Now, pick $\Lambda \subset\subset \mathcal{L}^{(\wp)}$ so that $V := \mathcal{O}^\ell_\wp \Lambda \supset J$. For $\eta \in \mathcal{X}$, we set

$$G_V(M^{(\ell)}_{V\setminus J}(\eta), \eta_{\mathcal{O}_\ell V^c}) := \mu\big(f(M^{(\ell)})\big|\mathcal{E}^{(\ell)}_{V\setminus J} \otimes \mathcal{F}_{\mathcal{O}_\ell V^c}\big)(\eta). \tag{6.44}$$

We shall prove that, μ–a.s.,

$$\lim_{V\uparrow\mathcal{L}^{(\ell)}} G_V(M^{(\ell)}_{V\setminus J}(\eta), \eta_{\mathcal{O}_\ell V^c}) = \sum_{m\in\mathcal{M}^{(\ell)}_J} q_J(M^{(\ell)}(\eta), m)\, f(m). \tag{6.45}$$

Therefore, by dominated convergence, we have

$$\begin{aligned} &\mu(f(M^{(\ell)})) = \mu(\mu(f(M^{(\ell)}|\mathcal{E}^{(\ell)}_{V\setminus J} \otimes \mathcal{F}_{\mathcal{O}_\ell V^c})) \\ &\overset{V\uparrow\mathcal{L}^{(\ell)}}{\longrightarrow} \int_{\mathcal{X}} \mu(d\eta) \sum_{m\in\mathcal{M}^{(\ell)}_J} q_J(M^{(\ell)}(\eta), m)\, f(m) \end{aligned} \tag{6.46}$$

so that (6.43) holds.

We finally prove (6.45) in the set of full measure $(M^{(\ell)})^{-1}(\bar{\mathcal{M}}^{(\ell)})$. By the Gibbs property of the original measure μ, for $\eta \in \mathcal{X}$, such that $M^{(\ell)}(\eta) \in \bar{\mathcal{M}}^{(\ell)}$, and $m' \in \mathcal{M}_{J^c}$, such that $m'_{J^c} = \big(M^{(\ell)}(\eta)\big)_{J^c}$, we have that

$$\begin{aligned} G_V(m'_{V\setminus J}, \eta_{\mathcal{O}_\ell V^c}) &= \mu^\eta_{\mathcal{O}_\ell V}\big(f(M^{(\ell)})\big|M^{(\ell)}_{V\setminus J} = m'_{V\setminus J}\big) \\ &= \frac{\displaystyle\sum_{\sigma\in\mathcal{X}_{\mathcal{O}_\ell V}} f(M^{(\ell)}(\sigma))\, e^{H_{\mathcal{O}_\ell V}(\sigma\eta_{\mathcal{O}_\ell V^c})}\, \mathbb{I}_{\{M^{(\ell)}_{V\setminus J}(\sigma)=m'_{V\setminus J}\}}}{\displaystyle\sum_{\sigma\in\mathcal{X}_{\mathcal{O}_\ell V}} e^{H_{\mathcal{O}_\ell V}(\sigma\eta_{\mathcal{O}_\ell V^c})}\, \mathbb{I}_{\{M^{(\ell)}_{V\setminus J}(\sigma)=m'_{V\setminus J}\}}} \\ &= \frac{\displaystyle\sum_{m\in\mathcal{M}^{(\ell)}_J} f(m) \sum_{\sigma\in\mathcal{X}_{\mathcal{O}_\ell V}} e^{H_{\mathcal{O}_\ell V}(\sigma\eta_{\mathcal{O}_\ell V^c})}\, \mathbb{I}_{\{M^{(\ell)}_V(\sigma)=mm'_{V\setminus J}\}}}{\displaystyle\sum_{m\in\mathcal{M}^{(\ell)}_J} \sum_{\sigma\in\mathcal{X}_{\mathcal{O}_\ell V}} e^{H_{\mathcal{O}_\ell V}(\sigma\eta_{\mathcal{O}_\ell V^c})}\, \mathbb{I}_{\{M^{(\ell)}_V(\sigma)=mm'_{V\setminus J}\}}} \\ &= \frac{\displaystyle\sum_{m\in\mathcal{M}^{(\ell)}_J} f(m)\, Z^{(\ell)}_{mm',V}(\mathcal{O}^\ell\eta)}{\displaystyle\sum_{m\in\mathcal{M}^{(\ell)}_J} Z^{(\ell)}_{mm',V}(\mathcal{O}^\ell\eta)}, \end{aligned} \tag{6.47}$$

see (4.7). Recall $V = \mathcal{O}^\ell_\wp\Lambda$ and set $\xi = \mathcal{O}^\wp_\ell\mathcal{O}^\ell\eta$, we have $Z^{(\ell)}_{mm',V}(\mathcal{O}^\ell\eta) = Z^{(\ell),\wp}_{mm',\Lambda}(\xi)$ for all $m \in \mathcal{M}^{(\ell)}_J$. By using the expansion (6.2), this is allowed because $mm' \in \bar{\mathcal{M}}^{(\ell)}$;

we thus get

$$
\begin{aligned}
&G_V(m'_{V\setminus J}, \eta_{\mathcal{O}_\ell V^c}) \\
&= \frac{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} f(m)\, e^{K_\Lambda^{(\wp)} - \frac{1}{2}\sum_{i\in V}(mm')_i^2 + \sum_{X\cap\Lambda\neq\emptyset}\left[\Psi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm')) + \Phi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm'))\right]}}{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} e^{K_\Lambda^{(\wp)} - \frac{1}{2}\sum_{i\in V}(mm')_i^2 + \sum_{X\cap\Lambda\neq\emptyset}\left[\Psi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm')) + \Phi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm'))\right]}} \\
&= \frac{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} f(m)\, e^{-\frac{1}{2}\sum_{i\in J} {m_i}^2 + \sum_{X\cap\mathcal{O}_\ell^\wp J\neq\emptyset}\left[\Psi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm')) + \Phi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm'))\right]}}{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} e^{-\frac{1}{2}\sum_{i\in J} {m_i}^2 + \sum_{X\cap\mathcal{O}_\ell^\wp J\neq\emptyset}\left[\Psi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm')) + \Phi_{X,\Lambda}^{(\ell),\wp}(\xi,\mathcal{O}_\ell^\wp(mm'))\right]}},
\end{aligned}
\tag{6.48}
$$

where in the second step we have used Theorem 6.2 to simplify the terms of the potential not intersecting $\mathcal{O}_\ell^\wp J$. By items 4 and 5 in Theorem 6.1 and by (6.28) we get

$$
\begin{aligned}
&\lim_{V\to\mathcal{L}^{(\ell)}} G_V(m'_{V\setminus J}, \eta_{\mathcal{O}_\ell V^c}) \\
&= \frac{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} f(m)\, e^{-\frac{1}{2}\sum_{i\in J} {m_i}^2 + \sum_{X\cap\mathcal{O}_\ell^\wp J\neq\emptyset}\left[\psi_X^{(\ell),\wp}(\mathcal{O}_\ell^\wp(mm')) + \phi_X^{(\ell),\wp}(\mathcal{O}_\ell^\wp(mm'))\right]}}{\displaystyle\sum_{m\in\mathcal{M}_J^{(\ell)}} e^{-\frac{1}{2}\sum_{i\in J} {m_i}^2 + \sum_{X\cap\mathcal{O}_\ell^\wp J\neq\emptyset}\left[\psi_X^{(\ell),\wp}(\mathcal{O}_\ell^\wp(mm')) + \phi_X^{(\ell),\wp}(\mathcal{O}_\ell^\wp(mm'))\right]}}.
\end{aligned}
\tag{6.49}
$$

By using definitions (6.29) and (6.30) the above expansion reduces to the renormalization scale ℓ. We thus get (6.45). □

Proof of Theorem 2.2. Item 1. Consider (6.35) and recall (6.9); the first term on the right–hand side of (6.35) tends to zero in the limit $\ell \to \infty$ by virtue of Item 6 of Theorem 5.1. The second term is estimated from above by the convergent series in (6.39); it is not difficult to prove that its sum tends to zero in the limit $\ell \to \infty$ as a consequence of (4.19)–(4.21).

Item 2. The statement is a straightforward consequence of (6.40) and of the expression of α_ℓ. □

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