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Interface Fluctuations in a Conserved System: Derivation and Long Time Behaviour

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Abstract. We study a simple model for interface fluctuations which can be seen as a simplified version of the stochastic phase field equations in one space dimension. In a suitable scaling limit, the front evolves according to a linear stochastic ODE with a long memory drift. We then study the long time behaviour of the limiting process proving an invariance principle; the latter can also be obtained directly from the original process. We note the model can be interpreted as a Brownian motion weakly coupled to a random environment whose evolution depends on the location of the Brownian motion. The limiting process is non-Markovian and exhibits aging effects.

KEYWORDS: stochastic equations, interface dynamics, invariance principle, random walk in random environment

AMS SUBJECT CLASSIFICATION: Primary 60H15, 60K35, Secondary 82C24

1. Introduction

Consider a fluid in a vessel, thermally isolated from the exterior, in a state where two phases coexist, for instance water and ice at the melting temperature. The equilibrium shape of the interface separating ice and water is given by the Wulff shape, however, if we "look more closely", we will see that the above

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description is valid "only in the average". Local fluctuations of temperature cause small portions of ice to melt and/or water to solidify. The latent heat produced by these local phase changes remains however in the system, which is thermally isolated, and influence the successive fluctuations: the analysis of the effect is the main goal of this paper.

Unfortunately, serious technical difficulties force us to drastic simplifications, starting from the most important one of a "planar symmetry" under which the system becomes one dimensional. Then, macroscopically, the interface is simply represented by a point, $X(t)$, with water and ice at its right and left. By fluctuations we will see small droplets of water forming in the ice and ice grains in the water and when such changes occur at the interface, they will cause a small displacement of $X(t)$. Our next assumption is that the effect of the local phase changes in the bulk is negligible and only those at the boundaries are relevant, so that our model will only admit the latter and not the former. A justification of the assumption can be found in [2], where it is proved that the feedback action of the latent heat will destroy the small droplets of the other phase on very short times, while its action on the interface dynamics is effective only on a much longer time scale. Actually the model we consider here can be viewed as a simplified version of the stochastic phase-field equation considered in [2], where the phase changes in the bulk are suppressed and only the interface dynamics is left.

We model the infinitesimal displacements of the interface as the sum of two contributions. The former is due to the Brownian nature of fluctuations, the latter as an interaction with the temperature field. The displacements of the interface produce a latent heat which, due to the assumption that the system is isolated, remains in the system and modifies the temperature in such a way to inhibit further displacements in the same direction. More precisely, the model is defined as follows. Let us denote by $\langle \cdot, \cdot \rangle$ the inner product in $L_2(\mathbf{R}, dx)$ and let $w(t)$ be a standard Brownian motion. The processes $X(\cdot) \in C(\mathbf{R}_+)$ (the interface position) and $h(t) = h(t,x)$, $h(t) \in C(\mathbf{R}_+; C(\mathbf{R}))$ (the external field) evolve according to

$$\begin{cases} dX(t) &= \lambda dw(t) + \gamma \langle \varphi_{X(t)}, h(t) \rangle \, dt, \\ dh(t) &= \dfrac{1}{2} \Delta h(t) \, dt - \varphi_{X(t)} dX(t), \end{cases} \tag{1.1}$$

where Δ is the Laplacian in $\mathbf{R}$, φ is a smooth positive function, $\varphi_X(x) := \varphi(x - X)$, the parameter $\lambda > 0$ is related to the thermal fluctuations, and $\gamma > 0$ is related to the latent heat. We shall consider the Cauchy problem for (1.1) with $X(0) = 0$, $h(0) = 0$.

Note that if $dw(t) > 0$ then, by the first equation in (1.1), $dX(t) > 0$ and by the second equation in (1.1), $dh(t) < 0$, so the drift $\langle \varphi_{X(t)}, h(t) \rangle$ has indeed the effect of restoring $X(t)$ towards its original value (recall that $\gamma > 0$). Due to the presence of the Laplacian in the evolution for $h(t)$ such a restoring effect is however delayed in time and we shall get, in a suitable scaling limit, that the evolution of $X(t)$ is non Markovian. Moreover we note that (1.1) can be also seen as a model for a random walk coupled to the random environment $h(t,x)$. The environment is however dynamically getting all its "randomness" from the random walk.

We shall discuss scaling limits of (1.1) as $\lambda \downarrow 0$ under the (much) simplifying assumption $\lambda = \gamma$ which means "weak" coupling with the external field. Referring to [2] (where an analogous assumption is made for the phase-field equations) for further comments on this point, we simply discuss our results. We prove that $X(\lambda^{-2}t)$ converges in law to a process $\xi(t)$ which solves

$$\begin{cases} d\xi(t) &= H(t)dt + db(t), \\ H(t) &= -\int_0^t \frac{d\xi(s)}{\sqrt{2\pi(t-s)}}, \end{cases} \tag{1.2}$$

where $b(t)$ is a Brownian motion. Note that (1.2) can be formally obtained from (1.1) by setting $\varphi_{X(t)} = \delta_0$ and $H(t) = h(t,0)$; it can be also seen as a linear Stefan problem. If we interpret, as before, $db(t)$ as a given external fluctuation, then $-d\xi(s)/\sqrt{2\pi(t-s)}$ is the contribution to the displacement of the interface at time t, due to the latent heat released at time s, which is depressed, due to one-dimensional diffusion, by the factor $1/\sqrt{2\pi(t-s)}$.

We shall discuss the long time asymptotics of $\xi(t)$ and show that, despite the simplicity of the model, it exhibits a rather rich and interesting structure with characteristic features as "slow growth" and "aging". In particular we obtain an invariance principle for $\xi(\varepsilon^{-1}t)$ and prove that a different scaling limit of (1.1) converges to the same limiting process.

We also introduce a model for the case of a one-dimensional droplet (a segment of ice in a one-dimensional ocean), the bump. We then have two processes $X_1(t)$ and $X_2(t)$ which describe, respectively, the variations from the initial positions of the first and of the second interface. We count as positive variations along the "outwards normal" and write an effective model for the evolution of $X_i(t)$, $i = 1, 2$, which is analogous to (1.1). Namely, by letting $w_1(t)$, $w_2(t)$ two independent Brownian motions, we have (remember we set $\lambda = \gamma$)

$$\begin{cases} dX_1(t) &= \lambda dw_1(t) + \lambda\langle \varphi_{X_1(t)}, h(t)\rangle\, dt, \\ dX_2(t) &= \lambda dw_2(t) - \lambda\langle \varphi_{X_2(t)}, h(t)\rangle\, dt, \\ dh(t) &= \frac{1}{2}\Delta h(t)\, dt - \varphi_{X_1(t)} dX_1(t) + \varphi_{X_2(t)} dX_2(t), \end{cases} \tag{1.3}$$

with initial condition $X_1(0) = 0$, $X_2(0) = \lambda^{-1}a$, $a > 0$, and $h(0) = 0$. This choice means that in the macroscopic interval $(0, a)$ there is a phase and outside the other one.

The scaling limit of $\big(X_1(\lambda^{-2}t), X_2(\lambda^{-2}t)\big)$ can be carried out as in the case of a single interface; the limiting process $\big(\xi_1(t), \xi_2(t)\big)$ solves

$$\begin{cases} d\xi_1(t) &= db_1(t) - \Big[\int_0^t \frac{d\xi_1(s)}{\sqrt{2\pi(t-s)}} + \int_0^t d\xi_2(s) \frac{\exp(-\frac{a^2}{2(t-s)})}{\sqrt{2\pi(t-s)}} \Big] dt, \\ d\xi_2(t) &= db_2(t) - \Big[\int_0^t d\xi_1(s) \frac{\exp(-\frac{a^2}{2(t-s)})}{\sqrt{2\pi(t-s)}} + \int_0^t \frac{d\xi_2(s)}{\sqrt{2\pi(t-s)}} \Big] dt, \end{cases} \tag{1.4}$$

with b_1 and b_2 independent Brownians.

Macroscopic deformations of the droplet are measured by the variable $\zeta(t) = \xi_1(t)+\xi_2(t)$ which gives the variation of the size of the droplet. The displacement of the center of the droplet is instead given by $\eta(t) = [\xi_2(t) - \xi_1(t)]/2$; recall we count as $\xi_i(t)$ positive in opposite directions. At very short times both are, approximately, independent Brownian motions. Then drifts enter into action and we shall prove that in the long times the deformation $\zeta(t)$ behaves as in the case of the single interface, having size of order $\sqrt{\log t}$, while $\eta(t)$ becomes a Brownian motion independent of $\xi(t)$.

2. Notation and results

We denote by $\|\cdot\|_p$, $p \in [1, +\infty]$, the norm in $L_p(\mathbf{R}) := L_p(\mathbf{R}, dx)$ and by $\langle \cdot, \cdot \rangle$ the inner product in $L_2(\mathbf{R})$. Let $b(t)$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathsf{P})$; we denote by E the expectation w.r.t. P. We assume the function φ in (1.1) to be a probability density in $\mathcal{S}(\mathbf{R})$, the Schwarz space of test functions on $\mathbf{R}$. We set

$$\begin{aligned} X_\lambda(t) &:= X(\lambda^{-2}t), \\ \varphi^{(\lambda)}(x) &:= \lambda^{-1}\varphi(\lambda^{-1}x), \\ \varphi_t^{(\lambda)} &:= \varphi^{(\lambda)}_{\lambda X_\lambda(t)} = \lambda^{-1}\varphi(\lambda^{-1}x - X_\lambda(t)), \\ h_\lambda(t,x) &:= \lambda^{-1}h(\lambda^{-2}t, \lambda^{-1}x). \end{aligned}$$

To keep notation simple we rewrite (1.1) in terms of the scaled variables X_λ, h_λ getting

$$\begin{cases} dX_\lambda(t) &= db(t) + \langle \varphi_t^{(\lambda)}, h_\lambda(t) \rangle \, dt, \\ dh_\lambda(t) &= \frac{1}{2}\Delta h_\lambda(t)\, dt - \varphi_t^{(\lambda)} dX_\lambda(t), \\ X_\lambda(0) &= 0, \\ h_\lambda(0) &= 0. \end{cases} \tag{2.1}$$

We consider the system (2.1) defined by its mild version

$$\begin{cases} X_\lambda(t) &= b(t) + \int_0^t ds\, \langle \varphi_s^{(\lambda)}, h_\lambda(s) \rangle, \\ h_\lambda(t) &= -\int_0^t db(s)\, p_{t-s}\varphi_s^{(\lambda)} - \int_0^t ds\, \langle \varphi_s^{(\lambda)}, h_\lambda(s)\rangle p_{t-s}\varphi_s^{(\lambda)}, \end{cases} \tag{2.2}$$

where $p_t := \exp(t\Delta/2)$ is the heat semigroup, whose kernel is given by $p_t(x) = \exp(-x^2/(2t))/\sqrt{2\pi t}$, and, for any $f \in L_p(\mathbf{R})$, $p_t f$ denotes the convolution $p_t * f$.

There exists a unique $\mathcal{F}_t$-adapted process $(X_\lambda, h_\lambda) \in C(\mathbf{R}_+; \mathbf{R} \times L_2(\mathbf{R}))$, which solves (2.2), see the Appendix for a sketch of the proof. The process $X_\lambda(t)$ is an $\mathcal{F}_t$-adapted, square integrable semimartingale; a straightforward computation shows

$$h_\lambda(t) = -\int_0^t dX_\lambda(s)\, p_{t-s}\varphi_s^{(\lambda)}. \tag{2.3}$$

We can rewrite the limiting equation (1.2) as

$$\xi(t) = b(t) - \int_0^t ds\, p_{t-s}(0)\xi(s). \tag{2.4}$$

Since Brownian motion is a.s. continuous, we will restrict to continuous $b(t)$ in (2.4); then existence and uniqueness of a solution to (2.4) follow by standard theory on singular Volterra equations with Abel kernel, [6, §1.12]. Our first result is the scaling limit of (2.1) to (2.4).

Theorem 2.1. *Let X_λ (resp. ξ) be the solution of* (2.2) *(resp.* (2.4)*). Then for each $N \in [1, \infty)$ there exists $\tau = \tau(N) > 0$ such that*

$$\lim_{\lambda \downarrow 0} \mathsf{E} \sup_{t \le \tau |\log \lambda|} |X_\lambda(t) - \xi(t)|^N = 0. \tag{2.5}$$

Note (2.5) implies that, as $\lambda \downarrow 0$, $X_\lambda \Longrightarrow \xi$ where $\Longrightarrow$ denotes weak convergence on $C(\mathbf{R}_+)$ endowed with the topology of uniform convergence in compacts. In addition (2.5) reduces the behaviour of $X_\lambda(t)$, solution of (2.2) up to times $t \approx |\log \lambda|$ to the long time asymptotics of (2.4).

It turns out that (2.4) is explicitly solvable. We introduce the associate Green function $F(t)$ as the solution of

$$F(t) = 1 - \int_0^t ds\, p_{t-s}(0)F(s). \tag{2.6}$$

Indeed, once we determine F, the unique solution of (2.4) is given by

$$\xi(t) = b(t) + \int_0^t F'(t-s)b(s)ds = \int_0^t F(t-s)db(s). \tag{2.7}$$

The second identity holds a.s. by integration by parts for stochastic integrals. The long time behaviour of (2.4) is determined by the following asymptotics of its Green function.

Theorem 2.2. *The Green function F is given by*

$$F(t) = e^{t/2}\,\mathrm{erfc}\Big(\sqrt{\frac{t}{2}}\Big), \tag{2.8}$$

where, for any $z \ge 0$,

$$\mathrm{erfc}(z) := 2\int_z^\infty dy\,\frac{e^{-y^2}}{\sqrt{\pi}}.$$

Furthermore

$$\lim_{t\to+\infty} \sqrt{t}F(t) = \sqrt{\frac{2}{\pi}}. \tag{2.9}$$

This result is known since Abel (1825), but for completeness and in view of next applications, we review some of its possible proofs in Section 4 below.

The explicit knowledge of the Green function as provided by Theorem 2.2, determines, via (2.7), the long time behaviour of the front $\xi(t)$. As already mentioned in the Introduction the feedback force on $\xi(t)$ represented by the integral term in (2.4) has a very important effect. The forcing random noise $b(t)$ typically grows proportionally to $\sqrt{t}$, but the front's magnitude only as $\sqrt{\log t}$. The fluctuations of $|b(t)|$ above $\sqrt{t}$ are responsible for such a growth, $\xi(t)$ would remain bounded if $|b(t)| \le c\sqrt{t}$. It is just because of the unboundedness of $b(t)/\sqrt{t}$ that $\xi(t)$ diverges as $t \to \infty$. If we kill the noise after a very long time t, the front $\xi(t+s)$ moves back towards the origin, restoring equilibrium, but the time s it takes has the same order as t. The system has therefore a very long memory which goes back to its whole previous history. This is a very simple example of aging, a property shared by many materials where non elastic, plastic effects are important and the pattern followed for restoring a deformation depends sensitively on the way it has been created.

Corollary 2.1. *We have that $\xi(t)$ is a mean zero Gaussian process, with*

$$\lim_{t\to\infty} \frac{1}{\log t}\,\mathsf{E}\;\xi(t)^2 = \frac{2}{\pi}. \tag{2.10}$$

Moreover, setting $\eta(t,s) := \mathsf{E}(\xi(t)\mid\mathcal{F}_s)$ for $0 < s < t$,

$$\lim_{\varepsilon\to 0} \mathsf{E}\;\eta(\varepsilon^{-1}t,\varepsilon^{-1}s)^2 = \frac{2}{\pi}\log\Big(1-\frac{s}{t}\Big). \tag{2.11}$$

We next formulate the asymptotic behaviour of $\xi(t)$ as an invariance principle. The aforementioned aging property of ξ will imply a scaling property of the limiting covariance, which turns out to be a function of s/t, see (2.13) below. Recalling (2.7), the asymptotics (2.9) suggests $\xi(\varepsilon^{-1}t)$ converges in law, as $\varepsilon \downarrow 0$, to a process whose formal expression is given by $x(t) = \sqrt{2/\pi}\int_0^t dw_s\,(t-s)^{-1/2}$, where w is a standard Brownian motion. Observe $x(t)$ is — formally — invariant under the scaling $t \mapsto \varepsilon^{-1}t$. Unfortunately, since $(t-s)^{-1/2}$ is not in $L_2([0,t],ds)$, $x(t)$ is not well defined as a process in $C(\mathbf{R}^+)$. Since it is Gaussian, there are several ways to define it on an appropriate distribution space. For our purposes it will be more convenient to define $x(t)$ as the image of the Wiener process under a suitable linear map.

For each $T > 0$ let $E := \{\psi \in C^1([0,T]) \,:\, \psi(T) = 0\}$ which is a separable Banach space under the norm $|\psi|_E := \sup_{t\in[0,T]} |\psi'(t)|$, and denote by E^* its dual space. Let $C_0([0,T]) := \{w \in C([0,T]) \,:\, w(0) = 0\}$ and $A : C_0([0,T]) \to E^*$ be the continuous linear map defined by

$$(Aw)(\psi) := -\int_0^T dt\, \psi'(t) \int_0^t ds\, \frac{\sqrt{2}}{\sqrt{\pi(t-s)}}\, w(s), \qquad \forall\, \psi \in E. \tag{2.12}$$

Let $\mathcal{P}$ be the Wiener measure on $C_0([0,T])$ and set $\mathcal{Q} := \mathcal{P} \circ A^{-1}$, which is a probability measure on $\big(E^*, \mathcal{B}(E^*)\big)$. Then $\mathcal{Q}$ is a Gaussian measure with covariance

$$\int_{E^*} \mathcal{Q}(dx)\, x(\psi)x(\varphi) = \int_0^T dt \int_0^T ds\, \psi(t)\, \kappa(t,s)\, \varphi(s), \tag{2.13}$$

where

$$\kappa(t,s) := \frac{2}{\pi}\int_0^{t\wedge s} du\, \frac{1}{\sqrt{(t-u)(s-u)}} = \frac{2}{\pi}\log\frac{\sqrt{t\vee s}+\sqrt{t\wedge s}}{\sqrt{t\vee s}-\sqrt{t\wedge s}}.$$

Theorem 2.3. *Fix $T > 0$ and let ξ be the solution of (2.4). We consider $\xi_\varepsilon(t) := \xi(\varepsilon^{-1}t)$, $t \in [0,T]$, as a random element of E^* by setting $\xi_\varepsilon(\psi) := \int_0^T dt\, \psi(t)\xi_\varepsilon(t)$, $\psi \in E$. Then $\xi_\varepsilon \Longrightarrow \mathcal{Q}$ as $\varepsilon \downarrow 0$ in the topology of E^*.*

Proof. By the scaling property of Brownian motion, $\mathcal{P}$ is the distribution of $b_\varepsilon(t) = \sqrt{\varepsilon}\, b(\varepsilon^{-1}t)$, $t \in [0,T]$, where b is the Brownian motion in (2.4). We also introduce $x_\varepsilon := A\, b_\varepsilon$ whose law is $\mathcal{Q}$ for any $\varepsilon > 0$. We shall prove the stronger statement

$$\lim_{\varepsilon\downarrow 0} \mathsf{E}\, \|\xi_\varepsilon - x_\varepsilon\|_{E^*}^2 = 0. \tag{2.14}$$

Let $F_\varepsilon(t) := \varepsilon^{-1/2}F(\varepsilon^{-1}t)$, F as in (2.8). From (2.7) and (2.12), by a straightforward computation,

$$\xi_\varepsilon(\psi) - x_\varepsilon(\psi) = -\int_0^T dt\, \psi'(t) \int_0^t ds\, \Big[F_\varepsilon(t-s) - \frac{\sqrt{2}}{\sqrt{\pi(t-s)}}\Big]\, b_\varepsilon(s).$$

By using (2.9) it is easy to show

$$\lim_{\varepsilon \downarrow 0} \int_0^T dt \int_0^t ds \left| F_\varepsilon(t-s) - \frac{\sqrt{2}}{\sqrt{\pi(t-s)}} \right| = 0,$$

from which (2.14) follows. □

From Theorems 2.1 and 2.3, we can get directly the scaling limit of X_λ to $\mathcal{Q}$.

Corollary 2.2. *Let $\gamma(\lambda)$ be a monotone positive function such that $\lim_{\lambda\to 0}\gamma(\lambda) = +\infty$ and $\lim_{\lambda\to 0}|\log\lambda|^{-1}\gamma(\lambda) = 0$. For each $T>0$ set $\bar{X}_\lambda(t) := X_\lambda(\gamma(\lambda)t)$, $t\in[0,T]$, where X_λ is the solution of (2.2). Then, as a random element in E^*, $\bar{X}_\lambda \Longrightarrow \mathcal{Q}$ as $\lambda \downarrow 0$.*

We next discuss the case with two interfaces. Let (X_1, X_2) be the solution of (1.3), setting $\xi_{1,\lambda}(t) := -X_1(\lambda^{-2}t)$, $\xi_{2,\lambda}(t) := X_2(\lambda^{-1}t) - \lambda^{-1}a$ we can prove a result analogous to Theorem 2.1, i.e. that $(\xi_{1,\lambda}, \xi_{2,\lambda})$ converges to the solution of

$$\begin{aligned} \xi_1(t) &= b_1(t) - \int_0^t ds \, \frac{\xi_1(s)}{\sqrt{2\pi(t-s)}} - \int_0^t ds \, \frac{\exp(-\frac{a^2}{2(t-s)})}{\sqrt{2\pi(t-s)}} \, \xi_2(s), \\ \xi_2(t) &= b_2(t) - \int_0^t ds \, \frac{\exp(-\frac{a^2}{2(t-s)})}{\sqrt{2\pi(t-s)}} \, \xi_1(s) - \int_0^t ds \, \frac{\xi_2(s)}{\sqrt{2\pi(t-s)}}. \end{aligned} \tag{2.15}$$

We omit the formal statement but discuss the long time behaviour of the limiting equation.

The Green function for the system (2.15) is the 2×2 matrix $\mathbf{F}(t)$ which satisfies

$$\mathbf{F}(t) = \mathbf{I} - \int_0^t ds \, \mathbf{G}(t-s)\mathbf{F}(s), \tag{2.16}$$

where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{G}(t) := \begin{pmatrix} p_t(0) & p_t(a) \\ p_t(a) & p_t(0) \end{pmatrix}. \tag{2.17}$$

Theorem 2.4. *The elements $F_{i,j}(t)$, $i,j\in\{0,a\}$, of the matrix $\mathbf{F}(t)$ are invariant under the exchange of 0 and a, namely*

$$F_{0,0} = F_{a,a}, \qquad F_{0,a} = F_{a,0}. \tag{2.18}$$

They are also decreasing functions of t and

$$0 \le F_{0,0} \le 1, \qquad 0 \le -F_{0,a} \le 1. \tag{2.19}$$

Finally the asymptotics for

$$F^+(t) := F_{0,0}(t) + F_{0,a}(t), \qquad F^-(t) := \frac{F_{0,0}(t) - F_{0,a}(t)}{2}, \tag{2.20}$$

are

$$\lim_{t\to+\infty} \sqrt{t}\, F^+(t) = \sqrt{\frac{1}{2\pi}}, \tag{2.21}$$

$$\lim_{t\to+\infty} F^-(t) = \frac{1}{2}\,\frac{1}{1+|a|}. \tag{2.22}$$

The solution of (2.15) can be represented, in terms of the Green function $\mathbf{F}(t)$, as

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} = \int_0^t \mathbf{F}(t-s) \begin{pmatrix} db_1(s) \\ db_2(s) \end{pmatrix}. \tag{2.23}$$

From Theorem 2.4 we deduce an invariance principle for the process (ξ_1, ξ_2).

Corollary 2.3. *Let*

$$\eta(t) := \frac{1}{2}\left[\xi_1(t) - \xi_2(t)\right], \qquad \zeta(t) := \xi_1(t) + \xi_2(t). \tag{2.24}$$

Then η and ζ are independent. Moreover $\eta_\varepsilon(t) := \sqrt{\varepsilon}\eta(\varepsilon^{-1}t)$ converges weakly in $C([0,T])$ to a Brownian motion with diffusion coefficient $[2(1+|a|)]^{-2}$. Finally ζ satisfies the asymptotics (2.10)–(2.11) *and the invariance principle in Theorem 2.3 with $2/\pi$ replaced by $1/(2\pi)$.*

The above asymptotics tells us that the displacement η of the droplet behaves as a Brownian motion whereas the "vibration mode" ζ is essentially bounded in time.

3. Convergence to the linear Stefan problem

In this section we prove Theorem 2.1. It will be convenient to prove convergence to one iteration of (2.4), i.e.

$$\xi(t) = b(t) - \int_0^t ds\, p_{t-s}(0)\, b(s) + \frac{1}{2}\int_0^t ds\, \xi(s), \tag{3.1}$$

where we used

$$\int_s^t ds'\, p_{t-s'}(0) p_{s'-s}(0) = \frac{1}{2}. \tag{3.2}$$

We define, for any $0 \le s \le t$,

$$P^{(\lambda)}_{t,s} := \left\langle \varphi^{(\lambda)}_t, p_{t-s}\varphi^{(\lambda)}_s \right\rangle, \qquad K^{(\lambda)}_{t,s} := \left\langle \varphi^{(\lambda)}_t, \int_s^t db(s')\, p_{t-s'}\varphi^{(\lambda)}_{s'} \right\rangle. \tag{3.3}$$

Plugging (2.3) into the first equation in (2.2) we obtain $X_\lambda(t)$ solves

$$X_\lambda(t) = b(t) + F_0^{(\lambda)}(t) + \int_0^t ds \int_0^s ds'\, P_{s,s'}^{(\lambda)} \left\langle \varphi_{s'}^{(\lambda)}, \int_0^{s'} dX_\lambda(s'')\, p_{s'-s''}\, \varphi_{s''}^{(\lambda)} \right\rangle, \quad (3.4)$$

$$F_0^{(\lambda)}(t) := -\int_0^t ds\, K_{s,0}^{(\lambda)}. \quad (3.5)$$

Note that in the kernel $K_{s,0}^{(\lambda)}$ the factor $\varphi_s^{(\lambda)}$ is the future w.r.t. the stochastic integrals.

The strategy of the proof consists in introducing

$$Y_\lambda(t) := X_\lambda(t) - b(t) - F_0^{(\lambda)}(t), \quad (3.6)$$

showing it is a.s. differentiable and proving (see Lemma 3.4 below) the convergence of $\dot{Y}_\lambda(t)$ to the derivative of $\xi(t) - b(t) - \int_0^T ds\, p_{t-s}(0) b(s)$. Together with an analysis of $F_0^{(\lambda)}$, this will imply Theorem 2.1. The process $Y_\lambda(t)$ solves

$$Y_\lambda(t) = \int_0^t ds \left[F_1^{(\lambda)}(s) + F_2^{(\lambda)}(s) \right] + \int_0^t ds \int_0^s ds'\, P_{s,s'}^{(\lambda)} \left\langle \varphi_{s'}^{(\lambda)}, \int_0^{s'} dY_\lambda(s'')\, p_{s'-s''}\, \varphi_{s''}^{(\lambda)} \right\rangle, \quad (3.7)$$

where

$$F_1^{(\lambda)}(t) := \int_0^t ds\, P_{t,s}^{(\lambda)}\, K_{s,0}^{(\lambda)}, \quad (3.8)$$

$$F_2^{(\lambda)}(t) := -\int_0^t ds \int_0^s ds'\, P_{t,s}^{(\lambda)}\, P_{s,s'}^{(\lambda)}\, K_{s',0}^{(\lambda)}. \quad (3.9)$$

It is easy to verify for each $\lambda > 0$ the process Y_λ is a.s. in $C^1(\mathbf{R}_+)$, therefore we can write the last stochastic integral in (3.7) as an ordinary one and get, after exchanging the order of integrations,

$$\dot{Y}_\lambda(t) = F_1^{(\lambda)}(t) + F_2^{(\lambda)}(t) + \int_0^t ds\, \dot{Y}_\lambda(s) \int_s^t ds'\, P_{t,s'}^{(\lambda)} P_{s',s}^{(\lambda)}. \quad (3.10)$$

We can now bound $|\dot{Y}_\lambda(t)|$ by the Gronwall lemma. To this end we need *a priori* bounds on $F_i^{(\lambda)}(t)$, $i = 1, 2$. A basic tool is the following lemma.

Lemma 3.1. *For each* $N \in [1, \infty)$ *and* $p, q, r \in (1, +\infty)$, *chosen so that* $p^{-1} + q^{-1} = 1$, $q > \max\{N, r\}$, *let*

$$\gamma := \frac{1}{2} - \frac{1}{2}\Big(\frac{1}{r} - \frac{1}{q}\Big), \qquad \alpha := \frac{Nq}{q - N}.$$

Then there is a constant $C = C(q,r)$ *such that*

$$\Big(\mathsf{E}\Big|\Big\langle\psi, \int_{t_1}^{t_2} db(s)\, p_{t_2-s}v(s)\Big\rangle\Big|^N\Big)^{1/N} \le C(t_2-t_1)^\gamma \big(\mathsf{E}\,\|\psi\|_p^\alpha\big)^{1/\alpha}$$
$$\times\Big(\mathsf{E}\sup_{t_1\le s\le t_2}\|v(s)\|_r^q\Big)^{1/q}, \qquad (3.11)$$

for any $t_1 \le t_2$ *and any random elements* ψ *on* $L_p(\mathbf{R})$, *and* $v(\cdot)$ *on* $C(\mathbf{R}_+; L_r(\mathbf{R}))$ *adapted w.r.t.* $\mathcal{F}_s$.

Note we did not assume any adaptability on ψ, see the remark after (3.5).

Proof. Applying Hölder inequality first on the measure dx (with exponents p and q) and then on the measure $d\,\mathsf{P}$ (with exponents αN^{-1} and $\alpha(\alpha-N)^{-1}$) we have

$$\begin{aligned}
&\Big(\mathsf{E}\Big|\Big\langle\psi, \int_{t_1}^{t_2} db(s)\, p_{t_2-s}v(s)\Big\rangle\Big|^N\Big)^{1/N} \\
&\le \Big[\mathsf{E}\Big(\|\psi\|_p^N \Big\|\int_{t_1}^{t_2} db(s)\, p_{t_2-s}v(s)\Big\|_q^N\Big)\Big]^{1/N} \\
&\le \big(\mathsf{E}\,\|\psi\|_p^\alpha\big)^{1/\alpha}\Big(\mathsf{E}\Big\|\int_{t_1}^{t_2} db(s)\, p_{t_2-s}v(s)\Big\|_q^q\Big)^{1/q}. \qquad (3.12)
\end{aligned}$$

Let us introduce

$$M_\tau^{(t_2)}(y) := \int_{t_1}^{\tau} db(s)\,[p_{t_2-s}v(s)](y), \qquad \tau\in[t_1,t_2), \qquad (3.13)$$

which is an $\mathcal{F}_\tau$-martingale with bracket

$$\big[M^{(t_2)}(y)\big]_\tau = \int_{t_1}^{\tau} ds\,[p_{t_2-s}v(s)]^2(y).$$

By the BDG inequality (see [5, IV, Corollary 4.2]) for each $q\in(1,+\infty)$ there is a universal constant $C = C(q)$ such that

$$\mathsf{E}\,\big|M_\tau^{(t_2)}(y)\big|^q \le C\,\mathsf{E}\Big(\int_{t_1}^{\tau} ds\,[p_{t_2-s}v(s)]^2(y)\Big)^{q/2}. \qquad (3.14)$$

By taking the limit $\tau \uparrow t_2$ we can thus bound the second factor on the r.h.s. of (3.12) as follows

$$
\begin{aligned}
\left(\mathsf{E} \Big\| \int_{t_1}^{t_2} db(s)\, p_{t_2-s} v(s) \Big\|_q^q \right)^{1/q}
&\le C \left(\mathsf{E} \Big\| \int_{t_1}^{t_2} ds\, [p_{t_2-s} v(s)]^2 \Big\|_{q/2}^{q/2} \right)^{1/q} \\
&\le C \left[\mathsf{E} \left(\int_{t_1}^{t_2} ds\, \| p_{t_2-s} v(s) \|_q^2 \right)^{q/2} \right]^{1/q} \\
&\le C \left[\mathsf{E} \left(\int_{t_1}^{t_2} ds\, \| p_{t_2-s} \|_{r'}^2 \| v(s) \|_r^2 \right)^{q/2} \right]^{1/q} .
\end{aligned}
\tag{3.15}
$$

In the last step we used the Young inequality, i.e. $\|f * g\|_q \le \|f\|_{r'} \|g\|_r$, where $(r')^{-1} + r^{-1} = 1 + q^{-1}$. To conclude the proof, we observe for each $r' \in (1, +\infty)$ there is a constant $C = C(r')$ such that

$$
\int_{t_1}^{t_2} ds\, \|p_{t_2-s}\|_{r'}^2 \le C (t_2 - t_1)^{1/r'},
$$

plugging this bound into (3.15) and using (3.12) we get (3.11). □

We can now get some *apriori* bounds on $X_\lambda(t)$.

Lemma 3.2. *For each $N \in [1, \infty)$ and $\zeta > 0$ there is a constant $C = C(N, \zeta)$ such that*

$$
\mathsf{E} \sup_{t \in [0,T]} |X_\lambda(t)|^N \le C e^{CT} \lambda^{-\zeta N}, \tag{3.16}
$$

$$
\sup_{t \in [0,T]} \mathsf{E} \sup_{t' \in [t, t+\tau]} |X_\lambda(t') - X_\lambda(t)|^N \le C e^{CT} \lambda^{-\zeta N} \tau^{N/2}, \tag{3.17}
$$

for any $\tau \in (0,1)$, $\lambda \in (0,1)$, and $T \in \mathbf{R}_+$.

Proof. Since $\varphi \in \mathcal{S}(\mathbf{R})$ is a probability density,

$$
|P_{t,s}^{(\lambda)}| \le p_{t-s}(0). \tag{3.18}
$$

Furthermore, by a change of variables, for each $p \in [1, +\infty]$, there is a constant $C = C(p)$ so that, for any $\lambda \in (0,1)$,

$$
|\varphi_t^{(\lambda)}\|_p \le C \lambda^{-1+1/p}. \tag{3.19}
$$

Hence, by applying Lemma 3.1 with $q^{-1} = \zeta/2$ and $r^{-1} > 1 - \zeta/2$ (which is allowed for ζ small enough) we get that for each ζ and N there is a constant $C = C(\zeta, N)$ so that, for any $\lambda \in (0,1)$,

$$\sup_{0\le s\le t\le T} \big(\mathsf{E}\,|K_{t,s}^{(\lambda)}|^N\big)^{1/N} \le C\sqrt{T}\lambda^{-\zeta}. \tag{3.20}$$

Using definitions (3.5), (3.8), (3.9), and (3.2), (3.18), (3.20) we conclude that

$$\begin{aligned} \mathsf{E}\sup_{t\in[0,T]} |F_0^{(\lambda)}(t)|^N &\le CT^{3N/2}\lambda^{-\zeta N}, \\ \sup_{t\in[0,T]} \mathsf{E}\,|F_i^{(\lambda)}(t)|^N &\le CT^{3N/2}\lambda^{-\zeta N}, \quad i=1,2. \end{aligned} \tag{3.21}$$

Combining (3.21) with (3.2), (3.18) and equation (3.10), by the Gronwall lemma we get

$$\begin{aligned} \sup_{t\in[0,T]} \mathsf{E}\,|\dot{Y}_\lambda(t)|^N &\le 4^{N-1} \sup_{t\in[0,T]} \big\{ \mathsf{E}\,|F_1^{(\lambda)}(t)|^N + \mathsf{E}\,|F_2^{(\lambda)}(t)|^N \big\} \\ &\quad + \frac{1}{2}(2T)^{N-1} \int_0^T ds\, e^{(N/2)(T-s)} \big\{ \mathsf{E}\,|F_1^{(\lambda)}(s)|^N + \mathsf{E}\,|F_2^{(\lambda)}(s)|^N \big\} \\ &\le Ce^{CT}\lambda^{-\zeta N}. \end{aligned} \tag{3.22}$$

Recalling that $X_\lambda(t) = Y_\lambda(t) + b(t) + F_0^{(\lambda)}(t)$, since $\mathsf{E}\sup_{t\le T} |b(t)|^N \le CT^{N/2}$, inequality (3.22), together with the first bound in (3.21), implies (3.16).

By using again (3.20) we show

$$\sup_{t\in[0,T];t+\tau\le T} \mathsf{E} \sup_{t'\in[t,t+\tau]} \big|F_0^{(\lambda)}(t') - F_0^{(\lambda)}(t)\big|^N \le CT^{N/2}(\lambda^{-\zeta}\tau)^N.$$

Then (3.17) follows from (3.22) and $\mathsf{E}\sup_{t'\in[t,t+\tau]} |b(t') - b(t)|^N \le C\tau^{N/2}$. □

Given $\delta \in (0,1)$, we decompose $F_i^{(\lambda)}(t)$ as follows

$$F_i^{(\lambda)}(t) = F_{i,L}^{(\delta)}(t) + F_{i,R}^{(\lambda,\delta)}(t), \qquad i = 0,1,2, \tag{3.23}$$

where

$$\begin{aligned} F_{0,L}^{(\delta)}(t) &:= -\,\mathbf{1}_{\{t\ge\delta\}} \int_\delta^t ds \int_0^{s-\delta} db(s')\, p_{s-s'}(0), \\ F_{1,L}^{(\delta)}(t) &:= \mathbf{1}_{\{t\ge\delta\}} \int_\delta^t ds\, p_{t-s}(0) \int_0^{s-\delta} db(s')\, p_{s-s'}(0), \\ F_{2,L}^{(\delta)}(t) &:= -\,\frac{1}{2}\,\mathbf{1}_{\{t\ge\delta\}} \int_\delta^t ds \int_0^{s-\delta} db(s')\, p_{s-s'}(0), \end{aligned} \tag{3.24}$$

and $F_{i,R}^{(\lambda,\delta)}(t)$, $i=0,1,2$, is defined by difference so that (3.23) holds. We need $\delta>0$ above because $\int_0^t db(s)\, p_{t-s}(0)$ is not well defined as a process in $C(\mathbf{R}_+)$.

Lemma 3.3. *For each $N\in[1,\infty)$ and $\zeta>0$ there exists a constant $C=C(N,\zeta)$ such that*

$$\Big(\mathsf{E}\sup_{t\in[0,T]}|F_{i,R}^{(\lambda,\delta)}(t)|^N\Big)^{1/N}\le Ce^{CT}\lambda^{-\zeta}[\delta^{(1-\zeta)/2}+\delta^{-1}\lambda^2+\lambda^{2/3}],\quad i=0,1,2, \tag{3.25}$$

for any $\lambda\in(0,1)$, $\delta\in(0,1)$, and $T\in\mathbf{R}_+$.

Proof. Setting

$$\Gamma_t^{(\lambda,\delta)}:=\Big\langle\varphi_t^{(\lambda)},\int_0^{t-\delta}db(s)\, p_{t-s}\varphi_s^{(\lambda)}\Big\rangle-\int_0^{t-\delta}db(s)\, p_{t-s}(0), \tag{3.26}$$

we have

$$\begin{aligned}
F_{0,R}^{(\lambda,\delta)}(t)&=-\int_0^{t\wedge\delta}ds\, K_{s,0}^{(\lambda)}-\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\,(\Gamma_s^{(\lambda,\delta)}+K_{s,s-\delta}^{(\lambda)}),\\
F_{1,R}^{(\lambda,\delta)}(t)&=\int_0^{t\wedge\delta}ds\, P_{t,s}^{(\lambda)}\,K_{s,0}^{(\lambda)}+\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\,(P_{t,s}^{(\lambda)}-p_{t-s}(0))K_{s,0}^{(\lambda)}\\
&\quad+\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\, p_{t-s}(0)(\Gamma_s^{(\lambda,\delta)}+K_{s,s-\delta}^{(\lambda)}),\\
F_{2,R}^{(\lambda,\delta)}(t)&=-\Big\{\int_0^{t\wedge\delta}ds\int_0^s ds'+\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\int_0^\delta ds'\Big\}P_{t,s}^{(\lambda)}\,P_{s,s'}^{(\lambda)}\,K_{s',0}^{(\lambda)}\\
&\quad-\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\,\Big(\int_s^t ds'\,P_{t,s'}^{(\lambda)}\,P_{s',s}^{(\lambda)}-\frac12\Big)K_{s,0}^{(\lambda)}\\
&\quad-\frac12\mathbf{1}_{\{t\ge\delta\}}\int_\delta^t ds\,(\Gamma_s^{(\lambda,\delta)}+K_{s,s-\delta}^{(\lambda)}).
\end{aligned} \tag{3.27}$$

The lemma is a consequence of the Hölder inequality, (3.27), (3.20), (3.18), and the following bounds: for each N and ζ, there exists a constant $C=C(N,\zeta)$ such that, for any $\lambda\in(0,1)$, $\delta\in(0,1)$, and $T\in\mathbf{R}_+$

$$\sup_{s\in[\delta,T]}\big(\mathsf{E}\,|\Gamma_s^{(\lambda,\delta)}|^N\big)^{1/N}\le Ce^{CT}\delta^{-1}\lambda^{2-\zeta}, \tag{3.28}$$

$$\Big(\mathsf{E}\sup_{t\in[\delta,T]}\Big|\int_\delta^t ds\,K^{(\lambda)}_{s,s-\delta}\Big|^N\Big)^{1/N}\le Ce^{CT}\big[\sqrt{\delta}+(\lambda^{-2}\delta)^\zeta(\delta\lambda^{-\zeta}+\sqrt{\delta})\big], \tag{3.29}$$

$$\Big(\mathsf{E}\sup_{t\in[\delta,T]}\Big|\int_\delta^t ds\,p_{t-s}(0)K^{(\lambda)}_{s,s-\delta}\Big|^N\Big)^{1/N}\le Ce^{CT}\big[\delta^{(1-\zeta)/2}+(\lambda^{-2}\delta)^\zeta(\delta\lambda^{-\zeta}+\sqrt{\delta})\big], \tag{3.30}$$

$$\Big[\mathsf{E}\Big(\sup_{0\le s<t\le T}(t-s)^{5/6}|P^{(\lambda)}_{t,s}-p_{t-s}(0)|\Big)^N\Big]^{1/N}\le Ce^{CT}\lambda^{2/3-\zeta}. \tag{3.31}$$

Proof of (3.28). By changing variables,

$$\Gamma_t^{(\lambda,\delta)}=\int dx\,\varphi(x-X_\lambda(t))\,I_t(x), \tag{3.32}$$

where

$$I_t(x):=\int_0^{t-\delta}db(s)\,p_{t-s}(0)\int dy\,\Big[\exp\Big\{-\frac{[\lambda(x-y-X_\lambda(s))]^2}{2(t-s)}\Big\}-1\Big]\varphi(y).$$

By the BDG inequality, see (3.13)–(3.14), for each N and ζ there exists a constant $C=C(N,\zeta)$ such that

$$\begin{aligned}\mathsf{E}\,|I_t(x)|^N&\le C\,\mathsf{E}\Big[\int_0^{t-\delta}ds\,p_{t-s}(0)^2\\&\qquad\times\Big(\int dy\,\Big|\exp\Big\{-\frac{[\lambda(x-y-X_\lambda(s))]^2}{2(t-s)}\Big\}-1\Big|\varphi(y)\Big)^2\Big]^{N/2}\\&\le Ce^{CT}\big\{\mathbf{1}_{\{|x|\le\lambda^{-\zeta}\}}\delta^{-1}\lambda^{2-2\zeta}+\mathbf{1}_{\{|x|>\lambda^{-\zeta}\}}|\log\delta|\big\}^N,\end{aligned} \tag{3.33}$$

where we used $1-\exp\{-z^2\}\le 1\wedge z^2$, and Lemma 3.2 in the last bound.

By Hölder inequality and (3.33) we then have

$$\begin{aligned}&\Big(\mathsf{E}\Big|\int_{\{|x|\le\lambda^{-\zeta}\}}dx\,\varphi(x-X_\lambda(t))I_t(x)\Big|^N\Big)^{1/N}\\&\le\Big\{\mathsf{E}\Big[\Big(\int_{\{|x|\le\lambda^{-\zeta}\}}dx\,\varphi(x-X_\lambda(t))^{N/(N-1)}\Big)^{N-1}\int_{\{|x|\le\lambda^{-\zeta}\}}dx\,|I_t(x)|^N\Big]\Big\}^{1/N}\\&\le\|\varphi\|_{N/(N-1)}\Big(\mathsf{E}\int_{\{|x|\le\lambda^{-\zeta}\}}dx\,|I_t(x)|^N\Big)^{1/N}\\&\le Ce^{CT}\delta^{-1}\lambda^{2-3\zeta}.\end{aligned} \tag{3.34}$$

Analogously, using again the Hölder inequality and (3.33), we get that, for each $M > 0$, N, and ζ, there is a constant $C = C(M, N, \zeta)$ so that

$$\begin{aligned}
&\Big(\mathsf{E}\Big|\int_{\{|x|>\lambda^{-\zeta}\}} dx\, \varphi(x - X_\lambda(t)) I_t(x)\Big|^N\Big)^{1/N} \qquad (3.35)\\
&\le \Big\{\mathsf{E}\Big[\Big(\int_{\{|x|>\lambda^{-\zeta}\}} dx\, \varphi(x - X_\lambda(t))^{2N/(2N-1)} (1 + x^2)^{1/(2N-1)}\Big)^{(2N-1)/2}\\
&\quad \times \Big(\int_{\{|x|>\lambda^{-\zeta}\}} dx\, \frac{1}{1 + x^2}\, |I_t(x)|^{2N}\Big)^{1/2}\Big]\Big\}^{1/N}\\
&\le \Big[\mathsf{E}\Big(\int_{\{|x|>\lambda^{-\zeta}\}} dx\, \varphi(x - X_\lambda(t))^{2N/(2N-1)}\, (1 + x^2)^{1/(2N-1)}\Big)^{2N-1}\Big]^{1/(2N)}\\
&\quad \times \Big(\mathsf{E}\int_{\{|x|>\lambda^{-\zeta}\}} dx\, \frac{1}{1 + x^2}\, |I_t(x)|^{2N}\Big)^{1/(2N)}\\
&\le C e^{CT} \lambda^{M\zeta}\, |\log \delta|,
\end{aligned}$$

where we used, in the last step, the decay (faster of any power) of $\varphi(x)$ as $x \to \infty$. Choosing M large enough, the bound (3.28) now follows from (3.32), (3.34), and (3.35).

Proof of (3.29). To prove (3.29) and (3.30), we decompose $K^{(\lambda)}_{t,t-\delta}$ as follows

$$\begin{aligned}
K^{(\lambda)}_{t,t-\delta} &= \Big\langle \varphi^{(\lambda)}, \int_{t-\delta}^{t} db(s)\, p_{t-s}\varphi^{(\lambda)} \Big\rangle\\
&\quad + \Big\langle \varphi^{(\lambda)}_{\lambda[X_\lambda(t)-X_\lambda(t-\delta)]} - \varphi^{(\lambda)}, \int_{t-\delta}^{t} db(s)\, p_{t-s}\varphi^{(\lambda)} \Big\rangle\\
&\quad + \Big\langle \varphi^{(\lambda)}_t, \int_{t-\delta}^{t} db(s)\, p_{t-s}(\varphi^{(\lambda)}_s - \varphi^{(\lambda)}_{t-\delta}) \Big\rangle, \qquad (3.36)
\end{aligned}$$

where we used

$$\Big\langle \varphi^{(\lambda)}_t, \int_{t-\delta}^{t} db(s)\, p_{t-s}\varphi^{(\lambda)}_{t-\delta} \Big\rangle = \Big\langle \varphi^{(\lambda)}_{\lambda[X_\lambda(t)-X_\lambda(t-\delta)]}, \int_{t-\delta}^{t} db(s)\, p_{t-s}\varphi^{(\lambda)} \Big\rangle.$$

It is now possible to switch the order of integrations (the remark after (3.5) forbids to do it directly on definition (3.3)); then, changing variables and integrating by parts, we have for $t \geq \delta$,

$$\begin{aligned}
&\int_\delta^t ds \left\langle \varphi^{(\lambda)}, \int_{s-\delta}^{s} db(s')\, p_{s-s'}\varphi^{(\lambda)} \right\rangle \\
&\quad = \int_0^t db(s') \int_{s'\vee\delta}^{(s'+\delta)\wedge t} ds\, \langle \varphi^{(\lambda)}, p_{s-s'}\varphi^{(\lambda)} \rangle \\
&\quad = -\int_0^t ds' b(s') \frac{\partial}{\partial s'} \int_{s'\vee\delta-s'}^{(s'+\delta)\wedge t-s'} ds\, \langle \varphi^{(\lambda)}, p_s\varphi^{(\lambda)} \rangle \\
&\quad = -\int_0^{\delta\wedge(t-\delta)} ds'\, b(s') \langle \varphi^{(\lambda)}, p_{\delta-s'}\varphi^{(\lambda)} \rangle \qquad (3.37) \\
&\qquad + \int_{t-\delta}^{t} ds'\, b(s') \langle \varphi^{(\lambda)}, p_{t-s'}\varphi^{(\lambda)} \rangle \\
&\qquad - \mathbf{1}_{\{t\in[\delta,2\delta]\}} \int_{t-\delta}^{\delta} ds'\, b(s') \langle \varphi^{(\lambda)}, p_{\delta-s'}\varphi^{(\lambda)} \rangle .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\mathsf{E} \sup_{t\in[\delta,T]} \Big| \int_\delta^t ds \left\langle \varphi^{(\lambda)}, \int_{s-\delta}^{s} db(s')\, p_{s-s'}\varphi^{(\lambda)} \right\rangle \Big|^N \right)^{1/N} \\
&\leq C \left(\mathsf{E} \sup_{t\in[0,T]} |b(t)|^N \right)^{1/N} \sup_{t\in[\delta,T]} \left(\int_0^\delta ds \frac{1}{\sqrt{s}} + \int_{t-\delta}^{t} ds \frac{1}{\sqrt{t-s}} \right) \\
&\leq C \sqrt{\delta T}, \qquad (3.38)
\end{aligned}$$

where we used the proof of (3.18).

Now we observe that, for any $a, b \in \mathbf{R}$ and $n \in [1,\infty]$,

$$\|\varphi_b - \varphi_a\|_n \leq \|\varphi'\|_n |b-a|. \qquad (3.39)$$

In fact for $n \in [1,\infty)$ we have

$$\begin{aligned}
\|\varphi_b - \varphi_a\|_n^n &= \int dx \Big| \int_{x-a\vee b}^{x-a\wedge b} dy\, \varphi'(y) \Big|^n \\
&\leq |b-a|^{n-1} \int dx \int_{x-a\vee b}^{x-a\wedge b} dy\, |\varphi'(y)|^n
\end{aligned}$$

$$= |b-a|^n \|\varphi'\|_n^n,$$

while (3.39) holds for $n=\infty$ because φ' is bounded.

Then, by Lemma 3.2 and (3.39), (we will choose ζ' at the next step),

$$\begin{aligned}
&\sup_{t\in[0,T]} \big(\mathsf{E}\, \|\varphi^{(\lambda)}_{\lambda(X_\lambda(t)-X_\lambda(t-\delta))} - \varphi^{(\lambda)}\|_p^\alpha \big)^{1/\alpha} \\
&\qquad \le C\lambda^{-1+1/p}\, \|\varphi'\|_p \sup_{t\in[0,T]} \{\mathsf{E}\, |X_\lambda(t) - X_\lambda(t-\delta)|^\alpha\}^{1/\alpha} \\
&\qquad \le Ce^{CT}\lambda^{-1+1/p}\, (\lambda^{-\zeta'}\delta + \sqrt{\delta}).
\end{aligned}$$

Hence, by Lemma 3.1 and (3.19)

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big\{ \mathsf{E} \Big| \Big\langle \varphi^{(\lambda)}_{\lambda(X_\lambda(t)-X_\lambda(t-\delta))} - \varphi^{(\lambda)}, \int_{t-\delta}^{t} db(s)\, p_{t-s}\varphi^{(\lambda)} \Big\rangle \Big|^N \Big\}^{1/N} \\
&\qquad \le C\delta^{[1-(\frac{1}{r}-\frac{1}{q})]/2} \sup_{t\in[0,T]} \big\{ \mathsf{E}\, \|\varphi^{(\lambda)}_{\lambda(X_\lambda(t)-X_\lambda(t-\delta))} - \varphi^{(\lambda)}\|_p^\alpha \big\}^{1/\alpha} \|\varphi^{(\lambda)}\|_r \\
&\qquad \le Ce^{CT}(\lambda^{-2}\delta)^\zeta(\lambda^{-\zeta}\delta + \sqrt{\delta}),
\end{aligned} \tag{3.40}$$

where we have chosen $r^{-1} - q^{-1} = 1 - 2\zeta$, $p^{-1} \ge 1 - 2\zeta$ (which is allowed for $\zeta \in (0, 1/2)$), and $\zeta' = p\zeta(1-2\zeta)$.

Similarly, by Lemma 3.2, for $q > r$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \big\{ \mathsf{E} \sup_{s\in[t-\delta,t]} \|\varphi^{(\lambda)}_s - \varphi^{(\lambda)}_{t-\delta}\|_r^q \big\}^{1/q} \\
&\qquad \le C\lambda^{-1+1/r}\, \|\varphi'\|_r \sup_{t\in[0,T]} \big\{ \mathsf{E} \sup_{s\in[t-\delta,t]} |X_\lambda(s) - X_\lambda(t-\delta)|^q \big\}^{1/q} \\
&\qquad \le Ce^{CT}\lambda^{-1+1/r}\, (\lambda^{-\zeta'}\delta + \sqrt{\delta}).
\end{aligned}$$

Hence, by Lemma 3.1 and (3.19),

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big\{ \mathsf{E} \Big| \Big\langle \varphi^{(\lambda)}_t, \int_{t-\delta}^{t} db(s)\, p_{t-s}(\varphi^{(\lambda)}_s - \varphi^{(\lambda)}_{t-\delta}) \Big\rangle \Big|^N \Big\}^{1/N} \\
&\qquad \le C\delta^{[1-(\frac{1}{r}-\frac{1}{q})]/2} \sup_{t\in[0,T]} \big\{ (\mathsf{E}\, \|\varphi^{(\lambda)}_t\|_p^\alpha)^{1/\alpha} (\mathsf{E} \sup_{s\in[t-\delta,t]} \|\varphi^{(\lambda)}_s - \varphi^{(\lambda)}_{t-\delta}\|_r^q)^{1/q} \big\} \\
&\qquad \le Ce^{CT}(\lambda^{-2}\delta)^\zeta(\lambda^{-\zeta}\delta + \sqrt{\delta}),
\end{aligned} \tag{3.41}$$

where we choose q, r as in (3.40), and $\zeta' = r\zeta(1-2\zeta)$. The bound (3.29) follows from (3.36), (3.38), (3.40), and (3.41).

The bound (3.30) can be proved by analogous computations which we omit.

Proof of (3.31). We use that $\varphi(\cdot)$ is a probability density, $1 - \exp\{-z^2\} \le |z|^{2/3}$, and Lemma 3.2 to get

$$\mathsf{E} \big[\sup_{\delta\le s<t\le T} (t-s)^{5/6} |P^{(\lambda)}_{t,s} - p_{t-s}(0)| \big]^N$$

$$\begin{aligned}
&\leq \mathsf{E} \sup_{\delta\leq s<t\leq T} (t-s)^{N/3} \int dx\, dy\, \varphi(x)\varphi(y) \\
&\quad \times \left| \exp\left\{ -\frac{[\lambda(x-y+X_\lambda(t)-X_\lambda(s))]^2}{2(t-s)} \right\} - 1 \right|^N \\
&\leq Ce^{CT}\, \lambda^{N(2/3-\zeta)}.
\end{aligned}$$

□

Lemma 3.4. *Let $\dot Y_\lambda$ be the solution of* (3.10) *and η the* (*unique*) *continuous solution of*

$$\eta(t) = \frac{1}{2}b(t) - \frac{1}{2}\int_0^t ds\, p_{t-s}(0)\, b(s) + \frac{1}{2}\int_0^t ds\, \eta(s). \tag{3.42}$$

Then for each $N \in [1,+\infty)$ and $\zeta > 0$ there exists $\tau = \tau(N,\zeta) > 0$ such that

$$\lim_{\lambda\downarrow 0} \lambda^{-(2/3-\zeta)}\, \mathsf{E} \sup_{t\leq\tau|\log\lambda|} |\dot Y_\lambda(t) - \eta(t)|^N = 0. \tag{3.43}$$

Proof. Recalling definitions (3.24), we claim that for each N there exists a constant $C = C(N)$ such that for any δ and T

$$\Big(\mathsf{E} \sup_{t\in[0,T]} \Big| F_{1,L}^{(\delta)}(t) - \frac{1}{2}b(t) \Big|^N \Big)^{1/N} \leq C\sqrt{\delta T}, \tag{3.44}$$

$$\Big(\mathsf{E} \sup_{t\in[0,T]} \Big| F_{2,L}^{(\delta)}(t) + \frac{1}{2}\int_0^t ds\, p_{t-s}(0) b(s) \Big|^N \Big)^{1/N} \leq C\sqrt{\delta T}. \tag{3.45}$$

Indeed, by switching the order of integrations in (3.24) we have

$$\begin{aligned}
F_{1,L}^{(\delta)}(t) - \frac{1}{2}b(t) &= -\frac{1}{2}\mathbf{1}_{\{t<\delta\}}\, b(t) \\
&\quad + \mathbf{1}_{\{t\geq\delta\}} \Big[\int_0^{t-\delta} db(s') \Big(\int_{s'+\delta}^t ds\, p_{t-s}(0) p_{s-s'}(0) - \frac{1}{2} \Big) \\
&\quad + \frac{1}{2}(b(t-\delta) - b(t)) \Big],
\end{aligned}$$

from which, recalling (3.2), (3.44) follows by a straightforward computation. Analogously

$$\begin{aligned}
F_{2,L}^{(\delta)}(t) + \frac{1}{2}\int_0^t ds\, p_{t-s}(0) b(s) &= \frac{1}{2}\mathbf{1}_{\{t<\delta\}} \int_0^t ds\, p_{t-s}(0) b(s) \\
&\quad + \frac{1}{2}\mathbf{1}_{\{t\geq\delta\}} \int_{t-\delta}^t ds\, p_{t-s}(0) b(s),
\end{aligned}$$

which implies (3.45).

We now choose $\delta = \lambda^{4/3}$. By Lemma 3.3 and (3.44)–(3.45), for each N and ζ, there exists a constant $C = C(N,\zeta)$ such that for any λ and T

$$\Big(\mathsf{E} \sup_{t\in[0,T]} \Big| F_1^{(\lambda)}(t) - \frac{1}{2} b(t) \Big|^N \Big)^{1/N} \le C e^{CT} \lambda^{2/3-\zeta}, \tag{3.46}$$

$$\Big(\mathsf{E} \sup_{t\in[0,T]} \Big| F_2^{(\lambda)}(t) + \frac{1}{2} \int_0^t ds\, p_{t-s}(0) b(s) \Big|^N \Big)^{1/N} \le C e^{CT} \lambda^{2/3-\zeta}. \tag{3.47}$$

Next from (3.22), (3.31), and Hölder inequality we get

$$\Big[\mathsf{E} \sup_{t\in[0,T]} \Big| \int_0^t ds\, \dot{Y}_\lambda(s) \Big(\int_s^t ds'\, P_{t,s'}^{(\lambda)} P_{s',s}^{(\lambda)} - \frac{1}{2} \Big) \Big|^N \Big]^{1/N} \le C e^{CT} \lambda^{2/3-\zeta}. \tag{3.48}$$

Let us define

$$\begin{aligned} R_\lambda(t) := F_1^{(\lambda)}(t) - \frac{1}{2} b(t) + F_2^{(\lambda)}(t) + \frac{1}{2} \int_0^t ds\, p_{t-s}(0) b(s) \\ + \int_0^t ds\, \dot{Y}_\lambda(s) \Big(\int_s^t ds'\, P_{t,s'}^{(\lambda)} P_{s',s}^{(\lambda)} - \frac{1}{2} \Big), \end{aligned} \tag{3.49}$$

then (3.46)–(3.48) imply that, for each N and ζ, there is $C = C(N,\zeta)$ such that for any λ and T

$$\big(\mathsf{E} \sup_{t\in[0,T]} |R_\lambda(t)|^N \big)^{1/N} \le C e^{CT} \lambda^{2/3-\zeta}. \tag{3.50}$$

By (3.10), (3.42), and (3.49) we have

$$\dot{Y}_\lambda(t) - \eta(t) = R_\lambda(t) + \frac{1}{2} \int_0^t ds\, (\dot{Y}_\lambda(s) - \eta(s)).$$

The lemma follows from (3.50) and the Gronwall lemma. □

Proof of Theorem 2.1. We recall $X_\lambda(t) = \int_0^t ds\, \dot{Y}_\lambda(s) + b(t) + F_0^{(\lambda)}(t)$ and observe that

$$\xi(t) = \int_0^t ds\, \eta(s) + b(t) - \int_0^t ds\, p_{t-s}(0) b(s),$$

where η (resp. ξ) solves (3.42) (resp. (3.1)). Therefore

$$X_\lambda(t) - \xi(t) = F_0^{(\lambda)}(t) + \int_0^t ds\, p_{t-s}(0) b(s) + \int_0^t ds\, (\dot{Y}_\lambda(s) - \eta(s)). \tag{3.51}$$

We note that, see (3.24), $F^{(\delta)}_{0,L}(t) = 2F^{(\delta)}_{2,L}(t)$. Hence, by Lemma 3.3 and (3.45) (recall we have chosen $\delta = \lambda^{4/3}$), for each N and ζ there exists a constant $C = C(N,\zeta)$ such that for any λ and T

$$\Big(\mathsf{E}\sup_{t\le T}\Big|F_0^{(\lambda)}(t) + \int_0^t ds\, p_{t-s}(0)b(s)\Big|^N\Big)^{1/N} \le Ce^{CT}\lambda^{2/3-\zeta}, \tag{3.52}$$

the limit (2.5) thus follows from Lemma 3.4, (3.51), and (3.52). □

4. The linear Stefan problem

Proof of Theorem 2.2. One iteration of the equation (2.6) solved by the Green function F leads to

$$F(t) = 1 - \int_0^t ds\, p_s(0) + \frac{1}{2}\int_0^t ds\, F(s), \tag{4.1}$$

which is the integral formulation of $F'(t) = F(t)/2 - p_t(0)$ with initial datum $F(0) = 1$. Its solution is (2.8) whose asymptotics yields (2.9), see [1].

The equation (2.6) can also be solved by successive iterations, giving rise to a series whose terms are multiple integrals that can be computed explicitly:

$$F(t) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{\Gamma(n/2+1)}\Big(\frac{t}{2}\Big)^{n/2}, \tag{4.2}$$

where Γ is the Euler Gamma function. One can then verify that the series on the r.h.s. of (4.2) is indeed the expansion of the r.h.s. of (2.8), see again [1].

In Proposition 4.1 below we will prove that

$$F(t) = \mathsf{E}_0(\exp\{-L^0(t)\}), \tag{4.3}$$

where P_x is the distribution of a Brownian motion $b(t)$ starting from x, E_x is the corresponding expectation, and $L^a(t)$ is the local time in a of $b(t)$. Note that $L^0(t)$ has the same law as $|w_t|$, for a Brownian motion w_t (see [5, VI, Corollary 2.2]); we can compute explicitly the r.h.s. of (4.3) and get a third proof of (2.8). □

We now formalize the probabilistic interpretation of the Green function $F(t)$ in terms of the local time of a Brownian motion. This interpretation will play a crucial role in the proof of the asymptotics for the two centers.

Definition 4.1. Let Q_x be the joint law of a Brownian motion $b(t)$ starting from x and of an independent time s with a mean 1 exponential distribution. A "Brownian motion starting from x and killed with a rate given by $L^0(t)$" is then the process $x(t)$ defined on the state space $\mathbf{R}\cup\{0'\}$ by

$$x(t) = \begin{cases} b(t) & \text{if } L^0(t) < s, \\ 0' & \text{if } L^0(t) \ge s, \end{cases} \tag{4.4}$$

where $0'$ denotes the cemetery point. For ease of reference we will sometimes call $x(t)$ a "Brownian motion killed with rate δ_0", and denote its law by $\tilde{Q}_x$.

Remark. Despite the fact that $L^0(t)$ depends on the whole past $\{b(u), u \le t\}$, $x(\cdot)$ is a Markov process. Indeed, conditioning on $\{x(u), u \le t\}$, if $x(t) = 0'$, then $x(t') = 0'$, for all $t' \ge t$, independently of the previous history; if instead $x(t) = z \in \mathbf{R}$, then $s > L^0(t)$ and the (conditional) law of the residual time $s - L^0(t)$ is again an exponential distribution of mean 1. The conditional law of $x(t + \cdot)$ is then $\tilde{Q}_z$, thus concluding the proof that $x(\cdot)$ is Markovian.

Proposition 4.1. *The transition probability of the process $x(t)$ restricted to $\mathbf{R}$ has a density $q_t(x, y)$ with respect to the Lebesgue measure, which is solution of*

$$q_t(x, y) = p_t(x - y) - \int_0^t ds\, p_{t-s}(x) q_s(0, y). \tag{4.5}$$

Furthermore $F(t)$ is the probability of survival at time t for the process $x(t)$ starting from 0, and (4.3) holds.

Proof. For any Borel set B in $\mathbf{R}$,

$$\begin{aligned} Q_x(x(t) \in B) &= \int_0^{+\infty} ds\, e^{-s}\ \mathsf{P}_x\left(b(t) \in B;\ L^0(t) < s\right) \\ &= \int_0^{+\infty} ds\, e^{-s} \int_B dy\, p_t(y - x)\ \mathsf{P}_{x,y;t}\left(L^0(t) < s\right) \\ &= \int_B dy\, p_t(y - x)\ \mathsf{E}_{x,y;t}(e^{-L^0(t)}), \end{aligned} \tag{4.6}$$

where in the last two terms, $\mathsf{P}_{x,y;t}$ is the law of a Brownian bridge $\beta(t)$ from x to y in time t, $\mathsf{E}_{x,y;t}$ is the corresponding expectation, and $L^0(t)$ is the local time of $\beta(t)$. Thus the restriction to $\mathbf{R}$ of the transition probability of $x(t)$ has a density $\rho_t(x, y) = p_t(y - x)\ \mathsf{E}_{x,y;t}(\exp(-L^0(t)))$. We are going to prove that $\rho_t(x, y)$ solves (4.5), thus $\rho_t(x, y) = q_t(x, y)$, which yields

$$q_t(x, y) = p_t(x - y)\, \mathsf{E}_{x,y;t}(\exp\{-L^0(t)\}). \tag{4.7}$$

We will see that (4.5) is the backward Kolmogorov equation for $x(t)$. We first define smoothened versions of (4.5) and of the killed Brownian motion. Let $J \in C_0^\infty(\mathbf{R})$ be a positive function with compact support in the positive half line and $\int dx\, J(x) = 1$; let $J_\varepsilon(x) := \varepsilon^{-1} J(\varepsilon^{-1} x)$, $\varepsilon > 0$, be the approximation of the Dirac measure δ_0; finally let

$$L_\varepsilon(t) := \int_0^t ds\, J_\varepsilon(b(s)) = \int dz\, J_\varepsilon(z) L^z(t), \tag{4.8}$$

where the last identity, which holds a.s., is the occupation times formula for local times, see [5, VI, Corollary 1.6]. We may and will consider a version of the process of Definition 4.1 where all $L^z(t)$ are continuous in z and t, see [5, VI, Theorem 1.7], so that $x^\varepsilon(t)$ defined by (4.4) with $L^0(t)$ replaced by $L_\varepsilon(t)$, converges pointwise as ε goes to 0 to $x(t)$. By proceeding as in (4.6), the transition probability of the process $x^\varepsilon(t)$ restricted to $\mathbf{R}$ has density

$$q_t^\varepsilon(x,y) = p_t(y-x)\,\mathsf{E}_{x,y;t}(\exp(-L^\varepsilon(t))). \tag{4.9}$$

By the Feynman – Kac formula $q_t^\varepsilon(x,y)$ is consequently the solution of

$$\begin{aligned} \frac{\partial}{\partial t} q_t^\varepsilon(x,y) &= \frac{1}{2}\frac{\partial^2}{\partial x^2} q_t^\varepsilon(x,y) - q_t^\varepsilon(x,y) J_\varepsilon(x), \\ q_0^\varepsilon(x,y) &= \delta(x-y). \end{aligned} \tag{4.10}$$

By taking its integral version we get a regularization of (4.5), namely

$$q_t^\varepsilon(x,y) = p_t(x-y) - \int_0^t ds \int dz\, p_{t-s}(x-z) J_\varepsilon(z) q_s^\varepsilon(z,y). \tag{4.11}$$

We next let ε go to 0. By [5, VI, Corollary 1.9] and the dominated convergence theorem, the r.h.s. of (4.9) has a limit as ε goes to 0 and, for each $t > 0$, the convergence is uniform for (x,y) on compacts. Therefore $q_t^\varepsilon(x,y)$, $t > 0$, converges to the continuous function $p_t(x-y)\,\mathsf{E}_{x,y;t}(\exp\{-L^0(t)\}) = \rho_t(x,y)$ which is by (4.6) the density of the killed Brownian motion of Definition 4.1; it remains to prove that this is also equal to $q_t(x,y)$.

Again by the above continuity properties, for any $s \in (0,t)$,

$$\lim_{\varepsilon\to 0} \int dz\, p_{t-s}(x-z) J_\varepsilon(z) q_s^\varepsilon(z,y) = p_{t-s}(x)\rho_s(0,y), \tag{4.12}$$

as $q_s^\varepsilon(z,y)$ converges to $\rho_s(z,y)$ uniformly on compacts. By (4.9) there is C so that for any $s \in (0,t)$

$$\int dz\, p_{t-s}(x-z) J_\varepsilon(z) q_s^\varepsilon(z,y) \le \int dz\, p_{t-s}(x-z) J_\varepsilon(z) p_s(z-y) \le \frac{C}{\sqrt{s(t-s)}}.$$

Then by dominated convergence and using (4.12), we can compute the limit as ε goes to 0 of (4.11). It yields that $\rho_t(x,y)$ satisfies (4.5), and it is therefore equal to $q_t(x,y)$. This concludes the proof of the first statement of the proposition, namely that the function $q_t(x,y)$, solution of (4.5), is also the transition probability density of the killed Brownian motion.

To prove (4.3), we take $B = \mathbf{R}$ in (4.6), observe that $Q_x(x(t) \neq 0') = Q_x(x(t) \in \mathbf{R})$, then by (4.7),

$$Q_x(x(t) \neq 0') = \int_{\mathbf{R}} dy\, q_t(x,y).$$

On the other hand, from the first identity in (4.6),

$$Q_x(x(t) \neq 0') = \int_0^\infty ds\, e^{-s}\ \mathsf{P}_x(L^0(t) < s) = \mathsf{E}_x(\exp\{-L^0(t)\}),$$

and (4.3) follows by comparing the last two equations. □

Remark. In the course of the latter proof, we have introduced the process $x^\varepsilon = \{x^\varepsilon(t),\ t \geq 0\}$ and shown that its transition probability density on $\mathbf{R}$, denoted by $q_t^\varepsilon(x,y)$, solves (4.10). Since $Q_x(x^\varepsilon(t) = 0')$ is simply $1 - \int dy q_t^\varepsilon(x,y)$, (4.10) completely determines the process. On the other hand (4.10) can be interpreted as the backward Chapman – Kolmogorov equation restricted to $\mathbf{R}$ for the Markov process with state space $S = \mathbf{R} \cup \{0'\}$ and generator

$$A^\varepsilon f(x) = \begin{cases} \frac{1}{2}\Delta f(x) + J_\varepsilon(x)[f(0') - f(x)] & \text{if } x \in \mathbf{R}, \\ 0 & \text{if } x = 0'. \end{cases} \tag{4.13}$$

The action of A^ε on $q_t^\varepsilon(x,y)$, with y held fixed, which is by definition the backward Chapman – Kolmogorov equation restricted to $\mathbf{R}$, is exactly (4.10), since $A^\varepsilon q_t^\varepsilon(0',y) = 0$.

The forward Chapman – Kolmogorov equation restricted to x and y in $\mathbf{R}$ gives again (4.10) but with x and y interchanged. Indeed, let $f(x)$ be a regular function for $x \in \mathbf{R}$, $f(0') = 0$, then

$$\frac{d}{dt}\mathsf{E}_x(f(x^\varepsilon(t))) = \int dy\, q_t^\varepsilon(x,y)\Big\{\frac{1}{2}f''(y) - J^\varepsilon(y)f(y)\Big\}.$$

After an integration by parts, since f is arbitrary,

$$\begin{aligned} \frac{\partial}{\partial t} q_t^\varepsilon(x,y) &= \frac{1}{2}\frac{\partial^2}{\partial y^2} q_t^\varepsilon(x,y) - J^\varepsilon(y) q_t^\varepsilon(x,y), \\ q_0^\varepsilon(x,y) &= \delta(x-y), \end{aligned}$$

which is the forward Chapman – Kolmogorov equation for the restriction to $\mathbf{R}$ of the transition probability. A comparison with (4.10) shows that $q_t^\varepsilon(x,y)$ depends symmetrically on x and y, a property therefore enjoyed also by its limit $q_t(x,y)$; this conclusion could also have been drawn directly from (4.7).

5. The linear Stefan problem with two centers

The matrix $\mathbf{F}(t)$ is related to an equation similar to (4.5) with traps at 0 and a.

Proposition 5.1. *Let $q_t(x,y)$ be the solution of*

$$q_t(x,y) = p_t(x-y) - \int_0^t ds\, [p_{t-s}(x) q_s(0,y) + p_{t-s}(x-a) q_s(a,y)]. \tag{5.1}$$

Then the Green function $\mathbf{F}(t)$*, solution of* (2.16)*, is given by*

$$\mathbf{F}(t) = \begin{pmatrix} 1 - \int_0^t ds\, q_s(0,0) & -\int_0^t ds\, q_s(a,0) \\ -\int_0^t ds\, q_s(0,a) & 1 - \int_0^t ds\, q_s(a,a) \end{pmatrix}. \tag{5.2}$$

Proof. We prove that the r.h.s. of (5.2) satisfies (2.16). Let

$$F_{0,0}(t) = 1 - \int_0^t ds\, q_s(0,0), \qquad F_{0,a}(t) = -\int_0^t ds\, q_s(a,0).$$

We have, using (5.1) and integration by parts

$$\begin{aligned} F_{0,0}(t) &= 1 - \int_0^t ds\, p_s(0) - \int_0^t ds \int_0^s ds'\, [p_{s-s'}(0) F'_{0,0}(s') + p_{s-s'}(a) F'_{0,a}(s')] \\ &= 1 - \int_0^t ds' [p_{t-s'}(0) F_{0,0}(s') + p_{t-s'}(a) F_{0,a}(s')]. \end{aligned} \tag{5.3}$$

The other terms are analogous. □

We will see in Proposition 5.2 that the function $q_t(x,y)$ is the transition probability of a killed Brownian motion with death at 0 and a. This will establish, via Proposition 5.1, a relation between $\mathbf{F}(t)$ and the killed Brownian motion, which will enable us to complete the proof of Theorem 2.4.

In this section we denote by Q_x the joint law of a Brownian motion $b(t)$ starting from x and of two times, s_0 and s_a, with independent exponential distributions of mean 1. Denoting by $\tau_t^z := \inf\{u > 0 : L^z(u) > t\}$ the inverse of the local time $L^z(t)$, hence $L^z(\tau_t^z) = t$, we define the Brownian motion starting from x and killed with rate δ_0 at 0 and δ_a at a, as

$$x(t) = \begin{cases} b(t) & \text{if } \tau^0_{s_0} > t,\ \tau^a_{s_a} > t, \\ 0' & \text{if } \tau^0_{s_0} \le t,\ \tau^a_{s_a} > \tau^0_{s_0}, \\ a' & \text{if } \tau^a_{s_a} \le t,\ \tau^0_{s_0} > \tau^a_{s_a}, \end{cases} \tag{5.4}$$

$0'$ and a' being the cemetery points of the Brownian motion when killed respectively at 0 and at a. An argument similar to the one after Definition 4.1 (which is omitted), would show that $x(\cdot)$ is a Markov process.

The following proposition can be proved as in the single trap case of Proposition 4.1.

Proposition 5.2. *The transition probability of the process* $x(t)$ *restricted to* $\mathbf{R}$ *has a density equal to* $q_t(x,y)$*, solution of* (5.1)*. Moreover*

$$q_t(x,y) = p_t(x-y)\, \mathsf{E}_{x,y;t}(\exp(-L^0(t) - L^a(t))), \tag{5.5}$$

and, for any $x \in \mathbf{R}$,

$$
\begin{aligned}
Q_x(x(t) = 0') &= \int_0^t ds\, q_s(x,0), \\
Q_x(x(t) = a') &= \int_0^t ds\, q_s(x,a).
\end{aligned}
\tag{5.6}
$$

Proof of Theorem 2.4. We first observe that from (5.2) we get (2.18). Indeed it is enough to note that $q_t(x,y) = q_t(x',y')$ where x' and y' are obtained from x and y by reflection around $a/2$. Propositions 5.1 and 5.2 imply (2.18) as well as the monotonicity of $F_{0,0}(t)$ and $F_{0,a}(t)$. The proof of the asymptotics (2.21) and (2.22) is obtained by using Laplace transforms.

Since $F^+(t)$ is monotone decreasing, we can find its asymptotics applying Tauberian theorems. By using (2.16)–(2.17), the Laplace transform $\widehat{F^+}(\lambda)$ of $F^+(t)$ is

$$
\widehat{F^+}(\lambda) := \int_0^\infty dt\, e^{-\lambda t} F^+(t) = \frac{\sqrt{2}\,\lambda^{-1/2}}{\sqrt{2\lambda} + 1 + e^{-|a|\sqrt{2\lambda}}}, \qquad \lambda > 0, \tag{5.7}
$$

from which, by applying the Tauberian theorem for densities, see e.g. [3, XIII.5, Theorem 4], (2.21) follows.

Analogously, the Laplace transform $\widehat{F^-}(\lambda)$ of $F^-(t)$ is

$$
\widehat{F^-}(\lambda) = \frac{1}{2} \frac{\sqrt{2}\,\lambda^{-1/2}}{\sqrt{2\lambda} + 1 - e^{-|a|\sqrt{2\lambda}}}. \tag{5.8}
$$

Since $F^-(t)$ converges as $t \to \infty$ (by the monotonicity of $F_{0,0}(t)$ and $F_{0,a}(t)$), its limit coincides with the limit as λ goes to 0 of $\lambda \widehat{F^-}(\lambda)$, which proves (2.22). A second proof of (2.22) is given below. □

Alternative proof of (2.22). By (5.6)

$$
\lim_{t\to\infty} Q_x(x(t) = a') = \int_0^\infty dt\, q_t(x,a) =: G(x). \tag{5.9}
$$

Since $\lim_{t\to\infty} L^y(t) = +\infty$ a.s. (see [5, VI, Corollary 2.4])

$$
\begin{aligned}
&\lim_{t\to\infty} Q_x(x(t) \in \mathbf{R}) = 0, \\
&\lim_{t\to\infty} Q_x(x(t) = 0') = 1 - G(x),
\end{aligned}
$$

and setting

$$
F^-(t,x) := \frac{1}{2}[1 - Q_x(x(t) = 0') + Q_x(x(t) = a')], \tag{5.10}
$$

(observe $F^-(t) = F^-(t,0)$) we have

$$\lim_{t\to\infty} F^-(t,x) = G(x). \tag{5.11}$$

From (5.9), (5.5) and continuity properties of local times, see [5, VI, Corollary 1.8], $G(x)$ is a positive, continuous and bounded function; furthermore, by routine computations that are omitted, $G(x)$ solves (in a distribution sense), for $a > 0$,

$$\Big[\frac{1}{2}\Delta - (\delta_0 + \delta_a)\Big] G = -\delta_a. \tag{5.12}$$

We can now conclude that (2.22) holds. Indeed the unique solution of (5.12) is

$$G(x) = \begin{cases} \dfrac{1}{2}\,\dfrac{1}{1+a} & \text{if} \quad x \le 0, \\[2ex] \dfrac{1}{2}\,\dfrac{1}{1+a} + \dfrac{x}{1+a} & \text{if} \quad x \in (0,a), \\[2ex] 1 - \dfrac{1}{2}\,\dfrac{1}{1+a} & \text{if} \quad x \ge a, \end{cases}$$

and, by (5.11), $\lim_{t\to\infty} F^-(t,0) = G(0)$. □

A. Existence and uniqueness

To prove existence and uniqueness of (2.2) we assume, without loss of generality, $\lambda = 1$. Let B be the Banach space whose elements are the vectors $(X,h) \in \mathbf{R} \times L_2(\mathbf{R})$ endowed with the norm

$$\|(X,h)\|_B := \sqrt{|X|^2 + \|h\|_2^2}.$$

For each $(X_0,h_0) \in B$ we consider the Cauchy problem

$$X(t) = X_0 + b(t) + \int_0^t ds\, \alpha(X(s),h(s)), \tag{A.1}$$

$$h(t) = p_t h_0 - \int_0^t db(s)\, p_{t-s}\varphi_{X(s)} - \int_0^t ds\, \alpha(X(s),h(s))\, p_{t-s}\varphi_{X(s)}, \tag{A.2}$$

where we recall $\varphi \in \mathcal{S}(\mathbf{R})$ is a probability density, and $\varphi_X(x) = \varphi(x-X)$. In the particular case

$$\alpha(X,h) = \langle \varphi_X, h\rangle, \tag{A.3}$$

the system (A.1)–(A.2) with initial conditions $(X_0,h_0) = (0,0)$ reduces to (2.2).

We first prove an existence and uniqueness theorem for the system (A.1)–(A.2) in the case the map α is bounded and globally Lipschitz.

Theorem A.1. *Assume the map $(X,h) \mapsto \alpha(X,h)$ from B to $\mathbf{R}$ is bounded and globally Lipschitz. Then, for each $(X_0,h_0) \in B$ there exists a solution of* (A.1)–(A.2) $(X(\cdot),h(\cdot)) \in C(\mathbf{R}_+;B)$ *such that*

$$\sup_{t\in[0,T]} \mathsf{E}\, \|(X(t),h(t))\|_B^2 < \infty \qquad \forall T>0. \tag{A.4}$$

This solution is unique in the following sense; if $(\tilde X(t),\tilde h(t))$ is another continuous solution for which (A.4) *holds, then*

$$\mathsf{P}\Big(\sup_{t\in[0,T]} \|(X(t),h(t)) - (\tilde X(t),\tilde h(t))\|_B^2 = 0 \Big) = 1 \qquad \forall T>0. \tag{A.5}$$

Proof. Let $\beta(X,h) := \alpha(X,h)\varphi_X$; by (3.39), the map $X \mapsto \varphi_X$ of $\mathbf{R}$ into $L_2(\mathbf{R})$ is bounded and globally Lipschitz. Then, for some $K>1$ and any (X,h), $(Y,g) \in B$,

$$|\alpha(X,h)|^2 + \|\beta(X,h)\|_2^2 + \|\varphi_X\|_2^2 \le K, \tag{A.6}$$

$$\begin{aligned} |\alpha(X,h) - \alpha(Y,g)| + \|\beta(X,h) - \beta(Y,g)\|_2 + \|\varphi_X - \varphi_Y\|_2 \\ \le K\|(X,h) - (Y,g)\|_B. \end{aligned} \tag{A.7}$$

We prove the existence result by Picard iterations. We set $(X^{(0)}(t),h^{(0)}(t)) = (X_0, p_t h_0)$ and, for $n \ge 1$,

$$X^{(n)}(t) = X_0 + b(t) + \int_0^t ds\, \alpha(X^{(n-1)}(s),h^{(n-1)}(s)), \tag{A.8}$$

$$h^{(n)}(t) = p_t h_0 - \int_0^t db(s)\, p_{t-s}\varphi_{X^{(n-1)}(s)} - \int_0^t ds\, p_{t-s}\beta(X^{(n-1)}(s),h^{(n-1)}(s)). \tag{A.9}$$

By the Cauchy – Schwarz inequality, for any $n\ge 1$,

$$\begin{aligned} &\mathsf{E}\, |X^{(n+1)}(t) - X^{(n)}(t)|^2 \\ &\le t \int_0^t ds\ \mathsf{E}\, \big|\alpha(X^{(n)}(s),h^{(n)}(s)) - \alpha(X^{(n-1)}(s),h^{(n-1)}(s))\big|^2. \end{aligned} \tag{A.10}$$

Analogously, by using first $(a+b)^2 \le 2(a^2+b^2)$ and then the Cauchy – Schwarz inequality,

$$\begin{aligned} \mathsf{E}\, \|h^{(n+1)}(t) - h^{(n)}(t)\|_2^2 \ \le\ & 2\ \mathsf{E} \int_0^t ds\, \big\|p_{t-s}[\varphi_{X^{(n)}(s)} - \varphi_{(X^{(n-1)}(s))}]\big\|_2^2 \\ & + 2t\ \mathsf{E} \int_0^t ds\, \big\|p_{t-s}\big[\beta(X^{(n)}(s),h^{(n)}(s)) \end{aligned}$$

$$- \beta(X^{(n-1)}(s), h^{(n-1)}(s))] \big\|_2^2. \tag{A.11}$$

Recalling (A.7) and using that p_t is a contraction in $L_2(\mathbf{R})$, from (A.10) and (A.11) we get, for $n \geq 1$,

$$\begin{aligned} &\mathsf{E}\, \|(X^{(n+1)}(t), h^{(n+1)}(t)) - (X^{(n)}(t), h^{(n)}(t))\|_B^2 \qquad \text{(A.12)} \\ &\leq 2K^2(1+t) \int_0^t ds\ \mathsf{E}\, \|(X^{(n)}(s), h^{(n)}(s)) - (X^{(n-1)}(s), h^{(n-1)}(s))\|_B^2. \end{aligned}$$

Analogously, in the case $n = 0$, from (A.6) we get

$$\mathsf{E}\, \|(X^{(1)}(t), h^{(1)}(t)) - (X^{(0)}(t), h^{(0)}(t))\|_B^2 \leq 2K^2(t+t^2). \tag{A.13}$$

By iterating (A.12) and using (A.13), we have, for any $n \geq 0$,

$$\mathsf{E}\, \|(X^{(n+1)}(t), h^{(n+1)}(t)) - (X^{(n)}(t), h^{(n)}(t))\|_B^2 \leq \frac{[2K^2(t+t^2)]^{n+1}}{n!}. \tag{A.14}$$

Thus the sequence $\{(X^{(n)}(t), h^{(n)}(t))\}$ converges in $L_2(d\mathsf{P})$ and also a.s. (by the Borel – Cantelli lemma). Moreover the convergence is uniform on each finite interval $[0, T]$. Let us denote by $(X(t), h(t))$ the limiting process. Taking the limit $n \to \infty$ in (A.8)–(A.9) we see that $(X(t), h(t))$ is a solution of (A.1)–(A.2), whereby $\mathcal{F}_t$-adapted. Moreover (A.4) follows from the definition of $(X(t), h(t))$ and (A.14).

Let us prove that the process $(X(\cdot), h(\cdot))$ is a.s. continuous. The continuity of $X(\cdot)$ is a straightforward consequence of (A.1) and (A.6). In fact the Brownian motion $b(t)$ is a.s. continuous and the map $s \mapsto \alpha(X(s), h(s))$ is bounded. Since $h(t)$ solves (A.2) and $t \mapsto p_t h_0$ is continuous, we are left with the continuity of the process $k(t) := h(t) - p_t h_0$. For any $t \geq 0$ and $\delta \in (0, 1]$,

$$\begin{aligned} -k(t+\delta) + k(t) = &\int_0^t db(s)\, (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)} \\ &+ \int_t^{t+\delta} db(s)\, p_{t+\delta-s} \varphi_{X(s)} \\ &+ \int_0^t ds\, \alpha(X(s), h(s)) (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)} \\ &+ \int_t^{t+\delta} ds\, \alpha(X(s), h(s)) p_{t+\delta-s} \varphi_{X(s)}. \end{aligned}$$

Then, recalling (A.6),

$$\mathsf{E}\, \|k(t+\delta) - k(t)\|_2^4 \leq 16K^2(A_1 + A_2) + 16(A_3 + A_4), \tag{A.15}$$

where

$$A_1 := \mathsf{E} \left\| \int_0^t ds\, (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)} \right\|_2^4,$$

$$A_2 := \mathsf{E} \left\| \int_t^{t+\delta} ds\, p_{t+\delta-s} \varphi_{X(s)} \right\|_2^4,$$

$$A_3 := \mathsf{E} \left\| \int_0^t db(s)\, (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)} \right\|_2^4,$$

$$A_4 := \mathsf{E} \left\| \int_t^{t+\delta} db(s)\, p_{t+\delta-s} \varphi_{X(s)} \right\|_2^4.$$

By the Cauchy – Schwarz inequality,

$$\begin{aligned} A_1 &\le t^2\, \mathsf{E} \Big(\int_0^t ds\, \|(p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)}\|_2^2 \Big)^2 \\ &= t^2 \Big(\int_0^t ds\, \|(p_{s+\delta} - p_s)\varphi\|_2^2 \Big)^2 \\ &= (t\delta)^2 \Big(\int_0^t ds \int_s^{s+\delta} d\tau\, \Big\| \frac{1}{2} p_\tau \varphi'' \Big\|_2^2 \Big)^2 \\ &\le \Big(\|\varphi''\|_2 \frac{t\delta}{2} \Big)^4, \end{aligned} \tag{A.16}$$

where in the first identity we used that $\|p_t \varphi_X\|_2 = \|p_t \varphi\|_2$ for all $X \in \mathbf{R}$. Similarly

$$A_2 \le \delta^2 \Big(\int_0^\delta ds\, \|p_s \varphi\|_2^2 \Big)^2 \le (\|\varphi\|_2 \delta)^4. \tag{A.17}$$

Setting $\psi(x) := \sqrt{1+|x|}$, by Hölder inequality, we have, for any $f \in L_2(\mathbf{R})$,

$$\|f\|_2^4 \le \|\psi^{-2}\|_2^2\, \|f\psi\|_4^4 \qquad \text{with} \quad \|\psi^{-2}\|_2^2 < +\infty. \tag{A.18}$$

By (A.18) and the BDG inequality (see (3.14)),

$$\begin{aligned} A_3 &\le \|\psi^{-2}\|_2^2\, \mathsf{E} \left\| \int_0^t db(s)\, \psi\, (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)} \right\|_4^4 \\ &\le C\, \mathsf{E} \left\| \int_0^t ds\, [\psi\, (p_{t+\delta-s} - p_{t-s}) \varphi_{X(s)}]^2 \right\|_2^2 \end{aligned}$$

$$\leq Ct \int_0^t ds \; \mathsf{E} \int dx \, \psi(x+X(s))^4 [(p_{t+\delta-s} - p_{t-s})\varphi](x)^4$$

$$\leq Ct\big(1+\mathsf{E} \sup_{u\in[0,t]} |X(u)|^2\big) \int_0^t ds \, \|\psi\,(p_{s+\delta}-p_s)\varphi\|_4^4$$

$$\leq Ct\delta^3\big(1+\mathsf{E} \sup_{u\in[0,t]} |X(u)|^2\big) \int_0^t ds \int_s^{s+\delta} d\tau \, \|\psi\, p_\tau \varphi''\|_4^4, \tag{A.19}$$

where we used, in the fourth inequality, $\psi(x+X)^4 \leq C(1+|X|^2)\psi(x)^4$. Since $\psi(x) \leq \psi(y) + \sqrt{|x-y|}$, we bound

$$\begin{aligned}
\|\psi p_\tau \varphi''\|_4^4 &\leq 16\,\|p_\tau(\psi\varphi'')\|_4^4 + 16 \int dx \Big(\int dy \, p_\tau(x-y)\sqrt{|x-y|}\varphi''(y)\Big)^4 \\
&= 16\,\|p_\tau(\psi\varphi'')\|_4^4 + 16\tau \int dx \Big(\int dz \, p_1(z)^{1/2}\sqrt{|z|}\, p_1(z)^{1/2}\varphi''(x+z\sqrt{\tau})\Big)^4 \\
&\leq 16\,\|\psi\varphi''\|_4^4 + 16\,\tau\,\|\varphi''\|_4^4 \Big[\int dz \, (p_1(z)^{1/2}\sqrt{|z|})^{4/3}\Big]^3 \\
&\quad \times \int dx \int dz \, p_1(z)^2 \varphi''(x+z\sqrt{\tau})^4 \;\leq\; C(1+\tau).
\end{aligned} \tag{A.20}$$

Moreover, since $X(t)$ is continuous, (A.4) implies $\mathsf{E}\sup_{u\in[0,t]} |X(u)|^2 < +\infty$. Then, from (A.19) and (A.20), there is $C_1(t) < +\infty$ such that

$$A_3 \leq C_1(t)\delta^4. \tag{A.21}$$

Analogously, for some $C_2(t) < +\infty$,

$$A_4 \leq C\delta\big(1+\mathsf{E} \sup_{u\in[0,t+\delta]} |X(u)|^2\big) \int_0^\delta ds \, \|\psi p_s \varphi\|_4^4 \leq C_2(t)\delta^2. \tag{A.22}$$

From (A.15)–(A.17), (A.21)–(A.22) and the Kolmogorov criterion, see [5, I, Theorem 2.2] follows the a.s. continuity of $k(t)$.

We are left with the uniqueness. By arguing as in getting existence, if $(\tilde{X}(t), \tilde{h}(t))$ is another solution of (A.1)–(A.2), we find $c(t)$, uniformly bounded on compacts, such that

$$\mathsf{E}\,\|(X(t),h(t)) - (\tilde{X}(t),\tilde{h}(t))\|_B^2 \leq c(t) \int_0^t ds \; \mathsf{E}\,\|(X(s),h(s)) - (\tilde{X}(s),\tilde{h}(s))\|_B^2,$$

whence, by the Gronwall lemma, $\mathsf{E}\,\|(X(t),h(t)) - (\tilde{X}(t),\tilde{h}(t))\|_B^2 = 0$. Since $(\tilde{X}(t),\tilde{h}(t))$ is continuous, (A.5) follows. □

In order to discuss the regularity of the map $x \mapsto h(t,x)$, $h(t)$ as constructed before, we need the following lemma.

Lemma A.1. *Let $Y(\cdot)$ be a $\mathcal{F}_t$-adapted process and*

$$I_t(x) := \int_0^t db(s)\, p_{t-s}\varphi_{Y(s)}(x).$$

Then $I_t(\cdot) \in C^\infty(\mathbf{R})$ a.s.

Proof. Let $I'_t(x) := \int_0^t db(s)\, p_{t-s}\varphi'_{Y(s)}(x)$ and

$$\xi_t(x,\delta) := \begin{cases} I'_t(x) & \text{if } \delta = 0, \\ \delta^{-1}[I_t(x+\delta) - I_t(x)] & \text{otherwise.} \end{cases}$$

By using the BDG inequality and the Taylor expansion one can show that there exists a constant $C(t)$ uniformly bounded on compacts such that

$$\mathsf{E}\, |\xi_t(x,\delta) - \xi_t(x',\delta')|^p \le C(t)(|x-x'|^p + |\delta-\delta'|^p) \qquad \forall\, x, x' \in \mathbf{R},\ \delta, \delta' \in [-1,1].$$

By Kolmogorov's criterion we get $I_t(\cdot) \in C^1(\mathbf{R})$ a.s. and its derivative is given by $I'_t(x)$. Since $\varphi' \in \mathcal{S}(\mathbf{R})$ as well, the lemma follows by induction. □

Since the map $x \mapsto \int_0^t ds\, \alpha(X(s), h(s)) p_{t-s}\varphi_{X(s)}$ is in $C^\infty(\mathbf{R})$, from the above lemma we get that $h(t,\cdot) \in C^\infty(\mathbf{R})$. A straightforward computation shows then that $h(\cdot)$ solves also the differential (in x) form of (A.2), i.e.

$$h(t) = h_0 + \int_0^t ds \left[\frac{1}{2}\Delta h(s) - \alpha(X(s), h(s))\varphi_{X(s)}\right] - \int_0^t db(s)\varphi_{X(s)}. \tag{A.23}$$

Theorem A.2. *Theorem* A.1 *holds also for* $\alpha(X,h)$ *as in* (A.3).

Since $\alpha(X,h)$ is locally Lipschitz and $|\alpha(X,h)| \le \|\varphi\|_2\, \|h\|_2$, we do not really need to exploit the "good sign" in the nonlinearity of (2.2). Finally by using Lemma A.1 we get that $h(t,\cdot) \in C^\infty(\mathbf{R})$ and satisfies also (A.23) with α as in (A.3).

Proof. Given $M \in \mathbf{N}$, we define the truncation $\chi_M : \mathbf{R}_+ \to [0,1]$ by

$$\chi_M(s) = \mathbf{1}_{\{s \le M\}}(s) + (M + 1 - s)\mathbf{1}_{\{M < s < M+1\}}(s), \qquad s \in \mathbf{R}_+,$$

and set

$$\alpha_M(X,h) := \alpha(X,h)\chi_M(\|h\|_2).$$

Since α_M is bounded and globally Lipschitz Theorem A.1 applies. We denote by (X_M, h_M) the solution of the truncated problem. We show next the sequence $\{(X_M, h_M)\}$ is fundamental with probability 1. The theorem will then follow by routine arguments, which are omitted, see e.g. [4, I, §6].

Let us introduce the stopping time $\tau_M := \inf\{t > 0 : \|h_M(t)\|_2 > M\}$. Since, for $M' > M$, $(X_M(t), h_M(t)) = (X_{M'}(t), h_{M'}(t))$ for $t \in [0, \tau_M]$, we have, for any $T > 0$,

$$\begin{aligned}
&\mathsf{P}\Big(\sup_{M'>M} \sup_{t\in[0,T]} \|(X_M(t), h_M(t)) - (X_{M'}(t), h_{M'}(t))\|_B > 0\Big) \\
&\le \mathsf{P}\left(\tau_M < T\right) \\
&= \mathsf{P}\Big(\sup_{t\in[0,T]} \|h_M(t)\|_2 > M\Big).
\end{aligned}$$

It is therefore enough to show the r.h.s. above goes to zero as $M \to +\infty$.

By using Itô's formula in (A.23) we have

$$\begin{aligned}
dh_M(t)^2 = &\Big\{2h_M(t)\Big[\frac{1}{2}\Delta h_M(t) - \varphi_{X_M(t)}\alpha_M(X_M(t)), h_M(t))\Big] + \varphi^2_{X_M(t)}\Big\}dt \\
&- 2h_M(t)\varphi_{X_M(t)}db(t),
\end{aligned}$$

from which we get, for $M > \|h_0\|_2$,

$$\begin{aligned}
\|h_M(t)\|_2^2 = &\|h(0)\|_2^2 - \int_0^t ds\, \left[\|\nabla h_M(s)\|_2^2 + 2\langle \varphi_{X_M(s)}, h_M(s)\rangle^2 \chi_M(\|h_M(s)\|_2)\right] \\
&+ \int_0^t ds\, \|\varphi_{X_M(s)}\|_2^2 - 2\int_0^t db(s)\, \langle \varphi_{X_M(s)}, h_M(s)\rangle \\
&\le \|h(0)\|_2^2 + \int_0^t ds\, \|\varphi\|_2^2 - 2\int_0^t db(s)\, \langle \varphi_{X_M(s)}, h_M(s)\rangle,
\end{aligned}$$

whence, by the Doob and the Cauchy – Schwarz inequalities,

$$\mathsf{E} \sup_{t\in[0,T]} \|h_M(t)\|_2^2 \le \|h(0)\|_2^2 + T\|\varphi\|_2^2 + \mathsf{E}\int_0^T ds\|\varphi\|_2^2\, \|h_M(s)\|_2^2,$$

which, by Gronwall's lemma, yields

$$\limsup_{M\to+\infty} \mathsf{E} \sup_{t\in[0,T]} \|h_M(t)\|_2^2 < +\infty, \tag{A.24}$$

and we are done. □

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