



Large Deviations for the Boundary Driven Symmetric Simple Exclusion Process *

L. BERTINI¹, A. DE SOLE², D. GABRIELLI³, G. JONA-LASINIO⁴ and C. LANDIM⁵

¹*Dipartimento di Matematica, Università di Roma La Sapienza, P.le A. Moro 2, 00185 Rome, Italy.*
e-mail: bertini@mat.uniroma1.it

²*Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, U.S.A.*
e-mail: desole@alum.mit.edu

³*Dipartimento di Matematica, Università dell'Aquila, 67100 Coppito, L'Aquila, Italy.*
e-mail: gabriell@univaq.it

⁴*Dipartimento di Fisica and INFN, Università di Roma La Sapienza, P.le A. Moro 2, 00185 Rome, Italy.*
e-mail: jona@roma1.infn.it

⁵*IMPA, Estrada Dona Castorina 110, J. Botânico, 22460 Rio de Janeiro, Brazil.*
CNRS UMR 6085, Université de Rouen, 76128 Mont-Saint-Aignan Cedex, France.
e-mail: landim@impa.br

(Received: 7 November 2002; accepted in final form: 6 December 2002)

Abstract. The large deviation properties of equilibrium (reversible) lattice gases are mathematically reasonably well understood. Much less is known in nonequilibrium, namely for nonreversible systems. In this paper we consider a simple example of a nonequilibrium situation, the symmetric simple exclusion process in which we let the system exchange particles with the boundaries at two different rates. We prove a dynamical large deviation principle for the empirical density which describes the probability of fluctuations from the solutions of the hydrodynamic equation. The so-called quasi potential, which measures the cost of a fluctuation from the stationary state, is then defined by a variational problem for the dynamical large deviation rate function. By characterizing the optimal path, we prove that the quasi potential can also be obtained from a static variational problem introduced by Derrida, Lebowitz, and Speer.

Mathematics Subject Classifications (2000): 82C22, 82C35, 60F10.

Key words: stationary nonreversible states, large deviations, boundary driven lattice gases.

1. Introduction

In previous papers [3, 4] we began the study of the macroscopic properties of stochastic nonequilibrium systems. Typical examples are stochastic lattice gases which exchange particles with different reservoirs at the boundary. In these systems, there is a flow of matter through the system and the dynamics is not reversible. The main difference with respect to equilibrium (reversible) states is

* Partially supported by Cofinanziamento MURST 2000 and 2001.

the following: in equilibrium the invariant measure, which determines the thermodynamic properties, is given for free by the Gibbs distribution specified by the Hamiltonian. On the contrary, in nonequilibrium states the construction of the appropriate ensemble, that is the invariant measure, requires the solution of a dynamical problem.

For equilibrium states, the thermodynamic entropy S is identified [6, 20, 22] with the large deviation rate function for the invariant measure. The rigorous study of large deviations has been extended to hydrodynamic evolutions of stochastic interacting particle systems [10, 17]. Developing the methods of [17], this theory has been extended to nonlinear hydrodynamic regimes [15]. In a dynamical setting one may ask new questions, for example what is the most probable trajectory followed by the system in the spontaneous emergence of a fluctuation or in its relaxation to equilibrium. In the physics literature, the Onsager–Machlup theory [23] gives the following answer under the assumption of time reversibility. In the situation of a linear macroscopic equation, that is, close to equilibrium, the most probable emergence and relaxation trajectories are one the time reversal of the other.

In [3, 4] we have heuristically shown how this theory has to be modified for nonequilibrium systems. At thermodynamic level, we do not need all the information carried by the invariant measure, but only its rate function S . This can be obtained, by solving a variational problem, from the dynamical rate function which describes the probability of fluctuations from the hydrodynamic behavior. The physical content of the variational problem is the following. Let ρ be the relevant thermodynamic variable, for instance the local density, whose stationary value is given by some function $\bar{\rho}(u)$. The entropy $S(\rho)$ associated to some profile $\rho(u)$ is then obtained by minimizing the dynamical rate function over all possible paths $\pi(t) = \pi(t, u)$ connecting $\bar{\rho}$ to ρ . We have shown that the optimal path $\pi^*(t)$ is such that $\pi^*(-t)$ is a solution of the hydrodynamic equation associated to the time reversed microscopic dynamics, which we call adjoint hydrodynamics. This relationship is the extension of the Onsager–Machlup theory to nonreversible systems. Moreover, we have also shown that S solves an infinite-dimensional Hamilton–Jacobi equation and how the adjoint hydrodynamics can be obtained once S is known.

In the present paper we study rigorously the symmetric one-dimensional exclusion process. In this model there is at most one particle for each site of the lattice $\{-N, \dots, N\}$ which can move to a neighboring site only if this is empty, with rate $1/2$ for each side. Moreover, a particle at the boundary may leave the system at rate $1/2$ or enter at rate $\gamma_-/2$, respectively $\gamma_+/2$, at the site $-N$, respectively $+N$. In this situation there is a unique invariant measure μ^N which reduces to a Bernoulli measure if $\gamma_- = \gamma_+$. On the other hand, if $\gamma_- \neq \gamma_+$, the measure μ^N exhibits long range correlations [7, 24] and it is not explicitly known. By using a matrix representation and combinatorial techniques, Derrida, Lebowitz, and Speer [8, 9] have recently shown that the rate function for μ^N can be obtained solving a nonlinear boundary value problem on the interval $[-1, 1]$. We here analyze

the macroscopic dynamical behavior of this system. The hydrodynamic limit for the empirical density has been proven in [12, 13]. We prove the associated dynamical large deviation principle which describes the probability of fluctuations from the solutions of the hydrodynamic equation. We then define the quasi-potential via the variational problem mentioned above. By characterizing the optimal path we prove that the quasi-potential can also be obtained from a static variational problem introduced in [8, 9]. Using the identification of the quasi-potential with the rate function for the invariant measure proven in [5], we finally obtain an independent derivation of the expression for the thermodynamic entropy found in [8, 9].

2. Notation and Results

For an integer $N \geq 1$, let $\Lambda_N := [-N, N] \cap \mathbb{Z} = \{-N, \dots, N\}$. The sites of Λ_N are denoted by x, y , and z while the macroscopic space variable (points in the interval $[-1, 1]$) by u, v , and w . We introduce the microscopic state space as $\Sigma_N := \{0, 1\}^{\Lambda_N}$ which is endowed with the discrete topology; elements of Σ_N , called configurations, are denoted by η . In this way $\eta(x) \in \{0, 1\}$ stands for the number of particles at site x for the configuration η .

The one-dimensional boundary driven simple exclusion process is the Markov process on the state space Σ_N with infinitesimal generator

$$L_N := L_{-,N} + L_{0,N} + L_{+,N}$$

defined by

$$(L_{0,N}f)(\eta) := \frac{N^2}{2} \sum_{x=-N}^{N-1} [f(\sigma^{x,x+1}\eta) - f(\eta)],$$

$$(L_{\pm,N}f)(\eta) := \frac{N^2}{2} [\gamma_{\pm} + (1 - \gamma_{\pm})\eta(\pm N)][f(\sigma^{\pm N}\eta) - f(\eta)]$$

for every function $f: \Sigma_N \rightarrow \mathbb{R}$. In this formula $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) := \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{if } z \neq x, y \end{cases}$$

and $\sigma^x\eta$ is the configuration obtained from η by flipping the configuration at x :

$$(\sigma^x\eta)(z) := \eta(z)[1 - \delta_{x,z}] + \delta_{x,z}[1 - \eta(z)],$$

where $\delta_{x,y}$ is the Kronecker delta. Finally, $\gamma_{\pm} \in (0, \infty)$ are the activities of the reservoirs at the boundary of Λ_N .

Notice that the generators are speeded up by N^2 ; this corresponds to the diffusive scaling. We denote by η_t the Markov process on Σ_N with generator L_N and

by \mathbb{P}_η its distribution if the initial configuration is η . Note that \mathbb{P}_η is a probability measure on the path space $D(\mathbb{R}_+, \Sigma_N)$, which we consider endowed with the Skorokhod topology and the corresponding Borel σ -algebra. Expectation with respect to \mathbb{P}_η is denoted by \mathbb{E}_η .

Our first main result is the dynamical large deviation principle for the measure \mathbb{P}_η . We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L_2([-1, 1], du)$ and let

$$\mathcal{M} := \{ \rho \in L_\infty([-1, 1], du) : 0 \leq \rho(u) \leq 1 \text{ a.e.} \} \tag{2.1}$$

which we equip with the topology induced by weak convergence, namely $\rho_n \rightarrow \rho$ in \mathcal{M} if and only if $\langle \rho_n, G \rangle \rightarrow \langle \rho, G \rangle$ for each continuous function $G: [-1, 1] \rightarrow \mathbb{R}$; we consider \mathcal{M} also endowed with the corresponding Borel σ -algebra. Let us define the map $\pi^N: \Sigma_N \rightarrow \mathcal{M}$ as

$$\pi^N(\eta) := \sum_{x=-N}^N \eta(x) \mathbf{1} \left\{ \left[\frac{x}{N} - \frac{1}{2N}, \frac{x}{N} + \frac{1}{2N} \right) \right\}, \tag{2.2}$$

where $\mathbf{1}\{A\}$ stands for the indicator function of the set A ; namely $\pi^N = \pi^N(\eta)$ is the empirical density obtained from the configuration η . Notice that $\pi^N(\eta) \in \mathcal{M}$, i.e. $0 \leq \pi^N(\eta) \leq 1$, because $\eta(x) \in \{0, 1\}$.

Let η^N be a sequence of configurations for which the empirical density $\pi^N(\eta^N)$ converges in \mathcal{M} , as $N \uparrow \infty$, to some function ρ , namely for each $G \in C([-1, 1])$

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \pi^N(\eta^N), G \rangle &= \lim_{N \rightarrow \infty} \sum_{x=-N}^N \eta^N(x) \int_{(-1) \vee (\frac{x}{N} - \frac{1}{2N})}^{1 \wedge (\frac{x}{N} + \frac{1}{2N})} du G(u) \\ &= \int_{-1}^1 du \rho(u) G(u), \end{aligned} \tag{2.3}$$

where we used the notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. If (2.3) holds we say that the sequence $\{\eta^N : N \geq 1\}$ is associated to the profile $\rho \in \mathcal{M}$.

For $T > 0$ and positive integers m, n we denote by $C_0^{m,n}([0, T] \times [-1, 1])$ the space of functions $G: [0, T] \times [-1, 1] \rightarrow \mathbb{R}$ with m continuous derivatives in time, n continuous derivatives in space and which vanish at the boundary: $G(\cdot, \pm 1) = 0$. Let also $D([0, T], \mathcal{M})$ be the Skorokhod space of paths from $[0, T]$ to \mathcal{M} equipped with its Borel σ -algebra. Elements of $D([0, T], \mathcal{M})$ will be denoted by $\pi(t) = \pi(t, u)$.

Let $\rho_\pm := \gamma_\pm / [1 + \gamma_\pm] \in (0, 1)$ be the density at the boundary of $[-1, 1]$ and fix a function $\rho \in \mathcal{M}$ which corresponds to the initial profile. For $H \in C_0^{1,2}([0, T] \times [-1, 1])$, let $J_{T,H,\rho} = J_H: D([0, T], \mathcal{M}) \rightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} J_H(\pi) &:= \langle \pi(T), H(T) \rangle - \langle \rho, H(0) \rangle - \int_0^T dt \langle \pi(t), \partial_t H(t) + \frac{1}{2} \Delta H(t) \rangle + \\ &\quad + \frac{\rho_+}{2} \int_0^T dt \nabla H(t, 1) - \frac{\rho_-}{2} \int_0^T dt \nabla H(t, -1) - \\ &\quad - \frac{1}{2} \int_0^T dt \langle \chi(\pi(t)), (\nabla H(t))^2 \rangle, \end{aligned} \tag{2.4}$$

where ∇ denotes the derivative with respect to the macroscopic space variable u , Δ is the Laplacian on $(-1, 1)$, and we have set $\chi(a) := a(1 - a)$. Let finally $I_T(\cdot | \rho): D([0, T], \mathcal{M}) \rightarrow [0, +\infty]$ be the functional defined by

$$I_T(\pi | \rho) := \sup_{H \in C_0^{1,2}([0, T] \times [-1, 1])} J_H(\pi). \tag{2.5}$$

Notice that, if $\pi(t)$ solves the heat equation with boundary condition $\pi(t, \pm 1) = \rho_{\pm}$ and initial datum $\pi(0) = \rho$, then $I_T(\pi | \rho) = 0$.

THEOREM 2.1. *Fix $T > 0$ and a profile $\rho \in \mathcal{M}$ bounded away from 0 and 1, namely such that there exists $\delta > 0$ with $\delta \leq \rho \leq 1 - \delta$ a.e. Consider a sequence η^N of configurations associated to ρ . Then the measure $\mathbb{P}_{\eta^N} \circ (\pi^N)^{-1}$ on $D([0, T], \mathcal{M})$ satisfies a large deviation principle with speed N and convex lower semi-continuous rate function $I_T(\cdot | \rho)$. Namely, for each closed set $\mathcal{C} \subset D([0, T], \mathcal{M})$ and each open set $\mathcal{O} \subset D([0, T], \mathcal{M})$,*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in \mathcal{C}] &\leq - \inf_{\pi \in \mathcal{C}} I_T(\pi | \rho), \\ \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in \mathcal{O}] &\geq - \inf_{\pi \in \mathcal{O}} I_T(\pi | \rho). \end{aligned}$$

It is possible to obtain a more explicit representation of the functional $I_T(\cdot | \rho)$, see Lemma 3.6 below. If the particle system is considered with periodic boundary conditions, i.e. Λ_N is replaced by the discrete torus of length N , this theorem has been proven in [17]. As we shall see later, the main difference with respect to the case with periodic boundary condition is the lack of translation invariance and the fact that the path $\pi(t, \cdot)$ is fixed at the boundary.

We now define precisely the variational problem mentioned in the Introduction. Let $\bar{\rho} \in \mathcal{M}$ be the linear profile $\bar{\rho}(u) := [\rho_-(1 - u) + \rho_+(1 + u)]/2, u \in [-1, 1]$, which is the density profile associate to the invariant measure μ^N , see Section 3 below. We then define $V: \mathcal{M} \rightarrow [0, +\infty]$ as the quasi potential for the rate function $I_T(\cdot | \bar{\rho})$:

$$V(\rho) := \inf_{T > 0} \inf_{\pi(\cdot): \pi(T) = \rho} I_T(\pi | \bar{\rho}) \tag{2.6}$$

which measures the minimal cost to produce the profile ρ starting from $\bar{\rho}$.

Let us first describe how the variational problem (2.6) is solved when $\gamma_- = \gamma_+ = \gamma$. In this case $\bar{\rho} = \gamma/(1 + \gamma)$ is constant and the process is reversible with respect to the Bernoulli measure with density $\bar{\rho}$. We have that $I_T(\pi | \rho_0) = 0$ if $\pi(t)$ solves the hydrodynamic equation which for this system is given by the heat equation:

$$\begin{aligned} \partial_t \rho(t) &= (1/2)\Delta \rho(t), \\ \rho(t, \pm 1) &= \rho_{\pm}, \\ \rho(0, \cdot) &= \rho_0(\cdot). \end{aligned} \tag{2.7}$$

Note that $\rho(t) \rightarrow \bar{\rho}$ as $t \rightarrow \infty$.

It can be easily shown that the minimizer for (2.6), defined on the time interval $(-\infty, 0]$ instead of $[0, +\infty)$ as in (2.6), is given by $\pi^*(t) = \rho(-t)$, where $\rho(t)$ is the solution of (2.7) with initial condition $\rho_0 = \rho$. This symmetry of the relaxation and fluctuation trajectories is the Onsager–Machlup principle mentioned before.

Moreover the quasi-potential $V(\rho)$ coincides with the entropy of the Bernoulli measure with density $\bar{\rho}$, that is, understanding $0 \log 0 = 0$,

$$V(\rho) = S_0(\rho) := \int_{-1}^1 du \left[\rho(u) \log \frac{\rho(u)}{\bar{\rho}} + [1 - \rho(u)] \log \frac{1 - \rho(u)}{1 - \bar{\rho}} \right]. \tag{2.8}$$

In the context of Freidlin–Wentzell theory [14] for diffusions in \mathbb{R}^n , the situation just described is analogous to the so called *gradient* case in which the quasi potential coincides with the potential. This structure reflects the reversibility of the underlying process. In general for *nongradient* systems, the solution of the dynamical variational problem, or of the associated Hamilton–Jacobi equation, cannot be explicitly calculated. The case $\gamma_+ \neq \gamma_-$ is analogous to a nongradient system, but for this particular model we shall prove that the quasi potential $V(\rho)$, as defined in (2.6), coincides with the functional $S(\rho)$ defined by a time independent variational problem introduced in [8, 9] which is stated below. This is the second main result of this paper.

Denote by $C^1([-1, 1])$ the space of once continuously differentiable functions $f: [-1, 1] \rightarrow \mathbb{R}$ endowed with the norm $\|f\|_{C^1} := \sup_{u \in [-1, 1]} \{|f(u)| + |f'(u)|\}$. Let

$$\mathcal{F} := \left\{ f \in C^1([-1, 1]) : f(\pm 1) = \rho_{\pm}, [\rho_+ - \rho_-]f'(u) > 0, \right. \\ \left. u \in [-1, 1] \right\}, \tag{2.9}$$

where f' denotes the derivative of f . Note that if $f \in \mathcal{F}$, then

$$0 < \rho_- \wedge \rho_+ \leq f(u) \leq \rho_- \vee \rho_+ < 1 \quad \text{for all } -1 \leq u \leq 1.$$

For $\rho \in \mathcal{M}$ and $f \in \mathcal{F}$ we set

$$\mathcal{G}(\rho, f) := \int_{-1}^1 du \left[\rho(u) \log \frac{\rho(u)}{f(u)} + [1 - \rho(u)] \log \frac{1 - \rho(u)}{1 - f(u)} + \right. \\ \left. + \log \frac{f'(u)}{[\rho_+ - \rho_-]/2} \right] \tag{2.10}$$

and

$$S(\rho) := \sup_{f \in \mathcal{F}} \mathcal{G}(\rho, f). \tag{2.11}$$

Theorem 4.5 below, which formalizes the arguments in [9], states that the supremum in (2.11) is uniquely attained for a function f which solves a nonlinear boundary value problem. We shall denote it by $F = F(\rho)$ to emphasize its dependence on ρ ; therefore $S(\rho) = \mathcal{G}(\rho, F(\rho))$.

THEOREM 2.2. *Let V and S as defined in (2.6) and (2.11). Then for each $\rho \in \mathcal{M}$ we have $V(\rho) = S(\rho)$.*

In the proof of the above theorem we shall construct a particular path $\pi^*(t)$ in which the infimum in (2.6) is almost attained. As recalled in the introduction, by the heuristic arguments in [4], $\pi^*(-t)$ is the solution of the hydrodynamic equation corresponding to the process with generator L_N^* , the adjoint of L_N in $L_2(\Sigma_N, d\mu^N)$ and initial condition ρ . In analogy to the Freidlin–Wentzell theory [14], we expect that the exit path from a neighborhood $\bar{\rho}$ to a neighborhood of ρ should, with probability converging to one as $N \uparrow \infty$, take place in a small tube around the path $\pi^*(t)$.

The optimal path can be described in a rather simple fashion. Recalling that we denoted by $F = F(\rho)$ the maximizer for (2.11), consider the heat equation in $[-1, 1]$ with boundary conditions ρ_{\pm} and initial datum F :

$$\begin{aligned} \partial_t \Phi(t) &= (1/2)\Delta \Phi(t), \\ \Phi(t, \pm 1) &= \rho_{\pm}, \\ \Phi(0, \cdot) &= F(\rho). \end{aligned} \tag{2.12}$$

We next define $\rho^*(t) = \rho^*(t, u)$ by

$$\rho^*(t) := \Phi(t) + \Phi(t)[1 - \Phi(t)] \frac{\Delta \Phi(t)}{(\nabla \Phi(t))^2}. \tag{2.13}$$

In view of (4.3) below, $\rho^*(0) = \rho$ and, by Lemma 5.6, $\lim_{t \rightarrow \infty} \rho^*(t) = \bar{\rho}$. The optimal path $\pi^*(t)$, defined on the time interval $(-\infty, 0]$ instead of $[0, +\infty)$ as in (2.6), is then given by $\pi^*(t) = \rho^*(-t)$.

From the dynamical large deviation principle we can obtain, by means of the quasi potential, the large deviation principle for the empirical density when the particles are distributed according to the invariant measure of the process η_t . Note that the finite state Markov process η_t with generator L_N is irreducible, therefore it has a unique invariant measure μ^N .

Let us introduce $\mathcal{P}_N := \mu^N \circ (\pi^N)^{-1}$ which is a probability on \mathcal{M} and describes the behavior of the empirical density under the invariant measure. In [7, 12, 13, 24] it is shown, see also Section 3 below, that \mathcal{P}_N satisfies the law of large numbers $\mathcal{P}_N \Rightarrow \delta_{\bar{\rho}}$ in which \Rightarrow stands for weak convergence of measures on \mathcal{M} and $\bar{\rho}$ is the linear profile already introduced.

Since $\bar{\rho}$ is globally attractive for (2.7), the quasi-potential with respect to $\bar{\rho}$ defined in (2.6) gives the rate function for the family \mathcal{P}_N . In [3, 4] we have heuristically derived this identification via a time reversal argument. For the present model a rigorous proof, in the same spirit of the Freidlin–Wentzell theory, is given in [5]; that is we have the following theorem.

THEOREM 2.3. *Let V as defined in (2.6). Then the measure \mathcal{P}_N satisfies a large deviation principle with speed N and rate function V .*

The identification of the rate function for \mathcal{P}_N with the functional S now follows from Theorems 2.1, 2.2 and 2.3.

COROLLARY 2.4. *Let S as defined in (2.11). The measure \mathcal{P}_N satisfies a large deviation principle on \mathcal{M} with speed N and convex lower semi-continuous rate function S . Namely for each closed set $\mathcal{C} \subset \mathcal{M}$ and each open set $\mathcal{O} \subset \mathcal{M}$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mu^N[\pi^N \in \mathcal{C}] \leq - \inf_{\rho \in \mathcal{C}} S(\rho),$$

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mu^N[\pi^N \in \mathcal{O}] \geq - \inf_{\rho \in \mathcal{O}} S(\rho).$$

As already mentioned, the rate function S has been first obtained in [8, 9] by using a matrix representation of the invariant measure μ^N and combinatorial techniques. By means of Theorems 2.1, 2.2, and 2.3 we prove here, independently of [8, 9], the large deviation principle by following the dynamical/variational route explained in [4] which is analogous to the Freidlin–Wentzell theory [14] for diffusions on \mathbb{R}^n .

We remark that it should be possible, modulo technical problems, to extend Theorems 2.1 and 2.3 to other boundary driven diffusive lattice gases, see [4] for a heuristic discussion. The characterization of the rate function for the invariant measure as the quasi potential allows to obtain some information on it directly from the variational problem (2.6). In particular, in Appendix, we discuss the symmetric simple exclusion in any dimension and get a lower bound on V in terms of the entropy S_0 of the equilibrium system. In the one-dimensional case, this bound has been proven in [8, 9] by using instead the variational problem (2.11).

Outline. In Section 3 we recall the hydrodynamic behavior of the boundary driven exclusion process and prove the associated large deviation principle described by Theorem 2.1. In Sections 4 and 5, which are more technical, we state and prove some properties of the functional S which is then shown to coincide with the quasi potential V . Finally, in Appendix, we consider the symmetric simple exclusion in any dimension and prove a lower bound on V .

3. Dynamical Behavior

In this section we study the dynamical properties of the empirical density for the boundary driven simple exclusion process in a fixed (macroscopic) time interval $[0, T]$. In particular, we review the hydrodynamic limit (law of large numbers) and prove the corresponding large deviation principle. This problem was considered before by Kipnis, Olla and Varadhan in [17] for the exclusion process with periodic boundary condition. For this reason, we present only the modifications needed in the argument and refer to [2, 16, 17] for the missing arguments.

As already stated, the invariant measure μ^N is not known explicitly but some of its properties have been derived. For example, the one-site marginals or the correlations can be computed explicitly. To compute the one-site marginals, which will be used later, let $\rho^N(x) = E_{\mu^N}[\eta(x)]$ for $-N \leq x \leq N$. Since μ^N is invariant, $E_{\mu^N}[L_N \eta(x)] = 0$. Computing $L_N \eta(x)$, we obtain a closed difference equation for $\rho^N(x)$:

$$\begin{aligned} (\Delta_N \rho^N)(x) &= 0 \quad \text{for } -N + 1 \leq x \leq N - 1, \\ \rho^N(N - 1) - \rho^N(N) + \gamma_+[1 - \rho^N(N)] - \rho^N(N) &= 0, \\ \rho^N(-N + 1) - \rho^N(-N) + \gamma_-[1 - \rho^N(-N)] - \rho^N(-N) &= 0. \end{aligned}$$

In this formula, Δ_N stands for the discrete Laplacian so that $(\Delta_N f)(x) = f(x + 1) + f(x - 1) - 2f(x)$. The unique solution of this discrete elliptic equation gives the one-site marginals of μ^N .

Denote by $\nu^N = \nu_{\gamma_-, \gamma_+}^N$ the product measure on Σ_N with marginals given by

$$\nu^N\{\eta : \eta(x) = 1\} = \rho^N(x)$$

and observe that the generators $L_{-,N}, L_{+,N}$ are reversible with respect to ν^N .

Denote by $\{\tau_x : x \in \mathbb{Z}\}$ the group of translations in $\{0, 1\}^{\mathbb{Z}}$ so that $(\tau_x \zeta)(z) = \zeta(x + z)$ for all x, z in \mathbb{Z} and configuration ζ in $\{0, 1\}^{\mathbb{Z}}$. Translations are extended to functions and measures in a natural way. Eyink *et al.* [12] and De Masi *et al.* [7] proved that

$$\lim_{N \rightarrow \infty} E_{\mu^N}[\tau_{[uN]} f] = E_{\nu_{\bar{\rho}(u)}}[f]$$

for every local function f and u in $(-1, 1)$. Here $\bar{\rho}$ is the unique solution of

$$(1/2)\Delta \rho = 0, \quad \rho(\pm 1) = \rho_{\pm},$$

namely $\bar{\rho}$ is the linear interpolation between ρ_- and ρ_+ and $\{\nu_{\alpha} : 0 \leq \alpha \leq 1\}$ stands for the Bernoulli product measure in $\{0, 1\}^{\mathbb{Z}}$ with density α and $\rho_{\pm} = \gamma_{\pm}/[1 + \gamma_{\pm}]$ is the density at the boundary of $[-1, 1]$.

3.1. HYDRODYNAMIC LIMIT

Recall that, for each configuration $\eta \in \Sigma_N$, we denote by $\pi^N = \pi^N(\eta) \in \mathcal{M}$ the empirical density obtained from η , see Equation (2.2). We say that a sequence of configurations $\{\eta^N : N \geq 1\}$ is associated to the profile γ if (2.3) holds for all continuous functions $G: [-1, 1] \rightarrow \mathbb{R}$. The following result is due to Eyink, Lebowitz and Spohn [13].

THEOREM 3.1. *Consider a sequence η^N associated to some profile $\rho_0 \in \mathcal{M}$. Then, for all $t > 0$, $\pi^N(t) = \pi^N(\eta_t)$ converges (in the sense (2.3)) in probability to $\rho(t, u)$, the unique weak solution of*

$$\begin{aligned}
\partial_t \rho &= (1/2) \Delta \rho, \\
\rho(t, \pm 1) &= \rho_{\pm}, \\
\rho(0, \cdot) &= \rho_0(\cdot).
\end{aligned} \tag{3.1}$$

By a weak solution of the Dirichlet problem (3.1) in the time interval $[0, T]$, we understand a *bounded* real function ρ which satisfies the following two conditions.

- (a) There exists a function $A(t, u)$ in $L^2([-1, 1] \times [0, T])$ such that

$$\begin{aligned}
&\int_0^t ds \int_{-1}^1 du \rho(s, u) (\nabla H)(u) \\
&= \{\rho_+ H(1) - \rho_- H(-1)\}t - \int_0^t ds \int_{-1}^1 du A(s, u) H(u)
\end{aligned}$$

for every smooth function $H: [-1, 1] \rightarrow \mathbb{R}$ and every $0 \leq t \leq T$. $A(t, u)$ will be denoted by $(\nabla \rho)(t, u)$.

- (b) For every function $H: [-1, 1] \rightarrow \mathbb{R}$ of class $C^1([-1, 1])$ vanishing at the boundary and every $0 \leq t \leq T$,

$$\begin{aligned}
&\int_{-1}^1 du \rho(t, u) H(u) - \int_{-1}^1 du \rho_0(u) H(u) \\
&= -(1/2) \int_0^t ds \int_{-1}^1 du (\nabla \rho)(s, u) (\nabla H)(u).
\end{aligned}$$

The classical H_{-1} estimates gives uniqueness of weak solutions of Equation (3.1). Note that here the weak solution coincides with the semi-group solution $\rho(t) = \bar{\rho} + e^{t\Delta^0/2}(\rho_0 - \bar{\rho})$, where $\bar{\rho}$ is the stationary profile and Δ^0 is the Laplacian with zero boundary condition.

3.2. SUPER-EXPONENTIAL ESTIMATE

We now turn to the problem of large deviations from the hydrodynamic limit. It is well known that one of the main steps in the derivation of a large deviation principle for the empirical density is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime. Essentially, the problem consists in bounding expressions such as $\langle V, f^2 \rangle_{\mu^N}$ in terms of the Dirichlet form $\langle -L_N f, f \rangle_{\mu^N}$. Here V is a local function and $\langle \cdot, \cdot \rangle_{\mu^N}$ indicates the inner product with respect to the invariant state μ^N .

In the context of boundary driven simple exclusion processes, the fact that the invariant state is not known explicitly introduces a technical difficulty. Following [19] we fix ν^N , the product measure defined in the beginning of this section, as reference measure and estimate everything with respect to ν^N . However, since ν^N is not an invariant state, there are no reasons for $\langle -L_N f, f \rangle_{\nu^N}$ to be positive. The first statement shows that this expression is almost positive.

For a function $f: \Sigma_N \rightarrow \mathbb{R}$, let

$$D_N(f) = \sum_{x=-N}^{N-1} \int [f(\sigma^{x,x+1}\eta) - f(\eta)]^2 d\nu^N(\eta).$$

LEMMA 3.2. *There exists a finite constant C_0 depending only on γ_{\pm} such that*

$$\langle L_{0,N}f, f \rangle_{\nu^N} \leq -\frac{N^2}{4} D_N(f) + C_0 N \langle f, f \rangle_{\nu^N}$$

for all functions $f: \Sigma_N \rightarrow \mathbb{R}$.

The proof of this lemma is elementary and left to the reader. Notice, on the other hand, that both $\langle -L_{+,N}f, f \rangle_{\nu^N}$, $\langle -L_{-,N}f, f \rangle_{\nu^N}$ are positive because ν^N is a reversible state by our choice of the profile ρ^N .

This lemma together with the computation presented in [2, p. 78] for non-reversible processes, permits to prove the super-exponential estimate. The statement of this result requires some notation. For a cylinder function Ψ , denote the expectation of Ψ with respect to the Bernoulli product measure ν_{α} by $\tilde{\Psi}(\alpha)$:

$$\tilde{\Psi}(\alpha) := E_{\nu_{\alpha}}[\Psi].$$

For a positive integer ℓ and $-N \leq x \leq N$, denote the empirical mean density on a box of size $2\ell + 1$ centered at x by $\eta^{\ell}(x)$, namely

$$\eta^{\ell}(x) = \frac{1}{|\Lambda_{\ell}(x)|} \sum_{y \in \Lambda_{\ell}(x)} \eta(y),$$

where

$$\Lambda_{\ell}(x) = \Lambda_{N,\ell}(x) = \{y \in \Lambda_N : |y - x| \leq \ell\}.$$

Let $H \in C([0, T] \times [-1, 1])$ and Ψ a cylinder function. For $\varepsilon > 0$, define also

$$V_{N,\varepsilon}^{H,\Psi}(t, \eta) = \frac{1}{N} \sum_x H(t, x/N) \{ \tau_x \Psi(\eta) - \tilde{\Psi}(\eta^{N\varepsilon}(x)) \},$$

where the summation is carried over all x such that the support of $\tau_x \Psi$ belongs to Λ_N . For a continuous function $G: [0, T] \rightarrow \mathbb{R}$, let

$$W_G^{\pm} = \int_0^T ds G(s) [\eta_s(\pm N) - \rho_{\pm}].$$

THEOREM 3.3. *Fix H in $C([0, T] \times [-1, 1])$, $G \in C([0, T])$, a cylinder function Ψ , and a sequence $\{\eta^N \in \Sigma_N : N \geq 1\}$ of configurations. For any $\delta > 0$ we have*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} \left[\left| \int_0^T V_{N,\varepsilon}^{H,\Psi}(t, \eta_t) dt \right| > \delta \right] = -\infty,$$

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} [|W_G^{\pm}| > \delta] = -\infty.$$

3.3. UPPER BOUND

The proof of the upper bound of the large deviation principle is essentially the same as in [17]. There is just a slight difference in the definition of the functionals J_H due to the boundary conditions.

For H in $C_0^{1,2}([0, T] \times [-1, 1])$ consider the exponential martingale M_t^H defined by

$$M_t^H = \exp \left\{ N \left[\langle \pi^N(t), H(t) \rangle - \langle \pi^N(0), H(0) \rangle - \frac{1}{N} \int_0^t e^{-N\langle \pi^N(s), H(s) \rangle} (\partial_s + N^2 L_N) e^{N\langle \pi^N(s), H(s) \rangle} ds \right] \right\}.$$

An elementary computation shows that

$$M_T^H = \exp N \{ J_H(\pi^N * \iota_\varepsilon) + \mathbb{V}_{N,\varepsilon}^H + C_H(\varepsilon) \},$$

where $\lim_{\varepsilon \rightarrow 0} C_H(\varepsilon) = 0$, ι_ε stands for the approximation of the identity $\iota_\varepsilon(u) = (2\varepsilon)^{-1} \mathbf{1}\{u \in [-\varepsilon, \varepsilon]\}$, $*$ stands for convolution,

$$\mathbb{V}_{N,\varepsilon}^H = \int_0^T V_{N,\varepsilon}^{H,\Psi_0}(t, \eta_t) dt + W_{\nabla H(\cdot, 1)}^+ - W_{\nabla H(\cdot, -1)}^-$$

and $\Psi_0(\eta) = \eta(0)[1 - \eta(1)]$.

Fix a subset A of $D([0, T], \mathcal{M})$ and write

$$\frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in A] = \frac{1}{N} \log \mathbb{E}_{\eta^N} [M_T^H (M_T^H)^{-1} \mathbf{1}\{\pi^N \in A\}].$$

Maximizing over π^N in A , we get from previous computation that the last term is bounded above by

$$- \inf_{\pi \in A} J_H(\pi * \iota_\varepsilon) + \frac{1}{N} \log \mathbb{E}_{\eta^N} [M_T^H e^{-N\mathbb{V}_{N,\varepsilon}^H}] - C_H(\varepsilon).$$

Denote by $\mathbb{P}_{\eta^N}^H$ the measure $\mathbb{P}_{\eta^N} M_T^H$. Since the martingale is bounded by $\exp\{CN\}$ for some finite constant depending only on H and T , Theorem 3.3 holds for $\mathbb{P}_{\eta^N}^H$ in place of \mathbb{P}_{η^N} . In particular, the second term of the previous formula is bounded above by $C_H(\varepsilon, N)$ such that $\lim_{\varepsilon \rightarrow 0} \limsup_{N \rightarrow \infty} C_H(\varepsilon, N) = 0$. Hence, for every $\varepsilon > 0$, and every H in $C_0^{1,2}([0, T] \times [-1, 1])$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in A] \leq - \inf_{\pi \in A} J_H(\pi * \iota_\varepsilon) + C'_H(\varepsilon),$$

where $\lim_{\varepsilon \rightarrow 0} C'_H(\varepsilon) = 0$.

Assume now that the set A is a compact set K . Since $J_H(\cdot * \iota_\varepsilon)$ is continuous for every H and $\varepsilon > 0$, we may apply the arguments presented in Lemma 11.3 of [25]

to exchange the supremum with the infimum. In this way we obtain that the last expression is bounded above by

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in K] \leq - \inf_{\pi \in K} \sup_{H, \varepsilon} \{J_H(\pi * \iota_\varepsilon) + C'_H(\varepsilon)\}.$$

First, letting $\varepsilon \downarrow 0$, since $J_H(\pi * \iota_\varepsilon)$ converges to $J_H(\pi)$ for every H in $C_0^{1,2}([0, T] \times [-1, 1])$, in view of the definition (2.5) of $I_T(\pi | \gamma)$, we deduce that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in K] \leq - \inf_{\pi \in K} I_T(\pi | \gamma),$$

which proves the upper bound for compact subsets.

To pass from compact sets to closed sets, we have to obtain ‘exponential tightness’ for the sequence $\mathbb{P}_{\eta^N}[\pi^N \in \cdot]$. The proof presented in [1] for the noninteracting zero-range process is easily adapted to our context.

3.4. HYDRODYNAMIC LIMIT OF WEAKLY ASYMMETRIC EXCLUSIONS

Fix a function H in $C_0^{1,2}([0, T] \times [-1, 1])$ and recall the definition of the martingale M_T^H . Denote by $\mathbb{P}_{\eta^N}^H$ the probability measure on $D([0, T], \Sigma_N)$ defined by $\mathbb{P}_{\eta^N}^H[A] = \mathbb{E}_{\eta^N}[M_T^H \mathbf{1}\{A\}]$. Under $\mathbb{P}_{\eta^N}^H$, the coordinates $\{\eta_t : 0 \leq t \leq T\}$ form a Markov process with generator $L_N^H = L_{+,N} + L_{0,N}^H + L_{-,N}$, where

$$\begin{aligned} (L_{0,N}^H f)(\eta) &= \frac{N^2}{2} \sum_{x=-N}^{N-1} e^{-\{H(t, [x+1]/N) - H(t, x/N)\} \{\eta(x+1) - \eta(x)\}} [f(\sigma^{x,x+1} \eta) - f(\eta)]. \end{aligned}$$

The next result is due to Eyink *et al.* [13]. Recall $\chi(\rho) = \rho(1 - \rho)$.

LEMMA 3.4. *Consider a sequence η^N associated to some profile $\gamma \in \mathcal{M}$ and fix H in $C_0^{1,2}([0, T] \times [-1, 1])$. Then, for all $t > 0$, $\pi^N(t) = \pi^N(\eta_t)$ converges in probability (in the sense (2.3)) to $\rho(t, u)$, the unique weak solution of*

$$\begin{aligned} \partial_t \rho &= (1/2) \Delta \rho - \nabla \{ \chi(\rho) \nabla H \}, \\ \rho(t, \pm 1) &= \rho_\pm, \\ \rho(0, \cdot) &= \gamma(\cdot). \end{aligned} \tag{3.2}$$

As in Subsection 3.1, by a weak solution of the Dirichlet problem (3.2) in the time interval $[0, T]$, we understand a *bounded* real function ρ which satisfies the following two conditions.

- (a) There exists a function $A(t, u)$ in $L^2([-1, 1] \times [0, T])$ such that

$$\int_0^t ds \int_{-1}^1 du \rho(s, u)(\nabla G)(u) = \{\rho_+ G(1) - \rho_- G(-1)\}t - \int_0^t ds \int_{-1}^1 du A(s, u)G(u)$$

for every smooth function $G: [-1, 1] \rightarrow \mathbb{R}$ and every $0 \leq t \leq T$. $A(t, u)$ will be denoted by $(\nabla \rho)(t, u)$.

(b) For every function $G \in C^1([-1, 1])$ vanishing at the boundary and every $t \geq 0$,

$$\int_{-1}^1 du \rho(t, u)G(u) - \int_{-1}^1 du \gamma(u)G(u) = \int_0^t ds \int_{-1}^1 du (\nabla G)(u) \{ - (1/2)(\nabla \rho)(s, u) + \chi(\rho(s, u))(\nabla H)(s, u) \}.$$

The classical H_{-1} estimates gives uniqueness of weak solutions of Equation (3.2).

3.5. THE RATE FUNCTION

We prove in this subsection some properties of the rate function $I_T(\cdot | \gamma)$. We first claim that this rate function is convex and lower semi-continuous. In view of the definition of $I_T(\cdot | \gamma)$, to prove this assertion, it is enough to show that J_H is convex and lower semi-continuous for each H in $C_0^{1,2}([0, T] \times [-1, 1])$. It is convex because $\chi(a) = a(1 - a)$ is a concave function. It is lower semi-continuous because for any positive, continuous function $G: [0, T] \times [-1, 1] \rightarrow \mathbb{R}$ and for any sequence π^n converging to π in $D([0, T], \mathcal{M})$,

$$\begin{aligned} \int_0^T dt \langle \chi(\pi(t)), G(t) \rangle &= \lim_{\varepsilon \rightarrow 0} \int_0^T dt \langle \chi(\pi(t) * \iota_\varepsilon), G(t) \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t) * \iota_\varepsilon), G(t) \rangle. \end{aligned}$$

Since χ is concave and G positive, a change of variables shows that this expression is bounded below by

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t)), G(t) * \iota_\varepsilon \rangle = \limsup_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t)), G(t) \rangle$$

because G is continuous and χ is bounded. This proves that J_H is lower semi-continuous for every H in $C_0^{1,2}([0, T] \times [-1, 1])$.

Denote by D_γ the subset of $D([0, T], \mathcal{M})$ of all paths $\pi(t, u)$ which satisfy the boundary conditions $\pi(0, \cdot) = \gamma(\cdot)$, $\pi(\cdot, \pm 1) = \rho_\pm$, in the sense that for every $0 \leq t_0 < t_1 \leq T$,

$$\lim_{\delta \rightarrow 0} \int_{t_0}^{t_1} dt \frac{1}{\delta} \int_{-1}^{-1+\delta} \pi(t, u) du = (t_1 - t_0)\rho_-$$

and a similar identity at the other boundary.

LEMMA 3.5. $I_T(\pi|\gamma) = \infty$ if π does not belong to D_γ .

Proof. Fix π in $D([0, T], \mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. We first show that $\pi(0, \cdot) = \gamma(\cdot)$. For $\delta > 0$, consider the function $H_\delta(t, u) = h_\delta(t)g(u)$, $h_\delta(t) = (1 - \delta^{-1}t)^+$, $g(\cdot)$ vanishing at the boundary ± 1 . Here a^+ stands for the positive part of a . Of course, H_δ can be approximated by smooth functions. Since π is bounded and since $t \rightarrow \pi(t, \cdot)$ is right-continuous for the weak topology,

$$\lim_{\delta \downarrow 0} J_{H_\delta}(\pi) = \langle \pi(0), g \rangle - \langle \gamma, g \rangle,$$

which proves that $\pi(0) = \gamma$ a.s. because $I_T(\pi|\gamma) < \infty$.

A similar argument shows that $\pi(t, \pm 1) = \rho_\pm$; to prove this statement we may consider the sequence of functions $H_\delta(t, u) = h(t)g_\delta(u)$, where $h(t)$ approximates the indicator of some time interval $[t_0, t_1]$ and where

$$g'_\delta(u) = \begin{cases} A - (A + b)(1 + u)/\delta, & \text{if } -1 \leq u \leq -1 + \delta, \\ -b, & \text{if } -1 + \delta \leq u \leq 1. \end{cases}$$

Here $A > 0$ is large and fixed and $b = b(A, \delta) > 0$ is chosen for the integral over $[-1, 1]$ of g'_δ to vanish. □

Fix π in D_γ and denote by $\mathcal{H}_1(\pi)$ the Hilbert space induced by $C_0^{1,2}([0, T] \times [-1, 1])$ endowed with the inner product $\langle \cdot, \cdot \rangle_\pi$ defined by

$$\langle H, G \rangle_\pi = \int_0^T dt \int_{-1}^1 du \chi(\pi)(\nabla G)(\nabla H).$$

LEMMA 3.6. Fix a trajectory π in D_γ and assume that $I_T(\pi|\gamma)$ is finite. There exists a function H in $\mathcal{H}_1(\pi)$ such that π is the unique weak solution of

$$\begin{aligned} \partial_t \pi &= (1/2)\Delta \pi - \nabla \{ \chi(\pi)\pi(1 - \pi)\nabla H \}, \\ \pi(t, \pm 1) &= \rho_\pm, \\ \pi(0, \cdot) &= \gamma(\cdot). \end{aligned} \tag{3.3}$$

Moreover,

$$I_T(\pi|\gamma) = (1/2) \int_0^T dt \int_{-1}^1 du \chi(\pi)(\nabla H)^2. \tag{3.4}$$

We refer the reader to [16, 17] for a proof. One of the consequences of this lemma is that every trajectory $t \mapsto \pi(t)$ with finite rate function is continuous in the weak topology, $\pi \in C([0, T]; \mathcal{M})$. Indeed, by the previous lemma, for π such that $I_T(\pi|\gamma) < \infty$, and every G in $C_0^2([-1, 1])$,

$$\begin{aligned} &\langle \pi(t), G \rangle - \langle \pi(s), G \rangle \\ &= (1/2) \int_s^t dr \langle \pi(r), \Delta G \rangle + \int_s^t dr \langle \chi(\pi(r)), \nabla G \nabla H \rangle - \\ &\quad - (1/2) \{ (\nabla G)(1) \rho_+ - (\nabla G)(-1) \rho_- \} (t - s) \end{aligned}$$

for some H in $\mathcal{H}_1(\pi)$. Since G is smooth and H belongs to $\mathcal{H}_1(\pi)$, the right-hand side vanishes as $|t - s| \rightarrow 0$.

3.6. LOWER BOUND

Denote by D_γ^0 the set of trajectories π in $D([0, T], \mathcal{M})$ for which there exists H in $C_0^{1,2}([0, T] \times [-1, 1])$ such that π is the solution of (3.3). For each π in D_γ^0 , and for each neighborhood \mathcal{N}_π of π

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}[\pi^N \in \mathcal{N}_\pi] \geq -I_T(\pi|\gamma).$$

This statement is proved as in the periodic boundary case, see [16]. To complete the proof of the lower bound, it remains to show that for every trajectory π such that $I_T(\pi|\gamma) < \infty$, there exists a sequence π_k in D_γ^0 such that $\lim_k \pi_k = \pi$, $\lim_k I_T(\pi_k|\gamma) = I_T(\pi|\gamma)$.

This is not too difficult in our context because the rate function is convex and lower semi continuous. We first show that any path π with finite rate function can be approximated by a path which is bounded away from 0 and 1. Fix a path π such that $I_T(\pi|\gamma) < \infty$. Fix $\delta > 0$ and denote by $\rho(t, u)$ the solution of the hydrodynamic equation (3.1) with initial condition γ instead of ρ_0 . Let $\pi_\delta = \delta\rho + (1 - \delta)\pi$. Of course, π_δ converges to π as $\delta \downarrow 0$. By lower semi-continuity, $I_T(\pi|\gamma) \leq \liminf_{\delta \rightarrow 0} I_T(\pi_\delta|\gamma)$. On the other hand, since $I_T(\cdot|\gamma)$ is convex, $I_T(\pi_\delta|\gamma) \leq (1 - \delta)I_T(\pi|\gamma)$ because ρ is the solution of the hydrodynamic equation and $I_T(\rho|\gamma) = 0$. This shows that $\lim_{\delta \rightarrow 0} \pi_\delta = \pi$, $\lim_{\delta \rightarrow 0} I_T(\pi_\delta|\gamma) = I_T(\pi|\gamma)$. Since $0 < \gamma < 1$, $0 < \rho_\pm < 1$, π_δ is bounded away from 0 and 1, proving the claim.

Fix now a path π with finite rate function and bounded away from 0 and 1. We claim that this trajectory may be approximated by a path in D_γ^0 . Since $I_T(\pi|\gamma) < \infty$, by Lemma 3.6, there exists H in $\mathcal{H}_1(\pi)$ satisfying (3.3). Since π is bounded away from 0 and 1, $\mathcal{H}_1(\pi)$ coincides with the usual Sobolev space H_1 associated to the Lebesgue measure. Consider a sequence of smooth functions $H_n: [0, T] \times [-1, 1] \rightarrow \mathbb{R}$ vanishing at the boundary and such that ∇H_n converges in $L^2([0, T] \times [-1, 1])$ to ∇H . Denote by π^n the solution of (3.2) with H_n instead of H . We claim that $\lim_{n \rightarrow \infty} \pi^n = \pi$, $\lim_{n \rightarrow \infty} I_T(\pi^n|\gamma) = I_T(\pi|\gamma)$.

The proof that π^n converges to π is divided in two pieces. We first show that the sequence is tight in $C([0, T], \mathcal{M})$ and then we prove that all limit points are solution of Equation (3.2). We start with a preliminary estimate which will be needed repeatedly. Recall that $\bar{\rho}$ is the stationary profile. Computing the time derivative of $\int_{-1}^1 du (\pi^n(t) - \bar{\rho})^2$, we obtain that

$$\int_0^T dt \int_{-1}^1 du (\nabla \pi^n(t))^2 \leq C \tag{3.5}$$

for some finite constant independent of n .

From the previous bound and since $\pi^n(t, u)$ belongs to $[0, 1]$, it is not difficult to show that the sequence π^n is tight in $C([0, T], \mathcal{M})$. To check uniqueness of limit points, consider any limit point β in $C([0, T], \mathcal{M})$. We claim that β is a weak solution of Equation (3.2). Of course β is positive and bounded above by 1. The existence of a function $A(s, u)$ in $L^2([-1, 1] \times [0, T])$ for which (a) holds follows from (3.5), which guarantees the existence of weak converging subsequences. The unique difficulty in the proof of identity (b) is to show that for any $0 \leq t \leq T$, G in $L^2([0, T] \times [-1, 1])$,

$$\lim_{n \rightarrow \infty} \int_0^t ds \langle \chi(\pi^n(s)), G(s) \rangle = \int_0^t ds \langle \chi(\beta(s)), G(s) \rangle \tag{3.6}$$

for any sequence π^n converging to β in $C([0, T], \mathcal{M})$ and satisfying (3.5). This identity holds because for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \int_0^t ds \langle \chi(\pi^n(s) * \iota_\delta), G(s) \rangle = \int_0^t ds \langle \chi(\beta(s) * \iota_\delta), G(s) \rangle$$

and because, by Schwartz inequality and $|\chi(a) - \chi(b)| \leq |a - b|$,

$$\begin{aligned} & \left(\int_0^t ds \langle \chi(\pi^n(s) * \iota_\delta) - \chi(\pi^n(s)), G(s) \rangle \right)^2 \\ & \leq \int_0^t ds \langle G(s)^2 \rangle \int_0^t ds \langle [\pi^n(s) * \iota_\delta - \pi^n(s)]^2 \rangle. \end{aligned}$$

It is not difficult to show, using estimate (3.5), that this term vanishes as $\delta \downarrow 0$, uniformly in n , proving (3.6). In conclusion, we proved that the sequence π^n is tight in $C([0, T], \mathcal{M})$ and that all its limit points are weak solutions of Equation (3.2). By uniqueness of weak solutions, π^n converges in $C([0, T], \mathcal{M})$ to π .

It remains to see that $I_T(\pi^n | \gamma)$ converges to $I_T(\pi | \gamma)$. Since $\pi^n \rightarrow \pi$ and $I_T(\cdot | \gamma)$ is lower semi-continuous, we just need to check that $\limsup_n I_T(\pi^n | \gamma) \leq I_T(\pi | \gamma)$. Here again the concavity and the boundness of χ help. Since ∇H^n converges in L^2 to ∇H and χ is bounded, the main problem is to show that

$$\limsup_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t)), (\nabla H(t))^2 \rangle \leq \int_0^T dt \langle \chi(\pi(t)), (\nabla H(t))^2 \rangle.$$

Since $\pi * \iota_\delta$ converges almost surely to π as $\delta \downarrow 0$,

$$\begin{aligned} & \int_0^T dt \langle \chi(\pi(t)), (\nabla H(t))^2 \rangle \\ & = \lim_{\delta \rightarrow 0} \int_0^T dt \langle \chi(\pi(t) * \iota_\delta), (\nabla H(t))^2 \rangle \\ & = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t) * \iota_\delta), (\nabla H(t))^2 \rangle. \end{aligned}$$

Since χ is concave, the previous expression is bounded below by

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t)) * \iota_\delta, (\nabla H(t))^2 \rangle.$$

Since χ is bounded and $(\nabla H)^2$ integrable, a change of variables shows that the previous expression is equal to

$$\limsup_{n \rightarrow \infty} \int_0^T dt \langle \chi(\pi^n(t)), (\nabla H(t))^2 \rangle,$$

concluding the proof of the lower bound.

4. The Rate Function for the Invariant Measure

In this section we discuss some properties of the functional $S(\rho)$ which are needed later. The results stated here are essentially contained in [9], but, for the sake of completeness, we review them and give more detailed proofs. Without any loss of generality, from now on we shall assume that $0 < \rho_- < \rho_+ < 1$. Recall the definitions of the set \mathcal{F} , (2.9), and of the functional $\mathcal{G}(\rho, f)$, (2.10).

The Euler–Lagrange equation associated to the variational problem (2.11) is given by the nonlinear boundary value problem

$$\begin{aligned} F'' &= (\rho - F) \frac{(F')^2}{F(1 - F)} \quad \text{in } (-1, 1), \\ F(\pm 1) &= \rho_\pm. \end{aligned} \tag{4.1}$$

We introduce the notation, which we will use throughout this section,

$$\mathcal{R}(u) = \mathcal{R}(\rho, F; u) = (\rho(u) - F(u)) \frac{F'(u)}{F(u)(1 - F(u))}. \tag{4.2}$$

Using this notation, Equation (4.1) takes the form

$$\begin{aligned} F'' &= F' \mathcal{R} \quad \text{in } (-1, 1), \\ F(\pm 1) &= \rho_\pm. \end{aligned} \tag{4.3}$$

In order to state and prove an existence and uniqueness result for $F \in \mathcal{F}$ we formulate (4.3) as the integro-differential equation

$$F(u) = \rho_- + (\rho_+ - \rho_-) \frac{\int_{-1}^u dv \exp\{\int_{-1}^v dw \mathcal{R}(\rho, F; w)\}}{\int_{-1}^1 dv \exp\{\int_{-1}^v dw \mathcal{R}(\rho, F; w)\}}. \tag{4.4}$$

We will denote its solution by $F = F(\rho)$ to emphasize its dependence on ρ . We observe that if $\rho = \bar{\rho}$ then $F = F(\bar{\rho}) = \bar{\rho}$ solves (4.4) and (4.3).

Notice that if $F \in C^2([-1, 1])$ is a solution of the boundary value problem (4.3) such that $F'(u) > 0$ for $u \in [-1, 1]$, then F is also a solution of the integro-differential equation (4.4). Conversely, if $F \in C^1([-1, 1])$ is a solution of (4.4), then $F'(u) > 0$, $F''(u)$ exists for almost every u and (4.3) holds almost everywhere. Moreover, if $\rho \in C([-1, 1])$, then $F \in C^2([-1, 1])$ and (4.3) holds everywhere.

Remark 4.1. There are nonmonotone solutions of Equation (4.3). For example, for the constant profile $\rho = 1/2$, it is easy to check that the functions

$$F(u) = \frac{1}{2}[1 + \sin(\lambda u + \varphi)];$$

satisfy Equation (4.3) for countably many choices of the parameters λ and φ (fixed in order to satisfy the boundary conditions in (4.3)). However only one such function is monotone. In fact, under the monotonicity assumption on F , we will prove uniqueness (and existence) of the solution of the boundary value problem (4.3).

The following theorem gives us the existence and uniqueness result for (4.4) together with a continuous dependence of the solution on ρ . Recall that we denote by $C^1([-1, 1])$ the Banach space of continuously differentiable functions $f: [-1, 1] \rightarrow \mathbb{R}$ endowed with the norm $\|f\|_{C^1} := \sup_{u \in [-1, 1]} \{|f(u)| + |f'(u)|\}$.

THEOREM 4.2. *For each $\rho \in \mathcal{M}$ there exists in \mathcal{F} a unique solution $F = F(\rho)$ of (4.4). Moreover:*

- (i) *if $\rho \in C([-1, 1])$, then $F = F(\rho) \in C^2([-1, 1])$ and it is the unique solution in $\mathcal{F} \cap C^2([-1, 1])$ of (4.3);*
- (ii) *if ρ_n converges to ρ in \mathcal{M} as $n \rightarrow \infty$, then $F_n = F(\rho_n)$ converges to $F = F(\rho)$ in $C^1([-1, 1])$;*
- (iii) *fix $T > 0$ and consider a function $\rho = \rho(t, u) \in C^{1,0}([0, T] \times [-1, 1])$. Then $F = F(t, u) = F(\rho(t, \cdot))(u) \in C^{1,2}([0, T] \times [-1, 1])$.*

The existence result in Theorem 4.2 will be proven by applying Schauder’s fixed point theorem. For each $\rho \in \mathcal{M}$ consider the map $\mathcal{K}_\rho: \mathcal{F} \rightarrow C^1([-1, 1])$ given by

$$\mathcal{K}_\rho(f)(u) := \rho_- + (\rho_+ - \rho_-) \frac{\int_{-1}^u dv \exp\{\int_{-1}^v dw \mathcal{R}(\rho, f; w)\}}{\int_{-1}^1 dv \exp\{\int_{-1}^v dw \mathcal{R}(\rho, f; w)\}}. \tag{4.5}$$

Let us also define the following closed, convex subset of $C^1([-1, 1])$:

$$\mathcal{B} := \{f \in C^1([-1, 1]) : f(\pm 1) = \rho_\pm, b \leq f'(u) \leq B\} \subset \mathcal{F}, \tag{4.6}$$

where, recalling we are assuming $\gamma_- < \gamma_+$,

$$b := \frac{\rho_+ - \rho_- \gamma_-}{2 \gamma_+}, \quad B := \frac{\rho_+ - \rho_- \gamma_+}{2 \gamma_-}.$$

LEMMA 4.3. *For each $\rho \in \mathcal{M}$, \mathcal{K}_ρ is a continuous map on \mathcal{F} and $\mathcal{K}_\rho(\mathcal{F}) \subset \mathcal{B}$. Furthermore $\mathcal{K}_\rho(\mathcal{B})$ has compact closure in $C^1([-1, 1])$. Hence, by Schauder’s fixed point theorem, for each $\rho \in \mathcal{M}$ Equation (4.4) has a solution $F = \mathcal{K}_\rho(F) \in \mathcal{B}$. Moreover, there exist a constant $C \in (0, \infty)$ depending on ρ_\pm such that for any $\rho \in \mathcal{M}$ and any $u, v \in [-1, 1]$ we have $|F'(u) - F'(v)| \leq C|u - v|$.*

Proof. It is easy to check that \mathcal{K}_ρ is continuous and $\mathcal{K}_\rho(f)(\pm 1) = \rho_\pm$. Let us define $g_\rho := \mathcal{K}_\rho(f)$, we have

$$g'_\rho(u) = (\rho_+ - \rho_-) \frac{\exp\{\int_{-1}^u dw \mathcal{R}(\rho, f; w)\}}{\int_{-1}^1 dv \exp\{\int_{-1}^v dw \mathcal{R}(\rho, f; w)\}}. \tag{4.7}$$

Since $\rho(w) - f(w) \leq 1 - f(w)$, $\rho(w) - f(w) \geq -f(w)$, and $f'(w) \geq 0$, we get

$$\frac{(1 - f)'}{1 - f} \leq \mathcal{R} \leq \frac{f'}{f}$$

which implies $b \leq g'_\rho(u) \leq B$ for all $u \in [-1, 1]$. In particular $\mathcal{K}_\rho(\mathcal{F}) \subset \mathcal{B}$.

To show that $\mathcal{K}_\rho(\mathcal{B})$ has a compact closure, by Ascoli–Arzela theorem, it is enough to prove that g'_ρ is Lipschitz uniformly for $f \in \mathcal{B}$. Indeed, by using (4.7), it is easy to check that there exists a constant $C = C(\rho_-, \rho_+) < \infty$ such that for any $u, v \in [-1, 1]$, any $f \in \mathcal{B}$, and any $\rho \in \mathcal{M}$ we have $|g'_\rho(u) - g'_\rho(v)| \leq C|u - v|$. \square

Proof of Theorem 4.2. The existence of solutions for (4.4) has been proven in Lemma 4.3; to prove uniqueness we follow closely the argument in [9]. Consider a solution $F \in \mathcal{F}$ of (4.4). Since it solves (4.3) almost everywhere, we get

$$F'(u) = F'(-1) + \int_{-1}^u dw F'(w) \mathcal{R}(\rho, F; w) \tag{4.8}$$

for all u in $[-1, 1]$. Moreover, taking into account that F is strictly increasing, we get from (4.3) that

$$\left(\frac{F(1 - F)}{F'}\right)' = 1 - F - \rho$$

holds a.e., so that

$$\frac{F(u)[1 - F(u)]}{F'(u)} = \frac{\rho_-[1 - \rho_-]}{F'(-1)} + \int_{-1}^u dv [1 - F(v) - \rho(v)] \tag{4.9}$$

for all u in $[-1, 1]$.

Let $F_1, F_2 \in \mathcal{F}$ be two solutions of (4.4). If $F'_1(-1) = F'_2(-1)$ an application of Gronwall inequality in (4.8) yields $F_1 = F_2$. We next assume $F'_1(-1) < F'_2(-1)$ and deduce a contradiction. Keep in mind that $F'_i > 0$ because F_i belongs to \mathcal{F} and recall (4.9). Let $\bar{u} := \inf\{v \in (-1, 1] : F_1(v) = F_2(v)\}$ which belongs to $(-1, 1]$ because $F_1(\pm 1) = F_2(\pm 1)$ and $F'_1(-1) < F'_2(-1)$. By definition of \bar{u} , $F_1(u) < F_2(u)$ for any $u \in (-1, \bar{u})$, $F_1(\bar{u}) = F_2(\bar{u})$ and $F'_1(\bar{u}) \geq F'_2(\bar{u})$. By (4.9), we also obtain

$$\frac{F_1(\bar{u})[1 - F_1(\bar{u})]}{F'_1(\bar{u})} > \frac{F_2(\bar{u})[1 - F_2(\bar{u})]}{F'_2(\bar{u})}$$

or, equivalently, $F_1'(\bar{u}) < F_2'(\bar{u})$, which is a contradiction and concludes the proof of the first statement of Theorem 4.2.

We turn now to statement (i). Existence follows from identity (4.3), which now holds for all points u in $[-1, 1]$ because ρ is continuous. Uniqueness follows from the uniqueness for the integro-differential formulation (4.4).

To prove (ii), let ρ_n be a sequence converging to ρ in \mathcal{M} and denote by $F_n = F(\rho_n)$ the corresponding solution of (4.4). By Lemma 4.3 and Ascoli–Arzela theorem, the sequence F_n is relatively compact in $C^1([-1, 1])$. It remains to show uniqueness of its limit points. Consider a subsequence n_j and assume that F_{n_j} converges to G in $C^1([-1, 1])$. Since ρ_{n_j} converges to ρ in \mathcal{M} and F_{n_j} converges to G in $C^1([-1, 1])$, by (4.5) $\mathcal{K}_{\rho_{n_j}}(F_{n_j})$ converges to $\mathcal{K}_\rho(G)$. In particular, $G = \lim_j F_{n_j} = \lim_j \mathcal{K}_{\rho_{n_j}}(F_{n_j}) = \mathcal{K}_\rho(G)$ so that, by the uniqueness result, $G = F(\rho)$. This shows that $F(\rho)$ is the unique possible limit point of the sequence F_n , and concludes the proof of (ii).

We are left to prove (iii). If $\rho(t, u) \in C^{1,0}([0, T] \times [-1, 1])$, we have from (i) and (ii) that $F(t, u) = F(\rho(t, \cdot))(u) \in C^{0,2}([0, T] \times [-1, 1])$. We then just need to prove that $F(t, u)$, as a function of t , is continuously differentiable. This will be accomplished by Lemma 4.4 below. \square

In order to prove the differentiability of $t \mapsto F(t, u) := F(\rho(t, \cdot))(u)$ it is convenient to introduce the new variable

$$\varphi(t, u) := \log \frac{F(t, u)}{1 - F(t, u)}, \quad (t, u) \in [0, T] \times [-1, 1]. \tag{4.10}$$

Note that $\varphi \in [\varphi_-, \varphi_+]$ where $\varphi_\pm := \log[\rho_\pm/(1 - \rho_\pm)] = \log \gamma_\pm$ and $u \mapsto \varphi(t, u)$ is strictly increasing. We remark that, as discussed in [4], while the function F is analogous to a density, the variable φ can be interpreted as a thermodynamic force. The advantage of using φ instead of F lies in the fact that, as a function of φ , the functional \mathcal{G} is concave. This property plays a crucial role in the sequel.

Let us fix a density profile $\rho \in C^{1,0}([0, T] \times [-1, 1])$. By (i)–(ii) in Theorem 4.2 and elementary computations, we have that $\varphi \in C^{0,2}([0, T] \times [-1, 1])$ and it is the unique strictly increasing (w.r.t. u) solution of the problem

$$\begin{aligned} \frac{\Delta\varphi(t, u)}{(\nabla\varphi(t, u))^2} + \frac{1}{1 + e^{\varphi(t, u)}} &= \rho(t, u), \quad (t, u) \in [0, T] \times (-1, 1), \\ \varphi(t, \pm 1) &= \varphi_\pm, \quad t \in [0, T]. \end{aligned} \tag{4.11}$$

Note also that, by Lemma 4.3, there exists a constant $C_1 = C_1(\rho_-, \rho_+) \in (0, \infty)$ such that

$$\frac{1}{C_1} \leq \nabla\varphi(t, u) \leq C_1, \quad \forall (t, u) \in [0, T] \times [-1, 1]. \tag{4.12}$$

LEMMA 4.4. *Let $\rho \in C^{1,0}([0, T] \times [-1, 1])$ and $\varphi = \varphi(t, u)$ be the corresponding solution of (4.11). Then $\varphi \in C^{1,2}([0, T] \times [-1, 1])$ and $\psi(t, u) := \partial_t \varphi(t, u)$ is the unique classical solution of the linear boundary value problem*

$$\nabla \left[\frac{\nabla \psi(t, u)}{(\nabla \varphi(t, u))^2} \right] - \frac{e^{\varphi(t, u)}}{(1 + e^{\varphi(t, u)})^2} \psi(t, u) = \partial_t \rho(t, u) \tag{4.13}$$

for $(t, u) \in [0, T] \times (-1, 1)$ with the boundary condition $\psi(t, \pm 1) = 0, t \in [0, T]$.

Proof. Fix $t \in [0, T]$, for $h \neq 0$ such that $t + h \in [0, T]$ let us introduce $\psi_h(t, u) := [\varphi(t + h, u) - \varphi(t, u)]/h$. Note that, by (i)–(ii) in Theorem 4.2, $\psi_h(t, \cdot) \in C^2([-1, 1])$. By using (4.11), we get that ψ_h solves

$$\begin{aligned} \nabla \left[\frac{\nabla \psi_h(t, u)}{\nabla \varphi(t, u) \nabla \varphi(t + h, u)} \right] - \frac{e^{\varphi(t, u)}}{(1 + e^{\varphi(t, u)})(1 + e^{\varphi(t + h, u)})} \frac{e^{h \psi_h(t, u)} - 1}{h} \\ = \frac{\rho(t + h, u) - \rho(t, u)}{h} \end{aligned} \tag{4.14}$$

for $(t, u) \in [0, T] \times (-1, 1)$ with the boundary condition $\psi_h(t, \pm 1) = 0, t \in [0, T]$.

Multiplying the above equation by $\psi_h(t, u)$ and integrating in du , after using the inequality $x(e^x - 1) \geq 0$ and an integration by parts, we get

$$\begin{aligned} \int_{-1}^1 du \frac{(\nabla \psi_h(t, u))^2}{\nabla \varphi(t, u) \nabla \varphi(t + h, u)} \\ \leq \left| \int_{-1}^1 du \psi_h(t, u) \frac{\rho(t + h, u) - \rho(t, u)}{h} \right| \\ \leq \varepsilon \int_{-1}^1 du \psi_h(t, u)^2 + \frac{1}{4\varepsilon} \int_{-1}^1 du \left(\frac{\rho(t + h, u) - \rho(t, u)}{h} \right)^2, \end{aligned}$$

where we used Schwartz inequality with $\varepsilon > 0$. Recalling the Poincaré inequality (with $f(\pm 1) = 0$)

$$\int_{-1}^1 du f(u)^2 \leq \frac{4}{\pi^2} \int_{-1}^1 du f'(u)^2$$

using (4.12) and choosing ε small enough we finally find

$$\limsup_{h \rightarrow 0} \int_{-1}^1 du [\nabla \psi_h(t, u)]^2 \leq C_2 \int_{-1}^1 du [\partial_t \rho(t, u)]^2$$

for some constant C_2 depending only on ρ_+, ρ_- .

Hence, by Sobolev embedding, the sequence $\psi_h(t, \cdot)$ is relatively compact in $C([-1, 1])$. By taking the limit $h \rightarrow 0$ in (4.14) it is now easy to show any limit point is a weak solution of (4.13). By classical theory on the one-dimensional elliptic problems, see, e.g., [21, IV, §2.1], there exists a unique weak solution of (4.13) which is in fact the classical solution since $\partial_t \rho(t, \cdot) \in C([-1, 1])$. This implies there exists a unique limit point $\psi(t, u)$ which is twice differentiable w.r.t. u . The continuity of $t \mapsto \psi(t, \cdot)$ follows from the continuous dependence (in the $C^2([-1, 1])$ topology) of the solution of (4.13) w.r.t. $\partial_t \rho(t, \cdot)$ (in the $C([-1, 1])$ topology). \square

The link between the boundary value problem (4.3) and the variational problem (2.11) is established by the following theorem.

THEOREM 4.5. *Let S be the functional on \mathcal{M} defined in (2.11). Then S is bounded, convex and lower semi-continuous on \mathcal{M} . Moreover, for each $\rho \in \mathcal{M}$, we have that $S(\rho) = \mathfrak{G}(\rho, F(\rho))$ where $F(\rho)$ is the solution of (4.4).*

Proof. For each $f \in \mathcal{F}$ we have that $\mathfrak{G}(\cdot, f)$ is a convex lower semi-continuous functional on \mathcal{M} . Hence the functional $S(\cdot)$ defined in (2.11), being the supremum of convex lower semi-continuous functionals, is a convex lower semi-continuous functional on \mathcal{M} . Furthermore, by choosing $f = \bar{\rho}$ in (2.11) we obtain that $0 \leq S_0(\rho) \leq S(\rho)$. Finally, by using the concavity of $x \mapsto \log x$, Jensen’s inequality, and $f(\pm 1) = \rho_{\pm}$, we get that $\mathfrak{G}(\rho, f)$ is bounded by some constant depending only on ρ_- and ρ_+ .

In order to show the supremum in (2.11) is uniquely attained when $f = F(\rho)$ solves (4.4), it is convenient to make, as in Lemma 4.4, the change of variables $\varphi = \phi(f)$ defined by $\varphi(u) := \log\{f(u)/[1 - f(u)]\}$. Note that $f(u) = e^{\varphi(u)}/[1 + e^{\varphi(u)}]$. We then need to show that the supremum of the functional

$$\begin{aligned} \tilde{\mathfrak{G}}(\rho, \varphi) &:= \mathfrak{G}(\rho, \phi^{-1}(\varphi)) \\ &= \int_{-1}^1 du \left\{ \rho(u) \log \rho(u) + [1 - \rho(u)] \log [1 - \rho(u)] + \right. \\ &\quad \left. + [1 - \rho(u)]\varphi(u) - \log [1 + e^{\varphi(u)}] + \log \frac{\varphi'(u)}{[\rho_+ - \rho_-]/2} \right\} \end{aligned} \tag{4.15}$$

for

$$\varphi \in \tilde{\mathcal{F}} := \phi(\mathcal{F}) = \{ \varphi \in C^1([-1, 1]) : \varphi(\pm 1) = \varphi_{\pm}, \varphi'(u) > 0 \}$$

is uniquely attained when $\varphi = \phi(F(\rho))$. We recall that $F(\rho)$ denotes the solution of (4.4).

Since the real functions $x \mapsto \log x$ and $x \mapsto -\log(1 + e^x)$ are strictly concave, for each $\rho \in \mathcal{M}$ the functional $\tilde{\mathfrak{G}}(\rho, \cdot)$ is strictly concave on $\tilde{\mathcal{F}}$. Moreover, it is easy to show that $\tilde{\mathfrak{G}}(\rho, \cdot)$ is Gateaux differentiable on $\tilde{\mathcal{F}}$ with derivative given by

$$\left\langle \frac{\delta \tilde{\mathfrak{G}}(\rho, \varphi)}{\delta \varphi}, g \right\rangle = \int_{-1}^1 du \left\{ \frac{g'(u)}{\varphi'(u)} + \left[\frac{1}{1 + e^{\varphi(u)}} - \rho(u) \right] g(u) \right\}.$$

By standard convex analysis, see, e.g., [11, I, Prop. 5.4], for any $\varphi \neq \psi \in \tilde{\mathcal{F}}$ we have

$$\tilde{\mathfrak{G}}(\rho, \psi) < \tilde{\mathfrak{G}}(\rho, \varphi) + \left\langle \frac{\delta \tilde{\mathfrak{G}}(\rho, \varphi)}{\delta \varphi}, \psi - \varphi \right\rangle.$$

By noticing that $\delta \tilde{\mathfrak{G}}(\rho, \varphi)/\delta \varphi = 0$ if φ solves (4.11) a.e. we conclude the proof that the supremum on $\tilde{\mathcal{F}}$ of $\tilde{\mathfrak{G}}(\rho, \cdot)$ is uniquely attained when $\varphi = \phi(F(\rho))$. \square

Remark 4.6. Given $\rho \in \mathcal{M}$, let us consider a sequence $\rho_n \in C^2([-1, 1]) \cap \mathcal{M}$ with $\rho_n(\pm 1) = \rho_{\pm}$, bounded away from 0 and 1, which converges to ρ a.e. Then, by dominated convergence and (ii) in Theorem 4.2, we have $S(\rho_n) = \mathcal{G}(\rho_n, F(\rho_n)) \rightarrow \mathcal{G}(\rho, F(\rho)) = S(\rho)$.

5. The Quasi Potential

In this section we show that the quasi potential for the one-dimensional boundary driven simple exclusion process, as defined by the variational problem (2.6), coincides with the functional $S(\rho)$ defined in (2.11). In the proof we shall also construct an optimal path for the variational problem (2.6).

Let us first recall the heuristic argument given in [4]. Taking into account the representation of the functional $I_T(\pi|\bar{\rho})$ given in Lemma 3.6, to the variational problem (2.6) is associated the Hamilton–Jacobi equation

$$\frac{1}{2} \left\langle \nabla \frac{\delta V}{\delta \rho}, \rho(1 - \rho) \nabla \frac{\delta V}{\delta \rho} \right\rangle + \left\langle \frac{\delta V}{\delta \rho}, \frac{1}{2} \Delta \rho \right\rangle = 0, \tag{5.1}$$

where ∇ denotes the derivative w.r.t. the macroscopic space coordinate $u \in [-1, 1]$. We look for a solution in the form

$$\frac{\delta V}{\delta \rho} = \log \frac{\rho}{1 - \rho} - \log \frac{f}{1 - f}$$

and obtain a solution of (5.1) provided f solves the boundary value problem (4.3), namely $f = F(\rho)$. On the other hand, by Theorem 4.5, we have

$$\begin{aligned} \frac{\delta S(\rho)}{\delta \rho} &= \left. \frac{\delta \mathcal{G}(\rho, f)}{\delta \rho} \right|_{f=F(\rho)} + \left. \frac{\delta \mathcal{G}(\rho, f)}{\delta f} \right|_{f=F(\rho)} \frac{\delta F(\rho)}{\delta \rho} \\ &= \log \frac{\rho}{1 - \rho} - \log \frac{F(\rho)}{1 - F(\rho)} \end{aligned}$$

since (4.3) is the Euler–Lagrange equation for the variational problem (2.11). We get therefore $V = S$ since we have $V(\bar{\rho}) = S(\bar{\rho}) = 0$.

Let $\pi^*(t) = \pi^*(t, u)$ be the optimal path for the variational problem (2.6) and define $\rho^*(t) := \pi^*(-t)$. By using a time reversal argument, in [4] it is also shown that $\rho^*(t)$ solves the hydrodynamic equation associated to the adjoint process (whose generator is the adjoint of L_N in $L_2(d\mu^N)$) which takes the form

$$\partial_t \rho^*(t) = -\frac{1}{2} \Delta \rho^*(t) + \nabla \left(\rho^*(t) [1 - \rho^*(t)] \nabla \frac{\delta S(\rho)}{\delta \rho} \Big|_{\rho=\rho^*(t)} \right). \tag{5.2}$$

We will not develop here a mathematical theory of the Hamilton–Jacobi equation (5.1). We shall instead work directly with the variational problem (2.6), making explicit computations for smooth paths and using approximation arguments to

prove that we have indeed $V = S$. Of course, the description of the optimal path will also play a crucial role.

To identify the quasi potential V with the functional S we shall prove separately the lower bound $V \geq S$ and the upper bound $V \leq S$. For this purpose we start with two lemmata, which connect S defined in (2.11) to the Hamilton–Jacobi equation (5.1), used for both inequalities. The bound $V \geq S$ will then be proven by choosing the right test field H in (2.4). To prove $V \leq S$ we shall exhibit a path $\pi^*(t) = \pi^*(t, u)$ which connects the stationary profile $\bar{\rho}$ to ρ in some time interval $[0, T]$ and such that $I_T(\pi^*|\bar{\rho}) \leq S(\rho)$. As outlined above, this path ought to be the time reversal of the solution of the adjoint hydrodynamic equation (5.2) with initial condition ρ . The adjoint hydrodynamic equation needs, however, infinite time to relax to the stationary profile $\bar{\rho}$. We have therefore to follow the time reversed adjoint hydrodynamic equation in a time interval $[0, T_1]$ to arrive at some profile $\rho^*(T_1)$, which is close to $\bar{\rho}$ if T_1 is large, and then interpolate, in some interval $[T_1, T_1 + T_2]$, between $\rho^*(T_1)$ and $\bar{\rho}$.

Recall that we are assuming $\rho_- < \rho_+$ and pick $\delta_0 > 0$ small enough for $\delta_0 \leq \rho_- < \rho_+ \leq 1 - \delta_0$. For $\delta \in (0, \delta_0]$ and $T > 0$, we introduce

$$\mathcal{M}_\delta := \{ \rho \in C^2([-1, 1]) : \rho(\pm 1) = \rho_\pm, \delta \leq \rho(u) \leq 1 - \delta \}, \tag{5.3}$$

$$D_{T,\delta} := \{ \pi \in C^{1,2}([0, T] \times [-1, 1]) : \pi(t, \pm 1) = \rho_\pm, \delta \leq \pi(t, u) \leq 1 - \delta \}. \tag{5.4}$$

LEMMA 5.1. *Let $\pi \in D_{T,\delta}$ and denote by $F(t, u) = F(\pi(t, \cdot))(u)$ the solution of the boundary value problem (4.3) with ρ replaced by $\pi(t)$. Set*

$$\Gamma(t, u) = \log \frac{\pi(t, u)}{1 - \pi(t, u)} - \log \frac{F(t, u)}{1 - F(t, u)}. \tag{5.5}$$

Then, for each $T \geq 0$,

$$S(\pi(T)) - S(\pi(0)) = \int_0^T dt \langle \partial_t \pi(t), \Gamma(t) \rangle. \tag{5.6}$$

Proof. Note that $F(t, \cdot)$ is strictly increasing for any $t \in [0, T]$ and $F \in C^{1,2}([0, T] \times [-1, 1])$ by (iii) in Theorem 4.2. Moreover, since $F(t, \pm 1) = \rho_\pm$, we have $\partial_t F(t, \pm 1) = 0$. By Theorem 4.5, dominated convergence, an explicit computation, and an integration by parts, we get

$$\begin{aligned} \frac{d}{dt} S(\pi(t)) &= \frac{d}{dt} \mathcal{G}(\pi(t), F(t)) \\ &= \langle \partial_t \pi(t), \Gamma(t) \rangle + \left\langle \partial_t F(t), \frac{1 - \pi(t)}{1 - F(t)} - \frac{\pi(t)}{F(t)} \right\rangle + \left\langle \frac{1}{\nabla F(t)}, \partial_t \nabla F(t) \right\rangle \\ &= \langle \partial_t \pi(t), \Gamma(t) \rangle + \left\langle \partial_t F(t), \frac{F(t) - \pi(t)}{F(t)[1 - F(t)]} + \frac{\Delta F(t)}{(\nabla F(t))^2} \right\rangle. \end{aligned}$$

The lemma follows by noticing that the last term above vanishes by (4.3). \square

LEMMA 5.2. *Let $\rho \in \mathcal{M}_\delta$, denote by $F(u) = F(\rho)(u)$ the solution of the boundary value problem (4.3), and set*

$$\Gamma(u) = \log \frac{\rho(u)}{1 - \rho(u)} - \log \frac{F(u)}{1 - F(u)}.$$

Then,

$$\langle \rho(1 - \rho), (\nabla \Gamma)^2 \rangle + \langle \Delta \rho, \Gamma \rangle = 0. \tag{5.7}$$

Proof. Note that $F \in \mathcal{M}_\delta$ by Theorem 4.2. After an integration by parts and simple algebraic manipulations (5.7) is equivalent to

$$-\left\langle \nabla \rho, \frac{\nabla F}{F(1 - F)} \right\rangle + \left\langle \rho(1 - \rho), \left(\frac{\nabla F}{F(1 - F)} \right)^2 \right\rangle = 0. \tag{5.8}$$

We rewrite the first term on the left-hand side as

$$-\left\langle \nabla F, \frac{\nabla F}{F(1 - F)} \right\rangle - \left\langle \nabla(\rho - F), \frac{\nabla F}{F(1 - F)} \right\rangle$$

which, by an integration by parts, is equal to

$$-\left\langle \nabla F, \frac{\nabla F}{F(1 - F)} \right\rangle + \left\langle \rho - F, \frac{\Delta F}{F(1 - F)} - \frac{(1 - 2F)(\nabla F)^2}{[F(1 - F)]^2} \right\rangle.$$

Hence, the left-hand side of (5.8) is given by

$$\begin{aligned} & \left\langle \rho - F, \frac{\Delta F}{F(1 - F)} \right\rangle - \left\langle \frac{(\nabla F)^2}{[F(1 - F)]^2}, (\rho - F)^2 \right\rangle \\ & = \left\langle \frac{\rho - F}{F(1 - F)}, \Delta F - (\rho - F) \frac{(\nabla F)^2}{F(1 - F)} \right\rangle = 0 \end{aligned}$$

thanks to (4.3). \square

Note that, for smooth paths, Lemma 5.1 identifies, in the sense given by Equation (5.6), Γ as the derivative of S . Lemma 5.2 then states that this derivative satisfies the Hamilton–Jacobi equation (5.1).

5.1. LOWER BOUND

We can now prove the first relation between the quasi potential V and the functional S .

LEMMA 5.3. *For each $\rho \in \mathcal{M}$ we have $V(\rho) \geq S(\rho)$.*

Proof. In view of the variational definition V , to prove the lemma we need to show that $S(\rho) \leq I_T(\pi|\bar{\rho})$ for any $T > 0$ and any path $\pi \in D([0, T]; \mathcal{M})$ which connects the stationary profile $\bar{\rho}$ to ρ in the time interval $[0, T]$: $\pi(0) = \bar{\rho}$, $\pi(T) = \rho$.

Fix such a path π and let us assume first that $\pi \in D_{T,\delta}$. Denote by $F(t) = F(\pi(t))$ the solution of the elliptic problem (4.3) with $\pi(t)$ in place of ρ . In view of the variational definition of $I_T(\pi|\bar{\rho})$ given in (2.5), to prove that $S(\rho) \leq I_T(\pi|\bar{\rho})$ it is enough to exhibit some function $H \in C_0^{1,2}([0, T] \times [-1, 1])$ for which $S(\rho) \leq J_{T,H,\bar{\rho}}(\pi)$. We claim that Γ given in (5.5) fulfills these conditions.

We have that $\Gamma \in C_0^{1,2}([0, T] \times [-1, 1])$ because: $\pi \in D_{T,\delta}$ by hypothesis, $F \in C^{1,2}([0, T] \times [-1, 1])$ by (iii) in Theorem 4.2, $\Gamma(t, \pm 1) = 0$ since $\pi(t, \cdot)$ and $F(t, \cdot)$ satisfy the same boundary conditions. Recalling (2.4) we get, after integration by parts,

$$J_{T,\Gamma,\bar{\rho}}(\pi) = \int_0^T dt \langle \partial_t \pi(t), \Gamma(t) \rangle - \frac{1}{2} \int_0^T dt [\langle \Gamma(t), \Delta \pi(t) \rangle + \langle \pi(t)[1 - \pi(t)], [\nabla \Gamma(t)]^2 \rangle].$$

By Lemmata 5.1 and 5.2 we then have $J_{T,\Gamma,\bar{\rho}}(\pi) = S(\rho)$.

Up to this point we have shown that $S(\rho) \leq I_T(\pi|\bar{\rho})$ for smooth paths π bounded away from 0 and 1. In order to obtain this result for general paths, we just have to recall the approximations performed in the proof of the lower bound of the large deviation principle. Fix a path π with finite rate function: $I_T(\pi|\bar{\rho}) < \infty$. In Section 3.6 we proved that there exists a sequence $\{\pi_n, n \geq 1\}$ of smooth paths such that π_n converges to π and $I_T(\pi_n|\bar{\rho})$ converges to $I_T(\pi|\bar{\rho})$. Let $\tilde{\pi}_n$ be defined by $(1 - n^{-1})\pi_n + n^{-1}\bar{\rho}$. Since π_n converges to π , $\tilde{\pi}_n$ converges to π . By lower semi-continuity of the rate function, $I_T(\pi|\bar{\rho}) \leq \liminf_{n \rightarrow \infty} I_T(\tilde{\pi}_n|\bar{\rho})$. On the other hand, by convexity, $I_T(\tilde{\pi}_n|\bar{\rho}) \leq (1 - n^{-1})I_T(\pi_n|\bar{\rho}) + n^{-1}I_T(\bar{\rho}|\bar{\rho}) = (1 - n^{-1})I_T(\pi_n|\bar{\rho})$ so that $\limsup_{n \rightarrow \infty} I_T(\tilde{\pi}_n|\bar{\rho}) \leq I_T(\pi|\bar{\rho})$. Since $\tilde{\pi}_n$ belongs to $D_{T,\delta}$ for some $\delta = \delta_n > 0$, each path π with finite rate function can be approximated by a sequence $\tilde{\pi}_n$ in D_{T,δ_n} , for some set of strictly positive parameters δ_n , and such that $I_T(\pi|\bar{\rho}) = \lim_n I_T(\tilde{\pi}_n|\bar{\rho})$. Therefore, by the result on smooth paths and the lower semi-continuity of S , we get

$$I_T(\pi|\bar{\rho}) = \lim_n I_T(\tilde{\pi}_n|\bar{\rho}) \geq \liminf_n S(\tilde{\pi}_n(T)) \geq S(\pi(T)),$$

which concludes the proof of the lemma. □

5.2. UPPER BOUND

The following lemma explains which is the right candidate for the optimal path for the variational problem (2.6).

LEMMA 5.4. Fix $\delta \in (0, \delta_0]$, a profile $\alpha \in \mathcal{M}_\delta$, and a path $\pi \in D_{T,\delta}$ with finite rate function, $I_T(\pi|\alpha) < \infty$. Denote by $F(t, u) = F(\pi(t, \cdot))(u)$ the solution of the boundary value problem (4.3) with ρ replaced by $\pi(t)$. Then there exists a function $K \in \mathcal{H}_1(\pi)$ such that π is the weak solution of

$$\begin{aligned} \partial_t \pi &= -\frac{1}{2} \Delta \pi + \nabla \left(\pi(1 - \pi) \nabla \left[\log \frac{F}{1-F} + K \right] \right), \\ (t, u) &\in [0, T] \times (-1, 1), \\ \pi(t, \pm 1) &= \rho_\pm, \quad t \in [0, T], \\ \pi(0, u) &= \alpha(u), \quad u \in [-1, 1]. \end{aligned} \tag{5.9}$$

Moreover,

$$I_T(\pi|\alpha) = S(\pi(T)) - S(\alpha) + \frac{1}{2} \int_0^T dt \langle \pi(t)[1 - \pi(t)], [\nabla K(t)]^2 \rangle. \tag{5.10}$$

The optimal path for the variational problem (2.6) will be obtained by taking a path π^* for which the last term on the right-hand side of the identity (5.10) (which is positive) vanishes, namely for a path π^* which satisfies (5.9) with $K = 0$. Then $\rho^*(t) = \pi^*(-t)$ will be a solution of (5.2).

Proof. Denote by H the function in $\mathcal{H}_1(\pi)$ introduced in Lemma 3.6, let Γ as defined in (5.5), and set $K := \Gamma - H$. Note that K belongs to $\mathcal{H}_1(\pi)$ because $\pi \in D_{T,\delta}$ by hypothesis, $F \in C^{1,2}([0, T] \times [-1, 1])$ by Theorem 4.2, and $\Gamma(t, \pm 1) = 0$. Then (5.9) follows easily from (3.3). To prove the identity (5.10), replace in (5.6) $\partial_t \pi(t)$ by the right-hand side of the differential equation in (5.9). After an integration by parts we obtain

$$\begin{aligned} S(\pi(T)) - S(\alpha) &= \int_0^T dt \left\{ \frac{1}{2} \langle \Gamma(t), \Delta \pi(t) \rangle + \langle \pi(t)[1 - \pi(t)], [\nabla \Gamma(t)]^2 \rangle - \right. \\ &\quad \left. - \langle \pi(t)[1 - \pi(t)], \nabla K(t) \nabla \Gamma(t) \rangle \right\} \\ &= \int_0^T dt \left\langle \pi(t)[1 - \pi(t)], \frac{1}{2} [\nabla \Gamma(t)]^2 - \nabla \Gamma(t) \nabla K(t) \right\rangle, \end{aligned}$$

where we used Lemma 5.2. Recalling $K = \Gamma - H$, we thus obtain

$$\begin{aligned} S(\pi(T)) - S(\alpha) &+ \frac{1}{2} \int_0^T dt \langle \pi(t)[1 - \pi(t)], [\nabla K(t)]^2 \rangle \\ &= \frac{1}{2} \int_0^T dt \langle \pi(t)[1 - \pi(t)], [\nabla H(t)]^2 \rangle, \end{aligned}$$

which concludes the proof of the lemma in view of (3.4). □

We write more explicitly the adjoint hydrodynamic equation (5.2). In the present paper, we shall use it only to describe a particular path which will be shown to be the optimal one. For $\rho \in \mathcal{M}$, consider the nonlocal differential equation

$$\begin{aligned} \partial_t \rho^* &= \frac{1}{2} \Delta \rho^* - \nabla \left(\rho^*(1 - \rho^*) \nabla \log \frac{F}{1 - F} \right), \quad (t, u) \in (0, \infty) \times [-1, 1], \\ F(t, u) &= F(\rho^*(t, \cdot))(u), \quad (t, u) \in (0, \infty) \times [-1, 1], \\ \rho^*(t, \pm 1) &= \rho_{\pm}, \quad t \in (0, \infty), \\ \rho^*(0, u) &= \rho(u), \quad u \in [-1, 1], \end{aligned} \tag{5.11}$$

where we recall that $F(t, u) = F(\rho^*(t, \cdot))(u)$ means that $F(t, u)$ has to be obtained from $\rho^*(t, u)$ by solving (4.4) with $\rho(u)$ replaced by $\rho^*(t, u)$. Since $\nabla \log[F/(1 - F)] > 0$, in (5.11) there is a positive drift to the right. Let us describe how it is possible to construct the solution of (5.11).

LEMMA 5.5. *For $\rho \in \mathcal{M}$ let $\Phi(t)$ be the solution of the heat equation (2.12) and define $\rho^* = \rho^*(t, u)$ by (2.13). Then $\rho^* \in C^{1,2}((0, \infty) \times [-1, 1]) \cap C([0, \infty); \mathcal{M})$ and solves (5.11). Moreover, if $\delta \leq \rho(u) \leq 1 - \delta$ a.e. for some $\delta > 0$, there exists $\delta' = \delta'(\rho_-, \rho_+, \delta) \in (0, 1)$, for which $\delta' \leq \rho^*(t, u) \leq 1 - \delta'$ for any $(t, u) \in (0, \infty) \times [-1, 1]$.*

Proof. Let $F(u) = F(\rho)(u)$, then, by Theorem 4.2, $F \in C^1([-1, 1])$ and, by Lemma 4.3, there is a constant $C \in (0, \infty)$ depending only on ρ_-, ρ_+ such that $C^{-1} \leq F'(u) \leq C$ for any $u \in [-1, 1]$. Since $\Phi(t, u)$ solves (2.12), there exists $C_1 = C_1(\rho_-, \rho_+) \in (0, \infty)$ such that $C_1^{-1} \leq (\nabla \Phi)(t, u) \leq C_1$ for any $(t, u) \in [0, \infty) \times [-1, 1]$. Moreover, $\Phi(t, \pm 1) = \rho_{\pm}$ so that $\Delta \Phi(t, \pm 1) = 2\partial_t \Phi(t, \pm 1) = 0$. Hence, ρ^* defined by (2.13) satisfies the boundary condition $\rho^*(t, \pm 1) = \Phi^*(t, \pm 1) = \rho_{\pm}$. Furthermore, $\rho^* \in C^{1,2}((0, \infty) \times [-1, 1])$.

For the reader's convenience, we reproduce below from [4, Appendix B] the proof that $\rho^*(t, u)$, as defined in (2.13), solves the differential equation in (5.11). From (2.13) we get that

$$\frac{\rho^*(1 - \rho^*)}{\Phi(1 - \Phi)} = 1 + (1 - 2\Phi) \frac{\Delta \Phi}{(\nabla \Phi)^2} - \Phi(1 - \Phi) \frac{(\Delta \Phi)^2}{(\nabla \Phi)^4}$$

recalling (2.12), by a somehow tedious computation of the partial derivatives which we omit, we get

$$\left(\partial_t - \frac{1}{2} \Delta \right) \left[\Phi(1 - \Phi) \frac{\Delta \Phi}{(\nabla \Phi)^2} \right] = -\nabla \left(\frac{\rho^*(1 - \rho^*)}{\Phi(1 - \Phi)} \nabla \Phi \right)$$

from which, by using again (2.13), we see that ρ^* satisfies the differential equation in (5.11).

To conclude the proof of the lemma, notice that ρ^* is the solution of

$$\begin{aligned} \partial_t \rho^* &= \frac{1}{2} \Delta \rho^* - \nabla \{ \rho^*(1 - \rho^*) \nabla H \}, \\ \rho^*(t, \pm 1) &= \rho_{\pm}, \\ \rho^*(0, \cdot) &= \rho(\cdot), \end{aligned}$$

for some function H in $C^{1,1}([0, \infty) \times [-1, 1])$ for which ∇H is uniformly bounded. Though H does not vanish at the boundary, we may use a weakly asymmetric

boundary driven exclusion process to prove the existence of a weak solution $\lambda(t, u)$, in the sense of Subsection 3.4, which takes values in the interval $[0, 1]$. Since ∇H is bounded, the usual H_{-1} method gives uniqueness so that $\lambda = \rho^*$ and $0 \leq \rho^* \leq 1$. In particular $\rho^* \in C([0, \infty); \mathcal{M})$.

Assume now that $\delta \leq \rho \leq 1 - \delta$ for some $\delta > 0$. Fix $t > 0$ and assume that $\rho^*(t, \cdot)$ has a local maximum at $-1 < u_0 < 1$. Since ρ^* is a smooth solution of (5.11), a simple computation gives that at (t, u_0)

$$(\partial_t \rho^*) = \frac{1}{2} \Delta \rho^* - \frac{\rho^*(1 - \rho^*)(\nabla F)^2}{F^2(1 - F)^2} (\rho^* + F - 1)$$

because

$$(\nabla \rho^*)(t, u_0) = 0 \quad \text{and} \quad \Delta \log \left\{ \frac{F}{1 - F} \right\} = (\nabla F)^2 \frac{(\rho^* + F - 1)}{F^2(1 - F)^2}.$$

Since u_0 is a local maximum, $\Delta \rho^* \leq 0$. On the other hand, assume that $\rho^*(t, u_0) > 1 - \rho_-$, in this case, since $\rho_- \leq F$, $\rho^* + F - 1 > 0$ so that $\partial_t \rho^* < 0$. In the same way we can conclude that $(\partial_t \rho^*)(t, u_1) > 0$ if u_1 is a minimum of $\rho^*(t, \cdot)$ and $\rho^*(t, u_1) \leq 1 - \rho_+$. These two estimates show that $\min\{\delta, 1 - \rho_+, \rho_-\} \leq \rho^*(t, u) \leq \max\{1 - \delta, 1 - \rho_-, \rho_+\}$, which concludes the proof of the lemma. \square

We now prove that the solution of (5.11), as constructed in Lemma 5.5, converges, as $t \rightarrow \infty$, to $\bar{\rho}$ uniformly with respect to the initial datum ρ . We use below the usual notation $\|f\|_\infty := \sup_{u \in [-1, 1]} |f(u)|$.

LEMMA 5.6. *Given $\rho \in \mathcal{M}$, let $\rho^*(t) = \rho^*(t, u)$ be the solution (5.11). Then,*

$$\lim_{t \rightarrow \infty} \sup_{\rho \in \mathcal{M}} \|\rho^*(t) - \bar{\rho}\|_\infty = 0.$$

Proof. Let us represent the solution $\Phi(t)$ of (2.12) in the form $\Phi(t, u) = \bar{\rho}(u) + \Psi(t, u)$. Then $\Psi(t) = P_t^0 \Psi(0)$ where P_t^0 is the semigroup generated by $(1/2)\Delta^0$, with Δ^0 the Dirichlet Laplacian on $[-1, 1]$. Since $\Psi(0) = F(\rho) - \bar{\rho}$ and since the solution $F(\rho)$ of (4.4) as well as $\bar{\rho}$ are contained in the interval $[\rho_-, \rho_+]$, we have that $\|\Psi(0)\|_\infty \leq |\rho_+ - \rho_-| < 1$. Therefore, by standard heat kernel estimates,

$$\lim_{t \rightarrow \infty} \sup_{\rho \in \mathcal{M}} \{ \|\Psi(t)\|_\infty + \|\nabla \Psi(t)\|_\infty + \|\Delta \Psi(t)\|_\infty \} = 0,$$

the lemma follows recalling that, by Lemma 5.5, $\rho^*(t)$ is given by (2.13). \square

Lemma 5.6 shows that we may join a profile ρ in \mathcal{M} to a neighborhood of the stationary profile by using Equation (5.11) for a time interval $[0, T_1]$ which at the same time regularizes the profile. On the other hand, from Lemma 5.4 we shall deduce that this path pays $S(\rho) - S(\rho^*(T_1))$. It thus remains to connect $\rho^*(T_1)$, which is a smooth profile close to the stationary profile $\bar{\rho}$ for large T_1 , to $\bar{\rho}$. In the next lemma we show this can be done by paying only a small price. We denote by $\|\cdot\|_2$ the norm in $L_2([-1, 1], du)$.

LEMMA 5.7. *Let $\alpha \in \mathcal{M}_{\delta_0}$ be a smooth profile such that $\|\alpha - \bar{\rho}\|_\infty \leq \delta_0/16$. Then there exists a smooth path $\hat{\pi}(t)$, $t \in [0, 1]$ with $\delta_0/2 \leq \hat{\pi} \leq 1 - \delta_0/2$, namely $\hat{\pi} \in D_{1, \delta_0/2}$, with $\hat{\pi}(0) = \bar{\rho}$, $\hat{\pi}(1) = \alpha$ and a constant $C = C(\delta_0) \in (0, \infty)$ such that*

$$I_1(\hat{\pi}|\bar{\rho}) \leq C\|\alpha - \bar{\rho}\|_2^2.$$

In particular $V(\alpha) \leq C\|\alpha - \bar{\rho}\|_2^2$.

We remark that by using the ‘straight path’ $\hat{\pi}(t) = \bar{\rho}(1 - t) + \alpha t$ one would get a bound in terms of the H_1 norm of $\alpha - \bar{\rho}$. Below, by choosing a more clever path, we get instead a bound only in term of the L_2 norm.

Proof. Let (e_k, λ_k) , $k \geq 1$ be the spectral basis for $-(1/2)\Delta^0$, where Δ^0 is the Dirichlet Laplacian on $[-1, 1]$, namely $\{e_k\}_{k \geq 1}$ is an complete orthonormal system in $L_2([-1, 1], du)$ and $-(1/2)\Delta^0 e_k = \lambda_k e_k$. Explicitly we have $e_k(u) = \cos(k\pi u/2)$ and $\lambda_k = k^2\pi^2/8$. We claim that the path $\hat{\pi}(t) = \hat{\pi}(t, u)$, $(t, u) \in [0, 1] \times [-1, 1]$ given by

$$\hat{\pi}(t) = \bar{\rho} + \sum_{k=1}^{\infty} \frac{e^{\lambda_k t} - 1}{e^{\lambda_k} - 1} \langle \alpha - \bar{\rho}, e_k \rangle e_k \tag{5.12}$$

fulfills the conditions stated in the lemma.

It is immediate to check that $\hat{\pi}(0) = \bar{\rho}$, $\hat{\pi}(1) = \alpha$ and $\hat{\pi}(t, \pm 1) = \rho_\pm$. Furthermore, by the smoothness assumption on α , we get that $\hat{\pi} \in C^{1,2}([0, 1] \times [-1, 1])$. In order to show that $\delta_0/2 \leq \hat{\pi} \leq 1 - \delta_0/2$, let us write $\hat{\pi}(t) = \bar{\rho} + q(-t)$, then $q(t) = q(t, u)$, $(t, u) \in [-1, 0] \times [-1, 1]$ solves

$$\begin{aligned} \partial_t q(t) &= \frac{1}{2} \Delta q(t) + g, \\ q(t, \pm 1) &= 0, \\ q(-1, u) &= \alpha(u) - \bar{\rho}(u), \end{aligned}$$

where $g = g(u)$ is given by

$$g = - \sum_{k=1}^{\infty} \frac{\lambda_k}{e^{\lambda_k} - 1} \langle \alpha - \bar{\rho}, e_k \rangle e_k.$$

Let us denote by $\|g\|_{H_1} := \|g'\|_2$ the H_1 norm in $[-1, 1]$; a straightforward computation shows

$$\begin{aligned} \|g\|_{H_1}^2 &= \sum_{k=1}^{\infty} 2\lambda_k \left(\frac{\lambda_k}{e^{\lambda_k} - 1} \right)^2 \langle \alpha - \bar{\rho}, e_k \rangle^2 \leq \frac{8}{\lambda_1} \sum_{k=1}^{\infty} \langle \alpha - \bar{\rho}, e_k \rangle^2 \\ &\leq \frac{8}{\lambda_1} \|\alpha - \bar{\rho}\|_2^2 \leq \left(\frac{8}{\pi} \right)^2 2 \left(\frac{\delta_0}{16} \right)^2 = \frac{1}{2\pi^2} \delta_0^2, \end{aligned}$$

where we used that, for $\lambda > 0$, we have $e^\lambda - 1 \geq \lambda^2/2$.

Let $P_t^0 = \exp\{t\Delta^0/2\}$ be the heat semigroup on $[-1, 1]$; since $\|g\|_\infty \leq \sqrt{2}\|g\|_{H_1}$, we have

$$\begin{aligned} & \sup_{t \in [-1, 0]} \|q(t)\|_\infty \\ &= \sup_{t \in [-1, 0]} \left\| P_{t+1}^0(\alpha - \bar{\rho}) + \int_{-1}^t ds P_{t-s}^0 g \right\|_\infty \leq \frac{\delta_0}{16} + \frac{1}{\pi} \delta_0 \leq \frac{7}{16} \delta_0, \end{aligned}$$

so that $\hat{\pi} \in D_{1, \delta_0/2}$.

We shall estimate $I_1(\hat{\pi}|\bar{\rho})$ by using the representation given in Lemma 3.6. To this end, let us define $h = h(t, u) \in C([0, 1] \times [-1, 1])$ by $h := -\partial_t \hat{\pi} + (1/2)\Delta \hat{\pi}$ and let $H = H(t, u)$ be the solution of

$$\begin{aligned} \nabla(\hat{\pi}[1 - \hat{\pi}]\nabla H) &= h, \\ H(t, \pm 1) &= 0, \end{aligned}$$

so that $\hat{\pi}$ solves (3.3) with H as above which belongs to $\mathcal{H}_1(\hat{\pi})$.

Let us denote by $\|\cdot\|_{H_{-1}}$ the usual negative Sobolev norm in $[-1, 1]$, namely

$$\|h\|_{H_{-1}}^2 := \sup_{f \neq 0, f(\pm 1) = 0} \frac{\langle f, h \rangle^2}{\langle \nabla f, \nabla f \rangle} = \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \langle h, e_k \rangle^2.$$

By using that $\hat{\pi}[1 - \hat{\pi}] \geq (\delta_0/2)^2$ a simple computation shows that

$$\int_0^1 dt \langle \hat{\pi}(t)[1 - \hat{\pi}(t)], (\nabla H(t))^2 \rangle \leq \frac{4}{\delta_0^2} \int_0^1 dt \|h(t)\|_{H_{-1}}^2.$$

By using the explicit expression for $\hat{\pi}$ we get

$$h(t) = - \sum_{k=1}^{\infty} \lambda_k \frac{2e^{\lambda_k t} - 1}{e^{\lambda_k} - 1} \langle \alpha - \bar{\rho}, e_k \rangle e_k,$$

hence, by a direct computation,

$$\begin{aligned} \|h(t)\|_{H_{-1}}^2 &= \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \left(\lambda_k \frac{2e^{\lambda_k t} - 1}{e^{\lambda_k} - 1} \right)^2 \langle \alpha - \bar{\rho}, e_k \rangle^2 \\ &\leq \sum_{k=1}^{\infty} 8\lambda_k e^{2\lambda_k(t-1)} \langle \alpha - \bar{\rho}, e_k \rangle^2, \end{aligned}$$

where we used that for $\lambda \geq \lambda_1$ we have $e^\lambda \geq 2$. We thus get

$$I_1(\hat{\pi}|\bar{\rho}) \leq \frac{2}{\delta_0^2} \int_0^1 dt \|h(t)\|_{H_{-1}}^2 \leq \frac{8}{\delta_0^2} \sum_{k=1}^{\infty} \langle \alpha - \bar{\rho}, e_k \rangle^2 = \frac{8}{\delta_0^2} \|\alpha - \bar{\rho}\|_2^2$$

which concludes the proof of the lemma. □

We can now prove the upper bound for the quasi potential and conclude the proof of Theorem 2.1.

LEMMA 5.8. For each $\rho \in \mathcal{M}$, we have $V(\rho) \leq S(\rho)$.

Proof. Fix $0 < \varepsilon < \delta_0/(32)$, $\rho \in \mathcal{M}$ and let $\rho^*(t, u)$ be the solution of (5.11) with initial condition ρ . By Lemma 5.6 there exists $T_1 = T_1(\varepsilon)$ such that $\|\rho^*(t) - \bar{\rho}\|_\infty < \varepsilon$ for any $t \geq T_1$. Let $\alpha := \rho^*(T_1)$ and let $\hat{\pi}$ be the path which connects $\bar{\rho}$ to α in the interval $[0, 1]$ constructed in Lemma 5.7.

Let $T := T_1 + 1$ and $\pi^*(t)$, $t \in [0, T]$ the path

$$\pi^*(t) = \begin{cases} \hat{\pi}(t), & \text{for } 0 \leq t \leq 1, \\ \rho^*(T - t), & \text{for } 1 \leq t \leq T. \end{cases} \tag{5.13}$$

By Remark 4.6, given $\rho \in \mathcal{M}$ as above, we can find a sequence $\{\rho_n, n \geq 1\}$ with $\rho_n \in \mathcal{M}_{\delta_n}$ for some $\delta_n > 0$ converging to ρ in \mathcal{M} and such that $S(\rho_n)$ converges to $S(\rho)$. Let us denote by $\rho^{n,*}$ the solution of (5.11) with initial condition ρ_n and set

$$\pi^{n,*}(t) = \begin{cases} \hat{\pi}^{n,*}(t), & \text{for } 0 \leq t \leq 1, \\ \rho^{n,*}(T - t), & \text{for } 1 \leq t \leq T, \end{cases} \tag{5.14}$$

where $\hat{\pi}^{n,*}(t)$ is the path joining $\bar{\rho}$ to $\alpha_n := \rho^{n,*}(T_1)$ in the time interval $[0, 1]$ constructed in Lemma 5.7. We claim that the path $\pi^{n,*}$ defined above converges in $D([0, T], \mathcal{M})$ to π^* , as defined in (5.13). Before proving this claim, we conclude the proof of the lemma.

By the lower semi continuity of the functional $I_T(\cdot|\bar{\rho})$ on $D([0, T], \mathcal{M})$ we have

$$I_T(\pi^*|\bar{\rho}) \leq \liminf_n I_T(\pi^{n,*}|\bar{\rho}). \tag{5.15}$$

On the other hand, by definition of the rate function and its invariance with respect to time shifts we get

$$I_T(\pi^{n,*}|\bar{\rho}) = I_1(\hat{\pi}^{n,*}|\bar{\rho}) + I_{T_1}(\rho^{n,*}(T_1 - \cdot)|\rho^{n,*}(T_1)). \tag{5.16}$$

By Theorem 4.2, $F_n := F(\rho_n)$ converges to $F = F(\rho)$ in $C^1([-1, 1])$ so that $\Phi_n(t)$, the solution of (2.9) with initial condition F_n , converges to $\Phi(t)$ in $C^2([-1, 1])$ for any $t > 0$. Hence, by (2.13), $\rho^{n,*}(T_1)$ converges to $\rho^*(T_1)$ in $C([-1, 1])$. Recalling that $\|\rho^*(T_1) - \bar{\rho}\|_\infty < \varepsilon \leq \delta_0/(32)$, we can find $N_0 = N_0(\delta_0)$ such that for any $n \geq N_0$ we have $\|\rho^{n,*}(T_1) - \bar{\rho}\|_\infty < \varepsilon \leq \delta_0/(16)$. We can thus apply Lemma 5.7 and get, for $n \geq N_0$

$$I_1(\hat{\pi}^{n,*}|\bar{\rho}) \leq C\|\rho^{n,*}(T_1) - \bar{\rho}\|_2^2 \tag{5.17}$$

for some constant $C = C(\delta_0)$.

By Lemma 5.5, $\rho^{n,*}(T_1 - t)$, $t \in [0, T_1]$ is smooth and bounded away from 0 and 1, namely it belongs to D_{T_1, δ_n} for some $\delta_n > 0$. We can thus apply Lemma 5.4 and conclude, as $\rho^{n,*}(T_1 - t)$ solves (5.9) with $K = 0$,

$$I_{T_1}(\rho^{n,*}(T_1 - \cdot)|\rho^{n,*}(T_1)) = S(\rho_n) - S(\rho^{n,*}(T_1)) \leq S(\rho_n). \tag{5.18}$$

From Equations (5.15)–(5.18), we now get

$$\begin{aligned} I_T(\pi^*|\bar{\rho}) &\leq \liminf_n [S(\rho_n) + C\|\rho^{n,*}(T_1) - \bar{\rho}\|_2^2] \\ &= S(\rho) + C\|\rho^*(T_1) - \bar{\rho}\|_2^2 \leq S(\rho) + 2C\varepsilon^2 \end{aligned}$$

and we are done by the arbitrariness of ε .

We are left to prove that $\pi^{n,*} \rightarrow \pi^*$ in $D([0, T], \mathcal{M})$. We show that $\pi^{n,*}$ converges to π^* in $C([0, T]; \mathcal{M})$. Pick $\varepsilon_1 \in (0, T_1]$; since $\rho^{n,*}(t)$ converges to $\rho^*(t)$ in $C([-1, 1])$ uniformly for $t \in [\varepsilon_1, T_1]$ we conclude easily that $\pi^{n,*}$ converges to π^* in $C([1, T - \varepsilon_1] \times [-1, 1])$. We recall that, by Lemma 4.3, $\nabla F_n(t)$ and $\nabla F(t)$ are uniformly bounded. Moreover, $\pi^{n,*}(T - t)$ and $\pi^*(T - t)$, $t \in [T - T_1, T]$ are weak solutions of (5.11); for each $G \in C([-1, 1])$ we thus get

$$\lim_{\varepsilon_1 \downarrow 0} \limsup_n \sup_{t \in [T - \varepsilon_1, T]} |\langle \pi^{n,*}(t), G \rangle - \langle \pi^*(t), G \rangle| = 0.$$

This concludes the proof that $\rho^{n,*}$ converges to ρ^* in $C([1, T]; \mathcal{M})$. Since $\rho^{*,n}(T_1)$ converges to $\rho^{*,n}(T_1)$ in $C^2([-1, 1])$ it is easy to show that $\hat{\pi}^{n,*}$ converges to $\hat{\pi}^*$ in $C([0, 1] \times [-1, 1])$. Hence $\pi^{n,*}$ converges to π^* in $C([0, T]; \mathcal{M})$. \square

Appendix: A Lower Bound on the Quasi-Potential ($d \geq 1$)

In this appendix we prove a lower bound for the quasi potential in the d -dimensional boundary driven simple exclusion process. For $d = 1$ this bound has been derived from (2.11) in [8, 9].

Let $\Lambda \subset \mathbb{R}^d$ be a smooth bounded open set and define $\Lambda_N := \mathbb{Z}^d \cap N\Lambda$. Let also $\gamma(u)$ be a smooth function defined in a neighborhood of $\partial\Lambda$. The d -dimensional boundary driven symmetric exclusion process is then the process on the state space $\Sigma_N := \{0, 1\}^{\Lambda_N}$ with generator

$$\begin{aligned} L_N f(\eta) &= \frac{N^2}{2} \sum_{\substack{\{x,y\} \subset \Lambda_N \\ |x-y|=1}} [f(\sigma^{x,y}\eta) - f(\eta)] + \\ &\quad + \frac{N^2}{2} \sum_{\substack{x \in \Lambda_N, y \notin \Lambda_N \\ |x-y|=1}} \left(\eta(x) + [1 - \eta(x)]\gamma\left(\frac{y}{N}\right) \right) [f(\sigma^x\eta) - f(\eta)], \end{aligned}$$

where $\sigma^{x,y}$ and σ^x have been defined in Section 2.

The hydrodynamic equation is given by the heat equation in Λ , namely

$$\begin{aligned} \partial_t \rho &= \frac{1}{2} \Delta \rho, \quad u \in \Lambda, \\ \rho(t, u) &= \alpha(u), \quad u \in \partial\Lambda, \\ \rho(0, u) &= \rho_0(u), \end{aligned}$$

where $\alpha(u) = \gamma(u)/[1 + \gamma(u)]$. We shall denote by $\bar{\rho} = \bar{\rho}(u)$, $u \in \Lambda$ the unique stationary solution of the hydrodynamic equation.

By the same arguments as the ones given in Section 3 it is possible to prove the dynamical large deviation principle for the empirical measure. The rate function is still given by the variational formula (2.5), but we now have

$$\begin{aligned}
 J_{T,H,\rho}(\pi) &:= \langle \pi(T), H(T) \rangle - \langle \rho, H(0) \rangle - \\
 &\quad - \int_0^T dt \langle \pi(t), \partial_t H(t) + \frac{1}{2} \Delta H(t) \rangle - \\
 &\quad - \frac{1}{2} \int_0^T dt \langle \chi(\pi(t)), (\nabla H(t))^2 \rangle + \\
 &\quad + \frac{1}{2} \int_0^T dt \int_{\partial\Lambda} d\sigma(u) \alpha(u) \partial_{\hat{n}} H(t, u),
 \end{aligned}$$

where $\partial_{\hat{n}} H(t, u)$ is the normal derivative of $H(t, u)$ (\hat{n} being the outward normal to Λ) and $\sigma(u)$ is the surface measure on $\partial\Lambda$.

Let us define the quasi potential $V(\rho)$ as in (2.6) and set

$$S_0(\rho) := \int_{\Lambda} du \left[\rho(u) \log \frac{\rho(u)}{\bar{\rho}(u)} + [1 - \rho(u)] \log \frac{1 - \rho(u)}{1 - \bar{\rho}(u)} \right].$$

THEOREM A.1. *For each $\rho \in \mathcal{M}$ we have $V(\rho) \geq S_0(\rho)$.*

Proof. We shall prove that $I_T(\pi | \bar{\rho}) \geq S_0(\rho)$ for any $\pi(\cdot)$ such that $\pi(0) = \bar{\rho}$ and $\pi(T) = \rho$. Let us assume first that $\pi \in C^{1,2}([0, T] \times \Lambda)$, $\pi(t, u) = \alpha(u)$ for $(t, u) \in [0, T] \times \partial\Lambda$, and π is bounded away from 0 and 1. Given such π we use the variational characterization of I_T and chose

$$H(t, u) = \log \frac{\pi(t, u)}{1 - \pi(t, u)} - \log \frac{\bar{\rho}(u)}{1 - \bar{\rho}(u)}.$$

Note that $H(t, u) = 0$ for $(t, u) \in [0, T] \times \partial\Lambda$ since π and $\bar{\rho}$ satisfy the same boundary condition. By dominated convergence and an explicit computation we get

$$S_0(\pi(T)) - S_0(\pi(0)) = \int_0^T dt \frac{d}{dt} S_0(\pi(t)) = \int_0^T dt \langle \partial_t \pi(t), H(t) \rangle.$$

Recalling that $J_{T,H,\bar{\rho}}(\pi)$ has been defined above, a simple computation shows

$$\begin{aligned}
 &J_{T,H,\bar{\rho}}(\pi) \\
 &= S_0(\pi(T)) + \frac{1}{2} \int_0^T dt \langle \nabla H(t), \nabla \pi(t) - \pi(t)[1 - \pi(t)] \nabla H(t) \rangle \\
 &= S_0(\pi(T)) + \frac{1}{2} \int_0^T dt \left\langle \frac{(\nabla_u \bar{\rho})^2}{[\bar{\rho}(1 - \bar{\rho})]^2} (\pi(t) - \bar{\rho})^2 \right\rangle,
 \end{aligned}$$

since the second term above is positive we conclude the proof of the lemma for smooth paths. To get the general result it is enough to repeat the approximation used in Lemma 5.3. \square

Acknowledgements

We are grateful to G. Dell'Antonio for a useful discussion on the variational problem which led to Lemma 4.4. We also thank T. Bodineau and G. Giacomin for stimulating discussions.

References

1. Benois, O.: Large deviations for the occupation times of independent particle systems, *Ann. Appl. Probab.* **6** (1996), 269–296.
2. Benois, O., Kipnis, C. and Landim, C.: Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes, *Stochastic Process. Appl.* **55** (1995), 65–89.
3. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G. and Landim, C.: Fluctuations in stationary nonequilibrium states of irreversible processes, *Phys. Rev. Lett.* **87** (2001), 040601.
4. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G. and Landim, C.: Macroscopic fluctuation theory for stationary nonequilibrium state, *J. Statist. Phys.* **107** (2002), 635–675.
5. Bodineau, T. and Giacomin, G.: From dynamic to static large deviations in boundary driven exclusion particles systems, Preprint, 2002.
6. Comets, F.: Grandes déviations pour des champs de Gibbs sur Z^d , *C.R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 511–513.
7. De Masi, A., Ferrari, P., Ianiro, N. and Presutti, E.: Small deviations from local equilibrium for a process which exhibits hydrodynamical behavior. II, *J. Statist. Phys.* **29** (1982), 81–93.
8. Derrida, B., Lebowitz, J. L. and Speer, E. R.: Free energy functional for nonequilibrium systems: an exactly solvable model, *Phys. Rev. Lett.* **87** (2001), 150601.
9. Derrida, B., Lebowitz, J. L. and Speer, E. R.: Large deviation of the density profile in the steady state of the open symmetric simple exclusion process, *J. Statist. Phys.* **107** (2002), 599–634.
10. Donsker, M. D. and Varadhan, S. R. S.: Large deviations from a hydrodynamic scaling limit, *Comm. Pure Appl. Math.* **42** (1989), 243–270.
11. Ekeland, I. and Temam, R.: *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
12. Eyink, G., Lebowitz, J. L. and Spohn, H.: Hydrodynamics of stationary nonequilibrium states for some lattice gas models, *Comm. Math. Phys.* **132** (1990), 253–283.
13. Eyink, G., Lebowitz, J. L. and Spohn, H.: Lattice gas models in contact with stochastic reservoirs: local equilibrium and relaxation to the steady state, *Comm. Math. Phys.* **140** (1991), 119–131.
14. Freidlin, M. I. and Wentzell, A. D.: *Random Perturbations of Dynamical Systems*, Springer, New York, 1998.
15. Jona-Lasinio, G., Landim, C. and Vares, M. E.: Large deviations for a reaction diffusion model, *Probab. Theory Related Fields* **97** (1993), 339–361.
16. Kipnis, C. and Landim, C.: *Scaling Limits of Interacting Particle Systems*, Grundlehren Math. Wiss. 320, Springer-Verlag, Berlin, 1999.
17. Kipnis, C., Olla, S. and Varadhan, S. R. S.: Hydrodynamics and large deviations for simple exclusion processes, *Comm. Pure Appl. Math.* **42** (1989), 115–137.

18. Landim, C.: Occupation time large deviations of the symmetric simple exclusion process, *Ann. Probab.* **20** (1992), 206–231.
19. Landim, C., Olla, S. and Volchan, S.: Driven tracer particle in one dimensional symmetric simple exclusion process, *Comm. Math. Phys.* **192** (1998), 287–307.
20. Lanford, O. E.: *Entropy and Equilibrium States is Classical Statistical Mechanics*, In: Lecture Notes in Phys. 20, Springer, New York, 1973.
21. Mikhaïlov, V. P.: *Partial Differential Equations*, 2nd edn, Nauka, Moscow, 1983.
22. Olla, S.: Large deviations for Gibbs random fields, *Probab. Theory Related Fields* **77** (1988), 343–357.
23. Onsager, L. and Machlup, S.: Fluctuations and irreversible processes, *Phys. Rev.* **91** (1953), 1505; *Phys. Rev.* **91** (1953), 1512.
24. Spohn, H.: Long range correlations for stochastic lattice gases in a nonequilibrium steady state, *J. Phys. A* **16** (1983), 4275–4291.
25. Varadhan, S. R. S.: *Large Deviations and Applications*, CBMS-NSF Regional Conf. Ser. Appl. Math. 46, SIAM, Philadelphia, 1984.