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On the long time behavior of the stochastic heat equation

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Abstract. We consider the stochastic heat equation in one space dimension and compute – for a particular choice of the initial datum – the exact long time asymptotic. In the Carmona-Molchanov approach to intermittence in non stationary random media this corresponds to the identification of the *sample Lyapunov exponent*. Equivalently, by interpreting the solution as the partition function of a directed polymer in a random environment, we obtain a weak law of large numbers for the quenched free energy. The result agrees with the one obtained in the physical literature via the replica method. The proof is based on a representation of the solution in terms of the weakly asymmetric exclusion process.

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1. Introduction and main result

We consider the following Ito stochastic partial differential equation (SPDE)

$$d\theta_t = \frac{1}{2}\nu\Delta\theta_t\,dt + \sqrt{\lambda}\theta_t\,dW_t \tag{1.1}$$

Key words and phrases: Stochastic heat equation – Sample Lyapunov exponent – Directed polymers in random environment – Simple exclusion processes

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where $\theta_t = \theta_t(r)$, $r \in \mathbf{R}$, $t \in \mathbf{R}^+$ is a scalar field, Δ is the Laplacian, W_t is a cylindrical Wiener process on $L_2(\mathbf{R})$ and $\nu, \lambda > 0$ are physical parameters. Equation (1.1) is often referred to as the stochastic heat equation (SHE). In this paper we compute – for a particular choice of the initial condition – the asymptotic behaviour of the solution and identify the precise constant, i.e. $\lim_{T\to\infty}(\log\theta_T)/T = -\lambda^2/(\nu 4!)$. Before stating more precisely our result we discuss briefly two different physical interpretations of equation (1.1).

When the space variable takes values in $\mathbf{Z}^d$, the analog of equation (1.1) has been proposed and analyzed in detail as a linearized model for a diffusion in non stationary random media [3, 10]. In particular the *p-moment Lyapunov exponents* have been studied and *intermittence* of the random field θ_t (see [10] for a definition) has been established in the case of $\theta_0 = 1$. For the continuum (and one dimensional) model we are dealing with, an exact calculation of the p-moment Lyapunov exponents is possible when p is an integer and one finds $\gamma_p = \lambda^2 p(p^2-1)/(\nu\, 4!)$, see [6, 8] for a physical derivation and [1] for a rigorous version.

The pathwise behavior of the solution is described through the so-called *sample Lyapunov exponent* [3, 10] $\gamma := \lim_{T\to\infty} T^{-1}\log\theta_T(r)$. For the SHE, sharp upper and lower bounds on the dependence of γ on ν are obtained in [4] in the d dimensional case ($r \in \mathbf{R}^d$) where, however, the cylindrical Wiener process is replaced by a noise which is regular in space and the stochastic differential is interpreted in the Stratonovich sense. Our result gives the exact value of the sample Lyapunov exponent for (1.1), i.e. $\gamma = -\lambda^2/(\nu\, 4!)$. This is very different from the case studied in [4] where the dependence on ν is instead given by $\gamma \asymp [\log\nu^{-1}]^{-1}$. We remark that we are forced to understand the SPDE (1.1) in the Ito sense because the difference between Ito and Stratonovich, in this case, is given only by an extra constant in the drift, which is however infinite for the cylindrical Wiener process. We note that the dependence on ν and λ, by rescaling the time, can be always reduced to only one parameter, even in the case of regular noise or on $\mathbf{Z}^d$. However in equation (1.1), thanks to the scaling properties of the cylindrical Wiener process, both ν and λ can be factorized: in particular $\gamma_p(\lambda, \nu) = \lambda^2\nu^{-1}\gamma_p(1, 1)$. We shall therefore set from now on $\lambda = \nu = 1$.

As a consequence of our result and the growth of the p-moment Lyapunov exponents the qualitative picture of the random field $\theta_t(r)$ agrees with the one described in [10] for the discrete case. The rapid growth of the moments indicates that most of the mass is concentrated in sparse peaks; since γ is negative we also have that between different peaks the field is exponentially small.

The second physical interpretation arises in the statistical mechanics of disordered systems. Equation (1.1) is in fact satisfied by the partition function of a system of directed polymers in a quenched random potential which is a two parameter white noise. This representation is obtained by writing a (formal) Feynman-Kac formula for θ_t, see [1, 6, 8].

The quenched free energy is then defined as $\log \theta_T(r)$ and one wants to show that in the thermodynamic limit $T \to \infty$ one has $T^{-1} \log \theta_T(r) \to \gamma$ for some (non random) γ. In the physical literature, see e.g. [6, 8], the above problem is solved via the replica method, i.e. one computes the expected value of γ as follows

$$\begin{aligned}\mathbf{E}(\gamma) &= \lim_{T\to\infty} \frac{\mathbf{E}(\log \theta_T(r))}{T} = \lim_{T\to\infty} \frac{1}{T} \lim_{p\to 0} \mathbf{E}\left(\frac{\theta_t(r)^p - 1}{p}\right) \\ &= \lim_{T\to\infty} \lim_{p\to 0} \frac{1}{T} \left[\frac{e^{(4!)^{-1} p(p^2-1)T} - 1}{p}\right] = -\frac{1}{4!} \ , \end{aligned} \tag{1.2}$$

where the dependence of the p-moment on p in a neighborhood of 0 is *inferred* from $p \in \mathbf{Z}^+$; compare with [6, Eq. (2.10)] and [8, §5.2.2]. It seems beyond any hope to justify the above formal steps and it is – in a sense – rather surprising that they do give the correct answer. We also remark that, at a physical level, the result in (1.2) is expected to hold essentially for every initial condition. Note finally that the annealed free energy vanishes, i.e. $T^{-1} \log [\mathbf{E}(\theta_T(r))] \to 0$.

There is a vast class of models in random environment whose long scale fluctuation behavior is expected to be described by the Kardar-Parisi-Zhang (KPZ) equation [7], a formal SPDE driven by space-time white noise. Continuum and discrete directed polymers belong to this universality class, as well as various driven systems and interface models. In [2] we have studied the fluctuations in a model of interface growth and we have shown that they have an anomalous (i.e. non CLT) behavior: they are described by the logarithm of the solution of the SHE, which we have therefore claimed to be a possible (and mathematically well defined) interpretation of the KPZ equation. The fact that the result on the SHE presented here coincides with the outcome of the replica method gives further support to our claim. The interface model studied in [2], which in turn is equivalent to an *asymmetric exclusion* particle system, and its relation to the SHE will also play a crucial role in the proof presented here.

We formulate the Cauchy problem for the SPDE (1.1) in the mild form

$$\theta_t = G_t * \theta_0 + \int_0^t G_{t-s} * \theta_s \, dW_s \tag{1.3}$$

where $G_t = G_t(r)$ is the heat kernel, $*$ denotes convolution in the space variable and θ_0 is the initial datum. In this paper we shall consider only the case $\theta_0(r) \overset{\text{Law}}{=} \exp\{B(r)\}$, where $B(r)$ is a bilateral brownian motion on $\mathbf{R}$, i.e. $B(r)$, $r \geq 0$ and $B(-r)$, $r \leq 0$ are independent brownians starting from the origin. For the formulation (1.3) is not difficult to prove existence and uniqueness of an adapted solution with a.s. continuous (in space and time) trajectories, see e.g. [1, 11] and references therein. Our main result is the weak (both probabilistically and in the space coordinate) law of large number for $(\log \theta_T)/T$. By $(\cdot, \cdot)$ we denote below the inner product in $L_2(\mathbf{R})$.

Theorem 1.1. *Let $\theta_t = \theta_t(r)$ be the solution of (1.3). For each $\varphi \in C_0^\infty(\mathbf{R})$*

$$\lim_{T\to\infty} \frac{(\log \theta_T, \varphi)}{T} = -\frac{1}{4!}(1,\varphi) \quad in \;\; L_2 \quad . \tag{1.4}$$

Remark. We shall actually prove the bound

$$\mathbf{E}\left((\log \theta_T, \varphi) + \frac{1}{4!}(1,\varphi)T \right)^2 \leq c_0 T^{\frac{3}{2}} \tag{1.5}$$

for some constant $c_0 = c_0(\varphi)$. The above estimate is however far from optimal; in fact the quantity on the left hand side of (1.5) is expected [6, 7, 8] to have the (highly nontrivial) scaling behavior $T^{2/3}$.

2. A microscopic representation

In this Section we recall the *weakly asymmetric simple exclusion process* (WASEP) and a microscopic analog of the Cole-Hopf transformation, first introduced in [5], which has been employed in [2] to prove a scaling limit to the SHE. The constant $1/4!$ will arise from this transformation.

For notation convenience we describe the particle model in terms of spin variables. The state space of the microscopic process is $\Omega := \{-1, 1\}^{\mathbf{Z}}$, its elements (*spin configurations*) are denoted by $\sigma = \{\sigma(x),\ x \in \mathbf{Z}\}$, where $\sigma(x) = +1$ (resp. -1) is interpreted as the site x being occupied (resp. empty).

The *weakly asymmetric simple exclusion process* (WASEP) is the process generated by

$$L_\varepsilon := \frac{1}{2}L^+ + \left(\frac{1}{2} + \sqrt{\varepsilon}\right) L^- \tag{2.1}$$

where $L^\pm$ are the generators of the totally asymmetric exclusion processes, defined by

$$L^\pm f(\sigma) := \sum_x \frac{1+\sigma(x)}{2} \frac{1-\sigma(x\pm 1)}{2} \left[f(\sigma^{x,x\pm 1}) - f(\sigma) \right] \tag{2.2}$$

in which f is a cylindrical function over Ω and given $x, y \in \mathbf{Z}$

$$\sigma^{x,y}(z) := \begin{cases} \sigma(x) & \text{if } z = y \\ \sigma(y) & \text{if } z = x \\ \sigma(z) & \text{otherwise} \end{cases} . \tag{2.3}$$

We stress in (2.1) we adopted the (unusual) convention of an asymmetry $\sqrt{\varepsilon}$ toward the left. The details on the construction of the process can be found in Liggett [9]. We shall consider the WASEP σ_t, $t \geq 0$ starting from the symmetric Bernoulli measure μ, i.e. the product measure on Ω with marginals $\mu\{\sigma(x)\} = 0$, which is an invariant measure for WASEP. We realize WASEP canonically on the Skorohod space $D(\mathbf{R}^+; \Omega)$ and denote by $\mathbf{P}^\varepsilon_\mu$ its law; the expectation with respect to $\mathbf{P}^\varepsilon_\mu$ is denoted by $\mathbf{E}^\varepsilon_\mu$.

Now we put a tag on the particle closest to the origin (on the positive semiaxis) at time zero and we follow its evolution under the WASEP process: its position will be denoted by x^0_t($x^0_0 := \min\{x \in \mathbf{Z} : x \geq 0,\ \sigma(x) = +1\}$). We define (see [5])

$$\zeta_t(x) := \begin{cases} \sum_{x^0_t < y \leq x} \sigma_t(y) - x^0_t & \text{if } x > x^0_t \\ -\sum_{x < y \leq x^0_t} \sigma_t(y) - x^0_t & \text{if } x < x^0_t \\ -x^0_t & \text{if } x = x^0_t \end{cases} \tag{2.4}$$

and

$$\xi_t(r) := \exp\{-\gamma_\varepsilon \zeta_t(r) + \lambda_\varepsilon t\} \tag{2.5}$$

where

$$\gamma_\varepsilon := \frac{1}{2}\log(1 + 2\varepsilon^{\frac{1}{2}}), \qquad \lambda_\varepsilon := 1 + \varepsilon^{\frac{1}{2}} - \sqrt{1 + 2\varepsilon^{\frac{1}{2}}} \tag{2.6}$$

and $\zeta_t(r)$ is defined by linear interpolation on the values $\zeta_t(x)$ in (2.4).

We next recall the scaling limit to the SHE, [2, Thm. 3.3]. We introduce the rescaled version of the process ξ by $\xi^\varepsilon_t(r) := \xi_{\varepsilon^{-2}t}(\varepsilon^{-1}r)$ for $t \in [0, T]$ and $r \in \mathbf{R}$. Therefore $\xi^\varepsilon \in D([0, T]; C(\mathbf{R}))$, where $C(\mathbf{R})$ is the space of continuous functions on the real line equipped with the topology of uniform convergence over compact sets. Below $\Longrightarrow$ denotes weak convergence in in $D\,([0, T]; C(\mathbf{R}))$.

Theorem 2.1. (2, 1.[Thm. 3.3]) *Let θ be the solution of* (1.3) *and ξ^ε as above. We then have*

$$\xi^\varepsilon \overset{\varepsilon\to 0}{\Longrightarrow} \theta \ . \tag{2.7}$$

Let us introduce $Z_t(x) := \zeta_t(x) - t\sqrt{\varepsilon}/2$ and the scaled process $Z^\varepsilon_t(r) := \sqrt{\varepsilon}\, Z_{\varepsilon^{-2}t}(\varepsilon^{-1}r)$. We also introduce the notation $(Z, \varphi)_\varepsilon := \varepsilon \sum_{x\in\mathbf{Z}} \varphi(\varepsilon x) Z(x)$.

Proposition 2.2. *For each $\varphi \in C_0^\infty(\mathbf{R})$ there is a constant $c_0 = c_0(\varphi)$ such that*

$$\varlimsup_{\varepsilon\to 0} \mathbf{E}_\mu^\varepsilon \left(\frac{1}{T} \left(Z_T^\varepsilon, \varphi \right)_\varepsilon \right)^2 \le c_0 \frac{1}{\sqrt{T}} \tag{2.8}$$

for any $T \geq 0$.

We will prove this Proposition in the next Section: here we show how, together with Theorem 2.1, it implies our main result.

Proof of Theorem 1.1. Recalling (2.5) and the definition of $Z_t^\varepsilon(r)$, we have

$$-\log \xi_T^\varepsilon(r) = \frac{\gamma_\varepsilon}{\sqrt{\varepsilon}} \left[Z_T^\varepsilon(r) + \frac{1}{4!} T + \sqrt{\varepsilon} \left(\upsilon_\varepsilon - \varepsilon^{-2} \frac{\lambda_\varepsilon}{\gamma_\varepsilon} \right) T \right] \tag{2.9}$$

where we have introduced $\upsilon_\varepsilon := \varepsilon^{-3/2}/2 - \varepsilon^{-1/2}/4!$.

Since $\lim_{\varepsilon\to 0} \gamma_\varepsilon/\sqrt{\varepsilon} = 1$ and $\lim_{\varepsilon\to 0} \sqrt{\varepsilon}\left(\upsilon_\varepsilon - \varepsilon^{-2}\lambda_\varepsilon/\gamma_\varepsilon\right) = 0$, by applying Proposition 2.2 we get that for each $\varphi \in C_0^\infty(\mathbf{R})$

$$\varlimsup_{\varepsilon\to 0} \mathbf{E}_\mu^\varepsilon \left(\frac{\left(\log \xi_T^\varepsilon, \varphi\right)_\varepsilon}{T} + \frac{(1,\varphi)}{4!} \right)^2 \le c_0 \frac{1}{\sqrt{T}} \quad . \tag{2.10}$$

On the other hand, by Theorem 2.1, the fact that $\theta_T(r) > 0$ a.s., see [11], and that $\theta \in C([0,T]; C(\mathbf{R}))$, we get

$$\left(\log \xi_T^\varepsilon, \varphi\right)_\varepsilon \overset{\varepsilon\to 0}{\Longrightarrow} (\log \theta_T, \varphi) \quad . \tag{2.11}$$

Take now a sequence of smooth functions $\{f_L\}_{L\in\mathbf{Z}^+}$, f_L: $\mathbf{R} \mapsto [0,1]$ such that $f_L(r) = 1$ if $|r| \le L$ and $f_L(r) = 0$ if $|r| \ge L+1$ and set $g_L(r) = r^2 f_L(r)$ for all $r \in \mathbf{R}$. Observe that

$$\begin{aligned} &\varlimsup_{\varepsilon\to 0} \mathbf{E}_\mu^\varepsilon \left(\frac{\left(\log \xi_T^\varepsilon, \varphi\right)_\varepsilon}{T} + \frac{(1,\varphi)}{4!} \right)^2 \\ &\quad \ge \varlimsup_{L\to\infty} \varlimsup_{\varepsilon\to 0} \mathbf{E}_\mu^\varepsilon \left[g_L \left(\frac{\left(\log \xi_T^\varepsilon, \varphi\right)_\varepsilon}{T} + \frac{(1,\varphi)}{4!} \right) \right] \\ &\quad = \varlimsup_{L\to\infty} \mathbf{E} \left[g_L \left(\frac{(\log \theta_T, \varphi)}{T} + \frac{(1,\varphi)}{4!} \right) \right] \\ &\quad = \mathbf{E} \left(\frac{(\log \theta_T, \varphi)}{T} + \frac{(1,\varphi)}{4!} \right)^2 \end{aligned} \tag{2.12}$$

in which we used first (2.11) and then the monotone convergence theorem. The bound (1.5) follows then from (2.10) and (2.12). □

3. Proof of Proposition 2.2

We start with a preliminary lemma, which provides a control of the variance of additive functionals of the WASEP by means of the symmetrized process SEP, i.e. the process with generator $L = (1+\sqrt{\varepsilon})(L^+ + L^-)/2$. For $\lambda > 0$, we will denote by $R^\varepsilon_\lambda := (\lambda - L_\varepsilon)^{-1}$ (resp. R_λ) the resolvent of L_ε (resp. L). This is a general comparison argument, communicated to us by H.T. Yau, between an asymmetric process and its symmetrized version.

Lemma 3.1. *There exists $C > 0$ such that*

$$\mathbf{E}^\varepsilon_\mu\left[\left(\int_0^T dt\ f(\sigma_t)\right)^2\right] \le CT\left(f, R_{1/T} f\right)_{L^2(\mu)} \quad , \tag{3.1}$$

for every $T \ge 1$, every $\varepsilon \in (0,1]$ and every local function f.

Proof. Consider the solution u_λ of the resolvent equation $u_\lambda = R^\varepsilon_\lambda f$, with $\lambda = 1/T$. Set

$$M(T) = u_\lambda(\sigma_T) - u_\lambda(\sigma_0) - \int_0^T dt\ L_\varepsilon u_\lambda(\sigma_t) \quad , \tag{3.2}$$

and it is immediate to see that $\mathbf{E}^\varepsilon_\mu[M(T)]^2 = 2T\|u_\lambda\|^2_{H_1}$, where $\|u_\lambda\|^2_{H_1} = -(u_\lambda, Lu_\lambda)_{L^2(\mu)}$. By using the fact that $L_\varepsilon u_\lambda = \lambda u_\lambda - f$, formula (3.2), stationariety of the process and the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}\mathbf{E}^\varepsilon_\mu\left[\left(\int_0^T dt\ f(\sigma_t)\right)^2\right] &\le 4\Bigg[2\,(u_\lambda, u_\lambda)_{L^2(\mu)} + \mathbf{E}^\varepsilon_\mu\left[\left(\int_0^T dt\ \lambda u_\lambda(\sigma_t)\right)^2\right] \\ &\qquad + \mathbf{E}^\varepsilon_\mu\left[(M(T))^2\right]\Bigg] \\ &\le 4\left[(2+\lambda^2 T)\,(u_\lambda, u_\lambda)_{L^2(\mu)} + 2T\|u_\lambda\|^2_{H_1}\right] \\ &\le 12T\left[\lambda\,(u_\lambda, u_\lambda)_{L^2(\mu)} + \|u_\lambda\|^2_{H_1}\right] \\ &= 12T\left(f, R^\varepsilon_\lambda f\right)_{L^2(\mu)} \quad . \end{aligned} \tag{3.3}$$

The result follows then from the fact that $\left(f, R^\varepsilon_\lambda f\right)_{L^2(\mu)} \le (f, R_\lambda f)_{L^2(\mu)}$, which can be proven by using the variational formula

$$(f, R_\lambda f)_{L^2(\mu)} = \sup_{h\in\mathscr{D}} \frac{(f,h)^2_{L^2(\mu)}}{(h, (\lambda - L)h)_{L^2(\mu)}} \quad . \tag{3.4}$$

where $\mathscr{D}$ is the set of local functions from Ω to $\mathbf{R}$, a common core for L and L_ε. Since

$$\sup_{h\in\mathscr{D}} \frac{(f,h)^2_{L^2(\mu)}}{(h,(\lambda-L)h)_{L^2(\mu)}} = \sup_{h\in\mathscr{D}} \frac{(f,h)^2_{L^2(\mu)}}{(h,(\lambda-L_\varepsilon)h)_{L^2(\mu)}} \quad , \tag{3.5}$$

the bound we claim follows by choosing $h = R^\varepsilon_\lambda f$ in the above formula. In general this h need not to be in $\mathscr{D}$ but, since $f \in \mathscr{D}$, we have that $R^\varepsilon_\lambda f$ is in the domain of L_ε, so we can anyway approximate it (in the graph norm) with a sequence $h_n \in \mathscr{D}$ and we are done. □

Proof of Proposition 2.2. A straightforward computation (see [2, 5]) shows that Z_t satisfies the semimartingale equation

$$\begin{aligned} dZ_t(x) \; &= \frac{1}{2}\left\{(1+\sqrt{\varepsilon})\,[\sigma_t(x+1)-\sigma_t(x)] - \sqrt{\varepsilon}\sigma_t(x)\sigma_t(x+1)\right\} dt \\ &\quad + dM_t(x) \end{aligned} \tag{3.6}$$

where the martingales $M_t(x)$, $x \in \mathbf{Z}$, have brackets

$$\begin{aligned} d\langle M(x), M(y)\rangle_t = dt\; \delta(x-y) \\ \times\Big\{1-\sigma_t(x)\sigma_t(x+1) + \sqrt{\varepsilon}\big[1+\sigma_t(x+1) \\ -\sigma_t(x) - \sigma_t(x)\sigma_t(x+1)\big]\Big\} \end{aligned} \tag{3.7}$$

where $\delta(x)$ is the Kronecker symbol.

By integrating (3.6) we get

$$\begin{aligned} \left(Z^\varepsilon_T,\varphi\right)_\varepsilon \; &= \left(Z^\varepsilon_0,\varphi\right)_\varepsilon + \varepsilon^{\frac{3}{2}}\int_0^{\varepsilon^{-2}T} dt\; f_1(\sigma_t) + \varepsilon\int_0^{\varepsilon^{-2}T} dt\; f_2(\sigma_t) \\ &\quad + \varepsilon^{\frac{1}{2}}\left(M^\varepsilon_T,\varphi\right)_\varepsilon + R^\varepsilon_T \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} f_1(\sigma) &:= -\frac{1}{2}(1+\sqrt{\varepsilon})\varepsilon\sum_{x\in\mathbf{Z}} (\nabla\varphi)\,(\varepsilon x)\,\sigma(x) \\ f_2(\sigma) &:= -\frac{1}{2}\varepsilon\sum_{x\in\mathbf{Z}} \varphi(\varepsilon x)\,\sigma(x)\sigma(x+1) \\ \left(M^\varepsilon_T,\varphi\right)_\varepsilon &:= \varepsilon\sum_{x\in\mathbf{Z}} \varphi(\varepsilon x) M_{\varepsilon^{-2}T}(x) \end{aligned} \tag{3.9}$$

$$R_T^\varepsilon := \varepsilon^{\frac{1}{2}} \int_0^{\varepsilon^{-2}T} dt \left\{ \frac{1}{2}(1+\sqrt{\varepsilon})\varepsilon \sum_{x\in\mathbf{Z}} [\varepsilon\,(\nabla\varphi)\,(\varepsilon x) - [\varphi(\varepsilon x) - \varphi(\varepsilon(x-1))]]\sigma_t(x) \right\} .$$

The Proposition will be proven bounding separately the variance of the various terms on the right hand side of (3.8). Let us first note that, by Taylor's theorem, we have trivially

$$\left|R_T^\varepsilon\right| \le \varepsilon^{\frac{1}{2}} T\, \|\Delta\varphi\|_{L_\infty(\mathbf{R})} \cdot \mathrm{Vol}(\mathrm{supp}\,\varphi) \xrightarrow{\varepsilon\to 0} 0 \tag{3.10}$$

where $\|\cdot\|_{L_\infty(\mathbf{R})}$ is the uniform norm, Vol is the Lebesgue measure on $\mathbf{R}$ and supp φ the support of φ.

To estimate the variance of the martingale part in (3.8), $\left(M_T^\varepsilon, \varphi\right)_\varepsilon$, we note that from (3.7) it follows

$$\frac{d}{dt}\langle M(x), M(y)\rangle_t \le 6\delta(x-y)$$

whence

$$\varepsilon\, \mathbf{E}_\mu^\varepsilon \left(\left(M_T^\varepsilon, \varphi\right)_\varepsilon\right)^2 \le 6\,\varepsilon^2 \int_0^{\varepsilon^{-2}T} dt\ \varepsilon \sum_{x\in\mathbf{Z}} \varphi(\varepsilon x)^2 \xrightarrow{\varepsilon\to 0} 6T \int dr\ \varphi(r)^2. \tag{3.11}$$

The variance of the initial condition in (3.8), $\left(Z_0^\varepsilon, \varphi\right)_\varepsilon$, is easily bounded recalling that μ is the symmetric Bernoulli measure. We have in fact

$$\begin{aligned} \mathbf{E}_\mu^\varepsilon \left(\left(Z_0^\varepsilon, \varphi\right)_\varepsilon\right)^2 &= \varepsilon^2 \sum_{x,y\in\mathbf{Z}} \varphi(\varepsilon x)\varphi(\varepsilon y)\varepsilon\mu\{\zeta(x)\zeta(y)\} \\ &\xrightarrow{\varepsilon\to 0} \int dr\, dr'\ \varphi(r)\varphi(r')\ \min\{|r|, |r'|\}. \end{aligned} \tag{3.12}$$

Call P_t be the Markov semigroup associated to SEP and observe that $(f, R_\lambda f)_{L^2(\mu)} = \int_0^\infty dt\ e^{-\lambda t}\ (f, P_t f)_{L^2(\mu)}$. By Lemma 3.1 and the definition of f_1, cf. (3.9), we have that

$$\begin{aligned} \mathbf{E}_\mu^\varepsilon \left(\int_0^T dt\ f_1(\sigma_t)\right)^2 &\le CT \int_0^\infty dt\ e^{-t/T}\ (f_1, P_t f_1)_{L^2(\mu)} \\ &= CT \int_0^\infty dt\ e^{-t/T} \varepsilon^2 \\ &\quad \times \sum_{x,y\in\mathbf{Z}} \nabla\varphi(\varepsilon x)\nabla\varphi(\varepsilon y)\,(\sigma(x), P_t\sigma(y))_{L^2(\mu)} \ . \end{aligned} \tag{3.13}$$

By using the self duality of SEP (see [9, VIII, §1]) one easily gets that for all $x, y \in \mathbf{Z}$

$$(\sigma(x), P_t\sigma(y))_{L_2(\mu)} = p_t^{(1)}(x; y) \leq ct^{-1/2} \quad , \tag{3.14}$$

where $p_t^{(1)}(x; y)$ denotes the transition probability of a simple random walk on $\mathbf{Z}$ (i.e. the kernel of the *1-particle* SEP process) and $c > 0$ is a universal constant. Therefore from (3.13) and (3.14) it follows that for any $\varphi \in C_0^\infty(\mathbf{R})$ there is a constant $C_1 = C_1(\varphi)$ such that

$$\mathbf{E}_\mu^\varepsilon \left(\varepsilon^{\frac{3}{2}} \int_0^{\varepsilon^{-2}T} dt \ f_1(\sigma_t) \right)^2 \leq C_1 T^{\frac{3}{2}} \tag{3.15}$$

for every $\varepsilon \in (0, 1]$ and $T \geq \varepsilon^2$.

We next proceed analogously to estimate the variance of the additive functional $\int_0^t ds \ f_2(\sigma_s)$. In what follows $p_t^{(2)}(x, x'; y, y')$ the symmetrized kernel of the 2-particle SEP process, that is the probability that the sites y and y' are occupied at time t, given that the sites x and x' are occupied (and all other sites are empty) at time zero (set $p_t^{(2)} = 0$ if $x = x'$ or $y = y'$). Therefore

$$(f_2, P_t f_2)_{L_2(\mu)} = \frac{\varepsilon^2}{4} \sum_{x,y \in \mathbf{Z}} \varphi(\varepsilon x)\varphi(\varepsilon y) \ p_t^{(2)}(y, y+1; x, x+1) \quad , \tag{3.16}$$

and we observe that, by Liggett decorrelation inequality, see [9, Cor. VIII.1.9], for any $x, x', y, y' \in \mathbf{Z}$ and $t \geq 0$

$$p_t^{(2)}(x, x'; y, y') \leq p_t^{(1)}(x; y) p_t^{(1)}(x'; y') + p_t^{(1)}(x; y') p_t^{(1)}(x'; y) \quad . \tag{3.17}$$

Insert (3.17) in (3.16) to obtain that there exists c' such that

$$(f_2, P_t f_2)_{L_2(\mu)} \leq c' \varepsilon t^{-1/2} \|\varphi\|_{L_\infty(\mathbf{R})} \|\varphi\|_{L_1(\mathbf{R})} \tag{3.18}$$

where we used again (3.14).

By using Lemma 3.1 and (3.18) we immediately obtain that for every $\varphi \in C_0^\infty(\mathbf{R})$ there is a constant $C_2 = C_2(\varphi)$, such that

$$\mathbf{E}_\mu^\varepsilon \left(\varepsilon \int_0^{\varepsilon^{-2}T} dt \ f_2(\sigma_t) \right)^2 \leq C_2 T^{\frac{3}{2}} \quad , \tag{3.19}$$

which completes the proof of the Proposition. □

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