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Coercive Inequalities for Gibbs Measures*

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We prove the Generalized Nash and Logarithmic Nash inequalities for Gibbs measures with Dirichlet form associated to the Kawasaki dynamics. © 1999 Academic Press

1. A STRATEGY FOR THE NASH INEQUALITIES

Let $\mathbb{Z}^d$ be the d-dimensional integer lattice with the Euclidean metric $\mathrm{d}(\cdot,\cdot)$. Let $\mathscr{F}$ be the family of finite sets in $\mathbb{Z}^d$. For a set $\Lambda \subset \mathbb{Z}^d$, by $|\Lambda|$ we denote its cardinality (volume) and we define the R-boundary of Λ by $\partial_R \Lambda \equiv \{ j \in \complement \Lambda : \mathrm{d}(j, \Lambda) \leqslant R \}$, where $\complement \Lambda \equiv \mathbb{Z}^d \backslash \Lambda$. Let $\Omega \equiv \mathbb{M}^{\mathbb{Z}^d}$ be the product space defined with a compact metric space $\mathbb{M}$. By Σ_Λ, $\Lambda \in \mathbb{Z}^d$, we denote the smallest σ-algebra of subsets in Ω with respect to which all the coordinate functions $\omega \mapsto \omega_i$, $i \in \Lambda$, are measurable and we set $\Sigma \equiv \Sigma_{\mathbb{Z}^d}$. For a probability measure μ on (Ω, Σ), we denote by $\mu(f) \equiv \mu f$ the corresponding expectation of the μ-integrable function f and we use the following notation $\mu(f; g) \equiv \mu fg - \mu f \mu g$ for the covariance of the functions f and g. By μ_0 we denote the free measure on (Ω, Σ), i.e. a product measure of uniform probability measures on $(\mathbb{M}, \mathscr{B}_{\mathbb{M}})$. The related conditional expectations with respect to $\Sigma_{\complement \Lambda}$ will be denoted by $\mu_{0,\Lambda}$, or in a special case $\mu_{0,\{i\}} \equiv \mu_{0,i}$. Given $x \in \mathbb{M}^\Lambda$ and $y \in \mathbb{M}^{\complement \Lambda}$, we define a configuration $x \bullet_\Lambda y \in \Omega$ as

$$(x \bullet_\Lambda y)_j \equiv \begin{cases} x_j & \text{if } \ j \in \Lambda \\ y_j & \text{if } \ j \in \complement \Lambda. \end{cases}$$

In particular if $\Lambda = \{i\}$, for some $i \in \mathbb{Z}^d$, we will use a simplified notation $x \bullet_{\{i\}} y \equiv x \bullet_i y$. If $\mathbb{M}$ is a smooth Riemannian manifold and for any $i \in \mathbb{Z}^d$

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and $\omega \in \Omega$ a function $\mathbb{M} \ni x \mapsto f(x \mid \omega) \equiv f(x \bullet_i \omega)$ is differentiable, we introduce the gradient ∇_i with respect to the coordinate ω_i, $i \in \mathbb{Z}^d$, by

$$\nabla_i f(\omega) \equiv (\partial_{\mathbb{M}} f(\cdot \mid \omega))(\omega_i),$$

where $\partial_{\mathbb{M}}$ denotes the corresponding gradient operator on the manifold $\mathbb{M}$ and we compute its length using the corresponding scalar product in the tangent space $\mathbb{T}_{\omega_i} \mathbb{M}$ as

$$|\nabla_i f(\omega)| \equiv ((\partial_{\mathbb{M}} f(\cdot \mid \omega))(\omega_i), (\partial_{\mathbb{M}} f(\cdot \mid \omega))(\omega_i))^{1/2}_{T_{\omega_i}\mathbb{M}}.$$

If $\mathbb{M}$ is a finite space, we define a discrete gradient

$$\nabla_i f(\omega) \equiv f(\omega) - (\mu_{0,i} f)(\omega)$$

and in this case its length is simply the absolute value of this expression. The gradient with respect to the coordinates in a set $\Lambda \subset \mathbb{Z}^d$ will be denoted by $\nabla_\Lambda f \equiv (\nabla_i f)_{i \in \Lambda}$ and in case when $\Lambda = \mathbb{Z}^d$, we simply set $\nabla_{\mathbb{Z}^d} f \equiv \nabla f$. We define the square of the gradient as

$$|\nabla_\Lambda f|^2 \equiv \sum_{i \in \Lambda} |\nabla_i f|^2.$$

We introduce the space $\mathscr{C}(\Omega)$ of continuous functions on Ω, which becomes a Banach space under the uniform norm $\|\cdot\|_u$, and a space $\mathscr{C}_q \equiv \mathscr{C}_q(\Omega)$, $q \in [1, \infty)$ of functions in $\mathscr{C}(\Omega)$ for which the following Lipschitz type semi-norm is finite

$$|||f|||_q \equiv \left(\sum_{i \in \mathbb{Z}^d} \|\nabla_i f\|_u^q \right)^{1/q}, \qquad \|\nabla_i f\|_u \equiv \sup_\omega |\nabla_i f(\omega)|.$$

One notes that $\mathscr{C}_q \supset \mathscr{C}_p$ if $q \geqslant p$.

We will use the following definitions.

Definition 1.1. • A probability measure μ satisfies *Standard Spectral Gap* inequality iff there is $M_\mu \in (0, \infty)$ such that

$$M_\mu \cdot \mu(f - \mu f)^2 \leqslant \mu\, |\nabla f|^2 \tag{SSG}$$

for any f for which the right hand side is finite.

• A probability measure μ satisfies *Standard Logarithmic Sobolev* inequality iff there is $c_\mu \in (0, \infty)$ such that

$$\mu \left(f \log \frac{f}{\mu f} \right) \leqslant c_\mu \cdot \mu\, |\nabla f^{1/2}|^2 \tag{SLS}$$

for any nonnegative function f for which the right hand side is finite.

Let $\mathscr{E} = \{\mu_\Lambda^\omega : \Lambda \in \mathscr{F}, \ \omega \in \Omega\}$ be a local (respectively of range R) specification on (Ω, Σ), i.e. a family consisting of probability kernels such that for any bounded measurable (respectively Σ_Λ-measurable) function f, the function $\omega \mapsto \mu_\Lambda^\omega(f)$ is $\Sigma_{\complement\Lambda}$ (respectively $\Sigma_{\partial_R\Lambda}$)-measurable and the following compatibility condition is satsified

$$\forall \Lambda_1 \subset \Lambda_2 \in \mathscr{F} \qquad \mu_{\Lambda_2}^\omega \mu_{\Lambda_1}^\bullet(f) = \mu_{\Lambda_2}^\omega(f).$$

A probability measure μ on (Ω, Σ) satisfying

$$\forall \Lambda \in \mathscr{F} \qquad \mu(\mu_\Lambda^\bullet(f)) = \mu(f) \tag{\textbf{DLR}}$$

for any bounded measurable function f, is called a *Gibbs measure* for $\mathscr{E}$. The (convex) set of all Gibbs measures for $\mathscr{E}$ will be denoted by $\mathscr{G}(\mathscr{E})$ and by $\partial\mathscr{G}(\mathscr{E})$ the set of its extremal points.

DEFINITION 1.2. Let $\mathscr{E} = \{\mu_\Lambda^\omega : \Lambda \in \mathscr{F}, \ \omega \in \Omega\}$ be a local specification of range R and let $\varphi_{jk} \equiv \varphi(j-k) \in [0, \infty)$, for $j, k \in \mathbb{Z}^d$, be such that for any cube Λ and any $j \in \partial_R \Lambda$ we have

$$\|\nabla_j \mu_\Lambda^\bullet f\|_u \leqslant \sum_{k \in \Lambda \cup j} \varphi_{jk} \cdot \|\nabla_k f\|_u.$$

• The local specification $\mathscr{E}$ will be called *Strongly Mixing* iff for any $j, k \in \mathbb{Z}^d$, we have

$$\varphi_{jk} \equiv \varphi(j-k) \leqslant \varphi_0 e^{-M_0 \operatorname{d}(j,k)} \tag{\textbf{SM}}$$

with some constants $\varphi_0, M_0 \in (0, \infty)$.

It is known that (**SM**) implies (**SSG**) and (**SLS**), respectively, (see e.g. [1] and [14]–[16], [8]–[10], [7], [6], ..., respectively).

Suppose for $X \in \mathscr{F}$, $\operatorname{diam}(X) \leqslant R$, and every $j \in \mathbb{Z}^d$ we are given a Markov generator L_{X+j} in $\mathscr{C}(\Omega)$ such that

(i) If f is Σ_{X+j}-measurable then $L_{X+j} f$ is $\Sigma_{(X \cup \partial_R X)+j}$-measurable, and

(ii) For any f, g in its domain $\mathscr{D}(L_{X+j})$ we have

$$\mu_{X+j}^\omega(f L_{X+j} g) = \mu_{X+j}^\omega(g L_{X+j} f).$$

For $\Lambda \in \mathscr{F}$, we introduce a finite volume Markov generator $\mathscr{L}_\Lambda$ as

$$\mathscr{L}_\Lambda \equiv \sum_\alpha \sum_{j : X_\alpha + j \subseteq \Lambda} L_{X_\alpha + j}$$

with the summation over a finite set of α's. Let $P_t^\Lambda \equiv e^{t\mathscr{L}_\Lambda}$ denotes the corresponding Markov semigroup on $\mathscr{C}(\Omega)$. We introduce also a densely defined on smooth cylinder function Markov pre-generator

$$\mathscr{L}f \equiv \sum_{\alpha,\, j \in \mathbb{Z}^d} L_{X_\alpha + j} f = \lim_{\Lambda \to \mathbb{Z}^d} \mathscr{L}_\Lambda f.$$

For $\mathscr{L}$ and $\mathscr{L}_\Lambda$ the corresponding Dirichlet form with the measure μ and μ_Λ^ω will be denoted by

$$\mathbf{D}_\mu(f) \equiv \mu(f(-\mathscr{L}f))$$

and

$$\mathbf{D}_{\Lambda,\omega}(f) \equiv \mathbf{D}_{\mu_\Lambda^\omega}(f) \equiv \mu_\Lambda^\omega(f(-\mathscr{L}_\Lambda f)),$$

respectively.

Under very general conditions, see e.g. [5, 9], it extends to the Markov generator (denoted later on by the same symbol) of the semigroup $P_t \equiv e^{t\mathscr{L}}$ and on cylinder functions we have

$$P_t f \equiv \lim_{\Lambda \to \mathbb{Z}^d} P_t^\Lambda f.$$

DEFINITION 1.3. • The family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ is called *locally conservative* iff for every cube $\Lambda \in \mathscr{F}$, the subspace $\mathscr{I}_\Lambda \subset \mathbb{L}_2(\mu_\Lambda^\omega)$ of Σ_Λ-measurable functions which satisfy

$$\mathscr{L}_\Lambda f = 0 \tag{$\circledast$}$$

is nontrivial, i.e., contains nonconstant functions.

• The family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ has a *local Spectral Gap property* iff, for every cube $\Lambda \in \mathscr{F}$, there is $m_\Lambda \in (0, \infty)$ such that

$$m_\Lambda \cdot \mu_\Lambda^\omega(f - \mu_\Lambda^\omega f)^2 \leqslant \mathbf{D}_{\Lambda,\omega}(f) \tag{$*$}$$

for every $w \in \Omega$ and any f belonging to the set $\aleph_\Lambda$ of Σ_Λ-measurable functions orthogonal to $\mathscr{I}_\Lambda$ for which the right hand side is finite.

• The family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ satsifies a *local Logarithmic Sobolev inequality* iff for every cube $\Lambda \in \mathscr{F}$ there is $c_\Lambda \in (0, \infty)$ such that

$$\mu_\Lambda^\omega\left(f \log \frac{f}{\mu_\Lambda^\omega f}\right) \leqslant c_\Lambda \cdot \mathbf{D}_{\Lambda,\omega}(f^{1/2}) \tag{$**$}$$

for every $\omega \in \Omega$ and any nonnegative function $f \in \aleph_\Lambda$ for which the right hand side is finite.

Later on we will consider nonnegative (nonlinear) convex functionals $\{A_\Lambda\}_{\Lambda\in\mathcal{F}}$ defined on a dense domain in $\mathscr{C}(\Omega)$ and vanishing on constants. Such functional will be called *subadditive* iff

$$\forall \Lambda_1, \Lambda_2 \in \mathcal{F}, \qquad \Lambda_1 \cap \Lambda_2 = \varnothing \qquad A_{\Lambda_1}(f) + A_{\Lambda_2}(f) \leqslant A_{\Lambda_1\cup\Lambda_2}(f). \tag{1.1}$$

We will restrict ourselves to the homogeneous functionals of degree 2, i.e. such that for every $\lambda \in \mathbb{R}^+$, we have

$$A_\Lambda(\lambda f) = \lambda^2 A_\Lambda(f). \tag{1.2}$$

Let A denote the functional defined by

$$A(f) \equiv \lim_{\Lambda \to \mathbb{Z}^d} A_\Lambda(f), \tag{1.3}$$

where the limit is taken along an increasing sequence of $\Lambda \in \mathcal{F}$ invading $\mathbb{Z}^d$. Note that, by subadditivity, the above limit always exists (possibly infinite) and do not depend on the sequence.

In our further considerations the following additional properties of the stochastic dynamic will play an important role. These properties abstract a scaling behaviour, which is relevant in order to prove a Generalized and Logarithmic Nash inequality of a local conservative dynamics satisfying $(*)$ and $(**)$, respectively. In the next Section we shall prove they hold for the Kawasaki dynamics.

Definition 1.4. • A locally conservative family $\{\mathcal{L}_\Lambda\}_{\Lambda\in\mathcal{F}}$ is called *asymptotically diffusive* iff for every cube $\Lambda \in \mathcal{F}$, there are $m_\Lambda \equiv m(|\Lambda|) \in (0, \infty)$, $m_\Lambda \xrightarrow{|\Lambda|\to\infty} 0$ and $\varepsilon_\Lambda \equiv \varepsilon(|\Lambda|) \in (0, \infty)$, $\varepsilon_\Lambda \xrightarrow{|\Lambda|\to\infty} 0$, such that we have

$$\mu_\Lambda^\omega(f - \mu_\Lambda^\omega f)^2 \leqslant m_\Lambda^{-1} \cdot \mathbf{D}_{\Lambda,\omega}(f) + \varepsilon_\Lambda \cdot A_\Lambda(f) \tag{$\circledast\circledast$}$$

with some subadditive functional $A_\Lambda(\cdot)$, for any $\omega \in \Omega$ and function f for which the right hand side is finite.

• A locally conservative family $\{\mathcal{L}_\Lambda\}_{\Lambda\in\mathcal{F}}$ is called *S-asymptotically diffusive* iff for every cube $\Lambda \in \mathcal{F}$, there are $c_\Lambda \in (0, \infty)$, $c_\Lambda \equiv c(|\Lambda|) \xrightarrow{|\Lambda|\to\infty} \infty$ and $\varepsilon_\Lambda \in (0, \infty)$, $\varepsilon_\Lambda \equiv \varepsilon(|\Lambda|) \xrightarrow{|\Lambda|\to\infty} 0$, such that we have

$$\mu_\Lambda^\omega\left(f \log \frac{f}{\mu_\Lambda^\omega f}\right) \leqslant c_\Lambda \cdot \mathbf{D}_{\Lambda,\omega}(f^{1/2}) + \varepsilon_\Lambda \cdot A_\Lambda(f^{1/2}) \tag{$\circledast\circledast\circledast$}$$

with some subadditive functional $A_\Lambda(\cdot)$ satisfying

$$A_\Lambda((Ef)^{1/2}) \leqslant A_\Lambda(f^{1/2}) \tag{1.4}$$

with any conditional expectation E, for any all nonnegative functions f for which the right hand side of (⊛⊛⊛) is finite and any $\omega \in \Omega$.

In this Section we prove the following general result.

THEOREM 1.1. *Suppose the local specification $\mathscr{E}$ of range R satisfies the* Strong Mixing condition (**SM**).

(I) *If a locally conservative family* $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ *is* asymptotically diffusive, *then the following inequality is true for every* $L \in \mathbb{N}$

$$\mu(f-\mu f)^2 \leqslant m(L^d)^{-1} \cdot \mathbf{D}_\mu(f) + \tilde{\varepsilon}(L) \cdot \tilde{A}(f) \tag{1.5}$$

with some $\tilde{\varepsilon}(L) \in (0, \infty)$, $\tilde{\varepsilon}(L) \xrightarrow{L \to \infty} 0$ *and a functional* $\tilde{A}(f) \equiv A(f) + |||f|||_2^2$, *for any function f for which the right hand side is finite.*

(II) *If a locally conservative family* $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ *is* S-asymptotically diffusive, *then the following inequality is true for every* $L \in \mathbb{N}$

$$\mu\left(f \log \frac{f}{\mu f}\right) \leqslant c(L^d) \cdot \mathbf{D}_\mu(f^{1/2}) + \tilde{\varepsilon}(L) \cdot \tilde{A}(f^{1/2}) \tag{1.6}$$

with some $\tilde{\varepsilon}(L) \in (0, \infty)$, $\tilde{\varepsilon}(L) \xrightarrow{L \to \infty} 0$ *and a functional* $\tilde{A}(f^{1/2}) \equiv A(f^{1/2}) + |||f^{1/2}|||_2^2$ *for any nonnegative function f for which the right hand side is finite.*

The bound (1.5) (respectively (1.6)) is called Generalized Nash (respectively Logarithmic Nash) inequality; we refer to [2] for an overview and a motivation in the context of infinite dimensional Markov semigroups, see also Section 3 for a further discussion.

Proof of Theorem 1.1 (I): Let Λ_0 be a reference cube of side L in $\mathbb{Z}^d$. For $i \in \mathbb{Z}^d$, let $\Lambda_i := \Lambda_0 + i(L+2R)$ be the translate of Λ_0 by a vector $i(L+2R) \in \mathbb{Z}^d$. It will be convenient to label all these translated cubes by a natural number; we thus obtain a family of cubes $\{\Lambda_l\}_1^\infty$ such that for $l \neq l'$, $\mathrm{d}(\Lambda_l, \Lambda_{l'}) \geqslant 2R$. Let $\{Y_l\}_0^\infty$ be the increasing sequence defined by $Y_0 := \emptyset$, $Y_l := \bigcup_{l' \leqslant l} \Lambda_{l'}$, if $l \geqslant 1$. Let $\Gamma_0 \equiv \bigcup_l Y_l$. To $\{Y_l\}_0^\infty$ we associate a family $\{E_l\}_0^\infty$ of conditional expectations defined by $E_0 f = f$ and for $l > 0$, $E_l f(\omega) := \mu_{Y_l}^\omega f$, for any $f \in \mathscr{C}(\Omega)$.

We then have

$$f(\omega) - \mu_{\Gamma_0}^\omega f = \sum_{l=1}^\infty (f_{l-1}(\omega) - f_l(\omega)), \tag{1.7}$$

where $f_l := E_l f = \mu^{\omega}_{\Lambda_l} f_{l-1}$. The series on the right hand side of (1.7) is actually a finite sum when f is a cylinder function (i.e. depends only on a finite number of coordinates) and is absolutely convergent if $f \in \mathscr{C}_1(\Omega)$.

Since the sequence f_l has orthogonal increments, we get

$$\mu(f-\mu f)^2 = \mu(\mu_{\Gamma_0} f - \mu f)^2 + \sum_{l=1}^{\infty} \mu(f_{l-1} - f_l)^2 \tag{1.8}$$

$$= \mu(\mu_{\Gamma_0} f - \mu f)^2 + \sum_{l=1}^{\infty} \mu(\mu_{\Lambda_l}(f_{l-1} - \mu_{\Lambda_l} f_{l-1})^2). \tag{1.9}$$

We estimate first every term in the infinite sum on the right hand side. We note that keeping the variables outside Λ_l fixed, we can use the asymptotic diffusivity inequality (⊛⊛) for the cube Λ_l (of side L) and the function f_{l-1} to get

$$\mu^{\omega}_{\Lambda_l}(f_{l-1} - \mu^{\omega}_{\Lambda_l} f_{l-1})^2 \leqslant m(L^d)^{-1} \cdot \mathbf{D}_{\Lambda_l, \omega}(f_{l-1}) + \varepsilon(L^d) \cdot A_{\Lambda_l}(f_{l-1}). \tag{1.10}$$

Using the fact that our cubes are separated by $2R$ and our local specification is of range R, we have

$$\mathbf{D}_{\Lambda_l, \omega}(f_{l-1}) = \mathbf{D}_{\Lambda_l, \omega}(\mu_{Y_{l-1}} f) \leqslant \mu^{\omega}_{Y_l}(\mathbf{D}_{\Lambda_l, \omega}(f)). \tag{1.11}$$

Also by convexity of our subadditive functionals we get

$$A_{\Lambda_l}(f_{l-1}) \leqslant A_{\Lambda_l}(f). \tag{1.12}$$

This together with (1.8) give

$$\mu(f-\mu f)^2 \leqslant \mu(\mu_{\Gamma_0} f - \mu f)^2 + \sum_{l=1}^{\infty} \{m(L^d)^{-1} \cdot \mu(\mathbf{D}_{\Lambda_l, \omega}(f)) + \varepsilon(L^d) \cdot A_{\Lambda_l}(f)\}$$

$$\leqslant m(L^d)^{-1} \cdot \mathbf{D}_{\mu}(f) + \varepsilon(L^d) \cdot A(f) + \mu(\mu_{\Gamma_0} f - \mu f)^2, \tag{1.13}$$

where in the last step we have used the definitions of our Dirichlet forms and our assumption about the subadditivity of the family $\{A_{\Lambda_l}\}_{l \in \mathbb{N}}$. To estimate the last term on the right hand side of (1.13) we note that, under the Strong Mixing condition, the measure μ satisfies (**SSG**) inequality, [1], ... Therefore we have

$$\mu(\mu_{\Gamma_0} f - \mu f)^2 \leqslant M_{\mu}^{-1} \cdot \mu\, |\nabla \mu_{\Gamma_0} f|^2. \tag{1.14}$$

We observe that

$$|\nabla \mu_{\Gamma_0} f|^2 = \sum_{j \in \complement \Gamma_0} |\nabla_j \mu_{\Gamma_0} f|^2 \tag{1.15}$$

and that by our construction for every $j \in \complement \Gamma_0$ there is a unique cube $\Lambda_{l(j)}$ such that $j \in \partial_R \Lambda_{l(j)}$ which implies

$$|\nabla_j \mu_{\Gamma_0} f|^2 \leqslant \|\nabla_j \mu_{\Lambda_{l(j)}} f\|_u^2. \tag{1.16}$$

Thus using (1.14)–(1.16) together with the Strong Mixing condition, we get

$$\mu(\mu_{\Gamma_0} f - \mu f)^2 \leqslant M_\mu^{-1} \cdot \sum_l \sum_{j \in \partial_R \Lambda_l} \left(\sum_{k \in \Lambda_l \cup j} \varphi_{jk} \cdot \|\nabla_k f\|_u \right)^2 \tag{1.17}$$

which with the use of Hölder inequality can be transformed to

$$\mu(\mu_{\Gamma_0} f - \mu f)^2 \leqslant M_\mu^{-1} \cdot \sup_{j \in \partial_R \Lambda_0} \left(\sum_{k \in \Lambda_0 \cup j} \varphi_{jk} \right) \sum_l \sum_{j \in \partial_R \Lambda_l} \sum_{k \in \Lambda_l \cup j} \varphi_{jk} \cdot \|\nabla_k f\|_u^2. \tag{1.18}$$

Using this and (1.13) we get

$$\mu(f - \mu f)^2 \leqslant m(L^d)^{-1} \cdot \mathbf{D}_\mu(f) + \varepsilon(L^d) \cdot A(f) + M_\mu^{-1} \cdot \sup_{j \in \partial_R \Lambda_0} \left(\sum_{k \in \Lambda_0 \cup j} \varphi_{jk} \right) \sum_l \sum_{j \in \partial_R \Lambda_l} \sum_{k \in \Lambda_l \cup j} \varphi_{jk} \cdot \|\nabla_k f\|_u^2. \tag{1.19}$$

Now we take advantage of the fact that the position of our reference cube Λ_0 was arbitrary and thus we can replace the inequality (1.19) by its average with respect to the translations $a = (a^1, ..., a^d)$ in the cube $\bar{\Lambda}_0$ of side $L + R$ centered at the origin. Using the Strong Mixing condition we have

$$\begin{aligned} &\frac{1}{(L+R)^d} \sum_{a \in \bar{\Lambda}_0} \sum_l \sum_{j \in \partial_R \Lambda_l + a} \sum_{k \in (\Lambda_l + a) \cup j} \varphi_{jk} \cdot \|\nabla_k f\|_u^2 \\ &\leqslant \sum_{k \in \mathbb{Z}^d} \frac{1}{(L+R)^d} \left\{ \sum_{a \in \bar{\Lambda}_0} \sum_{l:\, [(\Lambda_l \cup \partial_R \Lambda_l) + a] \ni k} \sum_{j \in \partial_R \Lambda_l + a} \varphi_{jk} \right\} \cdot \|\nabla_k f\|_u^2 \\ &\leqslant C \frac{1}{L} \cdot \sum_{k \in \mathbb{Z}^d} \|\nabla_k f\|_u^2 \end{aligned} \tag{1.20}$$

with

$$C \equiv \max_{L \in \mathbb{N}} \frac{1}{(L+R)^{d-1}} \cdot \left\{ \sum_{a \in \bar{\Lambda}_0} \sum_{l:\, [(\Lambda_l \cup \partial_R \Lambda_l) + a] \ni k} \sum_{j \in \partial_R \Lambda_l + a} \varphi_{jk} \right\}.$$

Thus we conclude that

$$\mu(f-\mu f)^2 \leqslant m(L^d)^{-1} \cdot \mathbf{D}_\mu(f) + \varepsilon(L^d) \cdot A(f) + C\frac{1}{L} \cdot \sum_{k \in \mathbb{Z}^d} \|\nabla_k f\|_u^2. \tag{1.21}$$

Choosing

$$\tilde{\varepsilon}(L) \equiv \max\left\{\varepsilon(L^d),\ C\frac{1}{L}\right\} \tag{1.22}$$

and recalling that

$$\tilde{A}(f) \equiv A(f) + \sum_{k \in \mathbb{Z}^d} \|\nabla_k f\|_u^2, \tag{1.23}$$

we get the first part of Theorem 1.1.

Proof of Theorem 1.1 (II). The proof of the second part is similar. We use first the following martingale decomposition of entropy:

$$\begin{aligned}\mu\left(f \log \frac{f}{\mu f}\right) = \mu\left(\mu_{\Gamma_0}\left(f \log \frac{\mu_{\Gamma_0} f}{\mu\mu_{\Gamma_0} f}\right)\right) \\ + \sum_{l=1}^{\infty} \mu\left(\mu_{\Lambda_l}\left(f_{l-1} \log \frac{f_{l-1}}{\mu_{\Lambda_l} f_{l-1}}\right)\right).\end{aligned} \tag{1.24}$$

We estimate each term in the sum on the right hand side using the S-asymptotic diffusivity property (⊛⊛⊛) as

$$\begin{aligned}\mu_{\Lambda_l}^{\omega}\left(f_{l-1} \log \frac{f_{l-1}}{\mu_{\Lambda_l}^{\omega} f_{l-1}}\right) &\leqslant c(L^d) \cdot \mathbf{D}_{\Lambda_l, \omega}(f_{l-1}^{1/2}) + \varepsilon(L^d) \cdot A_{\Lambda_l}(f_{l-1}^{1/2}) \\ &\leqslant c(L^d) \cdot \mathbf{D}_{\Lambda_l, \omega}(f^{1/2}) + \varepsilon(L^d) \cdot A_{\Lambda_l}(f^{1/2}),\end{aligned} \tag{1.25}$$

where in the second line we have used our assumption (1.4). This implies the following bound:

$$\sum_{l=1}^{\infty} \mu\left(\mu_{\Lambda_l}\left(f_{l-1} \log \frac{f_{l-1}}{\mu_{\Lambda_l} f_{l-1}}\right)\right) \leqslant c(L^d) \cdot \mathbf{D}_\mu(f^{1/2}) + \varepsilon(L^d) \cdot A(f^{1/2}). \tag{1.26}$$

The first term on the right hand side of (1.24) is estimated using the Standard Logarithmic Sobolev inequality for the measure μ. We get

$$\begin{aligned}\mu\left(\mu_{\Gamma_0}\left(f \log \frac{\mu_{\Gamma_0} f}{\mu(\mu_{\Gamma_0} f)}\right)\right) &\leqslant c_\mu \cdot \mu\, |\nabla(\mu_{\Gamma_0} f)^{1/2}|^2 \\ &\leqslant c_\mu \cdot \sum_{j \in \complement \Gamma_0} \|\nabla_j (\mu_{\Gamma_0} f)^{1/2}\|_u^2.\end{aligned} \tag{1.27}$$

Since one can show, see e.g. [8, 10], that under the Strong Mixing condition we have

$$|\nabla_j(\mu_\Lambda^\omega f)^{1/2}| \leqslant \sum_{j\in\partial_R\Lambda} \sum_{k\in\Lambda\cup j} \tilde{\varphi}_{jk} \|\nabla_k f^{1/2}\|_u \tag{1.28}$$

with

$$\sum_{j\in\partial_R\Lambda} \sum_{k\in\Lambda\cup j} \tilde{\varphi}_{jk} \leqslant C_1(\log L)^d \tag{1.29}$$

for some constant $C_1 \in (0, \infty)$, by the similar arguments as before one can get the estimate

$$\begin{aligned}\frac{1}{(L+R)^d} \sum_{a\in\bar{\Lambda}_0} \mu\left(\mu_{\Gamma_0+a} f \log \frac{\mu_{\Gamma_0+a} f}{\mu\mu_{\Gamma_0+a} f}\right)\\ \leqslant C_2 \frac{(\log L)^{2d}}{L} \cdot \sum_{k\in\mathbb{Z}^d} \|\nabla_k f^{1/2}\|_u^2\end{aligned} \tag{1.30}$$

with some constant $C_2 \in (0, \infty)$. Using (1.24)–(1.30) we get the desired inequality (1.6) with

$$\tilde{\varepsilon}(L) \equiv \max\left\{\varepsilon(L^d),\, C_2 \frac{(\log L)^{2d}}{L}\right\}. \tag{1.31}$$

We recall in fact that

$$\tilde{A}(f^{1/2}) \equiv A(f^{1/2}) + \sum_{k\in\mathbb{Z}^d} \|\nabla_k f^{1/2}\|_u^2. \tag{1.32}$$

This ends the proof of the second part and so of Theorem 1.1. ∎

Remark. In the continuous case the factor $(\log L)^{2d}$ can be omitted.

2. THE NASH INEQUALITIES FOR GIBBS MEASURES

In this Section we show that the strategy described in the previous Section can be applied in nontrivial situation of particle systems with non-zero interaction.

We choose the configuration space to be given by $\Omega \equiv \{0, 1\}^{\mathbb{Z}^d}$. Let $\Phi \equiv \{\Phi_X\}_{X\in\mathscr{F}}$ be a translation invariant interaction potential of a finite range $R > 0$, i.e. a family consisting of continuous real functions such that for every $X \in \mathscr{F}$ the function Φ_X is Σ_X-measurable and we have $\Phi_Y \equiv 0$ if $\operatorname{diam}(Y) > R$. Let $\|\Phi\| \equiv \Sigma_{X\ni 0} \|\Phi_X\|_u$. For the reasons which will be more clear later, we distinguish the one particle potential $\Phi^{(1)} \equiv \{\Phi_{\{i\}} \equiv -\lambda\omega_i\}_{i\in\mathbb{Z}^d}$, where $\lambda \in \mathbb{R}$ and ω_i are called the chemical potential and the coordinate function at the point $i \in \mathbb{Z}^d$, respectively. One defines a finite volume energy U_Λ in $\Lambda \in \mathcal{F}$ corresponding to the interaction potential Φ, by

$$U_\Lambda \equiv \sum_{X \in \mathcal{F} : X \cap \Lambda \neq \varnothing} \Phi_X$$

and a finite volume Gibbs measure μ_Λ^ω at $\Lambda \in \mathcal{F}$ with boundary conditions given by a configuration $\omega \in \Omega$ as

$$\mu_\Lambda^\omega(f) \equiv \frac{\mu_{0\,|\,\Lambda}(e^{-U(\tilde{\omega} \bullet_\Lambda \omega)} f(\tilde{\omega} \bullet_\Lambda \omega))}{\mu_{0\,|\,\Lambda}(e^{-U(\tilde{\omega} \bullet_\Lambda \omega)})}, \tag{2.1}$$

where $\mu_{0\,|\,\Lambda}$ denotes the integration with respect to symmetric product measure on Ω restricted to Σ_Λ. To stress the dependence of μ_Λ^ω on the one particle potential $\Phi^{(1)}$, we will also use a notation $\mu_{\Lambda,\lambda}^\omega = \mu_\Lambda^\omega$. It is standard that the family $\mathcal{E}_\lambda \equiv \{\mu_{\Lambda,\lambda}^\omega : \omega \in \Omega,\ \Lambda \in \mathcal{F}\}$ is a local specification. For the rest of this paper we will take on the following:

Assumption. *The local specification $\mathcal{E}_\lambda \equiv \{\mu_{\Lambda,\lambda}^\omega : \omega \in \Omega, \Lambda \in \mathcal{F}\}$ is Strongly Mixing uniformly in λ.*

Let

$$\delta_{ij} f(\omega) \equiv f(T_{ij}\omega) - f(\omega),$$

where T_{ij} is a measurable bijection on Ω defined by

$$(T_{ij}\omega)_l \equiv \begin{cases} \omega_j & \text{if } \ l = i \\ \omega_i & \text{if } \ l = j \\ \omega_l & \text{otherwise.} \end{cases}$$

For later purposes we note that

$$\delta_{ij} f(\omega) = [\omega_i(1-\omega_j) + \omega_j(1-\omega_i)] \cdot (\nabla_i - T_{ij}\nabla_j) f(\omega), \tag{2.2}$$

where, in this Section,

$$\nabla_i f(\omega) := f(\omega^i) - f(\omega)$$

with $\omega^i(j) := 1 - \omega(i)$, if $j = i$ and $\omega(j)$ otherwise.

We introduce the following elementary Markov operator

$$\mathcal{L}_{ij} f(\omega) \equiv c_{ij}(\omega)\, \delta_{ij} f(\omega), \tag{2.3}$$

where

$$c_{ij}(\omega) \equiv \frac{e^{\delta_{ij} U_{\{ij\}}}}{(1+e^{-\delta_{ij} U_{\{ij\}}})}. \tag{2.4}$$

We remark that $c_{ij}(\omega) = c_{ji}(\omega)$ and that these coefficients are independent of the one particle potential. Moreover we have

$$0 < (1 + e^{2 \sup_{ij} \|\delta_{ij} U_{\{ij\}}\|_u})^{-1} \leqslant c_{ji}(\omega) \leqslant 1 \tag{2.5}$$

the lower bound being also independent of the one particle potential. It is not difficult to see that for any $\Lambda \ni i, j$ we have

$$\mu^{\omega}_{\Lambda, \lambda}(g \mathscr{L}_{ij} f) = \mu^{\omega}_{\Lambda, \lambda}(f \mathscr{L}_{ij} g). \tag{2.6}$$

Let us introduce a Markov (pre-)generator $\mathscr{L}_\Lambda$, $\Lambda \subseteq \mathbb{Z}^d$ defined (on the dense set $\mathscr{C}_1$) as

$$\mathscr{L}_\Lambda \equiv \sum_{\langle ij \rangle \subset \Lambda} \mathscr{L}_{ij}, \tag{2.7}$$

where the summation is running over the nearest neighbors pairs $\langle ij \rangle$ of points contained in the set Λ. If $\Lambda = \mathbb{Z}^d$, we will suppress the corresponding subscript from the notation.

We note that the family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ is *locally conservative*. In fact it is not difficult to see that for any characteristic function

$$\chi_{\Lambda, n}(\omega) \equiv \chi(N_\Lambda(\omega) = n) \tag{2.8}$$

with $n = 0, ..., |\Lambda|$, and where

$$N_\Lambda(\omega) \equiv \sum_{i \in \Lambda} \omega_i$$

we have

$$\mathscr{L}_\Lambda \chi_{\Lambda, n} = 0. \tag{2.9}$$

Thus $\mathscr{L}_\Lambda$ vanishes on all functions which are measurable with respect to the σ-algebra $\Sigma(N_\Lambda) \subset \Sigma_\Lambda$ generated by N_Λ.

It is a standard matter to show that $\mathscr{L}_\Lambda$ extends to a Markov generator, [5], denoted later on by the same symbol. Let $P^{(\Lambda)}_t \equiv e^{t\mathscr{L}_\Lambda}$ and $P_t \equiv e^{t\mathscr{L}}$ be the corresponding Markov semigroup, respectively. Using the property (2.6) one can show that for any $\lambda \in \mathbb{R}$ and for any Gibbs measure $\mu \in \mathscr{G}(\mathscr{E}_\lambda)$ we have

$$\mu(g \mathscr{L} f) = \mu(f \mathscr{L} g) \tag{2.10}$$

for all $f, g \in \mathscr{C}_1$, and similarly for any finite set $\Lambda \in \mathscr{F}$ and any boundary condition $\omega \in \Omega$, we have

$$\mu^{\omega}_{\Lambda, \lambda}(g \mathscr{L}_{\Lambda} f) = \mu^{\omega}_{\Lambda, \lambda}(f \mathscr{L}_{\Lambda} g). \tag{2.11}$$

This in particular implies that the set of all invariant measures for P_t contains an uncountable set $\bigcup_{\lambda \in \mathbb{R}} \mathscr{G}(\mathscr{E}_\lambda)$. For more information about the structure of the set of invariant measures see [3, 11].

We will like to study the ergodic properties of the infinite volume Markov semi-group via the strategy based on general Nash coercive inequalities, (i.e. some lower bounds on the corresponding Dirichlet form of the generator). Under the condition (2.5), it is sufficient to study the following equivalent quadratic form

$$\mathbf{D}_\lambda(f) \equiv \tfrac{1}{2} \sum_{\langle ij \rangle} \mu_\lambda \, |\delta_{ij} f|^2, \tag{2.12}$$

where $\mu_\lambda \in \mathscr{G}(\mathscr{E}_\lambda)$ is—under the Strong Mixing assumption—the unique Gibbs measure for $\mathscr{E}_\lambda$. Respectively in a finite volume $\Lambda \in \mathscr{F}$, instead of the quadratic form of $\mathscr{L}_\Lambda$ in $\mathbb{L}_2(\mu^{\omega}_{\Lambda, \lambda})$, it will be more convenient to study the following equivalent form

$$\mathbf{D}^{\omega}_{\Lambda, \lambda}(f) \equiv \tfrac{1}{2} \sum_{\langle ij \rangle \subset \Lambda} \mu^{\omega}_{\Lambda, \lambda} \, |\delta_{ij} f|^2. \tag{2.13}$$

Using this forms give us the advantage that all our inequalities remain true for other generators constructred with rates given by

$$c'_{ij} = a_{ij} c_{ij},$$

where a_{ij} is symmetric in i and j, uniformly bounded and strictly positive functions independent of ω_i and ω_j.

To formulate the main result of this Section let us introduce the following semi-norm

$$|||f|||_{\Lambda, q} \equiv \left(\sum_{i \in \Lambda} \|\nabla_i f\|_u^q \right)^{1/q}. \tag{2.14}$$

In this Section we prove the following result.

THEOREM 2.1. (i) *The family* $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ *is asymptotically diffusive in the sense that for any* $\lambda \in \mathbb{R}$, $q \in [1, 2)$ *and any cube* $\Lambda \in \mathscr{F}$, $\omega \in \Omega$, *we have*

$$\mu^{\omega}_{\Lambda, \lambda}(f - \mu^{\omega}_{\Lambda, \lambda} f)^2 \leqslant m_\Lambda^{-1} \, \mathbf{D}^{\omega}_{\Lambda, \lambda}(f) + \varepsilon_\Lambda \, |||f|||^2_{\Lambda, q} \tag{2.15}$$

with

$$m_\Lambda \equiv m_0 \, |\Lambda|^{-2/d}$$

$$\varepsilon_\Lambda \equiv \varepsilon_0 \, |\Lambda|^{-((2/q)-1)}$$

for some constants m_0 and ε_0 independent on Λ, ω, q and any function f.

(ii) *The family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ is S-asymptotically diffusive in the sense that for any $\lambda \in \mathbb{R}$, $q \in [1, 2)$ and any cube $\Lambda \in \mathscr{F}$, $\omega \in \Omega$, we have*

$$\mu^\omega_{\Lambda,\lambda}\left(f \log \frac{f}{\mu^\omega_{\Lambda,\lambda} f}\right) \leqslant c_\Lambda \, \mathbf{D}^\omega_{\Lambda,\lambda}(f^{1/2}) + \hat{\varepsilon}_\Lambda \, |||f^{1/2}|||^2_{\Lambda,q} \tag{2.16}$$

with

$$c_\Lambda \equiv c_0 \, |\Lambda|^{1+(2/d)}$$

$$\hat{\varepsilon}_\Lambda \equiv \hat{\varepsilon}_0 \, |\Lambda|^{-((2/q)-1)}$$

for some constants c_0 and $\hat{\varepsilon}_0$ independent of Λ, ω, q and any function f.

By applying the general result proven in the previous Section we then conclude that the asymptotical diffusivity and S-asymptotical diffusivity, implies the Generalized and Logarithmic Nash inequality, respectively.

Corollary 2.2. *Let the local specification $\mathscr{E}_\lambda \equiv \{\mu^\omega_{\Lambda,\lambda} : \omega \in \Omega, \ \Lambda \in \mathscr{F}\}$ be Strongly Mixing uniformly in λ. Then*

(i) *For each $q \in [1, 2)$, $\lambda \in \mathbb{R}$ the (unique) Gibbs measure $\mu_\lambda \in \mathscr{G}(\mathscr{E}_\lambda)$ satisfies the following Generalized Nash inequality with respect to the Kawasaki dynamics*

$$\mu_\lambda(f - \mu_\lambda f)^2 \leqslant \mathbf{D}_\lambda(f)^\alpha \cdot \tilde{A}_q(f)^{1-\alpha}, \tag{2.17}$$

where $\alpha = (1/q - 1/2)(1/d + 1/q - 1/2)^{-1}$ if $1/q - 1/2 \leqslant 1/(2d)$, $\alpha = 1/3$ if $1/q - 1/2 > 1/(2d)$ and $\tilde{A}_q(f) = \bar{C} \, |||f|||_q$ for some constant $\bar{C} = \bar{C}(\Phi, \lambda, d, q)$, for any function $f \in \mathscr{C}_q$.

(ii) *For each $\delta > 0$ $q \in [1, 2)$, $\lambda \in \mathbb{R}$ the (unique) Gibbs measure $\mu_\lambda \in \mathscr{G}(\mathscr{E}_\lambda)$ satisfies the following Logarithmic Nash inequality with respect to the Kawasaki dynamics*

$$\mu_\lambda\left(f \log \frac{f}{\mu_\lambda f}\right) \leqslant \mathbf{D}_\lambda(f^{1/2})^{\bar{\alpha}} \cdot \tilde{A}_q(f^{1/2})^{1-\bar{\alpha}}, \tag{2.18}$$

where $\bar{\alpha} = (1/q - 1/2)(1/d + 1/q)^{-1}$ *if* $1/q - 1/2 < 1/(2d)$, $\bar{\alpha} = (d+3)^{-1} - \delta$ *if* $1/q - 1/2 \geqslant 1/(2d)$ *and* $\tilde{A}_q(f^{1/2}) = \hat{C}\,|||f|||_q$ *for some constant* $\hat{C} = \hat{C}(\Phi, \lambda, d, q, \delta)$, *for any nonnegative function* $f \in \mathscr{C}_q$.

Proof of Corollary 2.2. By Theorem 2.1 we have that the family $\{\mathscr{L}_\Lambda\}_{\Lambda \in \mathscr{F}}$ is asymptotically diffusive, respectively S-asymptotically diffusive. Hence, by Theorem 1.1, the Gibbs measure μ_λ satisfies inequality (1.5) with $m(L^d) = m_0 L^{-2}$, $\tilde{\varepsilon}(L) = \tilde{\varepsilon}_0 \max\{L^{-1}, L^{-d(2/q-1)}\}$, respectively (1.6) with $c(L^d) = c_0 L^{d+2}$, $\tilde{\varepsilon}(L) = \tilde{\varepsilon}_0 \max\{L^{-1}(\log L)^{2d}, L^{-d(2/q-1)}\} \leqslant \tilde{\varepsilon}'_0 \max\{L^{-1+\delta}, L^{-d(2/q-1)}\}$ where $\delta > 0$ is arbitrary. By using an (easy) a priori bound of Dirichlet form $\mathbf{D}_\lambda(f)$ in terms of the seminorm $|||\cdot|||_2$, see [2, Lemma 7], the inequality (2.17) and (2.18), follows from (1.5) and (1.6), respectively, by optimizing on L. ∎

Proof of Theorem 2.1. Asymptotic diffusivity. We begin by observing that

$$\mu^\omega_{\Lambda,\lambda}(f - \mu^\omega_{\Lambda,\lambda} f)^2 = \mu^\omega_{\Lambda,\lambda}(\mu^\omega_{\Lambda,\lambda}([f - \mu^\omega_{\Lambda,\lambda}(f \mid N_\Lambda)]^2 \mid N_\Lambda)) + \mu^\omega_{\Lambda,\lambda}(\mu^\omega_{\Lambda,\lambda}(f \mid N_\Lambda) - \mu^\omega_{\Lambda,\lambda}(f))^2 \tag{2.19}$$

with $\mu^\omega_{\Lambda,\lambda}(\cdot \mid N_\Lambda)$ denoting the conditional expectation knowing N_Λ associated to the measure $\mu^\omega_{\Lambda,\lambda}$ and is given by

$$\mu^\omega_{\Lambda,\lambda}(f \mid N_\Lambda = k)\,\frac{\mu^\omega_{\Lambda,\lambda}(\chi_{\Lambda,k} f)}{\mu^\omega_{\Lambda,\lambda}(\chi_{\Lambda,k})} \equiv \mu^{k,\omega}_\Lambda(f), \tag{2.20}$$

where the notation introduced on the right hand side emphasizes the fact that this conditional expectation is independent of the chemical potential λ. To estimate the first term on the right hand side of (2.19) we note that, [6], there is a constant $a_0 \in (0, \infty)$ such that for any $n = 1, ..., |\Lambda|$ we have

$$\mu^{k,\omega}_\Lambda(f - \mu^{k,\omega}_\Lambda f)^2 \leqslant a_0\,|\Lambda|^{2/d} \cdot \mathbf{D}^{k,\omega}_\Lambda(f), \tag{$\mathbf{SG}(\mu^{k,\omega}_\Lambda)$}$$

where

$$\mathbf{D}^{k,\omega}_\Lambda(f) \equiv \tfrac{1}{2} \sum_{\langle ij\rangle \subset \Lambda} \mu^{k,\omega}_\Lambda\,|\delta_{ij} f|^2.$$

Therefore we get

$$\mu^\omega_{\Lambda,\lambda}(\mu^\omega_{\Lambda,\lambda}([f - \mu^\omega_{\Lambda,\lambda}(f \mid N_\Lambda)]^2 \mid N_\Lambda)) \leqslant a_0\,|\Lambda|^{2/d} \cdot \mathbf{D}^\omega_{\Lambda,\lambda}(f). \tag{2.21}$$

To estimate the second term on the right hand side of (2.19) we note that under assumption of strong mixing the finite volume measures satisfy the following *Standard Spectral Gap inequality*

$$\mu^{\omega}_{\Lambda,\lambda}(g-\mu^{\omega}_{\Lambda,\lambda}g)^2 \leqslant M^{-1}\cdot \sum_{i\in\Lambda} \mu^{\omega}_{\Lambda,\lambda}\,|\nabla_i g|^2 \tag{\textbf{SSG}}$$

with some constant $M\in(0,\infty)$ independent of Λ, ω and g (in fact a weaker mixing property suffices, [1]). To apply this in our situation we will need the following simple lemma proven in [2, Lemma 18].

LEMMA 2.3 [2, Lemma 18]. *For any real function F and any finite set $\Lambda\subset\mathbb{Z}^d$ we have*

$$\begin{aligned}\sum_{i\in\Lambda} \mu^{\omega}_{\Lambda,\lambda}\,|\nabla_i F(\mu^{\omega}_{\Lambda,\lambda}(f \mid N_\Lambda))|^2 \\ = \sum_{k=1}^{|\Lambda|} |F(\mu^{k,\omega}_{\Lambda}(f))-F(\mu^{k-1,\omega}_{\Lambda}(f))|^2\,k\cdot\mu^{\omega}_{\Lambda,\lambda}(\chi_{\Lambda,k}) \\ + \sum_{k=0}^{|\Lambda|-1} |F(\mu^{k+1,\omega}_{\Lambda}(f))-F(\mu^{k,\omega}_{\Lambda}(f))|^2\,(|\Lambda|-k)\cdot\mu^{\omega}_{\Lambda,\lambda}(\chi_{\Lambda,k}).\end{aligned} \tag{2.22}$$

Using (**SSG**) together with (2.22) for $F(x)=x$, we obtain

$$\begin{aligned}M\mu^{\omega}_{\Lambda,\lambda}(\mu^{\omega}_{\Lambda,\lambda}(f \mid N_\Lambda)-\mu^{\omega}_{\Lambda,\lambda}(f))^2 \\ \leqslant \sum_{k=1}^{|\Lambda|} |\mu^{k,\omega}_{\Lambda}(f)-\mu^{k-1,\omega}_{\Lambda}(f)|^2\,k\cdot\mu^{\omega}_{\Lambda,\lambda}(\chi_{\Lambda,k}) \\ + \sum_{k=0}^{|\Lambda|-1} |\mu^{k+1,\omega}_{\Lambda}(f)-\mu^{k,\omega}_{\Lambda}(f)|^2\,(|\Lambda|-k)\cdot\mu^{\omega}_{\Lambda,\lambda}(\chi_{\Lambda,k}).\end{aligned} \tag{2.23}$$

The estimate of the right hand side will be based on the following lemma.

LEMMA 2.4. *There are constants a_1, a_2 depending only on Φ, such that for any cube Λ and any boundary condition ω, we have*

$$\begin{aligned}|\mu^{k,\omega}_{\Lambda}f-\mu^{k-1,\omega}_{\Lambda}f|^2 \leqslant a_1\frac{1}{\max(k,|\Lambda|-k)}\cdot\mu^{k,\omega}_{\Lambda}(f-\mu^{k,\omega}_{\Lambda}f)^2 \\ + a_2\cdot\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\|\nabla_i f\|_u\right)^2\end{aligned} \tag{2.24}$$

for any $k=1,\dots,|\Lambda|$.

We will prove this lemma later. Now assuming it, we see that using the estimate $\mathbf{SG}(\mu_\Lambda^{k,\omega})$, [6], and applying Lemma 2.4 to bound the first and the second sum from the right hand side (2.23), respectively, one easily gets the following estimate

$$\mu_{\Lambda,\lambda}^\omega(\mu_{\Lambda,\lambda}^\omega(f \mid N_\Lambda) - \mu_{\Lambda,\lambda}^\omega(f))^2$$
$$\leqslant 2a_0 a_1 M^{-1} \cdot |\Lambda|^{2/d} \cdot \mathbf{D}_{\Lambda,\lambda}^\omega(f) + a_2 M^{-1} \frac{1}{|\Lambda|} \left(\sum_{i \in \Lambda} \|\nabla_i f\|_u \right)^2 \tag{2.25}$$

Combining this together with (2.19)–(2.21) we arrive at the following inequality

$$\mu_{\Lambda,\lambda}^\omega(f - \mu_{\Lambda,\lambda}^\omega f)^2 \leqslant m_0^{-1} \cdot |\Lambda|^{2/d} \cdot \mathbf{D}_{\Lambda,\lambda}^\omega(f) + \varepsilon_0 \frac{1}{|\Lambda|} \left(\sum_{i \in \Lambda} \|\nabla_i f\|_u \right)^2 \tag{2.26}$$

with some constants m_0, $\varepsilon_0 \in (0, \infty)$. From this the general case with $q \in [1, 2)$ follows by a simple use of Hölder inequality. This ends the proof of asymptotic diffusivity estimate (2.15) assuming Lemma 2.4. ∎

Proof of Lemma 2.4. We begin from recalling a lemma, [6, Lemma 3.1], which allows us to compare the (mutually singular) measures $\mu_\Lambda^{k,\omega}$ and $\mu_\Lambda^{k+1,\omega}$. Let

$$G_i(\eta) := (1 - \eta_i) \exp\{-\nabla_i U_\Lambda(\eta \bullet_\Lambda \omega)\} \tag{2.27}$$

$$\hat{G}_i(\eta) := \eta_i \exp\{-\nabla_i U_\Lambda(\eta \bullet_\Lambda \omega)\}. \tag{2.28}$$

We have:

LEMMA 2.5 [6, Lemma 3.1]. *The following identities hold for any bounded* $\Lambda \subset \mathbb{Z}^d$ *and each* $\omega \in \Omega$

$$\text{(a)} \quad \mu_\Lambda^{k,\omega} f - \mu_\Lambda^{k+1,\omega} f = \frac{1}{k+1} \sum_{i \in \Lambda} \mu_\Lambda^{k+1,\omega}(\eta_i \nabla_i f)$$
$$- \sum_{i \in \Lambda} \mu_\Lambda^{k,\omega}(f; G_i) \Big/ \sum_{i \in \Lambda} \mu_\Lambda^{k,\omega}(G_i) \tag{2.29}$$

$$\text{(b)} \quad \mu_\Lambda^{k+1,\omega} f - \mu_\Lambda^{k,\omega} f = \frac{1}{|\Lambda| - k} \sum_{i \in \Lambda} \mu_\Lambda^{k,\omega}((1 - \eta_i) \nabla_i f)$$
$$- \sum_{i \in \Lambda} \mu_\Lambda^{k+1,\omega}(f; \hat{G}_i) \Big/ \sum_{i \in \Lambda} \mu_\Lambda^{k+1,\omega}(\hat{G}_i) \tag{2.30}$$

for any $k = 0, ..., |\Lambda| - 1$ *and all functions* $f \in \mathscr{C}(\Omega)$.

Now we note first that the first terms on the right hand sides of both cases in Lemma 2.5 have the same bound.

$$e^{4\,\|\Phi\|}\,\frac{1}{|\Lambda|}\cdot\sum_{i\in\Lambda}\|\nabla_i f\|_u$$

This is because we have

$$\begin{aligned}\left|\frac{1}{k+1}\sum_{i\in\Lambda}\mu_\Lambda^{k+1,\omega}(\eta_i\nabla_i f)\right| &\leqslant \frac{1}{k+1}\cdot\sup_{i\in\Lambda}\,(\mu_\Lambda^{k+1,\omega}(\eta_i))\cdot\sum_{i\in\Lambda}\|\nabla_i f\|_u\\ &\leqslant e^{4\,\|\Phi\|}\,\frac{1}{|\Lambda|}\cdot\sum_{i\in\Lambda}\|\nabla_i f\|_u \end{aligned}\tag{2.31}$$

and

$$\begin{aligned}\left|\frac{1}{|\Lambda|-k}\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}((1-\eta_i)\,\nabla_i f)\right| &\leqslant \frac{1}{|\Lambda|-k}\cdot\sup_{i\in\Lambda}\,(\mu_\Lambda^{k,\omega}(1-\eta_i))\cdot\sum_{i\in\Lambda}\|\nabla_i f\|_u\\ &\leqslant e^{4\,\|\Phi\|}\,\frac{1}{|\Lambda|}\cdot\sum_{i\in\Lambda}\|\nabla_i f\|_u, \end{aligned}\tag{2.32}$$

where the second step in these two inequalities is justified by the following lemma proven in [6, Lemma 3.3].

Lemma 2.6 [6, Lemma 3.3]. *For any bounded $\Lambda\subset\mathbf{Z}^d$ and each $\omega\in\Omega$*

$$e^{-4\,\|\Phi\|}\,\frac{k}{|\Lambda|}\leqslant\mu_\Lambda^{k,\omega}(\eta_i)\leqslant e^{4\,\|\Phi\|}\,\frac{k}{|\Lambda|}\tag{2.33}$$

and

$$e^{-4\,\|\Phi\|}\left(1-\frac{k}{|\Lambda|}\right)\leqslant\mu_\Lambda^{k,\omega}(1-\eta_i)\leqslant e^{4\,\|\Phi\|}\left(1-\frac{k}{|\Lambda|}\right)\tag{2.34}$$

for any $k=0, ..., |\Lambda|$ and all $i\in\Lambda$.

Now we need only to estimate the second term from the right hand side of (2.29) and (2.30), respectively, and finally, for a given k, choose the most convenient estimate. For this we note that by Hölder inequality, we have

$$\left(\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(f;G_i)\right)^2\leqslant\mu_\Lambda^{k,\omega}(f;f)\cdot\sum_{i,j\in\Lambda}\mu_\Lambda^{k,\omega}(G_i;G_j)\tag{2.35}$$

and

$$\left(\sum_{i\in\Lambda}\mu_\Lambda^{k+1,\omega}(f;\hat{G}_i)\right)^2\leqslant\mu_\Lambda^{k+1,\omega}(f;f)\cdot\sum_{i,j\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i;\hat{G}_j).\qquad(2.36)$$

Thus we need to estimate the following ratios:

$$\frac{\sum_{i,j\in\Lambda}\mu_\Lambda^{k,\omega}(G_i;G_j)}{(\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(G_i))^2}\quad\text{and}\quad\frac{\sum_{i,j\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i;\hat{G}_j)}{(\sum_{i\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i))^2}.\qquad(2.37)$$

To this end we note first that, using (2.27) and (2.28) together with Lemma 2.6, we have

$$e^{-6\,\|\Phi\|}(|\Lambda|-k)\leqslant\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(G_i)\qquad(2.38)$$

and

$$e^{-6\,\|\Phi\|}(k+1)\leqslant\sum_{i\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i).\qquad(2.39)$$

Thus to get the bounds of the ratios from (2.37), we will need the following lemma which is proven in the Appendix A.

LEMMA 2.7. *There is a constant* $C\in(0,\infty)$ *such that for any* $k=0,...,|\Lambda|-1$, *we have*

$$\sum_{i,j\in\Lambda}\mu_\Lambda^{k,\omega}(G_i;G_j)\leqslant C\cdot(|\Lambda|-k)\qquad(2.40)$$

and

$$\sum_{i,j\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i;\hat{G}_j)\leqslant C\cdot(k+1).\qquad(2.41)$$

With the above bounds we can now finish estimating the ratios given in (2.37). Using Lemma 2.6 together with (2.38), (respectively (2.39) in the second case), we get

$$\frac{\sum_{i,j\in\Lambda}\mu_\Lambda^{k,\omega}(G_i;G_j)}{(\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(G_i))^2}\leqslant Ce^{12\,\|\Phi\|}\,\frac{1}{|\Lambda|-k}\qquad(2.42)$$

and respectively in the second case

$$\frac{\sum_{i,j\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i;\hat{G}_j)}{(\sum_{i\in\Lambda}\mu_\Lambda^{k+1,\omega}(\hat{G}_i))^2}\leqslant Ce^{12\,\|\Phi\|}\,\frac{1}{k+1}.\qquad(2.43)$$

From this and (2.35)–(2.39) we obtain

$$2\left(\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(f;G_i)\Big/\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(G_i)\right)^2$$
$$\leqslant 2Ce^{12\,\|\Phi\|}\,\frac{1}{|\Lambda|-k}\cdot\mu_\Lambda^{k,\omega}(f-\mu_\Lambda^{k,\omega}f)^2 \tag{2.44}$$

and

$$2\left(\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}(f;\hat G_i)\Big/\sum_{i\in\Lambda}\mu_\Lambda^{k,\omega}\,\mu_\Lambda^{k,\omega}(\hat G_i)\right)^2$$
$$\leqslant 2Ce^{12\,\|\Phi\|}\,\frac{1}{k+1}\cdot\mu_\Lambda^{k,\omega}(f-\mu_\Lambda^{k,\omega}f)^2. \tag{2.45}$$

Combining these bounds together with (2.31)–(2.32) and recalling Lemma 2.5, we arrive at the following estimate

$$|\mu_\Lambda^{k,\omega}f-\mu_\Lambda^{k-1,\omega}f|^2\leqslant 2Ce^{12\,\|\Phi\|}\cdot\frac{1}{\max(k,\,|\Lambda|-k)}\cdot\mu_\Lambda^{k,\omega}(f-\mu_\Lambda^{k,\omega}f)^2$$
$$+\,2e^{12\,\|\Phi\|}\cdot\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\|\nabla_i f\|_u\right)^2 \tag{2.46}$$

This ends the proof of Lemma 2.4, hence of part (i) in Theorem 2.1. ∎

Proof of Theorem 2.1. S-Asymptotic diffusivity. We begin by observing that

$$\mu_{\Lambda,\lambda}^\omega\left(f\log\frac{f}{\mu_{\Lambda,\lambda}^\omega f}\right)=\mu_{\Lambda,\lambda}^\omega\left(\mu_{\Lambda,\lambda}^\omega\left(f\log\frac{f}{\mu_{\Lambda,\lambda}^\omega(f\mid N_\Lambda)}\,\middle|\,N_\Lambda\right)\right)$$
$$+\,\mu_{\Lambda,\lambda}^\omega\left(\mu_{\Lambda,\lambda}^\omega(f\mid N_\Lambda)\log\frac{\mu_{\Lambda,\lambda}^\omega(f\mid N_\Lambda)}{\mu_{\Lambda,\lambda}^\omega f}\right), \tag{2.47}$$

where, we recall

$$\mu_{\Lambda,\lambda}^\omega(f\mid N_\Lambda=k)=\frac{\mu_{\Lambda,\lambda}^\omega(\chi_{\Lambda,k}f)}{\mu_{\Lambda,\lambda}^\omega(\chi_{\Lambda,k})}\equiv\mu_\Lambda^{k,\omega}(f). \tag{2.48}$$

To estimate the first term on the right hand side of (2.47) we note that, [13], there is a constant $\bar c_0\in(0,\infty)$ such that for any $k=0,...,|\Lambda|$ we have

$$\mu_\Lambda^{k,\omega}\left(f\log\frac{f}{\mu_\Lambda^{k,\omega}f}\right)\leqslant\bar c_0\,|\Lambda|^{2/d}\cdot\mathbf{D}_\Lambda^{k,\omega}(f^{1/2}),\qquad (\mathbf{LN}(\mu_\Lambda^{k,\omega}))$$

where

$$\mathbf{D}_{\Lambda}^{k,\omega}(f^{1/2}) \equiv \tfrac{1}{2} \sum_{\langle ij \rangle \subset \Lambda} \mu_{\Lambda}^{k,\omega} |\delta_{ij} f^{1/2}|^2.$$

Therefore we get

$$\mu_{\Lambda,\lambda}^{\omega}\left(\mu_{\Lambda,\lambda}^{\omega}\left(f \log \frac{f}{\mu_{\Lambda,\lambda}^{\omega}(f \mid N_{\Lambda})} \,\middle|\, N_{\Lambda}\right)\right) \leqslant \bar{c}_0 \, |\Lambda|^{2/d} \cdot \mathbf{D}_{\Lambda,\lambda}^{\omega}(f^{1/2}). \tag{2.49}$$

To estimate the second term on the right hand side of (2.47) we note that under assumption of strong mixing the finite volume measures satisfy the following *Standard Logarithmic Sobolev inequality* [6–10, 14–16],

$$\mu_{\Lambda,\lambda}^{\omega}\left(g \log \frac{g}{\mu_{\Lambda,\lambda}^{\omega} g}\right) \leqslant \bar{c} \cdot \sum_{i \in \Lambda} \mu_{\Lambda,\lambda}^{\omega} |\nabla_i g^{1/2}|^2, \tag{\textbf{SLN}}$$

with some constant $\bar{c} \in (0, \infty)$ independent of Λ, ω, and g.

Applying (**SLN**) and using Lemma 2.3 with F denoting the square root we get

$$\begin{aligned}
&\mu_{\Lambda,\lambda}^{\omega}\left(\mu_{\Lambda,\lambda}^{\omega}(f \mid N_{\Lambda}) \log \frac{\mu_{\Lambda,\lambda}^{\omega}(f \mid N_{\Lambda})}{\mu_{\Lambda,\lambda}^{\omega} f}\right) \\
&\quad \leqslant \bar{c} \sum_{k=1}^{|\Lambda|} |(\mu_{\Lambda}^{k,\omega}(f))^{1/2} - (\mu_{\Lambda}^{k-1,\omega}(f))^{1/2}|^2 \, k \cdot \mu_{\Lambda,\lambda}^{\omega}(\chi_{\Lambda,k}) \\
&\qquad + \bar{c} \sum_{k=0}^{|\Lambda|-1} |(\mu_{\Lambda}^{k+1,\omega}(f))^{1/2} - (\mu_{\Lambda}^{k,\omega}(f))^{1/2}|^2 \, (|\Lambda| - k) \cdot \mu_{\Lambda,\lambda}^{\omega}(\chi_{\Lambda,k}).
\end{aligned} \tag{2.50}$$

The estimate of the right hand side will be based on the following lemma

Lemma 2.8. *There are constants b_1, b_2 dependent only on Φ, such that for any cube Λ and any boundary condition ω, we have*

$$\begin{aligned}
|(\mu_{\Lambda}^{k,\omega}(f))^{1/2} - (\mu_{\Lambda}^{k-1,\omega}(f))^{1/2}|^2 &\leqslant b_1 \cdot |\Lambda|^{2/d} \cdot [\mathbf{D}_{\Lambda}^{k,\omega}(f^{1/2}) + \mathbf{D}_{\Lambda}^{k-1,\omega}(f^{1/2})] \\
&\quad + b_2 \cdot \left(\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \|\nabla_i f^{1/2}\|_u\right)^2
\end{aligned} \tag{2.51}$$

for any $k = 1, ..., |\Lambda|$.

We will prove this lemma later. Now assuming it, we see that by applying Lemma 2.8 to bound the first and the second sum, respectively, from the right hand side of (2.50), one gets the estimate

$$\mu_{\Lambda,\lambda}^{\omega}\left(\mu_{\Lambda,\lambda}^{\omega}(f \mid N_\Lambda) \log \frac{\mu_{\Lambda,\lambda}^{\omega}(f \mid N_\Lambda)}{\mu_{\Lambda,\lambda}^{\omega} f}\right)$$

$$\leqslant 2b_1 \bar{c} \cdot |\Lambda|^{1+(2/d)} \cdot \mathbf{D}_{\Lambda,\lambda}^{\omega}(f) + 2b_2 \bar{c} \frac{1}{|\Lambda|} \left(\sum_{i \in \Lambda} \|\nabla_i f\|_u\right)^2. \tag{2.52}$$

Combining this together with (2.47)–(2.49) we arrive at the inequality

$$\mu_{\Lambda,\lambda}^{\omega}\left(f \log \frac{f}{\mu_{\Lambda,\lambda}^{\omega} f}\right) \leqslant c_0 \cdot |\Lambda|^{1+(2/d)} \cdot \mathbf{D}_{\Lambda,\lambda}^{\omega}(f)$$

$$+ \hat{\varepsilon}_0 \frac{1}{|\Lambda|} \left(\sum_{i \in \Lambda} \|\nabla_i f\|_u\right)^2 \tag{2.53}$$

with some constants c_0, $\hat{\varepsilon}_0 \in (0, \infty)$. From this the general case with $q \in [1, 2)$ follows by a simple use of Hölder inequality. This ends the proof of S-asymptotic diffusivity estimate (2.16) assuming Lemma 2.8. ▮

Proof of Lemma 2.8. We note first that we have

$$|(\mu_\Lambda^{k,\omega}(f))^{1/2} - (\mu_\Lambda^{k-1,\omega}(f))^{1/2}|$$

$$= |\mu_\Lambda^{k,\omega}(f) - \mu_\Lambda^{k-1,\omega}(f)| \cdot |(\mu_\Lambda^{k,\omega}(f))^{1/2} + (\mu_\Lambda^{k-1,\omega}(f))^{1/2}|^{-1} \tag{2.54}$$

and

$$\begin{aligned} |\mu_\Lambda^{k,\omega}(f) - \mu_\Lambda^{k-1,\omega}(f)| &= |\mu_\Lambda^{k,\omega} \otimes \tilde{\mu}_\Lambda^{k-1,\omega}(f - \tilde{f})| \\ &= |\mu_\Lambda^{k,\omega} \otimes \tilde{\mu}_\Lambda^{k-1,\omega}(f^{1/2} - \tilde{f}^{1/2})(f^{1/2} + \tilde{f}^{1/2})| \\ &\leqslant (\mu_\Lambda^{k,\omega} \otimes \tilde{\mu}_\Lambda^{k-1,\omega}(f^{1/2} - \tilde{f}^{1/2})^2)^{1/2} \\ &\quad \cdot ((\mu_\Lambda^{k,\omega}(f))^{1/2} + (\mu_\Lambda^{k-1,\omega}(f))^{1/2}), \end{aligned} \tag{2.55}$$

where $\tilde{f}^{1/2}$ is integrated with respect to the isomorphic copy $\tilde{\mu}_\Lambda^{k-1,\omega}$ of $\mu_\Lambda^{k-1,\omega}$. Using this we get

$$\begin{aligned} |(\mu_\Lambda^{k,\omega}(f))^{1/2} - (\mu_\Lambda^{k-1,\omega}(f))^{1/2}| &\leqslant (\mu_\Lambda^{k,\omega} \otimes \tilde{\mu}_\Lambda^{k-1,\omega}(f^{1/2} - \tilde{f}^{1/2})^2)^{1/2} \\ &\leqslant (\mu_\Lambda^{k,\omega}(f^{1/2} - \mu_\Lambda^{k,\omega} f^{1/2})^2)^{1/2} \\ &\quad + (\mu_\Lambda^{k-1,\omega}(f^{1/2} - \mu_\Lambda^{k-1,\omega} f^{1/2})^2)^{1/2} \\ &\quad + |\mu_\Lambda^{k,\omega} f^{1/2} - \mu_\Lambda^{k-1,\omega} f^{1/2}| \end{aligned} \tag{2.56}$$

and from this

$$\begin{aligned}
&|(\mu_\Lambda^{k,\omega}(f))^{1/2}-(\mu_\Lambda^{k-1,\omega}(f))^{1/2}|^2\\
&\quad\leqslant 3[\mu_\Lambda^{k,\omega}(f^{1/2}-\mu_\Lambda^{k,\omega}f^{1/2})^2+\mu_\Lambda^{k-1,\omega}(f^{1/2}-\mu_\Lambda^{k-1,\omega}f^{1/2})^2]\\
&\qquad+3\,|\mu_\Lambda^{k,\omega}f^{1/2}-\mu_\Lambda^{k-1,\omega}f^{1/2}|^2
\end{aligned}\tag{2.57}$$

Now we use the spectral gap inequality $\mathbf{SG}(\mu_\Lambda^{k,\omega})$ to estimate the first part from the right hand side of (2.57) as

$$\begin{aligned}
&3[\mu_\Lambda^{k,\omega}(f^{1/2}-\mu_\Lambda^{k,\omega}f^{1/2})^2+\mu_\Lambda^{k-1,\omega}(f^{1/2}-\mu_\Lambda^{k-1,\omega}f^{1/2})^2]\\
&\quad\leqslant 3a_0\,|\Lambda|^{2/d}\cdot[\mathbf{D}_\Lambda^{k,\omega}(f^{1/2})+\mathbf{D}_\Lambda^{k-1,\omega}(f^{1/2})].
\end{aligned}\tag{2.58}$$

The last part from the right hand side (2.57) can be estimated using Lemma 2.4 and $\mathbf{SG}(\mu_\Lambda^{k,\omega})$ as

$$\begin{aligned}
3\,|\mu_\Lambda^{k,\omega}f^{1/2}-\mu_\Lambda^{k-1,\omega}f^{1/2}|^2 &\leqslant 3a_1\frac{1}{\max(k,|\Lambda|-k)}\cdot\mu_\Lambda^{k,\omega}(f^{1/2}-\mu_\Lambda^{k,\omega}f^{1/2})^2\\
&\quad+3a_2\cdot\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_u\right)^2\\
&\leqslant 3a_1\frac{1}{\max(k,|\Lambda|-k)}\cdot a_0\,|\Lambda|^{2/d}\cdot\mathbf{D}_\Lambda^{k,\omega}(f^{1/2})\\
&\quad+3a_2\cdot\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_u\right)^2.
\end{aligned}\tag{2.59}$$

Combining (2.54)–(2.58) and (2.59), we arrive at the desired estimate. This ends the proof of Lemma 2.8. ∎

3. SOME FINAL REMARKS

In this paper we have shown that there is a systematic method of proving of coercive inequalities for a general class of nontrivial infinite dimensional models. In particular under general assumptions concerning the mixing property of a local specification which assures that the corresponding unique Gibbs measure satisfies the Standard Spectral Gap and the Standard Logarithmic Sobolev inequality, we have shown that also a family of Generalized Nash and Logarithmic Nash inequalities hold. The later type of inequalities provides us with new interesting bounds on entropy in terms of a Dirichlet forms related to some stochastic dynamics with a diffusive behaviour. On the other hand it can be considered as an interesting characterization of the domain of the generator of this dynamics.

To get another profit from our inequalities in the form of a control of the decay to equilibrium in $\mathbb{L}_2$ or entropy sense, one needs to get more information about monotonicity properties of the related A functionals. For this it would be useful to have more information about monotone or bounded functionals for a given stochastic dynamics. In general it is a difficult and wide open question how characterize them and should be studied in a future. Here we would like to point out that in fact to get some decay it is sufficient to have a weaker property than monotonicity or boundedness, and for example the following fact is true.

PROPOSITION 3.1. *Let μ satisfies the General Nash Inequality*

$$\mu(f-\mu f)^2 \leqslant \mathbf{D}(f)^\alpha A(f)^{1-\alpha} \tag{3.1}$$

for some $\alpha \in (0,1)$ and a functional A satisfying

$$A(P_t f) \leqslant \max\{1, t^\varepsilon\} B(f) \tag{3.2}$$

for some $\varepsilon \in [0, \alpha/(1-\alpha))$ and a functional B densely defined on some domain $\mathscr{D}(B)$. Let $\gamma = \alpha/(1-\alpha)$; then

$$\mu(P_t f - \mu f)^2 \leqslant \gamma^\gamma \frac{B(f)}{t^{\gamma-\varepsilon}} \tag{3.3}$$

for any $t \geqslant \max\{1, (2\varepsilon/(\gamma+\varepsilon))^{\gamma/(\gamma-\varepsilon)}\}$ and function $f \in \mathscr{D}(B) \cap \mathscr{Q}(\mathbf{D})$.

Proof. We can assume $\mu f = 0$. Let us define $F(t) := \mu(P_t f)^2$; the inequalities (3.1) and (3.2) imply

$$\frac{d}{dt} F(t) = -2\mathbf{D}(P_t f) \leqslant -2F(t)^{1/\alpha} (\max\{1, t^\varepsilon\})^{-1/\gamma} B(f)^{-1/\gamma}.$$

Solving the above differential inequality we get, for $t \geqslant 1$,

$$F(t) \leqslant B(f) \left(\frac{\gamma}{2}\right)^\gamma \left[1 + \frac{t^{1-\varepsilon/\gamma} - 1}{1-\varepsilon/\gamma}\right]^{-\gamma}$$

and elementary estimates yield (3.3). ∎

Finally we would like to indicate that our analysis of the product case [2] suggests that, in case of the Kawasaki dynamics, it should be possible to get the coercive inequalities of interest to use also with some functionals which could give some faster decay to equilibrium. This problem should also be studied in a future.

APPENDIX A: COVARIANCE ESTIMATES FOR CANONICAL GIBBS MEASURES

In this Appendix we prove the technical estimates used in the proof of Theorem 2.1. We shall need some mixing property for the canonical Gibbs measures $\mu_\Lambda^{k,\omega}$, which are formulated in the lemma below, see [6, A.2] and [13].

LEMMA A.1. *Let $\delta > 0$, there exist a function $\varphi\colon \mathbb{R} \mapsto [0, \infty)$, $\varphi(r) \leqslant \varphi_0 r^{-(d+\delta)}$ and a constant $B_0 \in (0, \infty)$ depending only on the interaction Φ, such that for any cube $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \Omega$*

$$|\mu_\Lambda^{k,\omega}(f; g)| \leqslant B_0 \cdot |\mathrm{supp}\, f|\ |\mathrm{supp}\, g|\ \|f\|_u\ \|g\|_u \times [|\Lambda|^{-1} + \varphi(\mathrm{d}(\mathrm{supp}\, f, \mathrm{supp}\, g))] \tag{A.1}$$

for any $k = 0, ..., |\Lambda|$ and all Σ_Λ-measurable functions f, g.

As a consequence we get the following estimate on the dependence of $\mu_\Lambda^{k,\omega}$ on the boundary condition ω, see [6, Lemma 3.2].

LEMMA A.2 ([6, Lemma 3.2]). *There is a constant $B_1 \in (0, \infty)$ dependent only on the interaction Φ, such that for any cube $\Lambda \subset \mathbb{Z}^d$, $\omega \in \Omega$ and each $i \in \partial_R \Lambda$*

$$|\mu_\Lambda^{k,\omega^i}(f) - \mu_\Lambda^{k,\omega}(f)| \leqslant B_1 \cdot |\mathrm{supp}\, f|\ \|f\|_u [|\Lambda|^{-1} + \varphi(\mathrm{d}(i, \mathrm{supp}\, f))] \tag{A.2}$$

for any $k = 0, ..., |\Lambda|$ and any Σ_Λ-measurable function f.

Remark. We note that, by changing the constants B_0, B_1, the above Lemmata A.1, A.2 holds also when the cube Λ is replaced by $\Lambda \backslash \{j\}$ (or $\Lambda \backslash \{j, j'\}$), where $j, j' \in \Lambda$. We shall therefore apply them also in the latter setting without further mention.

From the above estimates we deduce a sharp bound on the covariance between η_i and η_j.

LEMMA A.3. *There is a constants $B_2 \in (0, \infty)$ dependent only on the interaction Φ, such that for any cube $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \Omega$*

$$|\mu_\Lambda^{k,\omega}(\eta_i; \eta_j)| \leqslant B_2 \frac{k}{|\Lambda|} \left(1 - \frac{k}{|\Lambda|}\right) [|\Lambda|^{-1} + \varphi(\mathrm{d}(i, j))] \tag{A.3}$$

for any $k = 0, ..., |\Lambda|$ and all $i, j \in \Lambda$.

Remark. We note that if there is no interaction, $\Phi \equiv 0$, we have

$$\mu_\Lambda^k(\eta(i); \eta(j)) = -\frac{k}{|\Lambda|}\left(1 - \frac{k}{|\Lambda|}\right) \cdot \frac{1}{|\Lambda| - 1}$$

so that (A.3) catches the correct dependence on k, $|\Lambda|$.

Proof. We note first that we have the following representation of the covariance of interest to us

$$\begin{aligned}
\mu_\Lambda^{k,\omega}(\eta_i; \eta_j) &\equiv \mu_\Lambda^{k,\omega}(\eta_i \eta_j) - \mu_\Lambda^{k,\omega}(\eta_i) \cdot \mu_\Lambda^{k,\omega}(\eta_j) \\
&= \mu_\Lambda^{k,\omega}(\eta_j)\, \mu_{\Lambda\setminus j,\, \eta_j = 1}^{k-1,\omega}(\eta_i) - \mu_\Lambda^{k,\omega}(\eta_j) \\
&\quad \times (\mu_\Lambda^{k,\omega}(\eta_j)\, \mu_{\Lambda\setminus j,\, \eta_j = 1}^{k-1,\omega}(\eta_i) + \mu_\Lambda^{k,\omega}(1 - \eta_j)\, \mu_{\Lambda\setminus j,\, \eta_j = 0}^{k,\omega}(\eta_i)) \\
&= \mu_\Lambda^{k,\omega}(\eta_j)\, \mu_\Lambda^{k,\omega}(1 - \eta_j)(\mu_{\Lambda\setminus j,\, \eta_j = 1}^{k-1,\omega}(\eta_i) - \mu_{\Lambda\setminus j,\, \eta_j = 0}^{k,\omega}(\eta_i)).
\end{aligned} \tag{A.4}$$

The first two factors from the right hand side of (A.4) can be estimated using Lemma 2.6. To estimate the last factor on the right hand side of (A.4) we use the decomposition

$$\begin{aligned}
|\mu_{\Lambda\setminus j,\, \eta_j = 1}^{k-1,\omega}(\eta_i) - \mu_{\Lambda\setminus j,\, \eta_j = 0}^{k,\omega}(\eta_i)| &\leqslant |\mu_{\Lambda\setminus j,\, \eta_j = 1}^{k-1,\omega}(\eta_i) - \mu_{\Lambda\setminus j,\, \eta_j = 1}^{k,\omega}(\eta_i)| \\
&\quad + |\mu_{\Lambda\setminus j,\, \eta_j = 1}^{k,\omega}(\eta_i) - \mu_{\Lambda\setminus j,\, \eta_j = 0}^{k,\omega}(\eta_i)|.
\end{aligned}$$

The second term in the above inequality is estimated by applying Lemma A.2; to bound the first term we use Lemma 2.5 (a) or (b), dependent on whether $k \leqslant (|\Lambda|/2)$ or not. Since both cases are similar, we consider here only the case $k \leqslant (|\Lambda|/2)$; to simplify the notation we introduce $\Lambda' \equiv \Lambda \setminus j$ and $\omega' \equiv \omega \bullet_j \{\omega_j = 1\}$. We have

$$\begin{aligned}
\mu_{\Lambda'}^{k-1,\omega'}(\eta_i) - \mu_{\Lambda'}^{k,\omega'}(\eta_i) &= \frac{1}{k} \sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(\eta_l \nabla_l \eta_i) - \frac{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(\eta_i; G_l)}{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(G_l)} \\
&= \frac{1}{k} \cdot \mu_{\Lambda'}^{k,\omega'}(\eta_i(1 - 2\eta_i)) - \frac{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(\eta_i; G_l)}{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(G_l)}.
\end{aligned} \tag{A.5}$$

From Lemma 2.6 the first term on the right hand side of (A.5) has the estimate

$$\left|\frac{1}{k} \cdot \mu_{\Lambda'}^{k,\omega'}(\eta_i(1 - 2\eta_i))\right| \leqslant e^{4\,\|\Phi\|} \cdot \frac{1}{|\Lambda'|}.$$

Using Lemma A.1 together with estimate (2.38) we get

$$\left| \frac{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(\eta_i; G_i)}{\sum_{l \in \Lambda'} \mu_{\Lambda'}^{k,\omega'}(G_l)} \right|$$

$$\leqslant e^{6\|\Phi\|}(|\Lambda'|-k)^{-1} \sum_{l \in \Lambda'} B_0 \cdot R^d e^{2\|\Phi\|}[|\Lambda'|^{-1} + B_R \cdot \varphi(\mathrm{d}(i,l))]$$

$$\leqslant B_0 \cdot R^d e^{8\|\Phi\|} \left[1 + B_R \cdot \sum_{l \in \mathbb{Z}^d} \varphi(\mathrm{d}(l,0)) \right] (|\Lambda'|-k)^{-1}$$

$$\leqslant 2B_0 \cdot R^d e^{8\|\Phi\|} \left[1 + B_R \cdot \sum_{l \in \mathbb{Z}^d} \varphi(\mathrm{d}(l,0)) \right] \cdot |\Lambda'|^{-1}, \tag{A.6}$$

where $B_R \equiv \sup\{(\mathrm{d}(i,l')/\mathrm{d}(i,l)) : i \neq l \text{ and } l, l' : \mathrm{d}(l,l') \leqslant R\}$ and in the last step we have inserted our assumption $k \leqslant (|\Lambda|/2)$. This ends the proof of Lemma A.3. ∎

We can now prove Lemma 2.7 which has been used in the proof of the asymptotically diffusive inequality (2.16).

Proof of Lemma 2.7. We shall prove only the inequality (2.40), the proof of (2.41) being similar. We note that

$$\mu_\Lambda^{k,\omega}(G_i; G_j) = \mu_\Lambda^{k,\omega}(\mu_\Lambda^{k,\omega}(G_i; G_j \mid \eta_i, \eta_j))$$
$$+ \mu_\Lambda^{k,\omega}(\mu_\Lambda^{k,\omega}(G_i \mid \eta_i, \eta_j); \mu_\Lambda^{k,\omega}(G_j \mid \eta_i, \eta_j)). \tag{A.7}$$

Since $G_i \equiv (1-\eta_i) \exp\{-\nabla_i U_\Lambda(\eta \bullet_\Lambda \omega)\} \equiv (1-\eta_i)\, g_i$, for the first term on the right hand side of (A.7), we have

$$\sum_{i,j \in \Lambda} |\mu_\Lambda^{k,\omega}(\mu_\Lambda^{k,\omega}(G_i; G_j \mid \eta_i, \eta_j))|$$

$$= \sum_{i,j \in \Lambda} \mu_\Lambda^{k,\omega}((1-\eta_i)(1-\eta_j))\, |\mu_\Lambda^{k,\omega}(g_i; g_j \mid \eta_i = 0, \eta_j = 0)|$$

$$\leqslant \sum_{i,j \in \Lambda} e^{4\|\Phi\|} \cdot \left(1 - \frac{k}{|\Lambda|}\right) \cdot |\mu_\Lambda^{k,\omega}(g_i; g_j \mid \eta_i = 0, \eta_j = 0)|$$

$$\leqslant B_0' R^{2d} e^{8\|\Phi\|} \cdot \left(1 - \frac{k}{|\Lambda|}\right) \cdot \sum_{i,j \in \Lambda} [|\Lambda|^{-1} + \varphi(\mathrm{d}(i,j))]$$

$$\leqslant \left[\left(1 + \sum_{i \in \mathbb{Z}^d} \varphi(i)\right) B_0' R^{2d} e^{8\|\Phi\|} \right] \cdot (|\Lambda| - k), \tag{A.8}$$

where we have used Lemma 2.6, the definition of g_i together with the fact that the interaction Φ is of finite range R and Lemma A.1.

It will be useful to represent the second term on the right hand side of (A.7) as

$$\begin{aligned}
&\mu_\Lambda^{k,\omega}(\mu_\Lambda^{k,\omega}(G_i \mid \eta_i, \eta_j); \mu_\Lambda^{k,\omega}(G_j \mid \eta_i, \eta_j)) \\
&\qquad = \mu_\Lambda^{k,\omega}((1-\eta_i); (1-\eta_j)) \cdot \mu_\Lambda^{k,\omega}(g_i \mid \eta_i = 0, \eta_j = 0) \\
&\qquad\qquad \cdot \mu_\Lambda^{k,\omega}(g_j \mid \eta_i = 0, \eta_j = 0) - R_\Lambda(i, j),
\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}
R_\Lambda(i, j) \equiv{}& R_\Lambda^1(i, j)\, R_\Lambda^2(i, j)\, \mu_\Lambda^{k,\omega}(\eta_i(1-\eta_j))\, \mu_\Lambda^{k,\omega}((1-\eta_i)\,\eta_j) \\
&+ R_\Lambda^1(i, j)\, \mu_\Lambda^{k,\omega}(g_j \mid \eta_i = 0, \eta_j = 0)\, \mu_\Lambda^{k,\omega}((1-\eta_i)\,\eta_j)\, \mu_\Lambda^{k,\omega}(1-\eta_j) \\
&+ R_\Lambda^2(i, j)\, \mu_\Lambda^{k,\omega}(g_i \mid \eta_i = 0, \eta_j = 0)\, \mu_\Lambda^{k,\omega}(\eta_i(1-\eta_j))\, \mu_\Lambda^{k,\omega}(1-\eta_i)
\end{aligned}$$

in which

$$\begin{aligned}
R_\Lambda^1(i, j) &\equiv \mu_\Lambda^{k,\omega}(g_i \mid \eta_i = 0, \eta_j = 1) - \mu_\Lambda^{k,\omega}(g_i \mid \eta_i = 0, \eta_j = 0) \\
R_\Lambda^2(i, j) &\equiv \mu_\Lambda^{k,\omega}(g_j \mid \eta_i = 1, \eta_j = 0) - \mu_\Lambda^{k,\omega}(g_j \mid \eta_i = 0, \eta_j = 0).
\end{aligned}$$

Using Lemma A.3, since $\|g_i\|_u \leqslant \exp\{2\,\|\Phi\|\}$, we can bound the first term on the right hand side of (A.9) as

$$\begin{aligned}
&\sum_{i, j \in \Lambda} |\mu_\Lambda^{k,\omega}((1-\eta_i); (1-\eta_j)) \times \mu_\Lambda^{k,\omega}(g_i \mid \eta_i = 0, \eta_j = 0)| \\
&\qquad \leqslant B_2 e^{4\,\|\Phi\|} \cdot \frac{k}{|\Lambda|}\left(1 - \frac{k}{|\Lambda|}\right) \sum_{i, j \in \Lambda} [\,|\Lambda|^{-1} + \varphi(\mathrm{d}(i, j))] \\
&\qquad \leqslant \left(1 + \sum_{i \in \mathbb{Z}^d} \varphi(\mathrm{d}(i, 0))\right) B_2 e^{4\,\|\Phi\|} \cdot \frac{k}{|\Lambda|}\,(|\Lambda| - k).
\end{aligned} \tag{A.10}$$

It remains to consider the second term on the right hand side of (A.9). For this we proceed as in Lemma A.3,

$$|R_\Lambda^1(i, j)| \leqslant |\mu_{\tilde\Lambda}^{k-1, \tilde\omega}(g_i) - \mu_{\tilde\Lambda}^{k, \tilde\omega}(g_i)| + |\mu_{\tilde\Lambda}^{k, \tilde\omega^j}(g_i) - \mu_{\tilde\Lambda}^{k, \tilde\omega}(g_i)|,$$

where $\tilde\Lambda \equiv \Lambda \backslash \{ij\}$, $\tilde\omega \equiv \omega \bullet_{\{ij\}} \{\omega_i = 0,\ \omega_j = 1\}$. We then bound the first term using Lemma 2.5 (see (A.5)–(A.6)) and the second by applying Lemma A.2. The bound for $R_\Lambda^2(i, j)$ is analogous. We find

$$|R_\Lambda^l(i, j)| \leqslant B_4\left(\frac{1}{|\Lambda|} + \varphi(\mathrm{d}(i, j))\right) \qquad l = 1, 2$$

for some constant B_4 depending only on Φ. Hence

$$\begin{aligned}\sum_{i,\,j\in\Lambda}|R_\Lambda(i,j)| &\leqslant B_4\sum_{i,\,j\in\Lambda}\left(\frac{1}{|\Lambda|}+\varphi(\mathrm{d}(i,j))\right)\\ &\qquad\times[2\mu_\Lambda^{k,\omega}(1-\eta_i)\,\mu_\Lambda^{k,\omega}(\eta_i)+\mu_\Lambda^{k,\omega}(\eta_j)\,\mu_\Lambda^{k,\omega}(1-\eta_j)]\\ &\leqslant 3B_4e^{8\,\|\Phi\|}\sum_{i,\,j\in\Lambda}\left(\frac{1}{|\Lambda|}+\varphi(\mathrm{d}(i,j))\right)\cdot\frac{k}{|\Lambda|}\left(1-\frac{k}{|\Lambda|}\right)\\ &\leqslant 3B_4e^{8\,\|\Phi\|}\left(1+\sum_{i\in\mathbb{Z}^d}\varphi(\mathrm{d}(0,i))\right)\cdot\frac{k}{|\Lambda|}(|\Lambda|-k),\end{aligned}\tag{A.11}$$

where we used Lemma 2.6.

From (A.8)–(A.11) we deduce the estimate (2.40). The proof of (2.41) is similar. This ends the proof of Lemma 2.7. ∎

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