Markov Processes Relat. Fields 5, 125–162 (1999)

Markov
Processes
and
Related Fields

Coercive Inequalities for Kawasaki Dynamics. The Product Case

L. Bertini[1] and B. Zegarlinski[2]

[1] Department of Mathematics, University of Rome I, Rome, Italy
[2] Department of Mathematics, Imperial College, London, UK

Received September 29, 1998

Abstract. We prove the Generalized Nash and Logarithmic Nash inequalities for a product measure with Dirichlet form associated to the Kawasaki dynamics.

KEYWORDS: Generalized Nash, Logarithmic Nash inequalities, Bernoulli measures, Kawasaki dynamics

AMS SUBJECT CLASSIFICATION: Primary 60K35, 46N55; Secondary 82C22, 82C20

1. Introduction

Let (Ω, Σ) be a Polish space with its Borel σ-algebra and let $P_t \equiv e^{t\mathcal{L}}$ be a Markov semigroup on the space of continuous functions $\mathcal{C}(\Omega)$. Let μ be a probability measure on (Ω, Σ) which is P_t invariant.

We will say that we have *decay to equilibrium in* L_2 *sense* iff there is a positive function $\theta(t)$ decreasing to zero when $t \nearrow \infty$ and a functional A with a dense domain $\mathcal{D}(A) \subset \mathcal{C}(\Omega)$ such that for any $f \in \mathcal{D}(A)$ we have

$$\mu(P_t f - \mu f)^2 \le \theta(t) \cdot A(f). \tag{1.1}$$

We will say that we have *decay to equilibrium in the relative entropy sense* iff there is a positive function $\theta(t)$ decreasing to zero when $t \nearrow \infty$ and a functional A with a dense domain $\mathcal{D}(A) \subset \mathcal{C}(\Omega)$ such that for any $f \ge 0$, $f^{1/2} \in \mathcal{D}(A)$, we have

$$\mu\Big(P_t f \log \frac{P_t f}{\mu f}\Big) \le \theta(t) \cdot A(f^{1/2}). \tag{1.2}$$

In both cases we assume that the functional A is homogeneous of degree two and vanishing on constants.

If the generator $\mathcal{L}$ is self-adjoint in $\mathsf{L}_2(\mu)$ and has non-degenerate ground state, then using spectral theory one can show the following estimate of L_2 decay to equilibrium: for any positive function G, one has

$$\mu(P_t f - \mu f)^2 \leq \sup_{\lambda>0}(e^{-\lambda t} G(\lambda)^{-1}) \cdot \mu\,(G(\mathcal{L})(f - \mu f))^2 \tag{1.3}$$

provided $f \in \mathcal{D}(G(\mathcal{L}))$. In particular if the spectrum of the generator $\mathcal{L}$ is continuous at zero, to get a non-trivial information about the long time behaviour we have to choose a function G which is singular at zero. However in this case $G(\mathcal{L})$ can be defined possibly only on a dense set of functions. Clearly we will have different decay to equilibrium for different classes of functions with different spectral behaviour (i.e. behaviour of corresponding spectral measures in the neighborhood of zero). This gives us a general abstract way of describing the decay to equilibrium in L_2 sense. Unfortunately in practice it is very difficult to get a detailed description of the spectral measures of a self-adjoint unbounded operator or characterize a domain of its unbounded functions. Also this abstract way based only on the spectral theory does not need to give the optimal estimate on the rate function.

Using the fact that the first derivative of $\mu(P_t f - \mu f)^2$ is a decreasing function of time t, one can see that

$$\mu(f - \mu f)^2 - t\mathcal{E}(f) \leq \mu(P_t f - \mu f)^2$$

with $\mathcal{E}$ denoting the corresponding Dirichlet form. Hence, given L_2 decay to equilibrium, one can easily show that the following coercive inequality, called later on the *Generalized Nash* inequality, is true

$$\mu(f - \mu f)^2 \leq \inf_{t>0}(t\mathcal{E}(f) + \theta(t) \cdot A(f)). \tag{1.4}$$

Motivated by the above considerations one would like also to introduce the following entropy bound, called later on the *Logarithmic Nash* inequality

$$\mu\Big(f \log \frac{f}{\mu f}\Big) \leq \inf_{t>0}(t\mathcal{E}(f) + \theta(t) \cdot A(f^{1/2})), \tag{1.5}$$

although in general this coercive inequality does not need to follow from the decay in entropy sense (unless some extra conditions, as e.g. weak Bakry–Emery condition, implying the monotonicity of the derivative of the entropy are imposed).

Inequality (1.4) in case when $\theta(t) = a_\gamma t^{-\gamma}$, with some constants $a_\gamma \in (0,1)$ and $\gamma \in (0,\infty)$, can be written in the more familiar in the literature form, see e.g. [4,5,8,12,27],

$$\text{(GN)} \qquad \mu(f - \mu f)^2 \leq \mathcal{E}(f)^\alpha \cdot A(f)^{1-\alpha}$$

with $\alpha = \gamma/(1+\gamma)$. Similarly in the case of relative entropy bound, for the algebraic rate function $\theta(t)$ one gets

$$\text{(LN)} \qquad \mu\Big(f \log \frac{f}{\mu f}\Big) \leq \mathcal{E}(f)^\alpha \cdot A(f^{1/2})^{1-\alpha}.$$

In particular if one takes $A(f) = (\mu|f - \mu f|)^2$, one obtains the Classical Nash inequality, which implies the ultracontractivity property of the corresponding semigroup [5]. Alternatively when $\alpha = 1$ and A is replaced by a constant, or when $A(f)$ equals a constant multiplied by $\mu(f - \mu f)^2$, one obtains the Spectral Gap inequality. In the case of entropy bound the corresponding choices give us the Logarithmic Sobolev inequality, which is equivalent to the hypercontractivity property of the corresponding semigroup [9].

It is an interesting fact that the coercive inequalities (GN) and (LN) introduced above imply a corresponding decay to equilibrium (even without symmetry assumption about the semigroup), provided that the corresponding functional A satisfies additional properties. In the first case we need one of the following properties to be true:

A is a P_t-monotone functional, i.e. for any $t \geq 0$ we have $P_t\mathcal{D}(A) \subseteq \mathcal{D}(A)$ and

$$A(P_t f) \leq A(f);$$

A is a P_t-bounded functional, i.e. there is a functional B with dense domain satisfying $P_t\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and

$$A(P_t f) \leq B(f)$$

for any $t \geq 0$.

In the case relevant to the Logarithmic Nash inequality one needs similar properties for $A(f^{1/2})$. Given these properties, one can derive the corresponding decay to equilibrium by the classical method based on the differential inequalities for relevant quantities (and a property of the Dirichlet form in the case of entropy).

Clearly all the L_p norms, with $p \in [1,\infty]$, as well as functionals

$$f \longmapsto \mu(G(\mathcal{L})(f - \mu f))^2,$$

are monotone. In many particular examples it is possible to show existence of other interesting and useful monotone functionals, [1,11,12]. It is clear that if a functional A is P_t-bounded, then one can replace it by a monotone functional $\tilde{A}(f) \equiv \sup_{t\geq 0} A(P_t f)$. However such functional is highly non-local and from the practical point of view can be not very useful. In general, it is a rather challenging problem how to characterize a set of all monotone or bounded functionals for a given semigroups. It is also an important one, due to its relevance to the study of decay to equilibrium via the General Nash and Logarithmic Nash inequalities.

In the present paper we consider the symmetric exclusion process and demonstrate on this simple example that there is a systematic strategy for proving General Nash and Logarithmic Nash inequalities. Moreover, we show that in this case one can give a comprehensive description of the set of corresponding monotone functionals and thus our Nash inequalities imply the decay to equilibrium in L_2 as well as in the relative entropy sense, respectively. Let us mention that the decay to equilibrium in the L_2 sense for this model has been studied by a method strongly based on the use of couplings and attractivity property of the corresponding stochastic dynamics in [6], (see also references given there). Of course, by general arguments, this decay implies the corresponding Nash inequality. However our proof of Nash inequalities is free of extra properties of this model and, as we demonstrate in [3], it applies as well to the situations where a non-trivial interaction is present. Unfortunately in that more general situation at the moment it seems to be difficult to get optimal estimates and understand the corresponding decay to equilibrium. Therefore it is important to develop and understand a fully fledged theory of *generalized Sobolev spaces* in the free case.

We would like to mention also a work [12] in which a General Nash inequality has been proven for a nearest particle dynamics of a one-dimensional model with some long range interaction.

The results concerning the Logarithmic Nash inequality are new. This interesting inequality, which in certain sense is stronger than the General Nash inequality, certainly deserves further studies. As we show in [3], the Logarithmic Nash inequality is also true in non-product case (though again with less optimal exponents).

The paper is organized as follows. In Section 2 we introduce a basic notation and formulate (a part of) results of this work. Section 3 and Section 4 are devoted to the proof of General Nash and Logarithmic Nash inequality, respectively. The necessary study of the set of monotone functionals is done in Section 5. In the last section we demonstrate that one can extend our strategy to prove also a faster decay to equilibrium for appropriate classes of functions (by which we perform a more detailed analysis of the corresponding spectral properties).

2. Notation and results

The state space of the process is $\Omega := \{0,1\}^{\mathbf{Z}^d}$, which is endowed with the product topology. Its elements $\eta \equiv \{\eta(i) : i \in \mathbf{Z}^d\}$ will be called configurations of particles. We will say that a site $i \in \mathbf{Z}^d$ is occupied (empty) iff $\eta(i) = 1$ ($\eta(i) = 0$, respectively). The space of continuous functions on Ω, denoted by $\mathcal{C}(\Omega)$, becomes a Banach space under the uniform norm

$$\|f\|_u := \sup_{\eta\in\Omega} |f(\eta)|. \tag{2.1}$$

We introduce an unbounded operator $\mathcal{L}$ in $\mathcal{C}(\Omega)$ densely defined on the cylinder functions (i.e. functions depending only on finitely many coordinates) by

$$\mathcal{L}f(\eta) = \sum_{\langle i,j\rangle} \delta_{i,j} f(\eta), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ indicates the nearest neighbors pairs in $\mathbf{Z}^d$ and

$$\delta_{i,j} f(\eta) := T_{i,j} f(\eta) - f(\eta)$$

with $T_{i,j} f(\eta) \equiv f(\eta^{i,j})$,

$$\eta^{i,j}(k) := \begin{cases} \eta(i) & \text{if } k = j, \\ \eta(j) & \text{if } k = i, \\ \eta(k) & \text{otherwise,} \end{cases} \tag{2.3}$$

i.e. $\eta^{i,j}$ is the configuration obtained from η by exchanging the coordinates at the points i and j. By [11, I, Theorem 3.9] the closure of $\mathcal{L}$ (denoted later on by the same symbol) generates a Markov semigroup P_t on $\mathcal{C}(\Omega)$. We remark that the set of cylinder functions is a core for the Markov generator.

For $\rho \in [0,1]$, let μ_ρ be the Bernoulli measure with density ρ, i.e. the product measure on Ω with marginals satisfying $\mu_\rho(\eta(i)) = \rho$ and

$$\mu_\rho\,(\eta(i) - \rho)^2 = \rho(1-\rho) \equiv \mathbf{v}_\rho.$$

It is not difficult to verify that μ_ρ is invariant and reversible for P_t. It is also known (see [11, VIII, Theorem 1.44]) that the set of invariant measures for P_t consists of all convex linear combinations of μ_ρ, $\rho \in [0,1]$.

We can then define the semigroup P_t also on the space $L_2(\mu_\rho)$. By reversibility it is a self-adjoint contracting semigroup and the Dirichlet form associated to it is given by

$$\mathbf{D}_\rho(f) \equiv \frac{1}{2}\sum_{\langle i,j\rangle} \mu_\rho\,(\delta_{i,j} f)^2 \tag{2.4}$$

on some domain $\mathcal{Q}(\mathbf{D}_\rho)$. In particular, it is bounded if f is cylinder, and then we have

$$\mathbf{D}_\rho(f) = -\mu_\rho\,(f\mathcal{L}f).$$

We introduce a Lipschitz-type seminorm $|\!|\!|f|\!|\!|_{p,q}$, $p,q \in [1,\infty]$ defined as follows

$$|\!|\!|f|\!|\!|_{p,q} := \Big(\sum_{i\in\mathbf{Z}^d} \|\nabla_i f\|_p^q\Big)^{1/q}, \qquad \|f\|_p := (\mu_\rho|f|^p)^{1/p}, \tag{2.5}$$

where

$$\nabla_i f(\eta) := f(\eta^i) - f(\eta) \tag{2.6}$$

with $\eta^i \in \Omega$ being a configuration "flipped" at site $i \in \mathbf{Z}^d$, defined as follows

$$\eta^i(j) := \begin{cases} 1-\eta(i) & \text{if } j = i, \\ \eta(j) & \text{otherwise.} \end{cases} \tag{2.7}$$

Note that $|\!|\!| f |\!|\!|_{p,q} = 0$ iff f is constant μ_ρ-a.s. (and thus $|\!|\!| \cdot |\!|\!|_{p,q}$ is only a seminorm). Let us also define

$$\mathcal{C}_{p,q}(\Omega) := \{ f \in \mathsf{L}_p(\mu_\rho) : |\!|\!| f |\!|\!|_{p,q} < \infty \}. \tag{2.8}$$

Since for $q \geq q'$, $p' \geq p$ we have

$$|\!|\!| f |\!|\!|_{p,q} \leq |\!|\!| f |\!|\!|_{p',q'} \tag{2.9}$$

therefore also

$$\mathcal{C}_{p',q'}(\Omega) \subset \mathcal{C}_{p,q}(\Omega). \tag{2.10}$$

We note that we have the following useful relation between the operators $\delta_{i,j}$ and ∇_i

$$\delta_{i,j} = [\eta(i)(1-\eta(j)) + \eta(j)(1-\eta(i))] \cdot (\nabla_i - T_{i,j}\nabla_j). \tag{2.11}$$

Our first main result establishes the algebraic rate of convergence to the invariant measure in $\mathsf{L}_2(\mu_\rho)$.

Theorem 2.1. *For any $\rho \in (0,1)$ and $q \in [1,2)$ the following Generalized Nash inequality is true*

$$\text{(N)} \qquad \mu_\rho\,(f - \mu_\rho f)^2 \leq \mathbf{D}_\rho(f)^\alpha\, \mathbf{A}_q(f)^{1-\alpha}$$

for all $f \in \mathcal{C}_{2,q}(\Omega)$ with

$$\alpha \equiv \alpha(q,d) \equiv \Big(\frac{1}{q} - \frac{1}{2}\Big)\Big(\frac{1}{d} + \frac{1}{q} - \frac{1}{2}\Big)^{-1} \tag{2.12}$$

and a P_t-monotone functional

$$\mathbf{A}_q(f) \equiv a_q\, |\!|\!| f |\!|\!|_{2,q}^2 \tag{2.13}$$

with some constant $a_q \equiv a_q(d) \in (0,\infty)$. Therefore for any $t > 0$ we have

$$\mu_\rho\,(P_t f - \mu_\rho f)^2 \leq \Big(\frac{\gamma}{2}\Big)^\gamma \frac{\mathbf{A}_q(f)}{t^\gamma} \tag{2.14}$$

with $\gamma \equiv d\Big(\frac{1}{q} - \frac{1}{2}\Big)$.

Remark 2.1. The decay rate is the best possible. To see this, for example when $q = 1$, it is enough to notice that $\mathcal{L}$ defined in (2.2) preserves the subspace of linear functions. It is thus possible to compute explicitly $P_t f$ when $f = \eta(0)$. In this case inequality (2.14) is saturated.

Our second main result establishes the algebraic rate of convergence to equilibrium in the entropy sense

Theorem 2.2. *For any $\rho \in (0,1)$ and any $q \in [1,2)$ the following Logarithmic Nash inequality is true*

$$\text{(LN)} \qquad \mu_\rho\Big(f \log \frac{f}{\mu_\rho f}\Big) \leq \mathbf{D}_\rho(f^{1/2})^{\overline{\alpha}}\, \widetilde{\mathbf{A}}_q(f^{1/2})^{1-\overline{\alpha}}$$

for all $f \in \mathcal{C}_{2,q}(\Omega)$, $f \geq 0$, with

$$\overline{\alpha}(q) \equiv \Big(\frac{1}{q} - \frac{1}{2}\Big)\Big(\frac{1}{d} + \frac{1}{q}\Big)^{-1} \tag{2.15}$$

and

$$\widetilde{\mathbf{A}}_q(f^{1/2}) \equiv \overline{a}_q\, \mathbf{A}_q(f^{1/2}) \tag{2.16}$$

with some constant $\overline{a}_q \equiv \overline{a}_q(d,\rho) \in (0,\infty)$. Moreover, if $f > \varepsilon > 0$ for some constant $\varepsilon > 0$, then

$$\text{(LN}'\text{)} \qquad \mu_\rho\Big(f \log \frac{f}{\mu_\rho f}\Big) \leq \mathbf{D}_\rho(f^{1/2})^{\alpha}\, \mathbf{B}_q(f^{1/2})^{1-\alpha}$$

with a P_t-monotone functional

$$\mathbf{B}_q(f^{1/2}) \equiv \frac{a'_q}{4 \inf f}\, |\!|\!| f |\!|\!|_{2,q} \tag{2.17}$$

defined with some constant $a'_q \equiv a'_q(d,\rho) \in (0,\infty)$ and $\alpha = \alpha(q,d)$ given by (2.12). Therefore for any $t > 0$, we have

$$\mu_\rho\Big(P_t f \log \frac{P_t f}{\mu_\rho f}\Big) \leq \Big(\frac{\gamma}{4}\Big)^\gamma \frac{\mathbf{B}_q(f)}{t^\gamma} \tag{2.18}$$

with $\gamma \equiv \alpha/(1-\alpha)$ and any strictly positive function $f \in \mathcal{C}_{2,q}(\Omega)$.

Remark 2.2. If $f \in \mathsf{L}_2(\mu_\rho)$, $f \geq 0$, then from Theorem 2.1 we can conclude similar decay in the entropy sense, but otherwise if $f \in \mathsf{L}_p(\mu_\rho)$, $p \in [1,2)$, and $f \geq \varepsilon > 0$, $f \in \mathcal{C}_{2,q}(\Omega) \setminus \mathcal{C}_{2,1}(\Omega)$, $q > 1$, Theorem 2.2 gives us a non-trivial information.

3. Proof of Generalized Nash inequality

The proof will use finite dimensional approximations. Therefore we need to introduce some more notation.

For $\Lambda \subset \mathbf{Z}^d$ we denote by $\mu_{\Lambda,\rho}$ the product measure on $\Omega_\Lambda := \{0,1\}^\Lambda$ with density ρ, i.e. $\mu_{\Lambda,\rho}\,\eta(i) = \rho$, for any $i \in \Lambda$. Accordingly, for a function f on Ω_Λ, we introduce the finite volume Dirichlet form by

$$\mathbf{D}_{\Lambda,\rho}(f) := \frac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda} \mu_{\Lambda,\rho}(\delta_{ij} f)^2 .$$

For $k = 0, \ldots, |\Lambda|$, we introduce the canonical measures $\mu_\Lambda^k(\cdot)$ with k particles in a finite volume Λ, defined on Ω_Λ by requiring the following equality for the conditional expectation $\mu_{\Lambda,\rho}\ (\cdot\,|N_\Lambda)$ associated to the measure $\mu_{\Lambda,\rho}$, given the total number of particles $N_\Lambda := \sum_{i\in\Lambda} \eta(i)$ in Λ

$$\mu_{\Lambda,\rho}\ (f\,|N_\Lambda)\,(\eta) \equiv \sum_{k=0}^{|\Lambda|} \mu_\Lambda^k(f)\chi_k(\eta), \tag{3.1}$$

where $\chi_k(\eta) \equiv \chi\,(N_\Lambda(\eta) = k)$ denotes the characteristic function. Note that the measures μ_Λ^k on the right-hand side of (3.1) are independent of ρ.

It is known, [9], that the probability measures $\mu_{\Lambda,\rho}$, $\Lambda \subset \mathbf{Z}^d$, satisfy the following useful inequality.

Standard Spectral Gap inequality. *For any $\Lambda \subseteq \mathbf{Z}^d$ we have*

$$(\mathrm{SSG}(\mu_{\Lambda,\rho})) \qquad \mu_{\Lambda,\rho}\,(g - \mu_{\Lambda,\rho}\, g)^2 \le M(\rho)^{-1}\mathcal{E}_{\Lambda,\rho}(g)$$

for any function g for which the standard Dirichlet form

$$\mathcal{E}_{\Lambda,\rho}(g) \equiv \sum_{i\in\Lambda} \mu_{\Lambda,\rho}\ |\nabla_i g|^2$$

is finite, with a constant $M(\rho)^{-1} = \mathbf{v}_\rho \equiv \rho(1-\rho)$.

It is also known, see [7,13,16], that the measures μ_Λ^k satisfy the following

Finite Volume Spectral Gap inequality. *For any cube $\Lambda \subset \mathbf{Z}^d$ we have*

$$(\mathrm{SG}(\mu_\Lambda^k)) \qquad \mu_\Lambda^k(f - \mu_\Lambda^k f)^2 \le c_0 |\Lambda|^{2/d} \mathbf{D}_\Lambda^k(f)$$

with a constant $c_0 \in (0,\infty)$ independent of Λ, $k = 0, 1, \cdots, |\Lambda|$ and f; by $\mathbf{D}_\Lambda^k$ we denote the following Dirichlet form corresponding to the measure μ_Λ^k

$$\mathbf{D}_\Lambda^k(f) := \frac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda} \mu_\Lambda^k(\delta_{ij} f)^2 . \tag{3.2}$$

Our first technical ingredient necessary for the proof of Theorem 2.1 is the following (finite volume) bound.

Lemma 3.1. *For any $\rho \in (0,1)$, any $q \in [1,2)$ and any cube $\Lambda \subset \mathbf{Z}^d$, we have*

$$\mu_{\Lambda,\rho}\,(f - \mu_{\Lambda,\rho}\, f)^2 \le c_0|\Lambda|^{2/d}\mathbf{D}_{\Lambda,\rho}(f) + 2|\Lambda|^{-((2/q)-1)}\Big(\sum_{i\in\Lambda} \|\nabla_i f\|_2^q\Big)^{2/q} \tag{3.3}$$

for all functions f on Ω_Λ.

Proof. It is sufficient to show the statement for $q = 1$, because then the general case follows from the Holder inequality for any $q \in [1,\infty)$

$$\sum_{i\in\Lambda} \|\nabla_i f\|_2 \le |\Lambda|^{(1-(1/q))}\Big(\sum_{i\in\Lambda} \|\nabla_i f\|_2^q\Big)^{1/q} . \tag{3.4}$$

Let $p_{\Lambda,\rho}(k) := \mu_{\Lambda,\rho}\ (\chi_k)$. Recalling the definition of μ_Λ^k, we have

$$\mu_{\Lambda,\rho}\, f = \sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k) \cdot \mu_\Lambda^k f. \tag{3.5}$$

Hence

$$\begin{aligned} \mu_{\Lambda,\rho}(f - \mu_{\Lambda,\rho}\, f)^2 \;&=\; \sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k)\mu_\Lambda^k(f - \mu_\Lambda^k f)^2 \\ &\quad + \mu_{\Lambda,\rho}(\mu_{\Lambda,\rho}(f \mid N_\Lambda) - \mu_{\Lambda,\rho}\,\mu_{\Lambda,\rho}(f \mid N_\Lambda))^2 . \end{aligned} \tag{3.6}$$

Using the spectral gap inequality $(\mathrm{SG}(\mu_\Lambda^k))$ we can estimate the first part of the above sum as follows

$$\sum_{k=0}^{|\Lambda|} p_{\Lambda,\rho}(k)\, \mu_\Lambda^k(f - \mu_\Lambda^k f)^2 \le c_0\, |\Lambda|^{2/d}\, \mathbf{D}_{\Lambda,\rho}(f). \tag{3.7}$$

We next bound the second part of the right-hand side of (3.6) using $(\mathrm{SSG}(\mu_{\Lambda,\rho}))$. We have

$$\begin{aligned} &\mu_{\Lambda,\rho}\big(\mu_{\Lambda,\rho}\,(f \mid N_\Lambda) - \mu_{\Lambda,\rho}\,\mu_{\Lambda,\rho}\,(f \mid N_\Lambda)\big)^2 \\ &\qquad \le\; M(\rho)^{-1} \sum_{i\in\Lambda} \mu_{\Lambda,\rho}\, \big|\nabla_i \mu_{\Lambda,\rho}\,(f\,|N_\Lambda)\,\big|^2 . \end{aligned} \tag{3.8}$$

To estimate the right-hand side of (3.8), we use the following simple lemma proven in the Appendix.

Lemma 3.2. *For any real function F and any finite set $\Lambda \subset \mathbf{Z}^d$ we have*

$$\sum_{i\in\Lambda} \mu_{\Lambda,\rho}\left|\nabla_i F(\mu_{\Lambda,\rho}(f \mid N_\Lambda))\right|^2 = \frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|F(\mu_\Lambda^k(f)) - F(\mu_\Lambda^{k-1}(f))\right|^2 \cdot k\,\mu_{\Lambda,\rho}(\chi_k). \tag{3.9}$$

Now to finish the proof we need only to observe that the following inequality is true.

Lemma 3.3. *For any $\rho \in (0,1)$ and any finite set $\Lambda \subset \mathbf{Z}^d$ we have*

$$\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|\mu_\Lambda^k f - \mu_\Lambda^{k-1} f\right|^2 \cdot k\,\mu_{\Lambda,\rho}(\chi_k) \le \frac{2}{\mathbf{v}_\rho}\frac{1}{|\Lambda|}\Big(\sum_{i\in\Lambda}\|\nabla_i f\|_2\Big)^2. \tag{3.10}$$

Proof. Let us first note the following identities, [13], hold for any $k = 1,\ldots,|\Lambda|$

$$\mu_\Lambda^{k-1} f - \mu_\Lambda^k f = \frac{1}{k}\mu_\Lambda^k\Big(\sum_{i\in\Lambda}\nabla_i f\cdot\eta(i)\Big), \tag{3.11}$$

$$\mu_\Lambda^k f - \mu_\Lambda^{k-1} f = \frac{1}{|\Lambda|-(k-1)}\mu_\Lambda^{k-1}\Big(\sum_{i\in\Lambda}\nabla_i f\cdot[1-\eta(i)]\Big), \tag{3.12}$$

formula (3.11) is proven simply by noticing that

$$\mu_\Lambda^{k-1} f = \frac{1}{k}\sum_{i\in\Lambda}\mu_\Lambda^k(f(\eta^i)\cdot\eta(i)) \tag{3.13}$$

and analogously for formula (3.12).

We rewrite the left-hand side of (3.10) by using (3.11), respectively (3.12), if $k > [|\Lambda|/2]$, respectively $k \le [|\Lambda|/2]$. We get

$$\begin{aligned}
&\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}|\mu_\Lambda^k f - \mu_\Lambda^{k-1} f|^2\cdot k\,\mu_{\Lambda,\rho}(\chi_k)\\
&= \frac{1}{1-\rho}\sum_{k=1}^{[|\Lambda|/2]} p_{\Lambda,\rho}(k-1)\frac{1}{|\Lambda|-(k-1)}\Big[\mu_\Lambda^{k-1}\Big(\sum_{i\in\Lambda}\nabla_i f\cdot[1-\eta(i)]\Big)\Big]^2\\
&\quad + \frac{1}{\rho}\sum_{k=[|\Lambda|/2]+1}^{|\Lambda|} p_{\Lambda,\rho}(k)\frac{1}{k}\Big[\mu_\Lambda^k\Big(\sum_{i\in\Lambda}\nabla_i f\cdot\eta(i)\Big)\Big]^2,
\end{aligned}$$

where we used that

$$\frac{1}{\rho}p_{\Lambda,\rho}(k)\frac{k}{|\Lambda|-(k-1)} = \frac{1}{1-\rho}p_{\Lambda,\rho}(k-1).$$

Hence, by the Cauchy – Schwarz inequality,

$$\begin{aligned}
&\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}|\mu_\Lambda^k f - \mu_\Lambda^{k-1} f|^2\cdot k\,\mu_{\Lambda,\rho}(\chi_k)\\
&= \frac{1}{1-\rho}\frac{1}{[|\Lambda|/2]+1}\sum_{k=0}^{[|\Lambda|/2]-1} p_{\Lambda,\rho}(k)\mu_\Lambda^k\Big(\sum_{i\in\Lambda}|\nabla_i f|\Big)^2\\
&\quad + \frac{1}{\rho}\frac{1}{[|\Lambda|/2]+1}\sum_{k=[|\Lambda|/2]+1}^{|\Lambda|} p_{\Lambda,\rho}(k)\mu_\Lambda^k\Big(\sum_{i\in\Lambda}|\nabla_i f|\Big)^2\\
&\le \frac{2}{\mathbf{v}_\rho}\frac{1}{|\Lambda|}\mu_{\Lambda,\rho}\Big(\sum_{i\in\Lambda}|\nabla_i f|\Big)^2 \le \frac{2}{\mathbf{v}_\rho}\frac{1}{|\Lambda|}\Big(\sum_{i\in\Lambda}\|\nabla_i f\|_2\Big)^2
\end{aligned}$$

which ends the proof of Lemma 3.3. □

By applying Lemmas 3.2 and 3.3 to bound the right-hand side of (3.8) and recalling that $M(\rho)^{-1} = \mathbf{v}_\rho$, we conclude the proof of inequality (3.3) for $q = 1$ and thus the proof of Lemma 3.1. □

By using a martingale decomposition we next deduce from Lemma 3.1 the following (infinite volume) bound.

Proposition 3.1. *For any $q \in [1,2)$ and $L \in \mathbf{N}$*

$$\mu_\rho\,(f - \mu_\rho f)^2 \le c_0 L^2\,\mathbf{D}_\rho(f) + 2L^{-d((2/q)-1)}\,|\!|\!|f|\!|\!|_{2,q}^2 \tag{3.14}$$

for any $f \in \mathcal{C}_{2,q}(\Omega)$.

Proof. Let $\Lambda_0 = \{1,\cdots,L\}^d$ be a reference cube of side L in $\mathbf{Z}^d$. For $i \in \mathbf{Z}^d$, let $\Lambda_i := \Lambda_0 + iL$ be the translate of Λ_0 by a vector $iL \in \mathbf{Z}^d$. The family $\{\Lambda_i\}_{i\in\mathbf{Z}^d}$ defines then a partition of $\mathbf{Z}^d$. It will be more convenient to label its elements by a natural number; we thus obtain a family of cubes $\{\Lambda_\ell\}_1^\infty$such that for $\ell \ne \ell'$, $\Lambda_\ell \cap \Lambda_{\ell'} = \emptyset$ and $\bigcup_\ell \Lambda_\ell = \mathbf{Z}^d$. Let $\{Y_\ell\}_0^\infty$ be the increasing sequence defined by $Y_0 := \emptyset$, $Y_\ell := \bigcup_{\ell'\le\ell}\Lambda_{\ell'}$, if $\ell \ge 1$.

We associate to $\{Y_\ell\}_0^\infty$a family $\{\mathsf{E}_\ell\}_0^\infty$ of conditional expectations defined by $\mathsf{E}_0 f = f$ and for $\ell > 0$, $\mathsf{E}_\ell f := \mu_{Y_\ell,\rho} f$, for any $f \in \mathcal{C}(\Omega)$. We then have

$$f - \mu_\rho f = \sum_{\ell=1}^\infty (f_{\ell-1} - f_\ell), \tag{3.15}$$

where $f_\ell := \mathsf{E}_\ell f = \mu_{\Lambda_\ell,\rho} f_{\ell-1}$. The series on the right-hand side of (3.15) is convergent in $\mathrm{L}_2(\mu_\rho)$.

Since the sequence f_ℓ has orthogonal increments, we get

$$\begin{aligned}\mu_\rho\,(f-\mu_\rho\,f)^2 &= \sum_{\ell=1}^{\infty}\mu_\rho\,(f_{\ell-1}-f_\ell)^2\\ &= \sum_{\ell=1}^{\infty}\mu_\rho\,\mu_{\Lambda_\ell,\rho}\,(f_{\ell-1}-\mu_{\Lambda_\ell,\rho}f_{\ell-1})^2\,. \end{aligned}\tag{3.16}$$

Keeping the variables outside Λ_ℓ fixed, we apply Lemma 3.1 to the cube Λ_ℓ and the function $f_{\ell-1}$. We get

$$\begin{aligned}\mu_{\Lambda_\ell,\rho}(f_{\ell-1}-\mu_{\Lambda_\ell,\rho}f_{\ell-1})^2 &\le c_0L^2\,\mathbf{D}_{\Lambda_\ell,\rho}(f_{\ell-1})\\ &\quad + 2L^{-d((2/q)-1)}\Big(\sum_{i\in\Lambda_\ell}\|\nabla_i f_{\ell-1}\|_2^q\Big)^{2/q}.\end{aligned}\tag{3.17}$$

This bound can be simplified if we observe that for $i,j\in\Lambda_\ell$, by the Cauchy–Schwarz inequality, one has

$$\mu_{\Lambda_\ell,\rho}(\mu_{Y_{\ell-1},\rho}f(\eta^{i,j})-\mu_{Y_{\ell-1},\rho}f(\eta))^2\le\mu_{Y_\ell,\rho}(f(\eta^{i,j})-f(\eta))^2 \tag{3.18}$$

and that

$$\|\nabla_i f_{\ell-1}\|_2\le\|\nabla_i f\|_2. \tag{3.19}$$

Hence, using (3.17) and (3.16), we get

$$\begin{aligned}&\mu_\rho\,(f-\mu_\rho\,f)^2\\ &\le \sum_{\ell=1}^{\infty}\Big\{c_0L^2\frac{1}{2}\sum_{\langle i,j\rangle\subset\Lambda_\ell}\mu_\rho(\delta_{i,j}f)^2+2L^{-d((2/q)-1)}\Big(\sum_{i\in\Lambda_\ell}\|\nabla_i f\|_2^q\Big)^{2/q}\Big\}\\ &\le c_0L^2\mathbf{D}_\rho(f)+2L^{-d((2/q)-1)}\Big(\sum_{i\in\mathbf{Z}^d}\|\nabla_i f\|_2^q\Big)^{2/q},\end{aligned}$$

where in the last step we have used our assumption $q\in[1,2)$. This ends the proof of (3.14). □

Conclusion of the proof of Theorem 2.1.

To prove the Generalized Nash inequality we need only to optimize the bound in Proposition 3.1 with respect to the parameter L. For this we observe first that we have the following a priori bound

Lemma 3.4. *For any $q\in[1,2]$*

$$\mathbf{D}_\rho(f)\le 2d\cdot|\!|\!|f|\!|\!|_{2,q}^2. \tag{3.20}$$

Proof. To see this, we use definition (2.4) of the Dirichlet form and identity (2.11) to get

$$\begin{aligned}\mathbf{D}_\rho(f) &= \frac{1}{2}\sum_{\langle i,j\rangle}\mu_\rho([\nabla_i f(\eta)-\nabla_j f(\eta^{i,j})]^2\cdot[\eta(i)(1-\eta(j))+\eta(j)(1-\eta(i))])\\ &\le \sum_{\langle i,j\rangle}[\|\nabla_i f\|_2^2+\|\nabla_j f\|_2^2]=2d\sum_{i\in\mathbf{Z}^d}\|\nabla_i f\|_2^2\le 2d\cdot|\!|\!|f|\!|\!|_{2,q}^2.\end{aligned}$$

where in the last step we have used (2.9). □

By (3.20) we have that

$$x:=\left(\frac{2d\cdot|\!|\!|f|\!|\!|_{2,q}^2}{\mathbf{D}_\rho(f)}\right)^{\frac{1}{2+d((2/q)-1)}}\ge 1, \tag{3.21}$$

and so $[x]^2\le x^2$ and $[x]^{-d(2/q-1)}\le 2^{d(2/q-1)}x^{-d(2/q-1)}$. Thus setting $L=[x]$ in Proposition 3.1 we obtain the Generalized Nash inequality (N) with

$$\alpha=\Big(\frac{1}{q}-\frac{1}{2}\Big)\Big(\frac{1}{d}+\frac{1}{q}-\frac{1}{2}\Big)^{-1}$$

and

$$a_q={c_q}^{1/(1-\alpha)},\qquad c_q\equiv c_0\cdot(2\,d)^{1-\alpha}+2^{1+d((2/q)-1)}\cdot(2\,d)^{-\alpha}.$$

The proof of algebraic decay (2.14) given inequality (N) follows by standard arguments (see e.g. [4,5,12]) once we know that $|\!|\!|\cdot|\!|\!|_{2,q}$ is monotone, i.e. that $|\!|\!|P_tf|\!|\!|_{2,q}\le|\!|\!|f|\!|\!|_{2,q}$. We shall prove a (more) general result in Section 5. □

4. Proof of Logarithmic Nash inequality

The idea of the proof of the Logarithmic Nash inequality is similar to the one used before to show the Nash Inequality. The important role in our proof is now played by the following Logarithmic Sobolev inequalities.

Standard Logarithmic Sobolev inequality. *For any $\Lambda\subseteq\mathbf{Z}^d$*

$$(\mathrm{SLS}(\mu_{\Lambda,\rho}))\qquad \mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}f}\Big)\le C(\rho)\mathcal{E}_{\Lambda,\rho}(f^{1/2})$$

for any $f\ge 0$ for which the (standard) Dirichlet form

$$\mathcal{E}_{\Lambda,\rho}(f^{1/2})\equiv\sum_{i\in\Lambda}\mu_{\Lambda,\rho}\left|\nabla_i(f^{1/2})\right|^2$$

is finite, with a coefficient $C(\rho)\in(0,\infty)$ independent of f and Λ; see [9].

Finite volume Logarithmic Sobolev inequality, $N_\Lambda = k$. *For any cube $\Lambda \subset \mathbf{Z}^d$*

$$(\mathrm{LS}(\mu_\Lambda^k)) \qquad \mu_\Lambda^k\Big(f\log\frac{f}{\mu_\Lambda^k f}\Big) \le \bar{c}_0|\Lambda|^{2/d}\mathbf{D}_\Lambda^k(f^{1/2})$$

for any $f \ge 0$ for which

$$\mathbf{D}_\Lambda^k(f^{1/2}) \equiv \frac{1}{2}\sum_{(i,j)\in\Lambda}\mu_{\Lambda,\rho}|\delta_{ij}(f^{1/2})|^2$$

is finite, with a coefficient $\bar{c}_0 \in (0,\infty)$independent of f, Λ and $k = 0, 1, \cdots, |\Lambda|$; see [22, 23].

Our first technical ingredient necessary for the proof of Theorem 2.2 is the following (finite volume) bound.

Lemma 4.1. *For any $\rho \in (0,1)$, any $q \in [1,2)$ and any cube $\Lambda \subset \mathbf{Z}^d$, we have*

$$\begin{aligned}\mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}f}\Big) &\le c_1|\Lambda|^{(1+(2/d))}\mathbf{D}_{\Lambda,\rho}(f^{1/2})\\ &\quad + c_2|\Lambda|^{-((2/q)-1)}\Big(\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_2^q\Big)^{2/q} \qquad (4.1)\end{aligned}$$

for all functions f on Ω_Λ, $f \ge 0$, with the constants $c_1 \equiv (\bar{c}_0 + c_0\cdot(3C(\rho)/\mathbf{v}_\rho))$ and $c_2 \equiv 6C(\rho)/\mathbf{v}_\rho$, both independent of $f$and Λ.

Remark 4.1. As we indicate in the proof below, for $f > \varepsilon > 0$, the similar inequality is true with the last factor on the right-hand side of (4.1) replaced by $(\sum_{i\in\Lambda}\|\nabla_i f\|_2^q)^{2/q}/(4\inf f)$ and $c_1|\Lambda|^{1+(2/d)}$ replaced by $\bar{c}_0|\Lambda|^{2/d}$.

Proof. Similarly as in the proof of the Nash inequality it is sufficient to consider the case $q = 1$. We have

$$\begin{aligned}\mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}f}\Big) &= \mu_{\Lambda,\rho}\Big(\mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}(f|N_\Lambda)}N_\Lambda\Big)\Big)\\ &\quad + \mu_{\Lambda,\rho}\Big(\mu_{\Lambda,\rho}(f|N_\Lambda)\log\frac{\mu_{\Lambda,\rho}(f\,|N_\Lambda)}{\mu_{\Lambda,\rho}(\mu_{\Lambda,\rho}(f\,|N_\Lambda)\,)}\Big). \qquad (4.2)\end{aligned}$$

Since

$$\mu_{\Lambda,\rho}\Big(\mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}(f\,|N_\Lambda)}N_\Lambda\Big)\Big) = \sum_{k=0}^{|\Lambda|}p_{\Lambda,\rho}(k)\mu_\Lambda^k\Big(f\log\frac{f}{\mu_\Lambda^k f}\Big),$$

using the finite volume Logarithmic Sobolev inequality we can estimate the first part on the right-hand side of (4.2) as follows

$$\mu_{\Lambda,\rho}\Big(\mu_{\Lambda,\rho}\Big(f\log\frac{f}{\mu_{\Lambda,\rho}(f\,|N_\Lambda)}N_\Lambda\Big)\Big) \le \bar{c}_0|\Lambda|^{2/d}\mathbf{D}_{\Lambda,\rho}(f^{1/2}). \qquad (4.3)$$

To bound the second part of the right-hand side of (4.2), we use first the Standard Logarithmic Sobolev inequality to get

$$\begin{aligned}&\mu_{\Lambda,\rho}\Big(\mu_{\Lambda,\rho}(f\,|N_\Lambda)\log\frac{\mu_{\Lambda,\rho}(f\,|N_\Lambda)}{\mu_{\Lambda,\rho}(\mu_{\Lambda,\rho}(f\,|N_\Lambda)\,)}\Big)\\ &\quad\le C(\rho)\sum_{i\in\Lambda}\mu_{\Lambda,\rho}|\nabla_i(\mu_{\Lambda,\rho}(f\,|N_\Lambda))^{1/2}|^2 \qquad (4.4)\end{aligned}$$

with the constant $C(\rho) \in (0,\infty)$ independent of Λ and the function f. To estimate the right-hand side of (4.4), we use Lemma 3.2 with the function F equal to the square root, to get

$$\begin{aligned}&\sum_{i\in\Lambda}\mu_{\Lambda,\rho}|\nabla_i(\mu_{\Lambda,\rho}(f|N_\Lambda))^{1/2}|^2\\ &\quad\le \frac{1}{\rho}\sum_{k=1}^{|\Lambda|}|(\mu_\Lambda^k(f))^{1/2} - (\mu_\Lambda^{k-1}(f))^{1/2}|^2 k\cdot\mu_{\Lambda,\rho}\chi_k. \qquad (4.5)\end{aligned}$$

We have

$$\begin{aligned}&|(\mu_\Lambda^k(f))^{1/2} - (\mu_\Lambda^{k-1}(f))^{1/2}|\\ &\quad = |\mu_\Lambda^k(f) - \mu_\Lambda^{k-1}(f)|\cdot|(\mu_\Lambda^k(f))^{1/2} + (\mu_\Lambda^{k-1}(f))^{1/2}|^{-1}. \qquad (4.6)\end{aligned}$$

Remark 4.2. At this point, if $f > \varepsilon > 0$, we could estimate the right-hand side of (4.6) by $|\mu_\Lambda^k(f) - \mu_\Lambda^{k-1}(f)|/(2\inf f^{1/2})$ and using Lemma 3.3 arrive at the inequality indicated in the remark after Lemma 4.1.

Since

$$\begin{aligned}|\mu_\Lambda^k(f) - \mu_\Lambda^{k-1}(f)| &= |\mu_\Lambda^k\otimes\widetilde{\mu}_\Lambda^{k-1}(f - \widetilde{f})|\\ &= \Big|\mu_\Lambda^k\otimes\widetilde{\mu}_\Lambda^{k-1}\Big(f^{\frac12} - \widetilde{f}^{\frac12}\Big)\Big(f^{\frac12} + \widetilde{f}^{\frac12}\Big)\Big|\\ &\le \Big(\mu_\Lambda^k\otimes\widetilde{\mu}_\Lambda^{k-1}\Big(f^{\frac12} - \widetilde{f}^{\frac12}\Big)^2\Big)^{\frac12}\\ &\quad\times\Big[\Big(\mu_\Lambda^k(f)\Big)^{\frac12} + \Big(\mu_\Lambda^{k-1}(f)\Big)^{\frac12}\Big], \qquad (4.7)\end{aligned}$$

where $\widetilde{f}^{1/2}$ is integrated with respect to the isomorphic copy $\widetilde{\mu}_\Lambda^{k-1}$ of μ_Λ^{k-1}, we have

$$\left|(\mu_\Lambda^k(f))^{\frac12} - (\mu_\Lambda^{k-1}(f))^{\frac12}\right| \le \left(\mu_\Lambda^k \otimes \tilde\mu_\Lambda^{k-1}\left(f^{\frac12} - \tilde f^{\frac12}\right)^2\right)^{\frac12}$$
$$\le \left(\mu_\Lambda^k\left(f^{\frac12} - \mu_\Lambda^k f^{\frac12}\right)^2\right)^{\frac12} + \left(\mu_\Lambda^{k-1}\left(f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right)^2\right)^{\frac12}$$
$$+ \left|\mu_\Lambda^k f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right|. \tag{4.8}$$

From (4.5)–(4.8) we obtain the following bound

$$\sum_{i\in\Lambda} \mu_{\Lambda,\rho}\left|\nabla_i\left(\mu_{\Lambda,\rho}(f|N_\Lambda)\right)^{\frac12}\right|^2 \le 3\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left[\mu_\Lambda^k\left(f^{\frac12} - \mu_\Lambda^k f^{\frac12}\right)^2\right.$$
$$\left. + \mu_\Lambda^{k-1}\left(f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right)^2\right] k\cdot\mu_{\Lambda,\rho}\chi_k$$
$$+ 3\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|\mu_\Lambda^k f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right|^2 k\cdot\mu_{\Lambda,\rho}\chi_k. \tag{4.9}$$

Now we use the spectral gap inequality ($SG(\mu_\Lambda^k)$) to get

$$\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left[\mu_\Lambda^k\left(f^{\frac12} - \mu_\Lambda^k f^{\frac12}\right)^2 + \mu_\Lambda^{k-1}\left(f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right)^2\right] k\cdot\mu_{\Lambda,\rho}\chi_k$$
$$\le |\Lambda|^{(1+\frac{2}{d})}\frac{c_0}{\mathbf{v}_\rho}\mathbf{D}_{\Lambda,\rho}\left(f^{\frac12}\right). \tag{4.10}$$

The last part from the right-hand side of (4.9) can be estimated by applying Lemma 3.3. We get

$$\frac{1}{\rho}\sum_{k=1}^{|\Lambda|}\left|\mu_\Lambda^k f^{\frac12} - \mu_\Lambda^{k-1} f^{\frac12}\right|^2 k\cdot\mu_\rho\,\chi_k \le \frac{2}{\mathbf{v}_\rho}\frac{1}{|\Lambda|}\left(\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_2\right)^2. \tag{4.11}$$

Combining (4.9)–(4.11) with (4.4), we arrive at the following estimate

$$\mu_{\Lambda,\rho}\left(\mu_{\Lambda,\rho}(f\,|N_\Lambda)\log\frac{\mu_{\Lambda,\rho}(f\,|N_\Lambda)}{\mu_{\Lambda,\rho}(\mu_{\Lambda,\rho}(f\,|N_\Lambda))}\right)$$
$$\le 3C(\rho)\frac{|\Lambda|^{(1+(2/d))}}{\mathbf{v}_\rho}c_0\mathbf{D}_{\Lambda,\rho}(f^{1/2}) + \frac{6C(\rho)}{\mathbf{v}_\rho|\Lambda|}\left(\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_2\right)^2. \tag{4.12}$$

Inequality (4.12) together with (4.3) yields the following bound

$$\mu_{\Lambda,\rho}\left(f\log\frac{f}{\mu_{\Lambda,\rho}f}\right) \le \left(\bar c_0 + \frac{3C(\rho)}{\mathbf{v}_\rho}c_0\right)|\Lambda|^{(1+(2/d))}\mathbf{D}_{\Lambda,\rho}(f^{1/2})$$
$$+ \frac{6C(\rho)}{\mathbf{v}_\rho|\Lambda|}\left(\sum_{i\in\Lambda}\|\nabla_i f^{1/2}\|_2\right)^2. \tag{4.13}$$

This ends the proof of Lemma 4.1 in case $q = 1$ from which the general case easily follows. □

By using a martingale decomposition we next deduce from Lemma 4.1 the following (infinite volume) bound.

Proposition 4.1. *For any $\rho \in (0,1)$, $q \in [1,2)$ and $L \in \mathbf{N}$*

$$\mu_\rho\left(f\log\frac{f}{\mu_\rho\, f}\right) \le c_1 L^{d+2}\mathbf{D}_\rho(f^{1/2}) + c_2 L^{-d((2/q)-1)}\| f^{1/2}\|_{2,q}^2 \tag{4.14}$$

for any $f \ge 0$ such that $f^{1/2} \in \mathcal{C}_{2,q}(\Omega)$.

Proof. Let $\Lambda_0 = \{1,\cdots,L\}^d$ and $\{Y_\ell\}_0^\infty$ be the increasing sequence defined by $Y_0 := \emptyset$, and $Y_\ell := \bigcup_{\ell'\le\ell}\Lambda_{\ell'}$, if $\ell \ge 1$, where $\Lambda_\ell := \Lambda_0 + i_\ell L$, $i_\ell \in \mathbf{Z}^d$ form a partition of $\mathbf{Z}^d$. Let also $\{\mathsf{E}_\ell\}_0^\infty$ be the family of conditional expectations defined by $\mathsf{E}_0 f = f$ and for $\ell > 0$, $\mathsf{E}_\ell f := \mu_{Y_\ell,\rho} f$, for any $f \in \mathcal{C}(\Omega)$. We then have

$$\mu_\rho\left(f\log\frac{f}{\mu_\rho\, f}\right) = \sum_{\ell=1}^\infty \mu_\rho\left(\mu_{\Lambda_\ell,\rho}\left(f_{\ell-1}\log\frac{f_{\ell-1}}{\mu_{\Lambda_\ell,\rho}\, f_{\ell-1}}\right)\right), \tag{4.15}$$

where $f_\ell := \mathsf{E}_\ell f = \mu_{\Lambda_\ell,\rho}\; f_{\ell-1}$. For the fixed variables outside Λ_ℓ, we apply Lemma 4.1 to the cube Λ_ℓ and the function $f_{\ell-1}$. We get

$$\mu_{\Lambda_\ell,\rho}\left(f_{\ell-1}\log\frac{f_{\ell-1}}{\mu_{\Lambda_\ell,\rho}f_{\ell-1}}\right)$$
$$\le c_1 L^{d+2}\mathbf{D}_{\Lambda_\ell,\rho}((f_{\ell-1})^{1/2}) + c_2 L^{-d((2/q)-1)}\left(\sum_{i\in\Lambda_\ell}\|\nabla_i(f_{\ell-1})^{1/2}\|_2^q\right)^{2/q}. \tag{4.16}$$

This bound can be simplified if we observe that for $i,j \in \Lambda_\ell$, one has

$$\mu_{\Lambda_\ell,\rho}((\mu_{Y_{\ell-1},\rho}f)^{1/2}(\eta^{i,j}) - (\mu_{Y_{\ell-1},\rho}f)^{1/2}(\eta))^2$$
$$\le \mu_{Y_\ell,\rho}(f^{1/2}(\eta^{i,j}) - f^{1/2}(\eta))^2 \tag{4.17}$$

and that

$$\|\nabla_i(f_{\ell-1})^{1/2}\|_2 \le \|\nabla_i f^{1/2}\|_2. \tag{4.18}$$

Hence, using (4.15)–(4.18), we get

$$\mu_\rho\left(f\log\frac{f}{\mu_\rho f}\right) \le \sum_{\ell=1}^\infty\left\{c_1 L^{d+2}\frac12\sum_{\langle i,j\rangle\subset\Lambda_\ell}\mu_\rho(\delta_{i,j}f^{1/2})^2\right.$$
$$\left. + c_2 L^{-d((2/q)-1)}\left(\sum_{i\in\Lambda_\ell}\|\nabla_i f^{1/2}\|_2^q\right)^{2/q}\right\}$$
$$\le c_1 L^{d+2}\mathbf{D}_\rho(f^{1/2})$$
$$+ c_2 L^{-d((2/q)-1)}\left(\sum_{i\in\mathbf{Z}^d}\|\nabla_i f^{1/2}\|_2^q\right)^{2/q}. \tag{4.19}$$

where in the last step we have used our assumption $q \in [1,2)$. This ends the proof of (4.14). □

Conclusion of the proof of the Logarithmic Nash inequality.

To prove the Logarithmic Nash inequality from Theorem 2.2, we need only to optimize the bound in Proposition 4.1 with respect to the parameter L. For this we use our a priori bound Lemma 3.4

$$\mathbf{D}_\rho(f^{1/2}) \leq 2\,d \cdot \|\!|f^{1/2}|\!\|_{2,q}^2. \tag{4.20}$$

Thus for

$$x := \left(\frac{2\,d \cdot \|\!|f^{1/2}|\!\|_{2,q}^2}{\mathbf{D}_\rho(f^{1/2})}\right)^{\frac{1}{2+2d/q}} \geq 1$$

we have $[x]^{d+2} \leq x^{d+2}$ and $[x]^{-d(2/q-1)} \leq 2^{d(2/q-1)} x^{-d(2/q-1)}$. Using this and setting $L = [x]$ in Proposition 4.1, we obtain

$$\mu_\rho\Big(f \log \frac{f}{\mu_\rho\, f}\Big) \leq c_1 x^{d+2} \mathbf{D}_\rho(f^{1/2}) + c_2 2^{d(2/q-1)} x^{-d(2/q-1)} \|\!|f^{1/2}|\!\|_{2,q}^2 \tag{4.21}$$

from which (LN) easily follows with the characteristic exponent

$$\overline{\alpha} \equiv \Big(\frac{1}{q} - \frac{1}{2}\Big)\Big(\frac{1}{d} + \frac{1}{q}\Big)^{-1} \tag{4.22}$$

and with

$$\widetilde{\mathbf{A}}_q(f^{1/2}) \equiv \overline{a}_q \cdot \|\!|f^{1/2}|\!\|_{2,q}^2, \tag{4.23}$$

where

$$\overline{a}_q^{1-\alpha} \equiv c_1 \cdot (2\,d)^{1-\overline{\alpha}} + c_2 2^{d(2/q-1)} \cdot (2\,d)^{-\overline{\alpha}}. \tag{4.24}$$

This ends the proof of the Logarithmic Nash inequality (LN).

The proof of the inequality (LN′) proceeds in a similar way with the use of the finite volume inequality indicated in the remark after Lemma 4.1 and an observation that because of

$$(\delta_{ij} f^{1/2})^2 \leq \frac{(\delta_{ij} f)^2}{4 \inf f} \tag{4.25}$$

and Lemma 3.4 we have

$$\mathbf{D}_\rho(f^{1/2}) \leq 2d \frac{\|\!|f|\!\|_{2,q}^2}{4 \inf f}. \tag{4.26}$$

□

Algebraic decay to equilibrium in the entropy sense.

To prove of the algebraic decay entropy given the inequality (LN′), we note first that, if $f \geq \varepsilon > 0$, then also $f_t \equiv P_t f \geq \varepsilon$ and we see that

$$\mathbf{B}_q(f) \equiv \frac{a_q'}{4 \inf(f)} \|\!|f|\!\|_{2,q}^2 \tag{4.27}$$

is a P_t-monotone functional defined on a total set in $\mathcal{C}_{2,q}(\Omega)$ of strictly positive functions. The P_t-monotonicity of $\|\!|\cdot|\!\|_{p,q}$ will be in fact proven in Section 5.

Remark 4.3. Note that we have

$$\widetilde{\mathbf{A}}_q(f_t^{1/2}) \leq C \cdot \mathbf{B}_q(f) \tag{4.28}$$

with some positive constant C independent of f.

Now if (LN′) is satisfied, then we have also

$$\mu_\rho\Big(f_t \log \frac{f_t}{\mu_\rho\, f}\Big) \leq \mathbf{D}_\rho(f_t^{1/2})^\alpha\, \mathbf{B}_q(f)^{1-\alpha} \tag{4.29}$$

and so

$$\left(\frac{\mu_\rho(f_t \log(f_t/\mu_\rho f))}{\mathbf{B}_q(f)^{1-\alpha}}\right)^{1/\alpha} \leq \mathbf{D}_\rho(f_t^{1/2}), \tag{4.30}$$

Using this we get

$$\begin{aligned} \frac{d}{dt}\mu_\rho\Big(f_t \log \frac{f_t}{\mu_\rho f}\Big) &= \mu_\rho\Big((\mathcal{L} f_t) \log \frac{f_t}{\mu_\rho f}\Big) \\ &\leq -4\,\mathbf{D}_\rho(f_t^{1/2}) \leq -4 \left(\frac{\mu_\rho(f_t \log(f_t/\mu_\rho f))}{\mathbf{B}_q(f)^{1-\alpha}}\right)^{1/\alpha} \end{aligned} \tag{4.31}$$

where in the second step we have used a general inequality for Dirichlet forms. Solving this differential inequality we arrive at the following bound

$$\mu_\rho\Big(f_t \log \frac{f_t}{\mu_\rho f}\Big) \leq \Big(\frac{\gamma}{4}\Big)^\gamma \frac{\mathbf{B}_q(f)}{t^\gamma}, \tag{4.32}$$

where $\gamma \equiv \alpha/(1-\alpha)$. This ends the proof of the algebraic decay to equilibrium in the entropy sense. □

5. Monotonicity of seminorms

In this section we identify two families of seminorms which are contracted by the semigroup P_t. These families are relevant to control the algebraic decay to equilibrium, but the result may be of its own interest.

We denote by $\mathcal{F} \equiv \mathcal{F}(\mathbf{Z}^d)$ and $\mathcal{F}_n \equiv \mathcal{F}_n(\mathbf{Z}^d)$ the collection of all finite subsets of $\mathbf{Z}^d$ and the collection of the subsets $Y \subset \mathbf{Z}^d$ of cardinality n, respectively. For $Y \in \mathcal{F}$ we define an operator

$$\nabla_Y := \prod_{j\in Y} \nabla_j. \tag{5.1}$$

For $n \in \mathbf{N}$, $p, q \in [1, \infty]$, we define the following seminorms

$$\mathbf{V}^n_{p,q}(f) := \Big(\sum_{Y\in\mathcal{F}_n} (\mu_\rho |\nabla_Y f|^p)^{q/p} \Big)^{1/q}. \tag{5.2}$$

Note that if $p' \geq p$, $q \geq q'$, then

$$\mathbf{V}^n_{p,q}(f) \leq \mathbf{V}^n_{p',q'}. \tag{5.3}$$

On the other hand, for $n \neq \tilde{n}$ it is easy to check that $\mathbf{V}^n_{p,q}$ and $\mathbf{V}^{\tilde{n}}_{p,q}$ are non-equivalent in the sense there exists f such that $\mathbf{V}^n_{p,q}(f) < \infty$ and $\mathbf{V}^{\tilde{n}}_{p,q}(f) = \infty$.

We will show the following result.

Theorem 5.1. *For any $n \in \mathbf{N}$, $p, q \in [1, \infty]$, the seminorm $\mathbf{V}^n_{p,q}$ is contracted by P_t, i.e. for any $t \geq 0$ we have*

$$\mathbf{V}^n_{p,q}(P_t f) \leq \mathbf{V}^n_{p,q}(f). \tag{5.4}$$

This results generalizes [11, I, Theorem 3.9(d)] in which it is proven that $\mathbf{V}^{(1)}_{\infty,1}$ is contracted by P_t.

Proof. We will show that the contractivity property can be obtained as a result of some more general construction. For this we need to consider an extended configuration space $\Omega_n := \mathcal{F}_n \times \Omega$; we shall denote the points in this space by (Y, η). In $\mathcal{C}(\Omega_n)$ we can distinguish a subset of cylinder functions, i.e. functions which depend only on a finite number of coordinates η_j, $j \in \mathbf{Z}^d$, and vanishing if Y is not contained in some ball. In $\mathcal{C}(\Omega_n)$ we introduce an operator $\mathcal{L}_n$ densely defined on the set of cylinder functions $F = F(Y, \eta)$, by the following formula

$$\begin{aligned}\mathcal{L}_n F(Y,\eta) \ := & \sum_{\langle i,j\rangle : \langle i,j\rangle \cap Y = \emptyset} [F(Y, \eta^{i,j}) - F(Y,\eta)] \\ & + \sum_{i\in Y} \sum_{j\notin Y : |j-i|=1} [F((Y \setminus i) \cup j, \eta^{i,j}) - F(Y, \eta)].\end{aligned} \tag{5.5}$$

It is immediate to check that $\mathcal{L}_n$ is a Markov pregenerator, i.e. it satisfies the maximum principle. One can construct the corresponding semigroup $P^{(n)}_t$ on cylinder function as the limit

$$P^{(n)}_t F(Y,\eta) = \lim_{L\to\infty} P^{(n,L)}_t F(Y,\eta) \tag{5.6}$$

with $P^{(n,L)}_t \equiv e^{t\mathcal{L}_{n,L}}$ where $\mathcal{L}_{n,L}$ is a generator with periodic boundary conditions on the cube Λ_L of side L centered at the origin, which is defined in a similar way as in (5.5) except that $Y \subset \Lambda_L$ and the nearest neighbors pairs of points are understood as on the corresponding torus $\mathbf{T}^d_L$. The construction can be carried out in a similar way as in e.g. [18] (one needs then to introduce additional gradient operator $\partial_{X,\langle ij\rangle} F(Y,\eta) = F((Y\setminus i)\cup j, \eta^{i,j}) - F(Y,\eta)$, if $Y = X$, and zero otherwise). We remark that the restriction of $\mathcal{L}_n$ to the functions F independent of η equals precisely the generator Δ_n of the simple symmetric exclusion process on the lattice $\mathbf{Z}^d$ with n particles.

The idea of considering such objects is based on the observation that for $F(Y,\eta) = \nabla_Y f(\eta)$, $(Y,\eta) \in \Omega_n$, one has

$$\nabla_Y \mathcal{L} f(\eta) = \mathcal{L}_n F(Y,\eta) \tag{5.7}$$

and therefore

$$\nabla_Y P_t f(\eta) = P^{(n)}_t F(Y,\eta). \tag{5.8}$$

Thus, if for $p, q \in [1,\infty]$ we introduce Banach space $\mathsf{X}_{p,q}$ of functions on Ω_n, with a norm defined by

$$\|F\|_{p,q} := \Big(\sum_{Y\in\mathcal{F}_n} (\mu_\rho |F(Y,\cdot)|^p)^{q/p} \Big)^{1/q}, \tag{5.9}$$

and we show that $P^{(n)}_t$ extends to a contractive semigroup in this space, then to get our result it is sufficient to note that for $F(Y,\eta) = \nabla_Y f(\eta)$ we have

$$\mathbf{V}^n_{p,q}(f) = \|F\|_{p,q}. \tag{5.10}$$

(The idea of using a relation similar to (5.7) with a different purpose has been exploited in an interesting way e.g. in [2] in the context of Glauber dynamics.)

We note that $\mathsf{X}_{2,2}$ equals the space $\mathsf{L}_2(dY \times d\mu_\rho)$, where dY denotes the counting measure on $\mathcal{F}_n$, with an inner product defined by

$$\langle F, G\rangle := \sum_{Y\in\mathcal{F}_n} \int d\mu_\rho(\eta) F(Y,\eta)\, G(Y,\eta). \tag{5.11}$$

A simple computation shows that $\mathcal{L}_n$, as defined in (5.5), is essentially self-adjoint and non-positive on $\mathsf{L}_2(dY \times d\mu_\rho)$ and the similar statement is true for the generator $\mathcal{L}_{n,L}$ with periodic boundary conditions (on functions localized in Λ_L).

Given the finite volume approximation property (5.6), it is now sufficient to show the necessary contraction properties for $P^{(n,L)}_t \equiv e^{t\mathcal{L}_{n,L}}$ for the subspaces of functions localized in Λ_L. For this we will apply the Hille–Yosida theorem to show that $\mathcal{L}_{n,L}$ generates a contractive semigroup on the Banach space $\mathsf{X}_{p,q}$, for any $p, q \in \{1, \infty\}$. It is sufficient to show that for a function $G \in \mathsf{X}_{p,q}$ satisfying

$$\lambda G - \mathcal{L}_{n,L} G = F \tag{5.12}$$

for some $\lambda > 0$ and a given $F \in \mathsf{X}_{p,q}$, $p, q \in \{1, \infty\}$, localized in Λ_L, we have

$$\lambda \|G\|_{p,q} \le \|F\|_{p,q}. \tag{5.13}$$

The case $(p,q) = (\infty,\infty)$ is easy and therefore will be omitted, but we note that from the considerations in $\mathsf{X}_{\infty,\infty}$ follows that the corresponding semigroup $P_t^{(n,L)}$ is positivity preserving.

Next we consider the case $(p,q) = (\infty, 1)$. Let $\bar\eta = \bar\eta(Y)$ be such that $\sup_\eta |G(Y,\eta)| = G(Y,\bar\eta)$. (Since Ω is compact, eventually changing signs of F and G, one can see that such $\bar\eta$ exists.) Recalling the definition $\mathcal{L}_{n,L}$, we have

$$\begin{aligned}
\lambda\|G(Y,\cdot)\|_\infty &= \lambda G(Y,\bar\eta) \\
&= F(Y,\bar\eta) + \sum_{i\in Y}\sum_{j\notin Y:|j-i|=1} [G((Y\setminus i)\cup j, \bar\eta^{i,j}) - G(Y,\bar\eta)] \\
&\le \|F(Y,\cdot)\|_\infty + \sum_{i\in Y}\sum_{j\notin Y:|j-i|=1} [\|G((Y\setminus i)\cup j,\cdot)\|_\infty - \|G(Y,\cdot)\|_\infty]
\end{aligned}$$

and summing over $Y \in \mathcal{F}_n(\Lambda_L)$, we get (5.13) in the considered case $(p,q) = (\infty, 1)$ (due to translation invariance of our generator the second part on the right-hand side of the above inequality adds up to zero). The bound (5.13) then implies that the resolvent of $\mathcal{L}_{n,L}$ satisfies $\|(\lambda - \mathcal{L}_{n,L})^{-1}\| \le \lambda^{-1}$ for all $\lambda > 0$, uniformly in L. Therefore, by the Hille–Yosida theorem, $\mathcal{L}_{n,L}$ generates a contractive semigroup on $\mathsf{X}_{\infty,1}$.

To get the corresponding result for $(p,q) = (1,\infty)$, we note that there is $\overline{Y} \in \mathcal{F}_n(\Lambda_L)$ such that for the function G

$$\sup_{Y\in\mathcal{F}_n(\Lambda_L)} |\mu_\rho G(Y,\cdot)| = \mu_\rho G(\overline{Y},\cdot) \tag{5.14}$$

(if necessary we change the signs of F and G). Now we have

$$\begin{aligned}
\lambda\|G\|_{1,\infty} &= \lambda\mu_\rho G(\overline{Y},\eta) \\
&= \mu_\rho F(\overline{Y},\eta) + \sum_{i\in\overline{Y}}\sum_{j\notin\overline{Y}:|j-i|=1} [\mu_\rho G((\overline{Y}\setminus i)\cup j,\eta^{i,j}) - \mu_\rho G(\overline{Y},\eta)] \\
&= \mu_\rho F(\overline{Y},\eta) + \sum_{i\in\overline{Y}}\sum_{j\notin\overline{Y}:|j-i|=1} [\mu_\rho G((\overline{Y}\setminus i)\cup j,\eta) - \mu_\rho G(\overline{Y},\eta)] \\
&\le \mu_\rho F(\overline{Y},\eta) \le \|F\|_{1,\infty},
\end{aligned}$$

here $\|\cdot\|_{1,\infty}$, means the norm associated with $\Omega_{n,L} \equiv \mathcal{F}_n(\Lambda_L)\times\Omega$.

To consider the case of $\mathsf{X}_{1,1}$, taking F non-negative, we see that if for some $\lambda > 0$ the function G satisfies $\lambda G + \mathcal{L}_{n,L}G = F$, it is also non-negative and we have

$$\begin{aligned}
\lambda\|G\|_{1,1} &= \lambda\sum_Y \mu G(Y,\cdot) = \sum_Y \mu F(Y,\cdot) + \sum_Y \mu\mathcal{L}_{n,L}G(Y,\cdot) \\
&= \sum_Y \mu F(Y,\cdot) = \|F\|_{1,1},
\end{aligned}$$

where in the last step we have used the symmetry of $\mathcal{L}_{n,L}$ in $\mathsf{X}_{2,2}$ and the fact that $\mathcal{L}_{n,L}1 = 0$. This implies that the corresponding semigroup is also contractive in $\mathsf{X}_{1,1}$.

Since for the given $p \in [1,\infty]$, respectively $q \in [1,\infty]$, the family $\{\mathsf{X}_{p,r} : r \in [1,\infty]\}$, respectively $\{\mathsf{X}_{r,q} : r \in [1,\infty]\}$, forms a complex interpolating family, using the interpolation theorem we arrive at

$$\|P_t^{(n,L)}F\|_{p,q} \le \|F\|_{p,q} \tag{5.15}$$

for every $p,q \in [1,\infty]$. Using this and the finite volume approximation property for cylinder functions, we get the desired contractivity property for a dense set of functions in every space $\mathsf{X}_{p,q}$. By continuity, this implies the general result and together with (5.10) implies the statement of the theorem. □

For $Y \in \mathcal{F}(\mathbf{Z}^d)$ and $\rho \in (0,1)$ we define

$$\sigma_Y := \mathbf{v}_\rho^{-|Y|/2} \prod_{i\in Y} (\eta(i) - \rho), \tag{5.16}$$

where we recall that $\mathbf{v}_\rho \equiv \rho(1-\rho) = \mu_\rho(\eta(i)-\rho)^2$. We note that $\{\sigma_Y\}$, $Y \in \mathcal{F}_n(\mathbf{Z}^d)$, $n = 0, 1, \cdots$ forms an orthonormal basis in $\mathsf{L}_2(\mu_\rho)$. Therefore for any $f \in \mathsf{L}_2(\mu_\rho)$ we have the following expansion

$$f \equiv \sum_{n=0}^{\infty} f^{(n)} = \sum_{n=0}^{\infty}\sum_{Y\in\mathcal{F}_n} \mathrm{f}_Y^{(n)} \cdot \sigma_Y. \tag{5.17}$$

Later on the subspace of $\mathsf{L}_2(\mu_\rho)$ of functions satisfying $f = f^{(n)}$ will be denoted by $\mathcal{H}_n$. Using the expansion (5.17) one can see that in particular the following useful relation is true

$$\mu_\rho(\sigma_Y f) = (-2)^{-|Y|}\mu_\rho(\sigma_Y \nabla_Y f). \tag{5.18}$$

Now for $n \in \mathbf{N}$, $a \ge 0$ and $p \in [1,\infty]$ we define a Sobolev-type seminorm $\mathbf{H}^n_{-a,p}$ as follows

$$\mathbf{H}^n_{-a,p}(f) := 2^{-n} \sup_{0\ne\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(\sigma_Y\nabla_Y f)}{\|(-\Delta_n)^{a/2}\varphi\|_q}. \tag{5.19}$$

where $q^{-1} + p^{-1} = 1$, the operator Δ_n is defined as the restriction of $\mathcal{L}_n$ to functions independent of $\eta \in \Omega$, $\ell_0(\mathcal{F}_n)$ denotes the space of finite sequences over $\mathcal{F}_n$ and $\|\cdot\|_q$ denotes the norm in the space $\ell_q(\mathcal{F}_n)$ of sequences absolutely summable in the power q. Since Δ_n is self-adjoint in $\ell_2(\mathcal{F}_n)$, the operator

$(-\Delta_n)^{a/2}$ is well defined via the spectral theorem. We note that, using (5.18), one has

$$\mathbf{H}^n_{-a,p}(f) = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}\varphi\|_q} = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mathrm{f}^{(n)}_Y}{\|(-\Delta_n)^{a/2}\varphi\|_q} \equiv \|(-\Delta_n)^{-a/2}\mathrm{f}^{(n)}\|_p \quad (5.20)$$

where $\mathrm{f}^{(n)} \equiv \{\mathrm{f}^{(n)}_Y\}_{Y\in\mathcal{F}_n}$. We shall denote the corresponding Sobolev-type space by $W_{-a,p}(\mathcal{F}_n(\mathbf{Z}^d))$.

Theorem 5.2. *For any $n\in\mathbf{N}$, $p\in[1,\infty]$, $a\geq 0$, the seminorm $\mathbf{H}^n_{-a,p}$ is contracted by P_t, i.e. for any $t\geq 0$ we have*

$$\mathbf{H}^n_{-a,p}(P_t f) \leq \mathbf{H}^n_{-a,p}(f). \quad (5.21)$$

Proof. The proof is similar to that of Theorem 5.1. For $Y\in\mathcal{F}_n$, consider $F(Y,\eta) = 2^{-n}\sigma_Y\nabla^n_Y f$. Then using the relation (5.18) we see that

$$\mathbf{H}^n_{-a,p}(f) = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}\varphi\|_q} = \sup_{0\neq\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(F(Y,\cdot))}{\|(-\Delta_n)^{a/2}\varphi\|_q}. \quad (5.22)$$

For such F we have (similarly as in the proof of Theorem 5.1)

$$\sigma_Y\nabla_Y\mathcal{L}f(\eta) = \mathcal{L}_n F(Y,\eta),$$

where $\mathcal{L}_n$ is the operator defined in (5.5). Therefore

$$\sigma_Y\nabla_Y P_t f(\eta) = P^{(n)}_t F(Y,\eta).$$

Since $P^{(n)}_t$ is a self-adjoint semigroup on $\mathrm{L}_2(dY\times\mu_\rho)$, we obtain

$$\sum_{Y\in\mathcal{F}_n}\varphi(Y)\mu_\rho(P^{(n)}_t F(Y,\eta)) = \langle\varphi, P^{(n)}_t F\rangle = \langle P^{(n)}_t\varphi, F\rangle = \langle Q^{(n)}_t\varphi, F\rangle$$

because φ does not depend on η and on those functions $P^{(n)}_t$ coincides with $Q^{(n)}_t := \exp\{t\Delta_n\}$. Thus, for $p^{-1}+q^{-1}=1$ and $F(Y,\eta)=\sigma_Y\nabla_Y f$, we have

$$\begin{aligned}\mathbf{H}^n_{-a,p}(P_t f) &= \sup_{\varphi\in\ell_0(\mathcal{F}_n)} \frac{\langle Q^{(n)}_t\varphi, F\rangle}{\|(-\Delta_n)^{a/2}\varphi\|_q}\\ &= \sup_{\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}Q^{(n)}_t\varphi(Y)\cdot\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}Q^{(n)}_t\varphi\|_q}\cdot\frac{\|(-\Delta_n)^{a/2}Q^{(n)}_t\varphi\|_q}{\|(-\Delta_n)^{a/2}\varphi\|_q}\\ &\leq \sup_{\varphi\in\ell_0(\mathcal{F}_n)} \frac{\sum_{Y\in\mathcal{F}_n}Q^{(n)}_t\varphi(Y)\cdot\mu_\rho(\sigma_Y f)}{\|(-\Delta_n)^{a/2}Q^{(n)}_t\varphi\|_q} \leq \mathbf{H}^n_{-a,p}(f),\end{aligned}$$

where we used that, by ℓ_q-contractivity of the semigroup, $Q^{(n)}_t = \exp\{t\Delta_n\}$,

$$\|(-\Delta_n)^{a/2}Q^{(n)}_t\varphi\|_q \leq \|(-\Delta_n)^{a/2}\varphi\|_q.$$

□

6. Proof of faster decay

We notice there are functions for which the rate of convergence is faster than the one found in Theorem 2.1; a typical example is, in the one-dimensional case, the function $f=\eta(1)-\eta(0)$ which decays as $t^{-3/2}$. In this section we exploit the above idea and pursue further the spectral analysis of the product case. In particular we prove a faster decay for appropriate class of functions by using the Sobolev-type seminorms introduced at the end of the previous section.

We take advantage of the fact that our stochastic dynamics is locally conservative in the sense that for any set $\Lambda\in\mathcal{F}$ the null subspace $\mathcal{I}_{\Lambda,\rho}\subseteq\mathrm{L}_2(\mu_{\Lambda,\rho})$ of the Dirichlet form $\mathbf{D}_{\Lambda,\rho}$ is non-trivial (i.e. contains non-constant elements). In our case one knows that $\mathcal{I}_{\Lambda,\rho}$ is spanned by the following orthonormal functions

$$\mathbf{S}^{(n)}_\Lambda \equiv |\mathcal{F}_n(\Lambda)|^{-1/2}\cdot\sum_{Y\in\mathcal{F}_n(\Lambda)}\sigma_Y, \qquad n=0,\cdots,|\Lambda|, \quad (6.1)$$

where we recall σ_Y has been defined in (5.16) and $\mathcal{F}_n(\Lambda)$ denotes the family of the subsets of Λ with cardinality n. In fact it is not difficult to see that the value of this function is constant on any set $\{\eta\in\Omega : N_\Lambda(\eta)=k\}$, $k=0,\cdots,|\Lambda|$. Let $\aleph_{\Lambda,\rho}$ denote the orthogonal complement of $\mathcal{I}_{\Lambda,\rho}$ in $\mathrm{L}_2(\mu_{\Lambda,\rho})$. It follows from the proof of Lemma 3.1 (see (3.6) and (3.7)) that, in case when $\Lambda\in\mathcal{F}$ is a cube of side $L\in\mathbf{N}$, for any $f\in\aleph_{\Lambda,\rho}$ we have

$$\mu_{\Lambda,\rho}(f-\mu_{\Lambda,\rho}f)^2 \leq c_0 L^2\cdot\mathbf{D}_{\Lambda,\rho}(f). \quad (6.2)$$

Given a partition $\{\Lambda_\ell\}_1^\infty$ of $\mathbf{Z}^d$ by disjoint cubes of side $L\in\mathbf{N}$ used in the proofs of Propositions 3.1 and 4.1, we define the families $\{\wp_\ell\}_1^\infty$, $\{\mathcal{I}_\ell\}_1^\infty$ and $\{\aleph_\ell\}_1^\infty$ of orthogonal projectors respectively by setting

$$\wp_\ell f = \mathrm{E}_{\ell-1}f - \mathrm{E}_\ell f \quad (6.3)$$

(where we recall $\mathrm{E}_0 f\equiv f$ and $\mathrm{E}_\ell f := \mu_{Y_\ell,\rho}f$, for $\ell\in\mathbf{N}$, with $\{Y_\ell\}_1^\infty$ being the increasing sequence defined by $Y_\ell := \bigcup_{\ell'\leq\ell}\Lambda_{\ell'}$),

$$\mathcal{I}_\ell f \equiv \sum_{m=1}^{|\Lambda|}\mathcal{I}^{(m)}_\ell f \equiv \sum_{m=1}^{|\Lambda|}\mu_{\Lambda_\ell,\rho}(\mathbf{S}^{(m)}_{\Lambda_\ell}\cdot\wp_\ell f)\cdot\mathbf{S}^{(m)}_{\Lambda_\ell} \quad (6.4)$$

and

$$\aleph_\ell f \equiv (\mathbf{1}-\mathcal{I}_\ell)\wp_\ell f. \quad (6.5)$$

Using this notation it is easy to see that one has

$$\mu_\rho(f - \mu_\rho\, f)^2 = \sum_{\ell=1}^{\infty} \mu_\rho(\wp_\ell f)^2 = \sum_{\ell=1}^{\infty} \mu_\rho(\aleph_\ell f)^2 + \sum_{\ell=1}^{\infty} \mu_\rho(\mathcal{I}_\ell f)^2. \tag{6.6}$$

Since always $\mu_{\Lambda_\ell,\rho}\aleph_\ell f = 0$, the first part of the right-hand side of (6.6) can be estimated with the help of (6.2) as follows

$$\sum_{\ell=1}^{\infty} \mu_\rho(\aleph_\ell f)^2 = \sum_{\ell=1}^{\infty} \mu_\rho(\mu_{\Lambda_\ell,\rho}(\aleph_\ell f)^2) \le c_0 L^2 \cdot \sum_{\ell=1}^{\infty} \mathbf{D}_{\Lambda_\ell,\rho}(\aleph_\ell f). \tag{6.7}$$

To analyze the second part on the right-hand side of (6.6), we observe first that the function $\mathcal{I}_\ell f$ depends only on the variables in $\complement Y_\ell \equiv \mathbf{Z}^d \setminus Y_\ell$ and as such can be expanded in the following series of orthogonal elements in $\mathsf{L}_2(\mu_\rho)$

$$\mathcal{I}_\ell f = \mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f + \sum_{\ell'>\ell}^{\infty} \wp_{\ell'} \mathcal{I}_\ell f. \tag{6.8}$$

Thus we have

$$\begin{aligned}\mu_\rho(\mathcal{I}_\ell f)^2 &= \sum_{\ell'>\ell}^{\infty} \mu_\rho(\wp_{\ell'}\mathcal{I}_\ell f)^2 + \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2 \\ &= \sum_{\ell'>\ell}^{\infty} \mu_\rho(\aleph_{\ell'}\mathcal{I}_\ell f)^2 + \sum_{\ell'>\ell}^{\infty} \mu_\rho(\mathcal{I}_{\ell'}\mathcal{I}_\ell f)^2 \\ &\quad + \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2. \end{aligned} \tag{6.9}$$

Combining (6.6)–(6.9), we obtain the following inequality

$$\begin{aligned}\mu_\rho(f - \mu_\rho\, f)^2 &\le c_0 L^2 \cdot \sum_{\ell=1}^{\infty} \mathbf{D}_{\Lambda_\ell,\rho}(\aleph_\ell f) + \sum_{\ell=1}^{\infty}\sum_{\ell'>\ell}^{\infty} \mu_\rho(\aleph_{\ell'}\mathcal{I}_\ell f)^2 \\ &\quad + \sum_{\ell=1}^{\infty}\sum_{\ell'>\ell}^{\infty} \mu_\rho(\mathcal{I}_{\ell'}\mathcal{I}_\ell f)^2 + \sum_{\ell=1}^{\infty} \mu_\rho(\mu_{\complement Y_\ell,\rho}\mathcal{I}_\ell f)^2. \end{aligned} \tag{6.10}$$

Applying this argument inductively, we arrive at the following bound

$$\begin{aligned}\mu_\rho(f - \mu_\rho\, f)^2 \le c_0 L^2 \Big\{ &\sum_{\ell(1)=1}^{\infty} \mathbf{D}_{\Lambda_{\ell(1)},\rho}(\aleph_{\ell(1)} f) \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(2)},\rho}(\aleph_{\ell(2)}\mathcal{I}_{\ell(1)} f) + \cdots \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty}\cdots\sum_{\ell(m)>\ell(m-1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(m)},\rho}(\aleph_{\ell(m)}\mathcal{I}_{\ell(m-1)}\cdots\mathcal{I}_{\ell(1)} f) + \cdots \Big\} \\ &+ \Big\{ \sum_{\ell(1)=1}^{\infty} \mu_\rho(\mu_{\complement Y_{\ell(1)},\rho}\mathcal{I}_{\ell(1)} f)^2 + \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mu_\rho(\mu_{\complement Y_{\ell(2)},\rho}(\mathcal{I}_{\ell(2)}\mathcal{I}_{\ell(1)} f))^2 + \ldots \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty}\cdots\sum_{\ell(m)>\ell(m-1)}^{\infty} \mu_\rho(\mu_{\complement Y_{\ell(m)},\rho}(\mathcal{I}_{\ell(m)}\mathcal{I}_{\ell(m-1)}\cdots\mathcal{I}_{\ell(1)} f))^2 + \cdots \Big\}. \end{aligned}$$

We note that

$$\begin{aligned}\mathbf{D}_\rho(f) \ge &\sum_{\ell(1)=1}^{\infty} \mathbf{D}_{\Lambda_{\ell(1)},\rho}(\aleph_{\ell(1)} f) \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(2)},\rho}(\aleph_{\ell(2)}\mathcal{I}_{\ell(1)} f) + \cdots \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty}\cdots\sum_{\ell(m)>\ell(m-1)}^{\infty} \mathbf{D}_{\Lambda_{\ell(m)},\rho}(\aleph_{\ell(m)}\mathcal{I}_{\ell(m-1)}\cdots\mathcal{I}_{\ell(1)} f) + \cdots. \end{aligned}$$

Thus introducing the following notation

$$\begin{aligned}\mathcal{A}_\rho^{(L)}(f) \equiv &\sum_{\ell(1)=1}^{\infty} \mu_\rho(\mu_{\complement Y_{\ell(1)},\rho}\mathcal{I}_{\ell(1)} f)^2 \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty} \mu_\rho\left(\mu_{\complement Y_{\ell(2)},\rho}(\mathcal{I}_{\ell(2)}\mathcal{I}_{\ell(1)} f)\right)^2 + \cdots \\ &+ \sum_{\ell(1)=1}^{\infty}\sum_{\ell(2)>\ell(1)}^{\infty}\cdots\sum_{\ell(m)>\ell(m-1)}^{\infty} \mu_\rho(\mu_{\complement Y_{\ell(m)},\rho}(\mathcal{I}_{\ell(m)}\mathcal{I}_{\ell(m-1)}\cdots\mathcal{I}_{\ell(1)} f))^2 + \cdots \end{aligned}$$

we arrive at the following intermediate result.

Proposition 6.1. *For any $\rho \in (0,1)$, $L \in \mathbf{N}$, we have*

$$\mu_\rho(f - \mu_\rho f)^2 \le c_0 L^2 \cdot \mathbf{D}_\rho(f) + \mathcal{A}_\rho^{(L)}(f) \tag{6.11}$$

for any function $f \in \mathsf{L}_2(\mu_\rho)$ for which the right-hand side is finite.

The functional $\mathcal{A}_\rho^{(L)}$ is rather complicated and therefore we need to find a convenient bound for it. For this we remark that if $f \in \mathcal{H}_n$, i.e. when it is of the following form

$$f = \sum_{Y\in\mathcal{F}_n} \mathrm{f}_Y^{(n)} \cdot \sigma_Y \tag{6.12}$$

with some real numbers $f_Y^{(n)}$, $Y \in \mathcal{F}_n$, then we have

$$\mathcal{A}_p^{(L)}(f) = \sum_{m=1}^{n} \sum_{\ell(1)<\cdots<\ell(m)} \sum_{\substack{k_1+\cdots+k_m=n \\ 1\le k_1,\cdots,k_m\le|\Lambda_0|}} \frac{1}{\prod_{j=1}^m |\mathcal{F}_{k(j)}(\Lambda_0)|} \times \Big(\sum_{\substack{Y_1\subset\Lambda_{\ell(1)},\cdots,Y_m\subset\Lambda_{\ell(m)} \\ |Y_1|=k_1,\cdots,|Y_m|=k_m}} (f^{(n)}_{Y_1\cup\cdots\cup Y_m})^2 \Big)^2$$
$$\le \frac{\bar{b}_n}{|\mathcal{F}_n(\Lambda_0)|} \|\chi_L * f\|^2_{\ell_2(\mathbf{Z}^{nd})}, \tag{6.13}$$

where

$$\bar{b}_n \equiv \max_{\substack{m=1,..,n \\ k_1+..+k_m=n \\ 1\le k_1,\cdots,k_m\le|\Lambda_0|}} \frac{|\mathcal{F}_n(\Lambda_0)|}{\prod_{j=1}^m |\mathcal{F}_{k(j)}(\Lambda_0)|} \tag{6.14}$$

and where we have introduced a convolution type operator, by identifying $\mathcal{H}_n$ with a subspace $\ell_2(\mathcal{F}_n) \subset \ell_2(\mathbf{Z}^{dn})$, as follows

$$\chi_L * f(\mathbf{i}) \equiv \sum_{\mathbf{y}\in\mathbf{Z}^{dn}} \chi_L(\mathbf{y} - \mathbf{i}\cdot L)\cdot f^{(n)}_{\mathbf{y}} \tag{6.15}$$

where $\mathbf{i} \equiv \{i_1, ..., i_n\}$ and $\mathbf{y} \equiv \{y_1, \dots, y_n\}$, and

$$\chi_L(\mathbf{y} - \mathbf{i}\cdot L) \equiv \prod_{m=1}^{n} \mathbf{1}_{\Lambda_0}(y_m - i_m \cdot L) \tag{6.16}$$

with $\mathbf{1}_{\Lambda_0}$ being the characteristic function of the cube $\Lambda_0 \subset \mathbf{Z}^d$ of side L. Using (6.13) and the fact that $|\mathcal{F}_n(\Lambda_0)| = (L^d \cdots (L^d - n + 1))/n!$, we see that for $f \in \mathcal{H}_n$, $1 \le n \le L^d$, we have

$$\mu_\rho(f - \mu_\rho f)^2 \le c_0 L^2 \cdot \mathbf{D}_\rho(f) + \frac{b_n}{L^{dn}} \|\chi_L * f\|^2_{\ell_2(\mathbf{Z}^{nd})} \tag{6.17}$$

with some positive constant b_n independent of L. We will show the following estimate, in which we recall $W_{-a,q}$ has been defined in (5.20).

Proposition 6.2. *For each $n, s \in \mathbf{N}$, $q \in [1,2]$ and $\gamma > 0$ there exist $L_0 = L_0(n,s,d,\gamma)$ and $b = b(n,s,q,d,\gamma)$ such that the following estimate*

$$L^{-nd}\|\chi_L * J\|^2_{\ell_2(\mathbf{Z}^{nd})} \le b\cdot L^{-4s}\cdot L^{-nd(2/q-1)} \|J\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))} + \gamma \|J\|^2_{\ell_2(\mathcal{F}_n(\mathbf{Z}^d))} \tag{6.18}$$

holds for any $L > L_0$ and $J \in W_{-2s,q}(\mathcal{F}_n(Z^d))$.

We first recall Young's inequality for the convolution operators. For $\psi \in \ell_1(\mathbf{Z}^{nd})$) and $L \in \mathbf{N}$, let ψ_L be a convolution operator defined as follows

$$\psi_L * J\mathbf{i} \equiv \sum_{\mathbf{y}\in\mathbf{Z}^{nd}} \psi(\mathbf{y} - \mathbf{i}\,L)\, J(\mathbf{y}). \tag{6.19}$$

Then we have

Lemma 6.1. *For each $p \in [1,\infty]$, $\psi \in \ell_1(\mathbf{Z}^{nd})$) and $J \in \ell_p(\mathbf{Z}^{nd})$*

$$\|\psi_L * J\|_{\ell_p(\mathbf{Z}^{nd})} \le \|\psi\|^{(p-1)/p}_{\ell_1(\mathbf{Z}^{nd})} \cdot \Big(\sup_{\mathbf{y}\in\mathbf{Z}^{nd}} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} |\psi(\mathbf{y} - \mathbf{i}L)| \Big)^{1/p} \cdot \|J\|_{\ell_p(\mathbf{Z}^{nd})}. \tag{6.20}$$

Proof. Inequality (6.20) follows by applying the Riesz – Thorin interpolation theorem with use of the following two simple bounds

$$\|\psi_L * J\|_{\ell_\infty(\mathbf{Z}^{nd})} = \sup_{\mathbf{i}\in\mathbf{Z}^{nd}} \Big| \sum_{\mathbf{y}\in\mathbf{Z}^{nd}} \psi(\mathbf{y} - \mathbf{i}L)\cdot J(\mathbf{y}) \Big| \le \|\psi\|_{\ell_1(\mathbf{Z}^{nd})} \cdot \|J\|_{\ell_\infty(\mathbf{Z}^{nd})} \tag{6.21}$$

and

$$\|\psi_L * J\|_{\ell_1(\mathbf{Z}^{nd})} = \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} \Big| \sum_{\mathbf{y}\in\mathbf{Z}^{nd}} \psi(\mathbf{y} - \mathbf{i}L)\cdot J(\mathbf{y}) \Big|$$
$$\le \sup_{\mathbf{y}\in\mathbf{Z}^{nd}} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} |\psi(\mathbf{y} - \mathbf{i}L)| \cdot \|J\|_{\ell_1(\mathbf{Z}^{nd})}.$$

□

Proof of Proposition 6.2. Using Lemma 6.1 we will show that, given $s, n \in \mathbf{N}$ and any $\gamma > 0$, one can choose a "smooth" approximation ψ_L^ϵ of the operator χ_L so that

$$L^{-nd}\|(\chi_L - \psi_L^\epsilon) * J\|^2_{\ell_2(\mathbf{Z}^{nd})} \le \frac{\gamma}{2}\|J\|^2_{\ell_2(\mathcal{F}_n(\mathbf{Z}^d))} \tag{6.22}$$

and with some constant $a \in (0,\infty)$ we have

$$L^{-nd}\|\psi_L^\epsilon * J\|^2_{\ell_2(\mathbf{Z}^{nd})} \le \frac{a}{2}\cdot L^{-4s}\cdot L^{-nd(2/q-1)}\|J\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))} \tag{6.23}$$

for any $J \in \ell_2(\mathcal{F}_n(\mathbf{Z}^d)) \cap W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))$.

Step 1 (proof of (6.22)). For $\delta \in (0,1)$ we choose first $\psi_\delta \in C_0^\infty(\mathbf{R})$ such that $0 \le \psi_\delta(r) \le 1$, $\psi_\delta(r) = 1$ for $r \in [0,1]$ and $\psi_\delta(r) = 0$ for $r \ge 1 + \delta$ and $r \le -\delta$. For $\mathbf{x} \in \mathbf{Z}^{nd}$ define

$$\psi_{\delta,L}(\mathbf{x}) := \prod_{k=1}^{nd} \psi_\delta(L^{-1}x_k). \tag{6.24}$$

We want to show that the convolution operator involving χ_L can be well approximated by the one defined with the smooth function $\psi_{\delta,L}$. Since using the definition of χ_L we have

$$\psi_{\delta,L}(\mathbf{x}) - \chi_L(\mathbf{x}) = \sum_{k=1}^{nd} \prod_{m<k} \psi_\delta(L^{-1}x_m)\cdot[\psi_\delta(L^{-1}x_k) - \mathbf{1}_{[0,1]}(L^{-1}x_n)] \times \prod_{m>n} \mathbf{1}_{[0,1]}(L^{-1}x_m),$$

we get the following bound

$$\|\psi_{\delta,L} - \chi_L\|_{\ell_1(\mathbf{Z}^{nd})} \le nd\cdot[(1+2\delta)L]^{nd-1}[(1+2\delta)L - L] \le \delta\cdot 2nd\cdot(3L)^{nd}.$$

Also we have

$$\sup_{\mathbf{x}\in\mathcal{F}_n(\mathbf{Z}^d)} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} |\psi_{\delta,L}(\mathbf{x}-\mathbf{i}L) - \chi_L(\mathbf{x}-\mathbf{i}L)| \le \sup_{\mathbf{x}\in\mathcal{F}_n(\mathbf{Z}^d)} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} \psi_{\delta,L}(\mathbf{x}-\mathbf{i}L) \le 3^{nd}.$$

Therefore, applying Lemma 6.1 with $p=2$, we get

$$\|(\chi_L - \psi_{\delta,L}) * J\|^2_{\ell_2(\mathbf{Z}^{nd})} \le \delta\cdot 2nd\cdot 3^{2nd}\cdot L^{nd}\cdot\|J\|^2_{\ell_2(\mathcal{F}_n(\mathbf{Z}^d))}. \tag{6.25}$$

Now we modify further the function $\psi_{\delta,L}$ by introducing another cutoff ε. The purpose is to obtain a new truncation $\psi^\varepsilon_L \equiv \psi^\varepsilon_{\delta,L}$ with support bounded away from the hyperplanes $y_k = y_{k'}$, $k,k'\in\{1,\ldots,n\}$, where the exclusion condition applies.

Given $\varepsilon\in(0,1)$, we pick $\zeta_\varepsilon\in C^\infty(\mathbf{R})$ such that $0\le\zeta_\varepsilon(r)\le 1$, $\zeta_\varepsilon(r)=1$ for $|r|\ge 2\varepsilon$ and $\zeta_\varepsilon(r)=0$ for $|r|\le\varepsilon$. For $(y_1,\ldots,y_n)\in\mathbf{Z}^{nd}$ let us define

$$\psi^\varepsilon_L(y_1,\ldots,y_n) := \psi_{\delta,L}(y_1,\ldots,y_n)\prod_{(l,l')}\zeta_\varepsilon\Big(\frac{|y_l - y_{l'}|}{L}\Big), \tag{6.26}$$

where the product runs over the pairs $(l,l')\subset\{1,\ldots,n\}$ and for $y\in\mathbf{R}^d$ we have used the notation $|y| := \sup_{1\le l\le d}|y_l|$.

It is immediate to verify that

$$\|\psi_{\delta,L} - \psi^\varepsilon_L\|_{\ell_1(\mathbf{Z}^{nd})} \le \frac{n(n-1)}{2}(4\varepsilon L)^d[(1+2\delta)L]^{d(n-1)} \le \varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{nd}L^{nd}.$$

Since

$$\sup_{\mathbf{y}\in\mathcal{F}_n(\mathbf{Z}^d)} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} |\psi_{\delta,L}(\mathbf{y}-\mathbf{i}L) - \psi^\varepsilon_L(\mathbf{y}-\mathbf{i}L)| \le \sup_{\mathbf{y}\in\mathcal{F}_n(\mathbf{Z}^d)} \sum_{\mathbf{i}\in\mathbf{Z}^{nd}} \psi_{\delta,L}(\mathbf{y}-\mathbf{i}L) \le 3^{nd},$$

by applying again Lemma 6.1, we find

$$\|(\psi_{\delta,L} - \psi^\varepsilon_L) * J\|^2_{\ell_2(\mathbf{Z}^{nd})} \le \varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{2nd}L^{nd}\cdot\|J\|^2_{\ell_2(\mathcal{F}_k(\mathbf{Z}^d))}. \tag{6.27}$$

Thus combining (6.25) and (6.27) we arrive at (6.22) provided that we have chosen δ and ε so that

$$\delta\cdot 2nd\cdot 3^{2nd} + \varepsilon^d\cdot\frac{n(n-1)}{2}\cdot 4^{2nd} \le \frac{\gamma}{2}. \tag{6.28}$$

<u>Step 2</u> (proof of (6.23)). We now estimate the main contribution to

$$\|\chi_L * J\|_{\ell_2(\mathbf{Z}^d)}$$

by taking advantage of the specific form of ψ^ε_L, which will allow us to get the correct dependence on L.

Let us first note that, by the definition (6.26), $\psi^\varepsilon_L(y_1,\ldots,y_n)$ vanishes if there exist $(j,j')\subset\{1,\ldots,n\}$ such that $|y_j - y_{j'}|\le\varepsilon L$. For $L\ge\varepsilon^{-1}$ we can consider ψ^ε_L as a function on $\mathcal{F}_n(\mathbf{Z}^d)$. We also note that, for any $s\in\mathbf{N}$, $s\le\varepsilon L$, we have

$$(-\Delta_n)^s\psi^\varepsilon_L(y_1,\ldots,y_n) = (-\Delta_{\mathbf{Z}^{nd}})^s\psi^\varepsilon_L(y_1,\ldots,y_n), \tag{6.29}$$

where $\Delta_{\mathbf{Z}^{nd}}$ is the Laplacian on $\mathbf{Z}^{nd}$. In fact, due to the above support property of ψ^ε_L, the exclusion condition in Δ_n does not appear.

The hypotheses $J\in W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))$ means that J is in the range of $(-\Delta_n)^s$ (as an operator on $\ell_q(\mathcal{F}_n(\mathbf{Z}^d))$). We therefore have

$$\begin{aligned}\psi^\varepsilon_L * J\,(\mathbf{i}) &= \sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} \psi^\varepsilon_L(Y-\mathbf{i}L)(-\Delta_n)^s(-\Delta_n)^{-s}J(Y)\\ &= \sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} (-\Delta_{\mathbf{Z}^{nd}})^s\psi^\varepsilon_L(Y-\mathbf{i}L)(-\Delta_n)^{-s}J(Y),\end{aligned} \tag{6.30}$$

where we used the self-adjointness of $-\Delta_n$ and (6.29).

We note that $\psi^\varepsilon_L(y_1,\ldots,y_n)$ is of the form

$$\psi^\varepsilon_L(y_1,\cdots,y_n) = \psi^\varepsilon\Big(\frac{y_1}{L},\cdots,\frac{y_n}{L}\Big)$$

for a smooth (i.e. in $C_0^\infty(\mathbf{R}^{nd})$) function ψ^ε. By Taylor's theorem we thus have

$$\begin{aligned}\sup_{(y_1,\cdots,y_n)\in\mathbf{Z}^{nd}} |(-\Delta_{\mathbf{Z}^{nd}})^s\psi^\varepsilon_L(y_1,\cdots,y_n)| &\le L^{-2s}\cdot\sup_{\mathbf{r}\in\mathbf{R}^{nd}}|(-\Delta_{\mathbf{R}^{nd}})^s\psi^\varepsilon(\mathbf{r})|\\ &\le C\cdot L^{-2s}\end{aligned} \tag{6.31}$$

for some constant $C = C(n,s,\delta,\varepsilon)$. In the above $\Delta_{\mathbf{R}^{nd}}$ is the Laplacian on $\mathbf{R}^{nd}$.

On the other hand, we note the support of the function $(-\Delta_{\mathbf{Z}^{nd}})^s\psi^\varepsilon_L(Y-\mathbf{i})$ is contained in the region $|y_j - i_j|\le(1+\delta)L+s$, for all $j=1,\ldots,n$. This is because $(-\Delta_{\mathbf{Z}^{nd}})^s$ enlarges the support of s lattice units (one for each power).

Hence, by (6.30) and (6.31), denoting $Y = \{y_1, \cdots, y_n\}$,

$$\begin{aligned}
\|\psi_L^\varepsilon * J\|^2_{\ell_2(\mathbf{Z}^{nd})} &\leq \sum_{\mathbf{i}\in L\mathbf{Z}^{nd}} \Big[\sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} |(-\Delta_{\mathbf{Z}^{nd}})^s \psi_L^\varepsilon(Y-\mathbf{i})||(-\Delta_n)^{-s} J(Y)|\Big]^2 \\
&\leq C^2 \cdot L^{-4s} \sum_{\mathbf{i}\in L\mathbf{Z}^{nd}} \Big[\sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j - i_j|\leq(1+\delta)L+s} \cdot |(-\Delta_n)^{-s} J(Y)|\Big]^2 \\
&\leq C^2 \cdot L^{-4s} \cdot [4L+2s]^{nd(2-2/q)} \\
&\quad \times \sum_{\mathbf{i}\in L\mathbf{Z}^{nd}} \Big[\sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j - i_j|\leq 3L} \cdot |(-\Delta_n)^{-s} J(Y)|^q\Big]^{2/q} \\
&\leq C^2 \cdot L^{-4s} \cdot (6L)^{nd(2-2/q)} \\
&\quad \times \Big[\sum_{\mathbf{i}\in L\mathbf{Z}^{nd}} \sum_{Y\in\mathcal{F}_n(\mathbf{Z}^d)} \prod_{j=1}^n \mathbf{1}_{|y_j - i_j|\leq 3L} \cdot |(-\Delta_n)^{-s} J(Y)|^q\Big]^{2/q} \\
&\leq \frac{a}{2} \cdot L^{-4s} \cdot L^{nd(2-2/q)} \|J\|_{W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))}
\end{aligned}$$

where we used the Hölder inequality in the second step and $q \in [1,2]$ in the third. Here $a = a(n, s, \delta, \varepsilon)$ is independent of L.

This ends the proof of (6.23) and so of Proposition 6.2. □

As a consequence of our considerations we get the following result.

Proposition 6.3. *For any $n, s \in \mathbf{N}$, $q \in [1,2]$ there exist $K_n \in \mathbf{N}$ and a constant $C' \in (0,\infty)$ such that for any $L \geq K_n$ we have*

$$\mu_\rho(f-\mu_\rho f)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + C' \cdot L^{-4s} \cdot L^{-nd(2/q-1)} \cdot \|f\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))} \quad (6.32)$$

for any $f \in \mathcal{H}_n$ for which the right-hand side is finite.

Proof. In order to prove the estimate (6.32), we apply Proposition 6.2 to bound the second term on the right-hand side of (6.17). We thus choose $\gamma = b_n^{-1}/2$ in Proposition 6.2 and $K_n \equiv \max\{L_0, n^{1/d}\}$. By noticing that clearly in our situation

$$\|f\|^2_{\ell_2(\mathcal{F}_n(\mathbf{Z}^d))} = \mu_\rho(f-\mu_\rho f)^2 \quad (6.33)$$

and taking into account (6.17), one easily arrives at

$$\mu_\rho(f-\mu_\rho f)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + C' \cdot L^{-4s} \cdot L^{-nd(2/q-1)} \cdot \|f\|^2_{W_{-2s,q}(\mathcal{F}_n(\mathbf{Z}^d))} \quad (6.34)$$

for all $L \geq K_n$ with a constant $C' \equiv 2b_n \cdot b \in (0,\infty)$. □

For finite sequences $\mathbf{s} = \{s(l) \in \mathbf{N}\}_{l=1,\cdots,n}$, $\mathbf{q} = \{q(l) \in [1,2]\}_{l=1,\cdots,n}$, we define a space $W_{-\mathbf{s},\mathbf{q}} \equiv W_{-\mathbf{s},\mathbf{q}}(\mathcal{F}(\mathbf{Z}^d))$ of functions for which the following seminorm is finite

$$\|f\|^2_{W_{-\mathbf{s},\mathbf{q}}} \equiv \sum_{l=1}^n \|f\|^2_{W_{-s(l),q(l)}(\mathcal{F}_l(\mathbf{Z}^d))}, \quad (6.35)$$

where $\|f\|^2_{W_{-s(l),q(l)}(\mathcal{F}_l(\mathbf{Z}^d))}$ denotes the corresponding Sobolev-type seminorm for the projection of f onto $\mathcal{H}_l$; we recall it has been defined in (5.20). As the spaces $\mathcal{H}_l$ are orthogonal from Proposition 6.3 we get the following corollary.

Corollary 6.1. *For any n and finite sequences $\mathbf{s}, \mathbf{q}$ such that*

$$2s(l) + nd(2/q(l) - 1) = Nd \quad (6.36)$$

for some number $N \in (0,\infty)$, there exist $L_N \in \mathbf{N}$ and a constant $B_N \in (0,\infty)$ such that for any $L \geq L_N$ we have

$$\mu_\rho(f-\mu_\rho f)^2 \leq 2c_0 L^2 \cdot \mathbf{D}_\rho(f) + B_N \cdot L^{-Nd} \cdot \|f\|^2_{W_{-\mathbf{s},\mathbf{q}}} \quad (6.37)$$

for any $f \in \bigoplus_{l=1,\cdots,n} \mathcal{H}_l$ for which the right-hand side is finite.

Using this result we prove now the following theorem. We recall $\|\cdot\|_{W_{-\mathbf{s},\mathbf{q}}}$ has been defined in (6.35).

Theorem 6.2. *For any $\rho \in (0,1)$ and $N \in (0,\infty)$ the following Generalized Nash inequality is true*

$$\mu_\rho(f-\mu_\rho f)^2 \leq \mathbf{D}_\rho(f)^{\alpha_N} \mathbf{A}_{W_{-\mathbf{s},\mathbf{q}}}(f)^{1-\alpha_N} \quad (6.38)$$

with

$$\alpha_N \equiv \frac{Nd}{2+Nd} \quad (6.39)$$

for any $f \in \bigoplus_{l=1,\cdots,n} \mathcal{H}_l$ for which the following functional is finite

$$\mathbf{A}_{W_{-\mathbf{s},\mathbf{q}}}(f) \equiv a_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W_{-\mathbf{s},\mathbf{q}}} \quad (6.40)$$

with some constant $a_{\mathbf{s},\mathbf{q}} \in (0,\infty)$ and finite sequences $\mathbf{s} = \{s(l) \in \mathbf{N}\}_{l=1,\cdots,n}$ and $\mathbf{q} = \{q(l) \in [1,2]\}_{l=1,\cdots,n}$ such that

$$2s(l) + nd2/q(l) - 1) = Nd. \quad (6.41)$$

Moreover, for any $t > 0$, we have the following decay to equilibrium estimate

$$\mu_\rho(P_t f - \mu_\rho f)^2 \leq \frac{\mathbf{A}_{W_{-\mathbf{s},\mathbf{q}}}(f)}{(\gamma_N t)^{\gamma_N}} \quad (6.42)$$

with $\gamma_N \equiv \alpha_N/(1-\alpha_N)$.

Example. Let ϕ be a cylinder function and $f = \mathcal{L}\phi$. By spectral analysis we immediately get a decay of order t^{-1}; however, since f is also a cylinder function and $\mathrm{f}^{(n)} \in W_{-2,1}(\mathcal{F}_n(\mathbf{Z}^d))$, for each n, by applying Theorem 6.2 we get the better estimate $\mu_\rho(P_t f)^2 \le ct^{-2-d/2}$ for some constant c.

Proof. We observe first that for $f = \sum_{m\in\mathrm{N}} f^{(m)}$ with $f^{(m)} = \sum_{Y\in\mathcal{F}_m} \mathrm{f}_Y^{(m)}\sigma_Y$, we have

$$\mathbf{D}_\rho(f) = -\sum_{m\in\mathrm{N}} (\mathrm{f}^{(m)}, \Delta_m \mathrm{f}^{(m)})_{\ell_2(\mathcal{F}_m)}. \tag{6.43}$$

Since the operator Δ_m is bounded in $\ell_2(\mathcal{F}_m)$, we have for any $s \in \mathbf{N}$ and $q \in [1,2]$

$$\begin{aligned}(\mathrm{f}^{(m)}, -\Delta_m \mathrm{f}^{(m)})_{\ell_2(\mathcal{F}_m)} &\le \|(-\Delta_m)^{s+1}\| \cdot \|f\|^2_{W_{-s,2}(\mathcal{F}_m)} \\ &\le \|(-\Delta_m)^{s+1}\| \cdot \|f\|^2_{W_{-s,q}(\mathcal{F}_m)}.\end{aligned} \tag{6.44}$$

Using this it is easy to see that for any finite sequences $\mathbf{s}, \mathbf{q}$ as in the statement of the theorem there is a constant $C_{\mathbf{s},\mathbf{q}} \in (0,\infty)$ such that

$$\mathbf{D}_\rho(f) \le C_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W_{-\mathbf{s},\mathbf{q}}} \tag{6.45}$$

for any $f \in \oplus_{l=1,\cdots,n}\mathcal{H}_l$ for which the $W_{-\mathbf{s},\mathbf{q}}$ seminorm is finite. Using Corollary 6.1 we see that for any real number $x > L_N$, we have

$$\mu_\rho(f - \mu_\rho f)^2 \le 2c_0 x^2 \cdot \mathbf{D}_\rho(f) + 2^{Nd} B_N \cdot x^{-Nd} \cdot \|f\|^2_{W_{-\mathbf{s},\mathbf{q}}}. \tag{6.46}$$

Thus choosing

$$x = \left(\frac{C_{\mathbf{s},\mathbf{q}} \cdot \|f\|^2_{W_{-\mathbf{s},\mathbf{q}}}}{\mathbf{D}_\rho(f)}\right)^{1/(2+Nd)} \cdot L_N \tag{6.47}$$

we arrive at the desired Generalized Nash inequality with

$$\alpha_N \equiv \frac{Nd}{2+Nd} \tag{6.48}$$

and the corresponding functional $\mathbf{A}_{W_{-\mathbf{s},\mathbf{q}}}$ defined with the constant $a_{\mathbf{s},\mathbf{q}}$ given by

$$a_{\mathbf{s},\mathbf{q}}^{1/(1-\alpha_N)} \equiv 2c_0 L_N^2 (C_{\mathbf{s},\mathbf{q}})^{1-\alpha_N} + 2^{Nd} B_N \cdot L_N^{-Nd} \cdot (C_{\mathbf{s},\mathbf{q}})^{-\alpha_N} \tag{6.49}$$

Given our Generalized Nash inequality, the decay to equilibrium estimate follows by standard methods using the monotonicity of the seminorm $\|\cdot\|_{W_{-\mathbf{s},\mathbf{q}}}$ which has been proven in Section 5. This ends the proof of Theorem 6.2. □

Remark 6.1. One can extend these results to show a faster decay also for any function $f \in \mathcal{H}_{>N_0} = \oplus_{n>N_0}\mathcal{H}_n$.

For a finite sequence $\mathbf{p} = \{p(n) \in [1,2), n = N_0+1, \ldots, 2N_0+1\}$ and $f \in \mathcal{H}_{>N_0}$ we introduce $\|f\|_{2,\mathbf{p}}^{(N_0)} := \sum_{n=N_0+1}^{2N_0+1} \mathbf{V}^n_{2,p(n)}(f)$, where we recall $\mathbf{V}^n_{q,p}$ has been defined in (5.2). For any $\rho \in (0,1)$, $N_0 \in \mathbf{N}$ and $N \in (0,\infty)$ the following Generalized Nash inequality is true

$$\mu_\rho(f - \mu_\rho f)^2 \le \mathbf{D}_\rho(f)^{\alpha_N} \widetilde{\mathbf{A}}_{N_0,\mathbf{p}}(f)^{1-\alpha_N} \tag{6.50}$$

with α_N as defined in (6.39) for any $f \in \mathcal{H}_{>N_0}$ for which the following functional is finite $\widetilde{\mathbf{A}}_{N_0,\mathbf{p}}(f) \equiv \widetilde{a}_{N_0,\mathbf{p}} \|f\|_{2,\mathbf{p}}^{(N_0)}$ with some constant $\widetilde{a}_{N_0,\mathbf{p}} \in (0,\infty)$ and any finite sequence $\mathbf{p} = \{p(n) \in [1,2), n = N_0+1, \cdots, 2N_0+1\}$ such that $n[2/p(n) - 1] \ge N$ for each $n = N_0+1, \cdots, 2N_0+1$.

Remark 6.2. We mention also the work [21] where a condition of faster decay has been exploited to prove the central limit theorem for additive functionals of the symmetric exclusion process.

A. Appendix. Proof of Lemma 3.2

Lemma A.1. *For any real function F and any finite set $\Lambda \subset \mathbf{Z}^d$ we have*

$$\sum_{i\in\Lambda} \mu_\rho |\nabla_i Ft(\mu_{\Lambda,\rho}(f\,|N_\Lambda))|^2 = \frac{1}{\rho}\sum_{k=1}^{|\Lambda|} |F(\mu_\Lambda^k(f)) - F(\mu_\Lambda^{k-1}(f))|^2 k \cdot \mu_\rho \chi_k. \tag{A.1}$$

Proof. We note first that

$$\nabla_i F(\mu_{\Lambda,\rho}\ (f\,|N_\Lambda))(\eta) = \sum_{k=0}^{|\Lambda|} F(\mu_\Lambda^k(f)) \nabla_i \chi_k(\eta), \tag{A.2}$$

where $\chi_k \equiv \chi(N_\Lambda = k)$. Since

$$\begin{aligned}\nabla_i \chi_k(\eta) &\equiv \chi_k(\eta^i) - \chi_k(\eta) \\ &= \chi(\eta_i = 1)(\chi_k(\eta^i) - \chi_k(\eta)) \\ &\quad + \chi(\eta_i = 0)(\chi_k(\eta^i) - \chi_k(\eta))\end{aligned} \tag{A.3}$$

and

$$\chi(\eta_i = 1)\chi_k(\eta^i) = \begin{cases} \chi(\eta_i = 1)\chi_{k+1}(\eta) & \text{for } k = 0, \cdots, |\Lambda| - 1, \\ 0 & \text{for } k = |\Lambda|, \end{cases}$$

$$\chi(\eta_i = 0)\chi_k(\eta^i) = \begin{cases} \chi(\eta_i = 0)\chi_{k-1}(\eta) & \text{for } k = 1, \cdots, |\Lambda|, \\ 0 & \text{for } k = 0, \end{cases} \tag{A.4}$$

we get

$$\nabla_i F(\mu_{\Lambda,\rho}(f|N_\Lambda))(\eta) \quad = \quad \sum_{k=0}^{|\Lambda|-1} F(\mu_\Lambda^k(f))\chi(\eta_i = 1)(\chi_{k+1}(\eta) - \chi_k(\eta))$$

$$- F(\mu_\Lambda^{|\Lambda|}(f))\chi(\eta_i = 1)\chi_{|\Lambda|}(\eta) + \sum_{k=1}^{|\Lambda|} F(\mu_\Lambda^k(f))\chi(\eta_i = 0)(\chi_{k-1}(\eta) - \chi_k(\eta)) - F(\mu_\Lambda^0(f))\chi(\eta_i = 0)\chi_0(\eta).$$

Rearranging the sums, we obtain

$$\nabla_i F(\mu_{\Lambda,\rho}(f|N_\Lambda))(\eta) = \sum_{k=1}^{|\Lambda|}(F(\mu_\Lambda^{k-1}(f)) - F(\mu_\Lambda^k(f)))\chi(\eta_i = 1)\chi_k(\eta) + \sum_{k=0}^{|\Lambda|-1}(F(\mu_\Lambda^{k+1}(f)) - F(\mu_\Lambda^k(f)))\chi(\eta_i = 0)\chi_k(\eta).$$

Thence we get

$$\sum_{i\in\Lambda}\mu_\rho|\nabla_i F(\mu_{\Lambda,\rho}(f \mid N_\Lambda))|^2 = \sum_{i\in\Lambda}\mu_\rho\Big\{\sum_{k=1}^{|\Lambda|}|F(\mu_\Lambda^{k-1}(f)) - F(\mu_\Lambda^k(f))|^2\chi(\eta_i = 1)\chi_k(\eta) + \sum_{k=0}^{|\Lambda|-1}|F(\mu_\Lambda^{k+1}(f)) - F(\mu_\Lambda^k(f))|^2\chi(\eta_i = 0)\chi_k(\eta)\Big\}$$

which after changing the order of summation gives

$$\sum_{i\in\Lambda}\mu_\rho|\nabla_i F(\mu_{\Lambda,\rho}(f \mid N_\Lambda))|^2 = \sum_{k=1}^{|\Lambda|}|F(\mu_\Lambda^{k-1}(f)) - F(\mu_\Lambda^k(f))|^2 k\cdot\mu_\rho\chi_k + \sum_{k=0}^{|\Lambda|-1}|F(\mu_\Lambda^{k+1}(f)) - F(\mu_\Lambda^k(f))|^2(|\Lambda| - k)\cdot\mu_\rho\chi_k.$$

Until this place we did not use any specific assumptions about the measure μ_ρ or corresponding conditional measures. Now we observe that for $k = 0,\cdots,|\Lambda|-1$, we have

$$(|\Lambda| - k)\mu_\rho\chi_k = \frac{(1-\rho)}{\rho}(k+1)\mu_\rho\chi_{k+1}. \tag{A.5}$$

Using this together with the previous relation, after simple transformations, we arrive at

$$\sum_{i\in\Lambda}\mu_\rho|\nabla_i F(\mu_\Lambda^\rho(f|N_\Lambda))|^2 = \frac{1}{\rho}\sum_{k=1}^{|\Lambda|}|F(\mu_\Lambda^k(f)) - F(\mu_\Lambda^{k-1}(f))|^2 k\cdot\mu_\rho\chi_k. \tag{A.6}$$

□

Acknowledgments

We would like to acknowledge the support of EPSRC grant GR/K 76801.

References

[1] M. Aizenman and R. Holley (1987) Rapid convergence to equilibrium of stochastic Ising models in the Dobrushin – Shlosman regime. In: *Percolation. Theory and Ergodic Theory of Infinite Particle Systems*, H. Kesten (ed.), Springer-Verlag, 1–11.

[2] D. Bakry and D. Michel (1992) Sur les ineégalités FKG. In: *Seminare de Probabilite* XXVI, LNM, Springer, Berlin, 170–188.

[3] L. Bertini and B. Zegarlinski (1996) Coercive inequalities for Gibbs measures. IC Preprint.

[4] E.A. Carlen, S. Kusuoka and D.W. Stroock (1987) Upper bounds for symmetric transition functions. *Ann. Inst. H. Poincaré* **23**, 245–287.

[5] E.B. Davies (1989) *Heat Kernels and Spectral Theory*. Cambridge University Press.

[6] J.D. Deuschel (1991) Algebraic L^2 decay of attractive critical processes on the lattice. *Ann. Prob.* **22**, 264–283.

[7] P. Diaconis and L. Saloff-Coste (1995) Logarithmic Sobolev inequalities for finite Markov chains. Preprint.

[8] K. Golden, S. Goldstein and J.L. Lebowitz (1988) Nash estimates and the asymptotic behaviour of diffusions. *Ann. Prob.* **16**, 1127–1146.

[9] L. Gross (1976) Logarithmic Sobolev inequalities. *Amer. J. Math.* **97**, 1061–1083.

[10] E. Laroche (1995) Hypercontractivité pour des systèmes de spin de portée infinie. *Probab. Theory and Relat. Fields* **101**, 89–132.

[11] T.M. Liggett (1985) *Interacting Particles Systems*. Springer, Berlin.

[12] T.M. Liggett (1991) L_2 rates of convergence for attractive reversible nearest particle systems: the critical case. *Ann. Prob.* **19**, 935–959.

[13] S.L. Lu and H.T. Yau (1993) Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Commun. Math. Phys.* **156**, 399–433.

[14] F. Martinelli and E. Olivieri (1994) Approach to equilibrium of Glauber dynamics in the one phase region: I. The attractive case. II. The general case. *Commun. Math. Phys.* **161**, 447–486, 487–514.

[15] F. Martinelli, E. Olivieri and R.H. Shonmann (1994) For 2-D lattice spin systems weak mixing implies strong mixing. *Commun. Math. Phys.* **165**, 33–47.

[16] J. Quastel (1992) Diffusion of color in the simple exclusion process. *Comm. Pure and Appl. Math.* **45**, 623–679.

[17] D.W. Stroock and B. Zegarlinski (1992) The logarithmic Sobolev inequality for continuous spin systems on a lattice. *J. Func. Anal.* **104**, 299–326.

[18] D.W. Stroock and B. Zegarlinski (1992) The equivalence of the logarithmic Sobolev inequality and the Dobrushin–Shlosman mixing condition. *Commun. Math. Phys.* **144**, 303–323.

[19] D.W. Stroock and B. Zegarlinski (1992) The logarithmic Sobolev inequality for discrete spin systems on a lattice. *Commun. Math. Phys.* **149**, 175–193.

[20] D.W. Stroock and B. Zegarlinski (1995) On the ergodic properties of Glauber dynamics. *J. Stat. Phys.* **81**, 1007–1019.

[21] S. Sethuraman and L. Xu (1995) A central limit theorem for reversible exclusion and zero-range particle systems. Preprint.

[22] H.T. Yau (1996) Logarithmic Sobolev inequalities for generalized simple exclusion process. Preprint mp-arc 96-205.

[23] H.T. Yau (1996) Logarithmic Sobolev inequality for lattice gases with mixing conditions. *Commun. Math. Phys.* **181**, 367–408.

[24] B. Zegarlinski (1990) On log-Sobolev inequalities for infinite lattice systems. *Lett. Math. Phys.* **20**, 173–182.

[25] B. Zegarlinski (1990) Log-Sobolev inequalities for infinite one-dimensional lattice systems. *Commun. Math. Phys.* **133**, 147–162.

[26] B. Zegarlinski (1992) Dobrushin uniqueness theorem and logarithmic Sobolev inequalities. *J. Func. Anal.* **105**, 77–111.

[27] B. Zegarlinski (1995) Ergodicity of Markov semigroups. In: *Proc. of the Conference: Stochastic Partial Differential Equations, Edinburgh* 1994, A. Etheridge (ed.), LMS Lecture Notes, Cambridge University Press, **216**, 312–337.