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The two-dimensional stochastic heat equation: renormalizing a multiplicative noise

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Abstract. We study, in two space dimensions, the heat equation with a random potential that is a white noise in space and time. We introduce a regularization of the noise and prove that, by a suitable renormalization of the coupling coefficient, the covariance has a non-trivial limit when the regularization is removed. The limit is described in terms of a two-body Schrödinger operator with singular interaction.

1. Introduction

We consider the linear stochastic partial differential equation (SPDE)

$$\mathrm{d}\psi_t = \tfrac{1}{2}\Delta\psi_t\,\mathrm{d}t + \sqrt{\Gamma}\psi_t\,\mathrm{d}W_t \tag{1.1}$$

where $\psi_t = \psi_t(x)$, $t \geqslant 0$, $x \in \mathbb{R}^d$, is a scalar field on $\mathbb{R}^d$, Δ is the Laplacian, Γ is a positive constant and $W_t = W_t(x)$ is a cylindrical Wiener process whose covariance is

$$\boldsymbol{E}(W_t(x)W_{t'}(x')) = \min\{t, t'\}\delta(x - x'). \tag{1.2}$$

Equation (1.1), often called the stochastic heat equation (SHE), arises in several physical problems. It is satisfied by the partition function of a directed polymer in a $(d + 1)$-dimensional random medium modelled by the potential $\dot{W}_t(x)$. It is, furthermore, related via the Cole–Hopf transformation to the Kardar–Parisi–Zhang (KPZ) equation [10] for the random growth of interfaces and to the Burgers equation with conservative additive noise. The problem of directed polymers in a random medium is one of the simplest problems in the theory of disordered systems which undergo a phase transition from a strong- to a low-disorder regime. In dimensions $d > 2$ the existence of a low-disorder phase has been proven by showing that under a certain intensity of disorder, as the fluctuations of the partition function are small, the annealed and quenched free energy are equal [8]. The renormalization group analysis indicates that for $d = 1, 2$ even a weak random environment becomes effectively strong at long times. While in $d = 1$ the absence of a phase transition is expected [11], in $d = 2$ numerical simulations indicate the existence of a phase transition [6].

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A rigorous analysis of the SHE is not trivial as the singularity of the noise introduces small-scale singularities (*ultraviolet divergences*). We also point out that, since equation (1.1) contains non-trivial diffusion, the interpretation of the stochastic differential $\psi_t \, \mathrm{d}W_t$ presents the well known ambiguities, for example Ito or Stratonovich. The standard approach to treat the ultraviolet divergences is the introduction of a regularization followed by a suitable renormalization of the coupling coefficients. We thus introduce a mollification of the Wiener process and try to prove that the solution of the corresponding regularized (1.1), after a suitable renormalization, has a non-trivial limit when the regularization is removed. In one space dimension, $d = 1$, it is not too difficult to complete the above task. The regularized process can be expressed through a Feynman–Kac formula; after a simple renormalization (the Wick exponential), a meaningful expression is obtained when the cut-off is removed. The renormalized Feynman–Kac formula defines a process with continuous (in space and time) trajectories; it solves equation (1.1) when the stochastic differential is interpreted in the Ito sense [3]. In the analysis of (1.1) via the replica method [9] the Ito choice corresponds to the fact that the self-interactions are ignored.

In this paper we consider the two-dimensional case, $d = 2$, where stronger ultraviolet divergences are present. A renormalization of the coupling constant Γ is then needed. We exhibit a function $\Gamma = \Gamma(\varepsilon)$ (where ε is the cut-off), vanishing as $\varepsilon \to 0$, such that the covariance of the regularized field converges as the cut-off is removed. The limit is explicitly described and its long-time behaviour indicates that the partition function of the directed polymer has large fluctuations even for weak disorder, as predicted by the renormalization group analysis. The above results are obtained by using the theory of Schrödinger operators with point interaction in two dimensions [2].

2. Notation and results

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of C^∞ functions of rapid decrease; its topological dual, the Schwartz space of distributions, is $\mathcal{S}'(\mathbb{R}^n)$. We denote by $H_m(\mathbb{R}^n)$ the Sobolev space of order m; we recall that its norm is defined by $\|\phi\|_m := \|(1-\Delta)^{m/2}\phi\|$, where $\|\cdot\|$ is the norm in $L_2(\mathbb{R}^n)$. The canonical pairing between $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$ is denoted by $(\cdot,\cdot)$. The paring between $\mathcal{S}'(\mathbb{R}^4)$ and $\mathcal{S}(\mathbb{R}^4)$ is instead indicated by $((\cdot,\cdot))$. We finally define $\Omega := C(\mathbb{R}^+; \mathcal{S}'(\mathbb{R}^2))$.

Let W_t, $t \geqslant 0$, be the cylindrical Wiener process on $L_2(\mathbb{R}^2)$ which is canonically realized on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \boldsymbol{P})$. Here $\mathcal{F}$ is the Borel σ-algebra, $\mathcal{F}_t$ the canonical filtration and $\boldsymbol{P}$ is the Gaussian measure with covariance

$$\boldsymbol{E}(W_t(f)W_{t'}(g)) = t \wedge t'(f, g) \tag{2.1}$$

where $f, g \in \mathcal{S}(\mathbb{R}^2)$ are test functions and $a \wedge b = \min\{a, b\}$.

We introduce a mollification of W_t as follows. Let $j \in \mathcal{S}(\mathbb{R}^2)$ be a probability density and denote by $J = j * j$ the probability density for the sum of two independent j-distributed random variables. For $\varepsilon > 0$, we set $j_\varepsilon(x) := \varepsilon^{-2} j(\varepsilon^{-1}x)$ and define

$$W_t^\varepsilon(x) := W_t(j_\varepsilon(x - \cdot)) \tag{2.2}$$

its covariance is given by

$$\boldsymbol{E}(W_t^\varepsilon(x)W_{t'}^\varepsilon(x')) = t \wedge t' J_\varepsilon(x - x'). \tag{2.3}$$

For the *pink noise* $\dot{W}_t^\varepsilon$ the informal SHE (1.1) can be written as an Ito SPDE

$$\begin{aligned} \mathrm{d}\psi_t^\varepsilon &= \tfrac{1}{2}\Delta\psi_t^\varepsilon \, \mathrm{d}t + \sqrt{\Gamma}\psi_t^\varepsilon \, \mathrm{d}W_t^\varepsilon \\ \psi_0^\varepsilon &= \varphi. \end{aligned} \tag{2.4}$$

For simplicity we shall assume that the initial condition $\varphi \in L_2(\mathbb{R}^2)$.

As $W_t^\varepsilon(x)$ is regular in the space variable, it is easy to verify that the SPDE (2.4) has a unique global solution $\psi_t^\varepsilon = \psi_t^\varepsilon(x)$ which is a.s. smooth ($C^\infty(\mathbb{R}^2)$) in x for each $t > 0$. For $f \in \mathcal{S}(\mathbb{R}^2)$, by setting

$$\psi_t^\varepsilon(f) := \int \mathrm{d}x\, f(x)\psi_t^\varepsilon(x) \tag{2.5}$$

we regard ψ^ε, $\varepsilon > 0$ as a family of processes on the path space Ω. Our first result concerns the weak convergence of this family.

In order to obtain the convergence result, the 'coupling constant' Γ cannot be kept fixed, but has to vanish with an appropriate rate as $\varepsilon \to 0$. By *renormalization* we mean just this. There is still, however, a 'free' parameter which measures the intensity of the noise, which we call β. Accordingly, the *renormalizing function* Γ will be a function of two variables, ε and β.

Theorem 2.1. Let $\Gamma = \Gamma_\beta(\varepsilon)$ satisfy

$$\Gamma_\beta(\varepsilon) = \frac{1}{h_\varepsilon} + \left(\frac{1}{4\pi}\log\frac{\beta}{2} + A\right)\frac{1}{h_\varepsilon^2} + o\left(\frac{1}{h_\varepsilon^2}\right) \tag{2.6}$$

where $\beta > 0$, $h_\varepsilon := (2\pi)^{-1}\log\varepsilon^{-1}$ and

$$A = A(J) := \frac{1}{2\pi}\int \mathrm{d}x\,\mathrm{d}x'\, J(x)J(x')\left[\kappa + \log\left(\frac{|x-x'|}{\sqrt{2}}\right)\right] \tag{2.7}$$

where $\kappa = 0.577\,2157\ldots$ is the Euler constant.

Then the finite-dimensional distributions of $\{\psi^\varepsilon\}$ converge weakly along subsequences.

We note that, starting from the finite-dimensional distributions in theorem 2.1, one can obtain a process associated with the two-dimensional SHE via the standard Kolmogorov construction.

In theorem 3.2 (see the next section) we show that the covariance of the random field ψ_t^ε converges to a non-trivial limit as $\varepsilon \to 0$. In particular, any limit point of $\{\psi^\varepsilon\}$ has a second moment. The proof is based on an approximation of the Schrödinger operators with point interaction in terms of scaled short-range Hamiltonians [2, theorem I.5.5] in which expression (2.6) is found. We note, however, that the correct form of $\Gamma_\beta(\varepsilon)$ can also be 'guessed' from formal perturbation theory in a power series of the 'physical coupling constant' β. This perturbation theory is related to the renormalization of the local times of two-dimensional Brownian motion; see [7] where a detailed combinatoric analysis is performed. In [12] a simplified approach is presented. The latter uses, however, a very special J which is not of the form $J = j * j$.

3. The limiting covariance

In this section we show that the covariance of the process ψ_t^ε converges as $\varepsilon \to 0$. We stress here that the convergence does not depend on the subsequence. We will use the fact that the two-body Schrödinger operator with point interaction can be obtained as a norm resolvent limit of scaled short-range Hamiltonians [2].

We recall first some results on the theory of Schrödinger operators with point interaction in two dimensions [1, 2]. Let

$$G_\lambda := (-\Delta + \lambda)^{-1} \qquad \lambda > 0 \tag{3.1}$$

be the resolvent of the Laplacian Δ on $\mathbb{R}^2$. Its integral kernel is given by

$$G_\lambda(x) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t}\, p_{2t}(x) \qquad x \neq 0 \tag{3.2}$$

where, for $t > 0$, $x \in \mathbb{R}^2$

$$p_t(x) := \frac{1}{2\pi t} \exp\left\{-\frac{|x|^2}{2t}\right\} \tag{3.3}$$

is the heat kernel.

For each $\beta > 0$ we introduce the Schrödinger operator with singular interaction at the origin $-\Delta_\beta$ on $L_2(\mathbb{R}^2)$. In [2] this is constructed by classifying the (one-parameter) self-adjoint extensions of $-\Delta$ on the minimal domain $C_0^\infty(\mathbb{R}^2\backslash\{0\})$ of the infinite differential function with compact support away from the origin. According to [2, theorem I.5.3], its domain $\mathcal{D}(-\Delta_\beta) \subset L_2(\mathbb{R}^2)$ consists of elements g of the type

$$g(x) = f_\lambda(x) + \frac{4\pi}{\log(\lambda/\beta)} f_\lambda(0) G_\lambda(x) \qquad x \neq 0 \tag{3.4}$$

where $f_\lambda \in H_2(\mathbb{R}^2) = \mathcal{D}(-\Delta)$ and $\lambda \in \mathbb{R}^+\backslash\{\beta\}$. The above decomposition is unique. We recall that, by Sobolev embedding, $H_2(\mathbb{R}^2) \subset C(\mathbb{R}^2)$, so that $f_\lambda(0)$ is meaningful.

For $\beta > 0$, $g \in \mathcal{D}(-\Delta_\beta)$, the operator $-\Delta_\beta$ is then defined by

$$-\Delta_\beta g = -\Delta f_\lambda - \frac{4\pi\lambda}{\log(\lambda/\beta)} f_\lambda(0) G_\lambda. \tag{3.5}$$

Its spectrum is given by the isolated eigenvalue $-\beta$ and the positive half-axes. We warn the reader that our β is related to what is called α in [2] by $\alpha = -[\log(\sqrt{\beta}/2) + \kappa]/2\pi$.

We finally introduce the two body-Schrödinger operator with point interaction. Given $-\Delta_\beta$ it is enough to separate the free motion of the centre of mass. We adopt the so-called *passive viewpoint*. Let $R : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}^2 \times \mathbb{R}^2$ be defined by

$$R(x, y) := (\tfrac{1}{2}(x + y), x - y) \tag{3.6}$$

which induces the unitary operator U on $L_2(\mathbb{R}^2 \times \mathbb{R}^2)$

$$(UF)(x, y) = F(R(x, y)). \tag{3.7}$$

We then introduce H_β as the operator on $L_2(\mathbb{R}^4) = L_2(\mathbb{R}^2) \otimes L_2(\mathbb{R}^2)$ defined on the dense domain

$$\mathcal{D}^0(H_\beta) := U^{-1}\mathcal{D}(-\Delta) \otimes \mathcal{D}(-\Delta_\beta) \tag{3.8}$$

by

$$H_\beta := U^{-1}(-\tfrac{1}{4}\Delta \otimes 1 + 1 \otimes -\Delta_\beta)U. \tag{3.9}$$

We note that H_β is closable and essentially self-adjoint on $\mathcal{D}^0(H_\beta)$.

In the following proposition we summarize some results on the semigroup generated by Δ_β [1, 2].

Proposition 3.1. The operator $-\Delta_\beta$ generates a bounded self-adjoint semigroup $P_t^\beta = \exp\{t\Delta_\beta\}$ on $L_2(\mathbb{R}^2)$. It has an integral kernel $P_t^\beta(x, y)$ whose Laplace transform is given by

$$R_\lambda(x, y) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} P_t^\beta(x, y) = G_\lambda(x - y) + \frac{4\pi}{\log(\lambda/\beta)} G_\lambda(x) G_\lambda(y) \tag{3.10}$$

which is analytical in the complex half-plane $\mathrm{Re}\,\lambda > \beta$.

Furthermore, the inverse Laplace transform can be expressed, for $x, y \neq 0$, as

$$P_t^\beta(x,y) = p_{2t}(x-y) + \int_0^t \mathrm{d}\tau \, \frac{1}{2\pi\tau} \, \mathrm{e}^{-(|x|^2+|y|^2)/4\tau} K_0\left(\frac{|x\|y|}{2\tau}\right) \times \int_0^\infty \mathrm{d}u \frac{\beta^u (t-\tau)^{u-1}}{\Gamma_E(u)} \tag{3.11}$$

which is analytic for $t > 0$. Here Γ_E is the Euler Γ-function and K_0 is the modified Bessel function.

We introduce the semigroup generated by H_β on $L_2(\mathbb{R}^4)$ as

$$\mathrm{e}^{-tH_\beta} := U^{-1} \, \mathrm{e}^{t\Delta/4} \otimes \mathrm{e}^{t\Delta_\beta} U \tag{3.12}$$

where $\mathrm{e}^{t\Delta/4}$ is the heat semigroup and $\mathrm{e}^{t\Delta_\beta}$ has integral kernel given by (3.11).

The main result in this section is as follows.

Theorem 3.2. Let $\varphi \in L_2(\mathbb{R}^2)$ be the initial datum for (2.4), $F \in L_2(\mathbb{R}^4)$. Then for each $t \in \mathbb{R}^+$

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(((\psi_t^\varepsilon \otimes \psi_t^\varepsilon, F))) = ((\varphi \otimes \varphi, \, \mathrm{e}^{-tH_\beta} F)). \tag{3.13}$$

The convergence is, furthermore, uniform for t in compact subsets of $\mathbb{R}^+$.

Remark. Theorem 3.2 is related to the aforementioned problem of the existence of a *renormalized exponential moment* for the local time of planar Brownian motion. A simple approximation argument shows that (3.13) also holds for $\varphi = 1$, although this case is not included in the assumption $\varphi \in L_2(\mathbb{R}^2)$. By writing a Feynman–Kac representation for $\psi_t^\varepsilon(x)$, its covariance can then be expressed in terms of a local time's exponential moment, see [3]. Equations (3.13) and (3.11) then imply for $x \neq 0$

$$\begin{aligned}\lim_{\varepsilon \to 0} \boldsymbol{E}_{\varphi=1}(\psi_t^\varepsilon(0)\psi_t^\varepsilon(x)) &= \lim_{\varepsilon \to 0} \boldsymbol{E}\left(\exp\left\{ \Gamma_\beta(\varepsilon) \int_0^t \mathrm{d}s \, J_\varepsilon(x + \sqrt{2} B_s) \right\} \right) \\ &= 1 + 4\pi \int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}u \frac{\beta^u (t-\tau)^u}{\Gamma_E(u+1)} \, p_{2\tau}(x)\end{aligned} \tag{3.14}$$

where B_t, $t \geqslant 0$, is a planar Brownian motion started at the origin. The second term on the right-hand side of (3.14) diverges as $t \to \infty$ and, therefore, indicates that in the long-time limit the partition function of the directed polymer has large fluctuations even for small β.

Proof of theorem 3.2. Fix $\varepsilon > 0$. We note that $x \mapsto \psi_t^\varepsilon(x)$ is smooth (C^∞) if $t > 0$. Recalling that ψ_t^ε solves (2.4) we can apply Ito's formula and get

$$\mathrm{d}(\psi_t^\varepsilon(x)\psi_t^\varepsilon(y)) = -H_\beta^\varepsilon \psi_t^\varepsilon(x)\psi_t^\varepsilon(y) \, \mathrm{d}t + \mathrm{d}N_t^\varepsilon(x,y) \tag{3.15}$$

where $N_t^\varepsilon(x,y)$ is a martingale term and H_β^ε is the two-body Schrödinger operator on $L_2(\mathbb{R}^2 \times \mathbb{R}^2)$ given by

$$H_\beta^\varepsilon = -\tfrac{1}{2}\Delta_x - \tfrac{1}{2}\Delta_y - \Gamma_\beta(\varepsilon) J_\varepsilon(x-y) \tag{3.16}$$

which is essentially self-adjoint on $H_2(\mathbb{R}^4)$ since $J_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$. We recall that the function $\Gamma_\beta(\varepsilon)$ is given by (2.6).

Let

$$C_t^\varepsilon(x,y) := \boldsymbol{E}(\psi_t^\varepsilon(x)\psi_t^\varepsilon(y)). \tag{3.17}$$

By taking the expectation in (3.15) we find C_t^ε solves

$$\partial_t C_t^\varepsilon = -H_\beta^\varepsilon C_t^\varepsilon \qquad C_0^\varepsilon = \varphi \otimes \varphi. \tag{3.18}$$

By a change of coordinates, we have

$$H_\beta^\varepsilon = U^{-1}(-\tfrac{1}{4}\Delta \otimes 1 + 1 \otimes -\Delta_\beta^\varepsilon)U \tag{3.19}$$

where $-\Delta_\beta^\varepsilon$ is the Schrödinger operator given by

$$-\Delta_\beta^\varepsilon = -\Delta - \Gamma_\beta(\varepsilon) J_\varepsilon(x).$$

Hence

$$C_t^\varepsilon = U^{-1}(\varepsilon^{t\Delta/4} \otimes \mathrm{e}^{t\Delta_\beta^\varepsilon})UC_0^\varepsilon = \mathrm{e}^{-tH_\beta^\varepsilon}\varphi \otimes \varphi \tag{3.20}$$

and therefore

$$\boldsymbol{E}(((\psi_t^\varepsilon \otimes \psi_t^\varepsilon, F))) = ((\varphi \otimes \varphi, \mathrm{e}^{-tH_\beta^\varepsilon} F)). \tag{3.21}$$

In [2, theorem I.5.5] it is proven that $-\Delta_\beta^\varepsilon$ converges to $-\Delta_\beta$, as an operator on $L_2(\mathbb{R}^2)$, in the norm resolvent sense. Using the representation of the resolvent in terms of the semigroup and the fact that U and U^{-1} are bounded operators on $L_2(\mathbb{R}^2) \otimes L_2(\mathbb{R}^2)$ we get that H_β^ε converges to H_β in the norm resolvent sense on $L_2(\mathbb{R}^2) \otimes L_2(\mathbb{R}^2)$. This implies that $\mathrm{e}^{-tH_\beta^\varepsilon}$ converges strongly to e^{-tH_β} in $L_2(\mathbb{R}^4)$ uniformly for t in compact subsets of $\mathbb{R}^+$, see for example [13, problem VIII.21]. □

Remark. To extend the above argument to the n-point correlation functions ($n > 2$), a theory of n-particle Schrödinger operators with point interaction is needed. This has been constructed in [5] where it is also shown that the operators generate a bounded semigroup. Unfortunately, the (non-local) regularization scheme used in [5] is not compatible with our underlying stochastic equation (2.4).

4. Convergence of the finite-dimensional distributions

In this section we obtain some uniform bounds and conclude the proof of theorem 2.1.

Lemma 4.1. For each $\beta > 0$, $T > 0$ there exists a constant $c = c(\beta, T)$ such that for any $\varphi \in L_2(\mathbb{R}^2)$, $f \in H_2(\mathbb{R}^2)$

$$\limsup_{\varepsilon \to 0} \boldsymbol{E}\left(\sup_{t \leqslant T} \psi_t^\varepsilon(f)^2\right) \leqslant c\|\varphi\|^2 \|f\|_2^2. \tag{4.1}$$

Proof. From theorem 3.2 it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0} \sup_{t\leqslant T} \boldsymbol{E}(\psi_t^\varepsilon(f)^2) = \sup_{t\leqslant T}((\varphi \otimes \varphi, \mathrm{e}^{-tH_\beta} f \otimes f)) \leqslant \sup_{t\leqslant T} \|\varphi \otimes \varphi\| \\ \times \|\mathrm{e}^{-tH_\beta} f \otimes f\| \leqslant \mathrm{e}^{\beta T}\|\varphi\|^2 \|f\|^2\end{aligned} \tag{4.2}$$

where we used the fact that $\mathrm{e}^{t\Delta_\beta}$ has norm $\mathrm{e}^{\beta t}$ [2].

We now want to squeeze the sup inside the expectation. Let us define

$$\begin{aligned} A_t^\varepsilon(f) &:= \frac{1}{2}\int_0^t \mathrm{d}s(\psi_s^\varepsilon, \Delta f) \\ M_t^\varepsilon(f) &:= (\psi_t^\varepsilon, f) - A_t^\varepsilon(f) - (\varphi, f) \end{aligned} \tag{4.3}$$

and note that by Ito's formula $M_t^\varepsilon(f)$ is a square integrable martingale. By Doob's L_2 inequality we thus get

$$\begin{aligned}\boldsymbol{E}\left(\sup_{t\leqslant T}\psi_t^\varepsilon(f)^2\right) &\leqslant 3\left[(\varphi, f)^2 + \boldsymbol{E}\left(\sup_{t\leqslant T}A_t^\varepsilon(f)^2\right) + 4\boldsymbol{E}(M_T^\varepsilon(f)^2)\right]\\ &\leqslant 3\left[13(\varphi, f)^2 + \boldsymbol{E}\left(\sup_{t\leqslant T}A_t^\varepsilon(f)^2\right) + 12\boldsymbol{E}(\psi_T^\varepsilon(f)^2) + 12\boldsymbol{E}(A_T^\varepsilon(f)^2)\right]. \qquad (4.4)\end{aligned}$$

By Cauchy–Schwartz inequality we have

$$\sup_{t\leqslant T}\left[\int_0^t \mathrm{d}s(\psi_s^\varepsilon, \Delta f)\right]^2 \leqslant T\int_0^T \mathrm{d}s(\psi_s^\varepsilon, \Delta f)^2$$

and (4.1) follows by (4.2) and (4.4). □

Proof of theorem 2.1. Let $0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$ and $f_i \in \mathcal{S}(\mathbb{R}^2)$, $i = 1, \ldots, n$. From lemma 4.1 there exists a constant c depending on n, T and $\{f_i\}_{i=1}^n$ such that

$$\limsup_{\varepsilon\to 0}\boldsymbol{E}\left(\sup_{t\leqslant T}\sum_{i=1}^n \psi_t^\varepsilon(f_i)^2\right) \leqslant c$$

and, therefore, by Chebyschev inequality, $\psi_{t_i}^\varepsilon(f_i)$, $i = 1, \ldots, n$, is a tight family of random variables on $\mathbb{R}^n$. By Prohorov's theorem (see e.g. [4]) we can thus find a subsequence $\varepsilon_k \to 0$ such that the joint distribution of $\{\psi_{t_i}^{\varepsilon_k}(f_i)\}_{i=1}^n$ is weakly convergent. □

Our last result establishes the uniform (in ε) continuity (in a mean-square sense) of the map $t \mapsto \psi_t^\varepsilon(f)$ for a fixed $f \in \mathcal{S}(\mathbb{R}^2)$. More precisely, we have the following proposition.

Proposition 4.2. For each $\beta > 0$ and any initial datum $\varphi \in L_2(\mathbb{R}^2)$, $f \in \mathcal{S}(\mathbb{R}^2)$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{t\leqslant T}\boldsymbol{E}(\psi_{t+\delta}^\varepsilon(f) - \psi_t^\varepsilon(f))^2 = 0. \qquad (4.5)$$

Proof. Let us define

$$D_t^\varepsilon(F) := \int_0^t \mathrm{d}s((\psi_s^\varepsilon \otimes \psi_s^\varepsilon, H_\beta^\varepsilon F)).$$

Let $\delta > 0$. By using the fact that $M_t^\varepsilon(f)$ (defined in (4.3)) and $\psi_t^\varepsilon(f)^2 - \psi_0^\varepsilon(f)^2 + D_t^\varepsilon(f\otimes f)$ are martingale, we get

$$\begin{aligned}\boldsymbol{E}\{(\psi_{t+\delta}^\varepsilon(f) - \psi_t^\varepsilon(f))^2|\mathcal{F}_t\} &= \boldsymbol{E}\{-D_{t+\delta}^\varepsilon(f\otimes f) + D_t^\varepsilon(f\otimes f)\\ &\quad -2(A_{t+\delta}^\varepsilon(f) - A_t^\varepsilon(f))\psi_t^\varepsilon(f)|\mathcal{F}_t\}. \qquad (4.6)\end{aligned}$$

The second term on the right-hand side of (4.6) can be easily bounded by noticing that

$$|A_{t+\delta}^\varepsilon(f) - A_t^\varepsilon(f)|^2 \leqslant \frac{1}{2}\delta\int_0^{T+\delta}\mathrm{d}t(\psi_t^\varepsilon, \Delta f)^2$$

and then using (4.2).

We next bound the first term. Applying the Markov property of ψ_t^ε we have

$$\begin{aligned}\boldsymbol{E}\{D_{t+\delta}^\varepsilon(f\otimes f) - D_t^\varepsilon(f\otimes f)|\mathcal{F}_t\} &= \boldsymbol{E}_{\psi_t^\varepsilon}(D_\delta^\varepsilon(f\otimes f)) = \boldsymbol{E}_{\psi_t^\varepsilon}(\psi_\delta^\varepsilon(f)^2) - \psi_t^\varepsilon(f)^2\\ &= ((\psi_t^\varepsilon\otimes\psi_t^\varepsilon, [\mathrm{e}^{-\delta H_\beta^\varepsilon} - 1]f\otimes f)) \qquad (4.7)\end{aligned}$$

where we used again the fact that $\psi_t^\varepsilon(f)^2 - \psi_0^\varepsilon(f)^2 - D_t^\varepsilon(f\otimes f)$ is a martingale in the second step and the semigroup representation for the covariance of ψ_t^ε (3.21) in the last identity.

Taking the expectation value in (4.7), we finally get

$$\boldsymbol{E}\{D^{\varepsilon}_{t+\delta}(f) - D^{\varepsilon}_{t}(f)\} = ((\varphi \otimes \varphi, [\mathrm{e}^{-(t+\delta)H^{\varepsilon}_{\beta}} - \mathrm{e}^{-tH^{\varepsilon}_{\beta}}] f \otimes f)). \tag{4.8}$$

The result now follows using the strong convergence of the semigroup $\mathrm{e}^{-tH^{\varepsilon}_{\beta}}$ uniformly for t in compact subsets of $\mathbb{R}^+$ for the limit $\varepsilon \to 0$, the boundedness of the semigroup $\mathrm{e}^{-tH_{\beta}}$ and the dominated convergence theorem for the limit $\delta \to 0$. □

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