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Derivation of Cahn–Hilliard Equations from Ginzburg–Landau Models

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The generalized Cahn–Hilliard equation is obtained as the hydrodynamic limit from a stochastic Ginzburg–Landau model. The associated large-deviation principle is also proved. In the one-dimensional case, we prove a related result about the scaling limit of conservative Langevin dynamics of an SOS surface.

KEY WORDS: Cahn–Hilliard equation; Ginzburg–Landau models; hydrodynamic limits; large deviations; surface diffusion.

1. INTRODUCTION

Cahn–Hilliard equations have been proposed and studied widely as models for describing phase segregation phenomena in binary alloys. This equation has the form

$$\partial_t m = \Delta(F'(m) - \Delta m) \tag{1.1}$$

where $m = m(t, u)$ ($t \geqslant 0$ and $u \in \mathbb{R}^d$) is the order parameter. It has similar features as the reaction-diffusion equations but here $m(t, u)$ is locally conserved. Segregation of phases typically appears in the cases where F is a non-convex function (e.g. a double-well). This deterministic equation does not take into account random effects that maybe important, for example if we want to study the escape from a metastable state.

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The aim of this paper is derivation of Eq. (1.1) as the hydrodynamic scaling limit from a stochastic Ginzburg–Landau model and the computation of the associated large deviation functional, which plays a key role for the metastability problem. Since Eq. (1.1) has no scaling properties, we need to introduce the scaling parameter explicitly in the microscopic dynamics. More precisely, we introduce a reference process which is the gradient flow in $\mathscr{H}_2$ of a discrete quadratic Ginzburg–Landau functional, which produces the forth order term in (1.1); the second order term is then obtained by adding a suitable perturbation depending on the scaling parameter. The functional form of the non-linearity $F'(m)$ is obtained by (equilibrium) statistical mechanics considerations. Within the models introduced, we are able to derive more general equations than (1.1): we consider, in fact, general nonlinear forth order terms (cf. (2.10) below).

This approach is analogous to the derivation of reaction-diffusion equations as the hydrodynamic limit of a scale depending superposition of Glauber and Kawasaki dynamics, see [DFL]. In that case the large deviation functional has been computed by [JLV], see [GJLV] for further analysis. The physical motivation behind these kinds of models is an *a priori* separation into "fast" and "slow" modes.

We also mention that another class of models with similar features has been studied. The scale in the microscopic model is introduced by means of a Kac potential (local mean field). In the so-called mesoscopic limit, a non-local equation describing the phase segregation is obtained. See [DOPT] for the non-conservative case (reaction-diffusion type) and [GL] for the conservative case (Cahn–Hilliard type).

In one space dimension, the reference microscopic Ginzburg–Landau process can be viewed as an effective model (Solid-On-Solid) for conservative random interface motion as proposed by [S, CDG]. In this case $m(t, u)$ is interpreted as the gradient of a single valued function $\lambda(t, u)$ representing the interface height; the integral of $\lambda(t, u)$ is then conserved. Equation (1.1) (with $F' = 0$) is obtained in the Gaussian case; for a genuine interaction we get instead a particular forth order equation (see (3.4) below for $\tilde{\mathbb{V}} = 0$) which, as discussed in [CDG], can be identified with the one obtained from free energy considerations. In the non-conservative case but in any dimension the scaling limit has been proved by [FS].

In the interface interpretation of the model, the addition of the perturbation to the reference process corresponds to the introduction of also a non-conservative interface motion, of the type considered in [FS]. In the scaling limit we consider, both the motions coexist and we thus get a space integration of the Cahn–Hilliard equation, see (3.4) below. Of course, the model introduced in [CDG] can be viewed as a particular case.

From a technical point of view, the tools used in this paper are the ones introduced in the diffusive hydrodynamic limit of the so-called *gradient particle systems*. In particular we shall make use of the entropy method introduced in [GPV, DV] and the analysis of small perturbation as in [KOV]. We shall not repeat the details of the proofs, but rather quote the relevant results and explain the modifications needed.

2. NOTATION AND RESULTS

For a positive integer N, denote by $\mathbb{T}_N$ the discrete one-dimensional torus with N points and set $\mathbb{T}_N^d = (\mathbb{T}_N)^d$. The state space $\mathbb{R}^{\mathbb{T}_N^d}$ is denoted by $\mathscr{X}_N^d$ and configurations by the greek letter φ. In this way, for a d-dimensional integer x, φ_x denotes the charge at site x for the configuration φ.

Consider a twice continuously differentiable function $W: \mathbb{R} \to \mathbb{R}$, and a local smooth function (interaction) $V(\varphi)$ and define the Hamiltonians $H_V, H_W: \mathscr{X}_N^d \to \mathbb{R}$ by

$$H_V(\varphi) = \sum_{x \in \mathbb{T}_N^d} V(\tau_x \varphi), \qquad H_W(\varphi) = \sum_{x \in \mathbb{T}_N^d} W(\varphi_x)$$

where τ_x is the shift operator on $\mathbb{T}_N^d$. We assume that W is normalized:

$$\int_{\mathbb{R}} e^{-W(\varphi_0)}\, d\varphi_0 = 1$$

that $W'(\varphi_0)$ has all exponential moments finite:

$$\int_{\mathbb{R}} e^{\gamma W'(\varphi_0) - W(\varphi_0)}\, d\varphi_0 < \infty$$

for all γ in $\mathbb{R}$ and that there exists a function $\omega: \mathbb{R} \to \mathbb{R}$ that increases faster than linearly at the boundary ($\lim_{|u| \to \infty} \omega(u)/|u| = \infty$) such that

$$\int_{\mathbb{R}} e^{\omega(\varphi_0) - W(\varphi_0)}\, d\varphi_0 < \infty \tag{2.1}$$

To state the assumption made on V, denote by Λ_V the support of the cylinder function V and by $\mathbb{V}$ the cylinder function defined by

$$\mathbb{V}(\varphi) = \frac{\partial H_V}{\partial \varphi_0}$$

We shall assume that $\mathbb{V}$ increases slower than W, i.e., that there exists finite constants C_0, $C_1 < 1$ such that

$$|\mathbb{V}(\varphi)| \leqslant C_0 + C_1 \sum_{x \in \Lambda_V} \omega(\varphi_x)$$

and that $\mathbb{V}(\varphi)^2$ has finite exponential moments:

$$\int_{\mathbb{R}} e^{\alpha \mathbb{V}(\varphi)^2 - W(\varphi_0)} \, d\varphi_0 < \infty \tag{2.2}$$

for some $\alpha > 1$.

The configuration $\varphi(t)$ evolves in time as a diffusion process following the stochastic differential equations:

$$d\varphi_x(t) = \left(\Delta_N \frac{\partial H_V}{\partial \varphi}\right)_x dt - \left(\Delta_N^2 \frac{\partial H_W}{\partial \varphi}\right)_x dt + \sqrt{2}\,(\Delta_N \, dB)_x(t) \tag{2.3}$$

where $\{B_x(t),\ x \in \mathbb{T}_N^d\}$ are independent Brownian motions. Here and below, for a function defined on $\mathbb{R}^{\mathbb{T}_N^d}$, $(\nabla_N H)_x$ stands for the d-dimensional vector whose jth coordinate is equal to $N\{H(\varphi_{x+e_j}) - H(\varphi_x)\}$ and $(\Delta_N H)_x$ stands for the discrete Laplacian of H so that $(\Delta_N H)_x = N^2 \sum_{1 \leqslant j \leqslant d} \{H(\varphi_{x+e_j}) + H(\varphi_{x-e_j}) - 2H(\varphi_x)\}$. In particular,

$$\left(\Delta_N \frac{\partial H_V}{\partial \varphi}\right)_x = D_x^N H_V$$

where

$$D_x^N = \left(\Delta_N \frac{\partial}{\partial \varphi}\right)_x = N^2 \sum_{1 \leqslant j \leqslant d} \left\{\frac{\partial}{\partial \varphi_{x+e_j}} + \frac{\partial}{\partial \varphi_{x-e_j}} - 2\frac{\partial}{\partial \varphi_x}\right\}$$

The infinitesimal generator of this Markov process is given by

$$L_N = L_W + L_V$$

where

$$L_W = \sum_x \left\{(D_x^N)^2 - \left(\Delta_N \frac{\partial H_W}{\partial \varphi}\right)_x D_x^N\right\}$$

and

$$L_V = -\sum_x \left(\nabla_N \frac{\partial H_V}{\partial \varphi}\right)_x \cdot \left(\nabla_N \frac{\partial}{\partial \varphi}\right)_x$$

In this formula $(D_x^N)^2$ stands for the second order operator $D_x^N D_x^N$.

We now introduce the reference measures. Let $Z\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$Z(\gamma) = \int_{\mathbb{R}} e^{\gamma\varphi_0 - W(\varphi_0)} \, d\varphi_0 < \infty \qquad \text{for every} \quad \gamma \in \mathbb{R} \tag{2.4}$$

Notice that the finiteness of Z follows from assumptions (2.1). For $\gamma \in \mathbb{R}$, let $\bar{\nu}_\gamma^N$ be the translation invariant product measure on $\mathscr{X}_N^d$ defined by

$$\bar{\nu}_\gamma^N(d\varphi) = \prod_{x \in \mathbb{T}_N^d} e^{\gamma\varphi_x - W(\varphi_x) - \log Z(\gamma)} \, d\varphi_x \tag{2.5}$$

Let $\rho(\gamma)$ be te mean charge under the measure $\bar{\nu}_\gamma$:

$$\rho(\gamma) = E_{\bar{\nu}_\gamma^N}[\varphi_0] = \int_{\mathbb{R}} \varphi_0 e^{\gamma\varphi_0 - W(\varphi_0) - \log Z(\gamma)} \, d\varphi_0 = Z'(\gamma)/Z(\gamma) \tag{2.6}$$

It is easy to see that $\rho\colon \mathbb{R} \to \mathbb{R}$ is a smooth strictly increasing bijection. Since $\rho(\gamma)$ has a physical meaning as the mean charge under the measure $\bar{\nu}_\gamma^N$, instead of parametrizing the above family of measures by γ, we use the mean ρ as parameter. In this way if $\gamma\colon \mathbb{R} \to \mathbb{R}$ denotes the inverse of the function $\rho(\gamma(u) = \rho^{-1}(u))$, we write:

$$\nu_\rho^N = \bar{\nu}_{\gamma(\rho)}^N, \qquad \rho \in \mathbb{R} \tag{2.7}$$

To keep notation simple, we denote the measure ν_0^N by ν^N.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be the Legendre transform of the function $\log Z$:

$$h(u) = \sup_{\gamma \in \mathbb{R}} \{\gamma u - \log Z(\gamma)\}$$

Sice $\log Z$ is a strictly convex function, h inherit this property and thus h' is a strictly increasing function. This fact is important in the proof of the two blocks estimate. Since $\rho(\gamma) = \partial_\gamma \log Z(\gamma)$, a simple computation shows that the function γ is equal to h' defined above.

We shall consider the process generated by L_N as a perturbation of the process generated by L_W, which is reversible with respect to ν^N. Our first main concern is to investigate the macroscopic behaviour of $\varphi(t)$. We prove below that it is described by the solution of a generalized Cahn–Hilliard equation. In order to state this first result, we need to introduce some notation. For a cylinder function $\Psi(\varphi)$, denote by $\tilde{\Psi}: \mathbb{R} \to \mathbb{R}$ the smooth function defined by

$$\tilde{\Psi}(m) = \int \Psi(\varphi)\, e^{h'(m)\varphi - W(\varphi)}\, d\varphi$$

To prove uniqueness of weak solutions of the hydrodynamic equation we shall require $\tilde{\mathbb{V}}$ to satisfy the following estimate: there exists a finite constant C_2 such that

$$(\tilde{\mathbb{V}}(v) - \tilde{\mathbb{V}}(u))^2 \leqslant C_2(v-u)(h'(v) - h'(u)) \tag{2.8}$$

for all u, v in $\mathbb{R}$.

On the other hand, for each probability measure μ^N in $\mathscr{X}^d_N$, denote by $H_N(\mu^N)$ the relative entropy of μ^N with respect to ν^N:

$$H_N(\mu^N) = \sup_f \left\{ \int f(\varphi)\, \mu^N(d\varphi) - \log \int e^{f(\varphi)} \nu^N(d\varphi) \right\}$$

where the supremum is carried over all bounded cylinder functions f.

Denote by $\mathbb{T}^d$ the d-dimensional torus. Fix a smooth profile $m_0: \mathbb{T}^d \to \mathbb{R}$ and denote by μ^N a sequence of probability measures associated to m_0 is the sense that

$$\lim_{N\to\infty} \mu^N \left\{ \left| \frac{1}{N^d} \sum_x J(x/N)\, \varphi_x - \int du\, J(u)\, m_0(u) \right| > \delta \right\} = 0 \tag{2.9}$$

for every continuous function $J: \mathbb{T}^d \to \mathbb{R}$ and every $\delta > 0$. Denote by $\mathbb{P}_{\mu^N} = \mathbb{P}^N_{\mu^N}$ the probability measure on the path space $C(\mathbb{R}_+, \mathscr{X}^d_N)$ corresponding to the Markov process $\varphi(t)$ starting from μ^N.

Theorem 2.1. Fix a smooth profile $m_0: \mathbb{T}^d \to \mathbb{R}$ and consider a sequence of probability measures μ^N associated to m_0 and such that $H_N(\mu^N) \leqslant K_0 N^d$ for some finite constant K_0. Let $m(t, u)$ be the unique weak solution of the generalized Cahn–Hilliard equation on the torus $\mathbb{T}^d$:

$$\begin{cases} \partial_t m = \Delta(\tilde{\mathbb{V}}(m) - \Delta h'(m)) \\ m(0, u) = m_0(u) \end{cases} \tag{2.10}$$

Then, for every $t > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[\left| \frac{1}{N^d} \sum_x J(x/N)\, \varphi_x(t) - \int du\, J(u)\, m(t, u) \right| > \delta \right] = 0$$

for every continuous function $J: \mathbb{T}^d \to \mathbb{R}$ and every $\delta > 0$.

Remark. The classical Cahn–Hilliard equation (1.1) is obtained when the reference measure ν^N is Gaussian, i.e., for $W(\varphi_0) = (1/2)\,\varphi_0^2$. In this case the unperturbed process generated by L_W is just a linear Ginzburg–Landau process. On the other hand, classical $\mathscr{H}_{-1}$ arguments translated to the $\mathscr{H}_{-2}$ context guarantees the uniqueness of weak solutions in $\mathscr{H}_{-2}$. It is in this proof that we need assumption (2.8) on $\tilde{\mathbb{V}}$. The latter condition gives a restriction on the behaviour of $\tilde{\mathbb{V}}(m)$ for large $|m|$ and rules out the "classical" quartic double well, i.e., $h'(m) = m$, $\tilde{\mathbb{V}}(m) = am^3 - bm$, $a > 0$. However, there is no restriction (apart for the smoothness) on the behaviour of $\tilde{\mathbb{V}}(m)$ for finite $|m|$, i.e., on the local shape of the potential F in (1.1).

This first result can be interpreted as a law of large numbers for the empirical measure π^N_t defined by

$$\pi^N_t(du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \varphi_x(t)\, \delta_{x/N}(du) \tag{2.11}$$

where, for a real u, δ_u is the Dirac measure concentrated on u. Theorem 2.1 states that for a sequence of initial probability measures associated to a profile m_0 in the sense of (2.9) and with entropy of order $K_0 N^d$, the empirical measure π^N_t converges in probability to an absolutely continuous measure whose density is the solution of Eq. (2.10).

To address the question of large deviations of the empirical measure, for a positive integer ℓ, let $\mathscr{M}_\ell$ be the space of signed measures on $\mathbb{T}^d$ with total variation bounded by ℓ. Equipped with the weak* topology induced by $C(\mathbb{T}^d)$ via $\langle \pi, H \rangle = \int H\, d\pi$ for $H \in C(\mathbb{T}^d)$, $\pi \in \mathscr{M}_\ell$, $\mathscr{M}_\ell$ is a compact metric space. Set $\mathscr{M} = \bigcup_\ell \mathscr{M}_\ell$.

To fix ideas and keep notation simple, we shall investigate the large deviations for a system starting from the product measure ν^N. All the analysis goes through for any sequence of product probability measures associated to a profile ρ_0 in the sense (2.9).

Fix a time $T > 0$. For a positive integer N, denote by $\mathbb{Q}_N$ the probability measure induced on $C([0, T], \mathscr{M})$ by the empirical measure $\pi^N_\bullet$ and the probability measure $\mathbb{P}_{\nu^N}$.

We now introduce the ingredients needed for defining the rate functional of the large deviation principle. Throughout this particle, for positive integers m and n, we denote by $C^{m,n}([0, T] \times \mathbb{T}^d)$ the space of functions

$H: [0,T] \times \mathbb{T}^d \to \mathbb{R}$ with m continuous derivatives in time and n continuous derivatives in space. For $H \in C^{1,4}([0,T] \times \mathbb{T}^d)$, let $I_H: C([0,T], \mathcal{M}) \to \mathbb{R} \cup \{\infty\}$ be given by:

$$I_H(\pi) = \langle m_T, H_T \rangle - \langle m_0, H_0 \rangle - \int_0^T dt \, \langle m_t, \partial_t H_t \rangle$$

$$- \int_0^T dt \int du \, \tilde{\mathbb{V}}(m(t,u)) \, \Delta H_t(u)$$

$$- \int_0^T dt \int_{\mathbb{T}^d} du \, h'(m(t,u)) \, \Delta^2 H_t(u) - \int_0^T dt \int_{\mathbb{T}^d} du \, (\Delta H_t(u))^2$$

if $\pi \in C([0,T], \mathcal{M})$ and π_t is absolutely continuous with respect to the Lebesgue measure λ with density m: $\pi(t, du) = m(t,u)\, du$. $I_H(\cdot) = +\infty$ if m_t is not absolutely continuous with respect to λ for some $0 \leqslant t \leqslant T$. We then let $I: C([0,T], \mathcal{M}) \to [0, +\infty]$ be defined as

$$I(\pi) = \int_{\mathbb{T}^d} h(m(0,u))\, du + \sup_{H \in C^{1,4}([0,T] \times \mathbb{T}^d)} I_H(\pi)$$

$I(\cdot)$ is the rate functional appearing in the lower and upper bound large deviations. In Lemma 5.2 below we obtain an explicit formula for the rate function I.

Let $\mathcal{A}$ be the space of all profiles π such that the density $m_\bullet = (d\pi_\bullet / d\lambda)$ is the solutions of the PDE

$$\begin{cases} \partial_t m = \Delta\{\tilde{\mathbb{V}}(m) + \Delta h'(m) - \Delta H\} \\ m(0,\cdot) = m_0(\cdot) \end{cases} \tag{2.12}$$

for some H in $C^{1,4}([0,T] \times \mathbb{T}^d)$ and m_0 in $C^4(\mathbb{T}^d)$. In other words, a profile π is in $\mathcal{A}$ if there exists H in $C^{1,4}([0,T] \times \mathbb{T}^d)$ such that $m_\bullet = (d\pi_\bullet / d\lambda)$ is the solution of the above PDE.

We are now in a position to state the second main theorem of this article.

Theorem 2.2. For every closed subset $\mathcal{C}$ and every open subset $\mathcal{O}$ of $C([0,T], \mathcal{M})$ we have

$$\limsup_{N \to \infty} \frac{1}{N^d} \log \mathbb{Q}_N(\mathcal{C}) \leqslant - \inf_{\pi \in \mathcal{C}} I(\pi)$$

$$\liminf_{N \to \infty} \frac{1}{N^d} \log \mathbb{Q}_N(\mathcal{O}) \geqslant - \inf_{\pi \in \mathcal{O} \cap \mathcal{A}} I(\pi)$$

3. SCALING LIMIT FOR SURFACE DIFFUSION

We consider in this section a microscopic dynamics that describes the evolution of a one dimensional surface. Recall from Section 2 the definitions and the assumptions made on the Hamiltonians H_W and H_V. For a positive integer N, let $\mathbb{Z}_N$ denote the set $\{-N,\dots,N\}$ and denote by s_V the linear size of the support of the potential V, i.e., the smallest integer k so that the support of V is contained in $\{-k,\dots,k\}$. To keep notation simple, let $N_V = N - s_V$. Throughout this section $\tilde{\Delta}_N$ (resp. $\tilde{\Delta}_N^V$) stands for the discrete laplacian on $\mathbb{Z}_N$ (resp. $\mathbb{Z}_{N_V}$): $(\tilde{\Delta}_N f)(x) = N^2[f(x+1) - 2f(x) + f(x-1)]$ for $|x| \leqslant N-1$,

$$(\tilde{\Delta}_N f)(N) = N^2[f(N-1) - f(N)]$$

and

$$(\tilde{\Delta}_N f)(-N) = N^2[f(-N+1) - f(-N)]$$

We consider a diffusion on $\mathbb{R}^{\mathbb{Z}_N}$ with reflection at the boundary. The time evolution is described by the stochastic differential equations

$$d\varphi_x(t) = \left(\tilde{\Delta}_N^V \frac{\partial H_V}{\partial \varphi}\right)_x dt - \left(\tilde{\Delta}_N^2 \frac{\partial H_W}{\partial \varphi}\right)_x dt + \sqrt{2}(\tilde{\Delta}_N \, dB)_x(t) \tag{3.1}$$

Like in Section 2, we may write the infinitesimal generator of this diffusion process and investigate the hydrodynamic behaviour. With the reflection conditions imposed at the boundary the hydrodynamic equation is easy to deduce.

For each ρ in $\mathbb{R}$, denote by ν_ρ^N the product measure $\bar{\nu}_\gamma^N$ introduced in (2.5) with $\mathbb{Z}_N$ replacing the torus $\mathbb{T}_N^d$ and with γ chosen in such a way that the particles density is ρ. Denote by $H_N(\cdot)$ the relative entropy with respect to ν_0^N and recall from Section 2 the definition of a sequence of measures associated to a smooth profile $m_0: [-1,1] \to \mathbb{R}$. Denote by $\mathbb{P}_{\mu^N} = \mathbb{P}_{\mu^N}^N$ the probability measure on the path space $C(\mathbb{R}_+, \mathbb{R}^{\mathbb{Z}_N})$ corresponding to the Markov process $\varphi(t)$ starting from μ^N.

Theorem 3.1. Fix a smooth profile $m_0: [-1,1] \to \mathbb{R}$ and consider a sequence of probability measures μ^N associated to m_0 and such that $H_N(\mu^N) \leqslant K_0 N$ for some finite constant K_0. Let $m(t,u)$ be the unique weak solution of the generalized Cahn–Hilliard equation on $[-1,1]$ with reflection at the boundary:

$$\begin{cases} \partial_t m = \Delta(\tilde{\mathbb{V}}(m) - \Delta h'(m)) \\ m(0,u) = m_0(u) \\ \partial_h h'(m(t,-1)) = \partial_u h'(m(t,1)) = 0 & \text{for all} \quad t \geq 0 \\ \partial_u^3 h'(m(t,-1)) = \partial_u^3 h'(m(t,1)) = 0 & \text{for all} \quad t \geq 0 \end{cases} \tag{3.2}$$

Then, for every $t > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\left[\left|\frac{1}{N^d}\sum_x J(x/N)\,\varphi_x(t) - \int du\, J(u)\, m(t,u)\right| > \delta\right] = 0$$

for every continuous function $J: \mathbb{T}^d \to \mathbb{R}$ and every $\delta > 0$.

There is a technical difficulty in the proof of this result. To show that all limit points of the sequence $\mathbb{Q}_N$ are concentrated on absolutely continuous paths whose density are weak solutions of the Cahn–Hilliard equation (3.2), we need to prove that the time integral of $N[W'(\varphi_N(t)) - W'(\varphi_{N-1}(t))]$ vanishes as $N \uparrow \infty$. This is possible because the Dirichlet form at the boundary writes $N^4 \int [(\partial_{\varphi_N} - \partial_{\varphi_{N-1}})f]^2\, d\nu^N$.

We are now in a position to define and investigate the time evolution of a surface diffusion. For $-N \leq x \leq N$, define $\psi_x(t)$ by

$$\psi_x(t) = \sum_{y \leq x} \varphi_y(t)$$

so that $\psi_{-N}(t) = 0$ and $\partial_t \psi_N(t) = 0$ because the total charge is conserved by the φ-dynamics. The following stochastic differential equations describes the evolution of $\psi(t)$:

$$d\psi_x(t) = \left(\nabla_N^V \frac{\partial H_V}{\partial\varphi}\right)_x dt - \left(\nabla_N \tilde{\Delta}_N \frac{\partial H_W}{\partial\varphi}\right)_x dt + \sqrt{2}\, d(\nabla_N B)_x(t) \tag{3.3}$$

for $|x| \leq N-1$. Moreover, a simple computation shows that macroscopic evolution of the surface is given by the equation:

$$\begin{cases} \partial_t \lambda = \partial_u(\tilde{\mathbb{V}}(\partial_u\lambda) - \Delta h'(\partial_u\lambda)) \\ \lambda(0,u) = \lambda_0(u) \\ \lambda(t,-1) = 0, \quad \lambda(t,1) = \lambda_0(1) & \text{for all} \quad t \geq 0 \\ \Delta h'(\partial_u\lambda(t,-1)) = \Delta h'(\partial_u\lambda(t,1)) = 0 & \text{for all} \quad t \geq 0 \end{cases} \tag{3.4}$$

Remark. For $V = 0$ the microscopic evolution of the surface is the same model introduced in [CDG]: compare [CDG, Eq. (2.13)] with our (3.3). Accordingly, the macroscopic evolution is the one obtained in the above reference under the hypotheses of local equilibrium (which we prove to hold), compare (3.4) for $\tilde{\mathbb{V}} = 0$ with [CDG, Eq. (4.7)].

4. HYDRODYNAMIC LIMIT

We prove in this section the hydrodynamic behaviour of the process $\varphi(t)$ defined in the previous section. The proof follows closely the approach introduced by Guo, Papanicolaou and Varadhan in [GPV] to deduce the hydrodynamic equation of diffusive gradient interacting particle systems. We just point out the main technical differences of our model.

We shall consider our system as a small perturbation of the reversible (with respect to the measures ν^N) Markov process with infinitesimal generator L_W. The Dirichlet form associated to this latter process is:

$$\hat{\mathscr{D}}_N(f) = \sum_x \int \frac{1}{f}\left[\left(\Delta_N \frac{\partial f}{\partial\varphi}\right)_x\right]^2 \nu^N(d\varphi)$$

We first obtain a lower bound for this Dirichlet form in terms of a simpler one.

Proposition 4.1. Denote by $\mathscr{D}_N$ the Dirichlet form defined by

$$\mathscr{D}_N(f) = \sum_x \int \frac{1}{f}\left[\left(\nabla_N \frac{\partial f}{\partial\varphi}\right)_x\right]^2 \nu^N(d\varphi)$$

There exists a finite constant C depending only on the dimension d such that for every density f with respect to ν^N,

$$\mathscr{D}_N(f) \leq C\hat{\mathscr{D}}_N(f)$$

Proof. Let $u(x)$ be a function in T_N^d. By Schwarz inequality,

$$\sum_x \left(u(x) - N^{-d}\sum_z u(z)\right)^2 \leq C(d)\, N^{-1} \sum_x \{(\nabla_N u)(x)\}^2$$

for some constant $C(d)$ depending only on the dimension. A summation by parts permits to rewrite $N^{-1}\sum_x\{(\nabla_N u)(x)\}^2$ as

$$\begin{aligned} &-N^{-1}\sum_x u(x)(\Delta_N u)(x) \\ &\leq N^{-1}\left\{\sum_x \left(u(x) - N^{-d}\sum_z u(z)\right)^2\right\}^{1/2} \left\{\sum_x \{(\Delta_N u)(x)\}^2\right\}^{1/2} \\ &\leq C(d)\left\{\sum_x \{(\nabla_N u)(x)\}^2\right\}^{1/2} \left\{\sum_x \{(\Delta_N u)(x)\}^2\right\}^{1/2} \end{aligned}$$

Therefore,

$$\sum_x \{(\nabla_N u)(x)\}^2 \leqslant C(d) \sum_x \{(\Delta_N u)(x)\}^2$$

what concludes the proof of the proposition. ∎

In possession of this estimate we may repeat the large deviations arguments of [DV], [KOV] to prove a superexponential one and two blocks estimates that allows the replacement, at the level of large deviations, of cylinder functions by functions of the empirical density.

For each positive integer ℓ and each site x, denote by φ_x^ℓ the empirical mean charge on a box of linear size ℓ centered at x:

$$\varphi_x^\ell = \frac{1}{(2\ell+1)^d} \sum_{|y-x| \leqslant \ell} \varphi_y$$

Denote by $\mathbb{P}_{\nu^N}^W$ the probability measure on the path space $C(\mathbb{R}_+, \mathscr{X}_N^d)$ corresponding to the Markov process with generator L_W starting from ν^N.

Lemma 4.2. Let Φ a bounded cylinder function. For each positive integer ℓ, denote by $\Omega_\ell^\Phi(\varphi)$ the cylinder function defined by

$$\Omega_\ell^\Phi(\varphi) = \frac{1}{(2\ell+1)^d} \sum_{|x| \leqslant \ell} \tau_x \Phi(\varphi) - \tilde{\Phi}(\varphi_0^\ell)$$

For every $t > 0$ and $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} N^{-d} \log \mathbb{P}_{\nu^N}^W \left[\int_0^t ds\, N^{-d} \sum_x |\tau_x \Omega_{\varepsilon N}^\Phi(\varphi(s))| > \delta \right] = -\infty$$

Proof. By the reversibility of the dynamics here considered, the Feynman–Kac formula and the variational representation for the largest eigenvalue of a symmetric operator, the problem boils down to prove that for every $a > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_f \{\langle \Omega_{\varepsilon N}^\Phi(\varphi) f \rangle - a N^{-1} \hat{\mathscr{D}}_N(f)\} = 0$$

where the supremum is taken over all probability densities with respect to ν^N and $\langle \cdot \rangle$ indicates integration with respect to ν^N. Thanks to Proposition 4.1, it is enough to prove that

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_f \{\langle \Omega_{\varepsilon N}^\Phi(\varphi) f \rangle - a N^{-1} \mathscr{D}_N(f)\} = 0$$

for every $a > 0$ and this is proved in [GPV]. ∎

This estimate can be extended to the full process repeating the arguments of Lemma 5.1 of [KOV]. It is in this extension that assumption (2.2) is required. Theorem 2.1 follows from this replacement lemma and the entropy method introduced in [GPV].

5. LARGE DEVIATIONS

The proof of large deviations from the hydrodynamic limit follows closely the approach presented in [DV] to investigate the large deviations of Ginzburg–Landau lattice models. We just point out below the main differences.

Upper Bound. For each smooth function $J: \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ and continuous function $G: \mathbb{T}^d \to \mathbb{R}$, the expression

$$\begin{aligned} M_t = M_t^{G,J} &= \exp N^d \{\langle \pi_0^N, G \rangle - \langle \lambda_N, \log Z(G) \rangle\} \\ &\times \exp \left\{ \sqrt{2} \sum_x \int_0^t (\Delta_N J)(s, x/N)\, dB_x(s) \right. \\ &\left. - \sum_x \int_0^t [(\Delta_N J)(s, x/N)]^2\, ds \right\} \end{aligned}$$

is a mean one, positive martingale. In this equation λ_N stands for the discrete approximation of the Lebesgue measure and $Z(\cdot)$ is the partition function defined in (2.4). By formula (2.3) and by an integration by parts, we may rewrite this martingale as the exponential of

$$\begin{aligned} & N^d \left(\langle \pi_t^N, J_t \rangle - \langle \pi_0^N, J_0 \rangle - \int_0^t ds\, \langle \pi_s^N, \partial_s J_s \rangle + \langle \pi_0^N, G \rangle - \langle \lambda_N, \log Z(G) \rangle \right) \\ & - \sum_x \int_0^t ds\, (\Delta_N J)(s, x/N) \frac{\partial H_V}{\partial \varphi_x}(\varphi(s)) - \sum_x \int_0^t ds\, (\Delta_N^2 J)(s, x/N)\, W'(\varphi_x(s)) \\ & - \sum_x \int_0^t ds\, [(\Delta_N J)(s, x/N)]^2 \end{aligned} \tag{5.1}$$

Denote by $N^d \ell_{J,G}(\pi^N)$ the first line of this expression. By definition of $\mathbb{V}$, the difference

$$N^{-d} \sum_x (\Delta_N J)(s, x/N) \frac{\partial H_V}{\partial \varphi_x}(\varphi(s)) - N^{-d} \sum_x (\Delta_N J)(s, x/N)\, \tau_x \mathbb{V}(\varphi(s)) \tag{5.2}$$

is of order $O(N^{-1})$ because J is a smooth function. Denote by $R_{J,\varepsilon,N}$ the difference between the second line of (5.1) and

$$-\sum_x \int_0^t ds\,(\Delta_N J)(s, x/N)\,\tilde{\mathbb{V}}(\varphi_x^{\varepsilon N}(s)) - \sum_x \int_0^t ds\,(\Delta_N^2 J)(s, x/N)\,h'(\varphi_x^{\varepsilon N}(s))$$

Fix a set K of $C([0, T], \mathcal{M})$. With the notation just introduced, for every $q>1$, $\mathbb{Q}_N[K]$ which is equal to $\mathbb{P}_{\nu^N}[\pi^N \in K]$ can be rewritten as

$$\begin{aligned}&\mathbb{E}_{\nu^N}[\mathbf{1}\{\pi^N \in K\}\; M_T^{1/q} M_T^{-1/q}]\\ &\quad\leqslant \exp -\{\inf_{\pi\in K}(1/q)\,\psi_{J,G,\varepsilon,N}\}\;\mathbb{E}_{\nu^N}[\mathbf{1}\{\pi^N\in K\}\; M_T^{1/q} e^{R_{J,\varepsilon,N}}]\end{aligned}$$

where $\psi_{J,G,\varepsilon,N}(\pi)$ is equal to

$$\begin{aligned}&N^d \ell_{J,G}(\pi) - \sum_x \int_0^T ds\,(\Delta_N J)(s, x/N)\,\tilde{\mathbb{V}}((2\varepsilon)^{-d}\,\pi(s, [(x/N)-\varepsilon, (x/N)+\varepsilon]))\\ &\quad - \sum_x \int_0^t ds\,(\Delta_N^2 J)(s, x/N)\,h'((2\varepsilon)^{-d}\,\pi(s, [(x/N)-\varepsilon, (x/N)+\varepsilon]))\\ &\quad - \sum_x \int_0^t ds\,[(\Delta_N J)(s, x/N)]^2\end{aligned}$$

Let p stands for the conjugate of q. By Schwarz inequality, $\mathbb{E}_{\nu^N}[\mathbf{1}\{\pi^N\in K\}\; M_T^{1/q}\exp\{R_{J,\varepsilon,N}\}]$ is bounded above by $\mathbb{E}_{\nu^N}[\exp\{pR_{J,\varepsilon,N}\}]^{1/p}$ because M_t is a mean one positive martingale. Denote by $R_{q,J,\varepsilon}$ the limit sup as $N\uparrow\infty$ of the logarithm of this expression divided by N^d.

Minimizing over q, J, ε and G, $\limsup_{N\to\infty} N^{-d}\log\mathbb{P}_{\nu^N}[\pi^N\in K]$ is bounded above by

$$-\sup_{q,J,G,\varepsilon}\{\inf_{\pi\in K}(1/q)\,\psi_{J,G,\varepsilon}(\pi) + R_{q,J,\varepsilon}\}$$

where $\psi_{J,G,\varepsilon}(\pi)$ is the limit as $N\uparrow\infty$ of $N^{-d}\psi_{J,G,\varepsilon,N}(\pi)$. By the super-exponential estimate stated in Lemma 4.2, for every $q>1$ and smooth J, the limit as $\varepsilon\downarrow 0$ of $R_{q,J,\varepsilon}$ vanishes.

Assume now that the set K is compact. Since $\psi_{J,G,\varepsilon}$ is continuous for every J, G and $\varepsilon>0$, we may apply the arguments presented in Lemma 11.3 of [V] to exchange the supremum with the infimum. Hence, last expression is equal to

$$-\inf_{\pi\in K}\;\sup_{q,J,G,\varepsilon}\{(1/q)\,\psi_{J,G,\varepsilon}(\pi) + R_{q,J,\varepsilon}\}$$

Notice that for a path $\pi_t(du)$ which is not absolutely continuous the supremum over J and $\varepsilon>0$ diverges. On the other hand, for an absolutely continuous path, letting $\varepsilon\downarrow 0$, then $q\downarrow 1$, by definition of I, we obtain that the last variational formula is bounded above by $-\inf_{\pi\in K} I(\pi)$, what concludes the proof of the upper bound for compact sets. To extend it to closed sets, we need to prove the exponential tightness of the sequence $\mathbb{Q}_N$. The arguments presented in [DV], [BKL] apply to the present context.

Lower Bound. The first result shows that from the point of view of large deviations, we may restrict our attention to paths $\pi(t, du)$ that are absolutely continuous with respect to the Lebesgue measure:

Lemma 5.1. Let π^* be a path in $C([0, T], \mathcal{M})$, $\varepsilon>0$ and $J\colon \mathbb{T}^d\to\mathbb{R}$ be a continuous function and consider the neighborhood $V_{J,\varepsilon}$ defined by

$$V_{t,J,\varepsilon} = \left\{\pi, \left|\int J(u)\,\pi(t, du) - \int J(u)\,\pi^*(t, du)\right| < \varepsilon\right\}$$

Assume that

$$\limsup_{N\to\infty} N^{-d}\log\mathbb{Q}_N[V_{t,J,\varepsilon}] \geqslant -\gamma$$

for every t, $\varepsilon>0$ and smooth function J. Then, π^* is absolutely continuous: $\pi^*(t, du) = m^*(t, u)\,du$ and

$$\sup_{0\leqslant t\leqslant T}\int h(m^*(t, u))\,du \leqslant \gamma$$

The proof is close to the one of Lemma 2.3 of [DV]. There is just an additional difficulty from the fact that ν^N is not invariant for the process. This can be overcome by considering the process as a small perturbation from the one with generator L_W.

We now obtain an explicit formula for the rate function I.

Lemma 5.2. Let $\pi(t, du) = m(t, u)\,du$ be an absolutely continuous path with finite rate function: $I(\pi)<\infty$. There exists a function G in $\mathcal{H}_2$ such that

$$\partial_t m = \Delta(\tilde{\mathbb{V}} + \Delta(h' + G))$$

Moreover,

$$I(\pi) = \int_{\mathbb{T}^d} h(m(0,u))\,du + \int_0^T dt \int_{\mathbb{T}^d} du\,(\Delta G)^2$$

$$= \int_{\mathbb{T}^d} h(m(0,u))\,du + \int_0^T dt\, \|\partial_t m - \Delta(\tilde{\mathbb{V}}(m_t) + \Delta h'(m_t))\|_{-2}^2$$

The proof if omitted. The reader is referred to Lemma 2.4 of [DV] or Lemma 5.1 of [KOV] for a similar statement.

The proof of the lower bound follows from these two lemmata and a law of large numbers for the empirical measure under a small perturbation of the dynamics that we now state (cf. [BKL]).

Fix a smooth function $J: \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ and a continuous function $G: \mathbb{T}^d \to \mathbb{R}$, and consider the probability measure $\mathbb{P}_N^{J,G}$ on the path space $C([0,T], \mathscr{X}_N^d)$ induced by $\mathbb{P}_{\nu^N}$ and the martingale $M_t^{G,J}$ through the formula $\mathbb{P}_N^{J,G}[\,\cdot\,] = \mathbb{E}_{\nu^N}[\,\cdot\, M_t^{G,J}]$.

Lemma 5.3. Under the probability measure $\mathbb{P}_N^{J,G}$, the empirical measure π_t^N converges in probability to the absolutely continuous path $\pi(t,du) = m(t,u)\,du$ whose density is the solution of the equation

$$\begin{cases} \partial_t m = \Delta(\tilde{\mathbb{V}}(m) - \Delta h'(m) - \Delta J) \\ m(0,u) = \rho(G(u)) \end{cases}$$

where ρ is the function defined in (2.6).

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