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DYNAMICAL FLUCTUATIONS AT THE CRITICAL POINT: CONVERGENCE TO A NONLINEAR STOCHASTIC PDE*

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Abstract. We consider an Ising spin system with Glauber dynamics and Kac interactions in one dimension at the critical temperature. We study the fluctuation filed of the magnetization density in a scaling limit which involves space, time and the range of the interaction. We prove that for a suitable choice of the scalings the normalized fluctuations field converges to the solution of a one-dimensional (nonlinear) Ginzburg–Landau equation perturbed by a white noise process.

Key words. Kac potentials, critical fluctuations, stochastic quantization of field theories

1. Introduction. In this paper we study a discrete approximation scheme for the Cauchy problem relative to the stochastic semilinear parabolic equation

$$dm_t = \left(\frac{D}{2}\,\Delta m_t - V'(m_t)\right)dt + dz_t, \tag{1.1}$$

where m_t is a scalar field in a bounded region of $\mathbf{R}^d$, $D > 0$, Δ is the laplacian, dz_t/dt a white noise process in space and time and $V(m) = am^2 + bm^4$ (we will eventually consider $d = 1$, $a = 0$ and $b > 0$, but, for the moment, it is convenient to retain only the latter condition). The precise meaning of (1.1) in $d = 1$ will be specified in the next section.

Among the many possibilities to discretize (1.1) we choose one based on point processes, where the probabilistic structure is intrinsically related to the approximation scheme. One of the motivations for this choice is the hope that the intrinsic validity of the approximating model may help to establish properties of the limit equation, maybe its same existence. A good example of the advantages of this approach is provided by the analysis of superprocesses in terms of branching Brownian motions. In our case the approximating scheme has its own interest, with a clear physical interpretation, related to the space-time structure of the fluctuations at the critical temperature in a stochastic Ising spin system.

Equation (1.1) is much studied in physics. Here we mention two important applications, concerning the stochastic quantization of the Euclidean field theory (see for instance, [2]) and the theory of critical phenomena. While in perspective the former is our main objective, at present we do not yet have much to say about it, as the interesting cases are for $d \geqq 2$, while our results are restricted to $d = 1$, which corresponds to the stochastic quantization of the nonharmonic oscillator. The step from one to more than one dimension, on the other hand, is not an easy one, because for $d \geqq 2$ the typical trajectories of the free process (defined with $V = 0$ in (1.1)) are only distributions. One then faces the characteristic problem of quantum field theory to give a sense to $V'(m)$ when this is nonlinear. By suitable regularization procedures, eventually lifted, it is, however, possible, for $d = 2$, to define processes, which "solve" in some weak sense (1.1) (see [18] and, more recently, [1], [29]). For $d = 3$, to our knowledge, the problem is still open and for $d > 3$ there are serious doubts about the possibility of defining the theory. As already mentioned, our analysis covers the case $d = 1$, where the above difficulties (ultraviolet divergences) disappear, since the trajectories are continuous in space and time. In this case a fairly good and complete theory of the equation is available (see [6], [8], [9], [15], and [20]).

To explain our discretization scheme of (1.1), it is useful to go back to its interpretation in terms of the Ginzburg–Landau theory. Observe that by varying the parameter a in V, we change it from a double well, $a < 0$, to a single well potential, $a \geqq 0$. When $a < 0$, the Cauchy problem for (1.1) with initial datum $m_0 = 0$ is used to model the onset of instabilities, as for instance, in problems of phase separation, where the unstable phase, $m = 0$, develops into the stable ones, which correspond to the values of m, where the potential V has its minima. The role of the noise is then fundamental, as in the purely deterministic equation the state $m = 0$ is stationary. It is thus the noise which triggers the whole phenomenon. There is plenty of evidence of stochastic effects in physics besides phase separation, where even macroscopic observations are intrinsically random, as the appearance of random patterns in crystals growth and in Rayleigh–Benard experiments. The noise, however, in all the above cases, is only effective in the very first stage of the evolution and is overcome, afterward, by the linear drift responsible for the instabilities. The nonlinear terms become important much later and the effect of the noise is by then negligible, even though its role has been essential. As the nonlinear drifts and the white noise are not competing on the same level, this is not the right physical situation for deriving (1.1). At the critical point, on the other hand, the linear instability disappears and only the nonlinear drifts can contrast the noise, hence we expect that on a suitable space-time regime they will be both present. This is the case studied in the present paper, and, from now on, $a = 0$, so that (1.1) is the critical Ginzburg–Landau equation with noise.

It is common wisdom in statistical physics that the critical phenomena have a universal character, in particular, that the Gibbs processes at the critical point have all similar behaviors. At the dynamical level, this would imply that an Ising system with Glauber dynamics at the critical temperature behaves as (1.1), with $a = 0$. As is well known, there is no phase transition for $d = 1$, but this refers only to "short range interactions." We thus consider Kac potentials, whose range depends on a parameter γ, becoming infinite when $\gamma \to 0$. In this limit, the phase diagram is as in the Van der Waals theory, it thus has a phase transition and a critical temperature (see [19], [22]). We put ourselves right at that temperature and study the Glauber dynamics with finite γ. The main result in this paper is that the magnetization fluctuations, suitably normalized, converge, as $\gamma \to 0$, to a process which solves (1.1). The precise statements are given in the next section, the proofs in the remaining ones. An outline of the content of the paper may be found at the end of the next section.

2. Definitions and main results. We consider the following Cauchy problem on $\mathcal{T}$, the $d = 1$ torus of length 1,

$$dm_t = \left(\frac{D}{2}\,\frac{\partial^2 m_t}{\partial r^2} - \frac{1}{3}\,m_t^3\right)dt + dz_t, \tag{2.1a}$$

$$m_0 = m^*, \tag{2.1b}$$

where m_t, $t \geqq 0$, is a $C(\mathcal{T})$-valued process and $m^* \in C(\mathcal{T})$. We assume that $D > 0$

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and z_t, $t \geqq 0$, is the Wiener process. We start by defining the latter and then we will recall the precise meaning of (2.1).

DEFINITION 2.1 (The Wiener process). The Wiener process z_t, $t \geqq 0$, is a process on $C(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$, namely its trajectories are continuous valued distributions on $\mathcal{T}$. Its law Q is Gaussian and it is characterized by the following conditions. For any φ and $\psi \in \mathcal{S}(\mathcal{T})$, (z_t, φ) and (z_t, ψ) are Brownian motions and, for any $s \geqq t \geqq 0$,

$$\int dQ\,(z_t,\varphi)(z_s,\psi) = 2t(\psi,\varphi), \tag{2.2}$$

where (ψ, φ) is the scalar product in $L_2(dx, \mathcal{T})$.

A reference for the characterization statement in Definition 2.1, to which we will refer several other times in the sequel, is [16], [33], [34].

We next write the problem (2.1) in integral form and against test functions: let $\varphi \in \mathcal{S}(\mathcal{T})$, then

$$(\varphi, m_t) = (\varphi, m^*) + \int_0^t ds \left\{ \frac{D}{2}(\varphi'', m_s) - \frac{1}{3}(\varphi, m_s^3) \right\} + (\varphi, z_t), \tag{2.3}$$

where φ'' is the second derivative of φ. In § 5 we prove the following result.

THEOREM 2.2. *There is a unique process m_t, $t \geqq 0$, in $C(\mathbf{R}_+, C(\mathcal{T}))$ which can be realized in the space of the Wiener process and which solves almost surely (2.3) for all $\varphi \in \mathcal{S}(\mathcal{T})$ and all $t \geqq 0$. The process m_t, when realized in this way, is adapted to z_t.*

We denote by $\mathcal{P}$ the law of $(m_t)_{t \geqq 0}$.

Recall that m_t is adapted to z_t if, for any $t \geqq 0$, the process m_t is measurable with respect to the σ-algebra generated by z_s, $s \leqq t$. There are several other ways to define the process m_t (see [15], [20]). In § 5 we also define (2.1) as a martingale problem which we show to be equivalent to (2.3). There we also prove the Girsanov formula to represent the process m_t as a Radon–Nikodym density with respect to the free process m_t^0, which is defined by (2.3) without the term m^3.

We turn next to the main issue in the paper, namely the derivation of the process m_t as the limit of Markov processes on finite state spaces. We start by discretizing the torus $\mathcal{T}$.

DEFINITION 2.3 (The discretization of $\mathcal{T}$). For any $\gamma > 0$ we denote by $\mathcal{T}_\gamma$ the finite interval in $\mathbf{Z}$ with identified endpoints $\pm L_\gamma$, where L_γ is the integer part of $\gamma^{-1-1/3}/2$. We also identify $\mathcal{T}_\gamma$ to $\mathbf{Z}$ modulo $2L_\gamma$.

For each $\gamma > 0$ we set

$$\mathcal{H}_\gamma := \{-1, 1\}^{\mathcal{T}_\gamma} \tag{2.4}$$

to be "the spin configurations space," whose elements are denoted by $\sigma = \{\sigma(x), x \in \mathcal{T}_\gamma\}$, and are called spin configurations; $\sigma(x)$ is thus the spin at x. We next define a Markov jump process on $\mathcal{H}_\gamma$.

DEFINITION 2.4 (The Glauber dynamics). For any $\gamma > 0$ we denote by $\mathcal{L}_\gamma$ the operator which acts on the real-valued functions f on $\mathcal{H}_\gamma$ as

$$\mathcal{L}_\gamma f(\sigma) = \sum_{x \in \mathcal{T}_\gamma} c_\gamma(x,\sigma)\left[f(\sigma^{(x)}) - f(\sigma)\right], \tag{2.5}$$

where

$$c_\gamma(x,\sigma) = e^{-\sigma(x)h_\gamma(x)}\left[e^{-h_\gamma(x)} + e^{h_\gamma(x)}\right]^{-1}, \tag{2.6}$$

$\sigma^{(x)}$ being the configuration obtained from σ by flipping the spin at x;

$$h_\gamma(x) = \sum_{y \neq x} J_\gamma(x,y)\sigma(y) =: (J_\gamma \circ \sigma)(x), \tag{2.7}$$

$$J_\gamma(x,y) = n_\gamma \gamma J(\gamma|x-y|), \qquad n_\gamma^{-1} = \gamma \sum_x J(\gamma|x|), \tag{2.8}$$

where function $J \in C^4(\mathbf{R}_+)$, is positive, with support in $[0,1]$ and such that $\int_{\mathcal{T}} dr\, J(|r|) = 1$.

The process generated by $\mathcal{L}_\gamma$ is in $D(\mathbf{R}_+, \mathcal{H}_\gamma)$ and its elements are denoted by σ_t, $t \geqq 0$. Its law is denoted by $\mathbf{P}_\mu^\gamma$, if the distribution of σ_0 is μ ($D(\mathbf{R}_+, \mathcal{H}_\gamma)$ is the Skorokhod space with cadlag trajectories [28]).

We will comment later on the meaning of this process in the context of statistical mechanics; here we just remark that in the continuum limit $\gamma \to 0$ the size of $\mathcal{T}_\gamma$ becomes much larger than the range γ^{-1} of J_γ, so that, in this limit, each spin interacts with an infinitesimal small fraction of the system. This is required if we want to derive the local equation (2.1). We still need a definition before establishing the relation with (2.1).

DEFINITION 2.5 (The fluctuation fields). For any $\varphi \in \mathcal{S}(\mathcal{T})$ and any $t \geqq 0$ we set

$$X_t^\gamma(\varphi) = \gamma \sum_x \varphi(\gamma^{1+1/3}x)\, \sigma_{\gamma^{-2/3}t}(x). \tag{2.9}$$

We then introduce the space $D(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$ and denote by $X_t(\varphi)$ its canonical coordinates: namely, if $\omega \equiv (\omega_s)_{s \geqq 0} \in D(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$, then

$$(X_t(\varphi))(\omega) = (\omega_t, \varphi), \quad \text{where } (\omega_t, \varphi) \text{ is the value of } \omega_t \text{ at } \varphi.$$

We finally call $\mathcal{P}^\gamma$ the law in $D(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$ which gives to the coordinates $X_t(\varphi)$ the same distribution that the variables $X_t^\gamma(\varphi)$ have in the Markov process of Definition 2.4 which starts from ν_0^γ, the product measure on $\mathcal{H}_\gamma$ such that the average of any spin $\sigma(x)$ is 0.

Notice the "anomalous normalization" in (2.9) with γ instead of the "normal" factor $\gamma^{1/2+1/6}$, inversely proportional to the square root of the number of terms which appear in the sum, if φ is everywhere positive. Our main result is the following theorem.

THEOREM 2.6. *For any $T > 0$ the law $\mathcal{P}^\gamma$, defined in Definition 2.5, restricted to $D([0,T], \mathcal{S}'(\mathcal{T}))$, converges weakly to the restriction to the same space of a law $\mathcal{P}$ on $D(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$. Moreover, $\mathcal{P}$ is supported by distributions with continuous densities with respect to the Lebesgue measure on $\mathcal{T}$, and, identifying the distributions with their densities, it is equal to the law $\mathcal{P}$ of Theorem 2.2, if D is set equal to*

$$D = \int dr\, J(|r|)r^2, \qquad \hat{J}(0) := \int dr\, J(|r|) = 1, \tag{2.10}$$

and if m^* is identically equal to 0.

Our analysis may be easily extended to the case when the initial measure is $\nu_{m^*}^\gamma$, i.e., the product measure on $\mathcal{H}_\gamma$ such that for all $x \in \mathcal{T}_\gamma$

$$\mathbf{E}_{\nu_{m^*}^\gamma}(\sigma(x)) = \gamma^{1/3} m^*(\gamma^{1+1/3}x)$$

and $m^* \in C(\mathcal{T})$. In this case the limit measure is the law of the solution of (2.3). For simplicity we have only considered the case $m^* \equiv 0$, as in Theorem 2.6.

Relations with statistical mechanics. To relate the process in Definition 2.4 to statistical mechanics we argue as follows. We first observe that for any $\gamma > 0$ there is a unique invariant measure μ_γ, as it follows from general theorems on Markov chains with finitely many states [27]. The special form of the flip rates c_γ allows us to compute μ_γ explicitly: the μ_γ-probability of a configuration σ is in fact

$$\mu_\gamma(\sigma) = \frac{1}{Z_\gamma} \exp\left\{\frac{1}{2}\sum_x \sigma(x) h_\gamma(x)\right\},$$

where Z_γ is the normalization factor. Then it is easy to see that the generator $\mathcal{L}_\gamma$ is self-adjoint with respect to μ_γ and the corresponding process is reversible. Furthermore, by changing J_γ into βJ_γ, $\beta > 0$, we obtain, respectively, new rates, a new generator, and a new invariant measure

$$\mu_\gamma^\beta(\sigma) = \frac{1}{Z_\gamma^\beta} \exp\left\{\frac{\beta}{2}\sum_x \sigma(x) h_\gamma(x)\right\}.$$

This is the expression for the Gibbs measure at the inverse temperature β and with spin-spin interaction $J_\gamma(x,y)$. The limit $\gamma \to 0$ is that considered by Kac et al. and by Lebowitz and Penrose to derive the Van der Waals theory of phase transition [19], [22].

As $\gamma \to 0$, the Gibbs measure μ_γ^β converges (in the sense of the joint distributions of finitely many spins) to the Bernoulli measure ν_0 if $\beta \le 1$, whereas, for $\beta > 1$, it converges to $\frac{1}{2}\nu_{m_\beta} + \frac{1}{2}\nu_{-m_\beta}$, where ν_a, $|a| \le 1$, denotes the Bernoulli measure of mean a; m_β is the positive solution of the mean field equation

$$m_\beta = \tanh\{\beta m_\beta\}.$$

This is interpreted, in the context of statistical mechanics, as a phase transition and the case considered here, $\beta = 1$, corresponds to the critical temperature. We are thus studying, in Theorem 2.6, the critical fluctuations.

Critical scalings. As mentioned in the introduction the derivation of a nonlinear evolution with noise requires a balance between the nonlinear drift and the noise. Here we see this with more precision, but for reasons of space we omit the proofs. For $0 \le b \le \frac{1}{3}$, in analogy with Definition 2.5, we define

$$Y_t^\gamma(\varphi) = \gamma^{b+(1+b)/2} \sum_{x \in \mathcal{T}_\gamma} \varphi(\gamma^{1+b} x)\, \sigma_{\gamma^{-2b}t}(x) \tag{2.11}$$

and consider as initial measure the product measure with 0 averages. For $b = \frac{1}{3}$ this is the fluctuation field considered in this paper, for which Theorem 2.6 applies. For $0 < b < \frac{1}{3}$, the process Y_t^γ, thought of as a process in $D(\mathbf{R}_+, \mathcal{S}'(\mathbf{R}))$, converges to the solution of (2.3) (with $m^* \equiv 0$) in the whole space and with the nonlinear term missing. If $b = 0$, as is proved in [11], the limit process $\xi_t \equiv \xi_t(r)$, $r \in \mathbf{R}$, solves the equation

$$d\xi_t = (J \star \xi_t - \xi_t)\, dt + dz_t, \tag{2.12}$$

z_t being here the Wiener process in the whole space. Thus, when b increases past 0, we first obtain a local equation, namely the first two terms on the right-hand side of (2.12) are replaced by ξ_t'', the second derivative of $\xi_t(r)$. Moreover, as b increases, the fluctuations become larger, as the exponent in the normalization factor in (2.11) increases. However, the limit equation is always the same, till b reaches its critical value, $b = \frac{1}{3}$. At this point the nonlinear drift has the same order as the noise and they both coexist as in (2.1).

Bibliographical remarks. The case when the length of $\mathcal{T}_\gamma$ is $\gamma^{-1}L$, with $L > 0$ fixed independently of γ, is studied in [7] for a general class of jump processes, which includes ours. With reference to our dynamics, in [7] it is proved that the total magnetic fluctuation

$$M_t^\gamma = \gamma^{1-1/4} \sum_x \sigma(x, \gamma^{-1/2} t) \tag{2.13}$$

converges, as $\gamma \to 0$, to a Markov process in $D(\mathbf{R}_+, \mathbf{R})$, which solves the stochastic ordinary differential equation

$$dm = -\tfrac{1}{3} m^3 dt + db. \tag{2.14}$$

In (2.14) the spatial structure is completely missing, due to the reduced length of $\mathcal{T}_\gamma$ and, for the same reason, the time scale and the normalization are different.

A similar analysis is done for Brownian motions which interact via a mean field potential [10]. The limit equation is again of the form (2.12). An analogous problem is studied in [5], where, however, the spin evolution (the so called Glauber + Kawasaki dynamics) involves both spin flips and spin exchanges. The choice of the rates is such that the nonlinear term in the final equation is positive and the solution explodes after a finite, random time. Convergence is proven till that time.

A physical discussion on the derivation of stochastic reaction diffusion equations from other discrete models can be found in [17]. We finally refer to [11] for an extensive discussion on the fluctuation fields in relation to Euclidean field theory and phase separation phenomena for Glauber dynamics with Kac potentials.

Outline of the paper. In § 3 we introduce a version of the voter model and prove that it converges weakly to the free process solution of (2.3), but with the nonlinear term missing. We also give some properties of the free process that will be used in the sequel. In § 4 we express the Glauber dynamics as the Radon–Nikodym derivative of the voter model. We compute its leading terms and take the macroscopic limit $\gamma \to 0$. Using a result in [20], we then characterize the process obtained in that limit. In § 5 we prove the equivalence of some different definitions of the process solution of (2.1), in particular, we give another proof of the result in [20], used earlier. We then identify the limit process of the previous section as the process of Theorem 2.2. In two appendices we report more technical estimates on the voter model and on the local central limit theorem, used earlier.

3. The voter model and the free process. This section is divided in two parts; in the first one we introduce a version of the voter model obtained by modifying the rates $c_\gamma(x,\sigma)$ in (2.6). We then prove, following the Holley and Stroock theory [16], [33], [34], that this voter model converges, as $\gamma \to 0$, to a process solution of a martingale problem. In the second part of the section we show that the solution of the martingale problem is the free process, i.e., the solution of (2.3) with the nonlinear term missing. We also recall the definition of the stochastic integral of the Wiener process and the corresponding Itô calculus, that will be extensively used in the next sections.

3.1. The voter model and its macroscopic limit. We start from the remark that if we expand the flip rates (2.6) to first order in "the small quantity" $h_\gamma(x)$ we get

$$c^0_\gamma(x,\sigma) := \tfrac{1}{2}\left[1-\sigma(x)h_\gamma(x)\right]. \tag{3.1}$$

While it is hopeless to try a perturbative expansion of the evolution in terms of this approximation, we observe that the right-hand side of (3.1) is still the intensity of a jump Markov process. Indeed, recalling (2.7), we can rewrite

$$c^0_\gamma(x,\sigma) = \frac{1}{2}\sum_y J_\gamma(x,y)\left[1-\sigma(x)\sigma(y)\right] = \sum_y J_\gamma(x,y)\mathbf{1}_{\{\sigma(x)\neq\sigma(y)\}}, \tag{3.2a}$$

so that the flip intensity at x is the J_γ-weighted sum over all the spins $\sigma(y)$ which disagree with (are opposite to) $\sigma(x)$: the process with the intensities (3.2a) is, therefore, a version of the well-known voter model.

In this section we prove the preliminary result that the voter model converges to the free process as $\gamma \to 0$. We then compare (see § 4) the Glauber and the voter model dynamics by computing the corresponding Radon–Nikodym derivative. In §§ 4 and 5 we complete the proof of Theorem 2.6 by showing convergence of the Radon–Nikodym derivatives to a limit, which is identified to the Radon–Nikodym derivative of the interacting process with respect to the free process.

The voter model is the process on $D(\mathbf{R}_+,\mathcal{H}_\gamma)$ whose generator $\mathcal{L}^0_\gamma$ is defined as in (2.5), with c_γ replaced by c^0_γ:

$$\mathcal{L}^0_\gamma f(\sigma) = \sum_{x\in\mathcal{T}_\gamma} c^0_\gamma(x,\sigma)\left[f(\sigma^{(x)}) - f(\sigma)\right]. \tag{3.2b}$$

We call $\mathbf{P}^\gamma_0$ its law when starting from ν^γ_0, the product measure on $\mathcal{H}_\gamma$ with zero averages. Its image in $D(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$ defined via the fluctuation fields (2.9) is then denoted by $\mathcal{P}^\gamma_0$.

THEOREM 3.1. *$\mathcal{P}^\gamma_0$ converges weakly to $\mathcal{P}_0$, which is the unique solution of the martingale problem* FMP *stated below.*

DEFINITION 3.2. (The free martingale problem FMP). The law $\mathcal{P}_0$ on $D(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$, whose canonical coordinates are denoted by m^0_t, solves the martingale problem FMP if (a), (b) and (c) below hold:

(a) $\mathcal{P}_0$ is supported by distributions which vanish at time 0;

(b) $\mathcal{P}_0$ is supported by $C(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$;

(c) for all φ and $t \geqq 0$, all stopping times τ such that there is a constant k for which

$$|(m^0_{t\wedge\tau},\varphi)| + |(m^0_{t\wedge\tau},\varphi'')| < k$$

the conditions (c₁) and (c₂) below hold:

(c₁) $(m^0_{t\wedge\tau},\varphi) - \int_0^{t\wedge\tau} ds\, \frac{D}{2}(m^0_s,\varphi'') =: Z_t(\varphi)$ is a $\mathcal{P}_0$-martingale,

(c₂) $Z_t(\varphi)^2 - 2(t\wedge\tau)(\varphi,\varphi)$ is a $\mathcal{P}_0$-martingale,

alternatively to (c₁) and (c₂), for any $f \in C^\infty_0(\mathbf{R})$, bounded with its derivatives

(c₃) $f\big((m^0_{t\wedge\tau},\varphi)\big) - \int_0^{t\wedge\tau} ds\, f'\big((m^0_s,\varphi)\big)\frac{D}{2}(m^0_s,\varphi'')$
$- \int_0^{t\wedge\tau} ds\, f''\big((m^0_s,\varphi)\big)(\varphi,\varphi)$ is a $\mathcal{P}_0$-martingale.

Our strategy in the proof of Theorem 3.1 is classic. We first show that the family $\mathcal{P}^\gamma_0$ is tight, so that it converges by subsequences. Then we prove that any of its limit points solves the problem FMP. By the Holley and Stroock theory [16] we know that the problem FMP has a unique solution, which is the same either for the formulation (c₁), (c₂), or (c₃). Therefore, all the limit points are the same and the proof will be concluded. We start recalling the following theorem.

THEOREM 3.3. (A criterion for tightness). *The laws $\mathcal{P}^\gamma_0$ on $D([0,T],\ \mathcal{S}'(\mathcal{T}))$, $T>0$, are tight if for any $\varphi \in \mathcal{S}(\mathcal{T})$ there is c such that for all $t \leqq T$*

$$\mathbf{E}^\gamma_0\left(\Gamma^\gamma_{1,t}(\varphi)^2 + \Gamma^\gamma_{2,t}(\varphi)^2 + X^\gamma_t(\varphi)^2\right) \leqq c, \tag{3.3}$$

where

$$\Gamma^\gamma_{1,t}(\varphi) = \gamma^{-2/3}\mathcal{L}^0_\gamma X^\gamma_t(\varphi), \tag{3.4a}$$

$$\Gamma^\gamma_{2,t}(\varphi) = \gamma^{-2/3}\{\mathcal{L}^0_\gamma X^\gamma_t(\varphi)^2 - 2X^\gamma_t(\varphi)\mathcal{L}^0_\gamma X^\gamma_t(\varphi)\}. \tag{3.4b}$$

This theorem is stated in [13] (see also [30] for applications to hydrodynamical limits.) The proof is based on results in [16], [24], [25], [26], [33], where several generalizations are discussed.

We postpone the proof that (3.3) is verified and proceed with the proof of Theorem 3.1. We want to show that any limit law $\mathcal{P}_0$ (by tightness we know of their existence) solves the problem FMP. We will see that there are martingale relations for $\mathcal{P}^\gamma_0$ which are very similar to those in the problem FMP, with an error which vanishes in probability, as $\gamma \to 0$. One would then be tempted to take a limit along a converging subsequence to prove the validity of the final martingale relation. There is, however, a difficulty, due to the fact that this relation involves the fluctuation fields $X_t(\varphi)$, which are not continuous in $D(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$. Weak convergence does not imply, therefore, the convergence of the expectation of functions of these fields. As shown in [16], this difficulty may be circumvented by showing that any limit law is supported by $C(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$. This property is implied, see [16], [33], [34], and Theorem 2.7.8 of [13], by the following statement: for any φ, $T>0$, and $\delta>0$

$$\lim_{\gamma\to 0}\sup_{t\leqq T}\mathbf{P}^\gamma_0\left(|X^\gamma_{t_+}(\varphi) - X^\gamma_{t_-}(\varphi)| > \delta\right) = 0, \tag{3.5}$$

where $t_\pm$ shorthands the limit for $s \to t$, from above and, respectively, from below. By the definition (2.9), we readily see that the difference in (3.5) is bounded by $2\gamma\|\varphi\|_\infty$, hence that (3.5) holds.

At this point, with the above support property of the limit process, we can use a general theorem (see [3]) to improve the convergence.

PROPOSITION 3.4. *Let F be a bounded measurable function on $D(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$, which is continuous on $C(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$. Then if $\mathcal{P}^\gamma_0$ converges weakly to $\mathcal{P}_0$ (along a subsequence) and $\mathcal{P}_0$ is supported by $C(\mathbf{R}_+,\mathcal{S}'(\mathcal{T}))$,*

$$\lim_{\gamma\to 0}\int d\mathcal{P}^\gamma_0\, F = \int d\mathcal{P}_0\, F. \tag{3.6}$$

Proof of Theorem 3.1. *Preliminary computations.* To prove tightness and to establish the martingale relations for a finite γ we need a few computations. Let $f \in C^3(\mathbf{R})$ be bounded with its derivatives, but the considerations below apply as well to $f(x) = x$ and to $f(x) = x^2$. Recalling that $\mathcal{L}^0_\gamma$ is the generator of the voter model, we have

$$M^\gamma_t(\varphi) := f(X^\gamma_t(\varphi)) - \gamma^{-2/3} \int_0^t ds\, \mathcal{L}^0_\gamma f(X^\gamma_s(\varphi)) \tag{3.7}$$

is a $\mathbf{P}^\gamma_0$-martingale. Let $X^\gamma(\varphi)$ be as in (2.9) at a generic and not explicited time and let f stand for $f(X^\gamma(\varphi))$. Then, from (3.2) there exists c_1 independent of γ such that

$$\begin{aligned}\gamma^{-2/3}\Big|\mathcal{L}^0_\gamma f - \sum_{x\in\mathcal{T}_\gamma} c^0_\gamma(x,\sigma)\{-2\gamma\varphi(\gamma^{1+1/3}x)\sigma(x)f' \\ + 2\gamma^2\varphi^2(\gamma^{1+1/3}x)f''\}\Big| \le c_1\gamma.\end{aligned} \tag{3.8}$$

Moreover, there exists c_2 such that

$$\left|\gamma^{-2/3}\sum_{x\in\mathcal{T}_\gamma}\frac{1}{2}[1-\sigma(x)h_\gamma(x)][-2\gamma\varphi(\gamma^{1+1/3}x)\sigma(x)] - \frac{D}{2}X^\gamma(\varphi'')\right| \le c_2\gamma^{1/3}, \tag{3.9}$$

where D is defined in (2.10). To prove (3.9) we observe that

$$\left|\sum_{y\neq x} J_\gamma(x,y)\gamma^2|y-x|^2 - D\right| \le c_2'\gamma, \tag{3.10a}$$

hence

$$\left|\sum_{x\neq y} J_\gamma(x,y)[\varphi(\gamma^{1+1/3}y) - \varphi(\gamma^{1+1/3}x)] - \gamma^{2/3}\frac{D}{2}\varphi''(\gamma^{1+1/3}x)\right| \le c_2\gamma^{4/3}; \tag{3.10b}$$

(3.10b) is obtained by a power expansion to fourth order, observing that the odd terms vanish by symmetry.

The term in (3.8) with f'' may be written as:

$$\begin{aligned}\gamma^{-2/3}\sum_{x\in\mathcal{T}_\gamma} c^0_\gamma(x,\sigma)2\gamma^2\varphi^2(\gamma^{1+1/3}x)f'' \\ = \gamma^{1+1/3}\sum_{x\in\mathcal{T}_\gamma}\varphi(\gamma^{1+1/3}x)^2 f'' + R^\gamma(\varphi),\end{aligned} \tag{3.11a}$$

$$R^\gamma(\varphi) := -\gamma^{1+1/3}\sum_{x\in\mathcal{T}_\gamma}\sigma(x)h_\gamma(x)\varphi(\gamma^{1+1/3}x)^2 f''. \tag{3.11b}$$

Calling $R^\gamma_t(\varphi)$ the expression (3.11b) at time $\gamma^{-2/3}t$, we have

$$\mathbf{E}^\gamma_0\left(|R^\gamma_t(\varphi)|\right) \le c_3 \sup_{x\in\mathcal{T}_\gamma}(\mathbf{E}^\gamma_0 h_\gamma(x,t)^2)^{1/2}; \tag{3.12}$$

here $h_\gamma(x,t)$ is the function $h_\gamma(x)$ defined in (2.7) and computed at $\sigma = \sigma_{\gamma^{-2/3}t}$. To obtain (3.12) we have used the Cauchy–Schwartz inequality. We have

$$\begin{aligned}\mathbf{E}^\gamma_0(h_\gamma(x,t)^2) = \sum_{y\neq z} J_\gamma(x,y)J_\gamma(x,z)\,\mathbf{E}^\gamma_0(\sigma(y,\gamma^{-2/3}t)\,\sigma(z,\gamma^{-2/3}t)) \\ + \sum_y J_\gamma(x,y)^2 \le c_4(\gamma^{2/3}+\gamma),\end{aligned} \tag{3.13}$$

where we have used the bound on the expectation of a product of spins proved in Appendix A.

Tightness. Recalling (3.4a) and letting $f(x) = x$ in (3.8), we have ($\sigma = \sigma_{\gamma^{-2/3}t}$)

$$\Gamma^\gamma_{1,t}(\varphi) = -\gamma^{-2/3}\sum_{x\in\mathcal{T}_\gamma}\tfrac{1}{2}[1-\sigma(x)h_\gamma(x,t)]2\gamma\varphi(\gamma^{1+1/3}x)\sigma(x),$$

hence, by (3.9),

$$|\Gamma^\gamma_{1,t}(\varphi)| \le \frac{D}{2}|X^\gamma_t(\varphi'')| + c_2\gamma^{1/3}. \tag{3.14}$$

Recalling (3.4b), from (3.8) we get

$$\Gamma^\gamma_{2,t}(\varphi) = \gamma^{-2/3}\sum_{x\in\mathcal{T}_\gamma} c^0_\gamma(x,\sigma)\,4\gamma^2\varphi(\gamma^{1+1/3}x)^2$$

is bounded uniformly in γ since c^0_γ is bounded. To prove the tightness criterion it remains to prove that, for arbitrary φ there exists c such that

$$\mathbf{E}^\gamma_0\left(X^\gamma_t(\varphi)^2\right) \le c, \tag{3.15}$$

for all $t \le T$ and all γ. But this is easily obtained proceeding as in (3.13) and using the bound from Appendix A on the expectation of a product of spins.

The martingale relations. We have from (3.7), (3.8), (3.9), and (3.11)

$$\lim_{\gamma\to 0}\mathbf{E}^\gamma_0\left(|\Lambda^\gamma(t) - M^\gamma_t(\varphi)|\right) = 0, \tag{3.16a}$$

where

$$\Lambda^\gamma(t) = f(X^\gamma_t(\varphi)) - \int_0^t ds\left\{\frac{D}{2}X^\gamma_s(\varphi'')f'(X^\gamma_s(\varphi)) - \frac{1}{2}2(\varphi,\varphi)f''(X^\gamma_s(\varphi))\right\}. \tag{3.16b}$$

We remark that we have replaced $\gamma^{4/3}\sum_x \varphi^2(\gamma^{4/3}x)$ by (φ,φ) as their difference vanishes when $\gamma \to 0$.

We then consider a converging subsequence $\mathbf{P}^\gamma_0$. For any $0 \le s < t$ and any bounded function F continuous on $C(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$ and measurable on $D([0,s], \mathcal{S}'(\mathcal{T}))$, we have, along the subsequence, using (3.16), that

$$\lim_{\gamma\to 0}\int d\mathcal{P}^\gamma_0\,[\Lambda(t) - \Lambda(s)]F = 0, \tag{3.17a}$$

where

$$\Lambda(t) := f(X_t(\varphi)) - \int_0^t ds\left\{\frac{D}{2}X_s(\varphi'')f'(X_s(\varphi)) - \frac{1}{2}2(\varphi,\varphi)f''(X_s(\varphi))\right\}. \tag{3.17b}$$

Observe that $\Lambda(t)$, $\Lambda(s)$ and F are all continuous functions on $C(\mathbf{R}_+, \mathcal{S}'(\mathcal{T}))$ and, even though the second term in $\Lambda(\cdot)$ is not bounded, we can nevertheless apply Proposition 3.4 with the help of (3.3), already proved in the paragraph on tightness. Then (3.17a) holds for the limit law $\mathcal{P}_0$. By the arbitrarily of s, t, and F this shows that $\mathcal{P}_0$ solves the problem FMP. Theorem 3.1 is thus proved.

3.2. The free process, stochastic integrals and Itô calculus. The free process m_t^0, $t \geqq 0$, is defined as the solution of the stochastic differential equation

$$dm_t^0 = \frac{D}{2}\frac{\partial m_t^0}{\partial r^2} + dz_t. \tag{3.18a}$$

The precise meaning of (3.18a) is specified in the following definitions.

DEFINITION 3.5 (The free process.) The free process is a process m_t^0 adapted to z_t, the Wiener process of Definition 2.1, supported by $C(\mathbf{R}_+, C(\mathcal{T}))$ and which verifies for all $\varphi \in \mathcal{S}(\mathcal{T})$ and all $t \geqq 0$:

$$(\varphi, m_t^0) = \int_0^t ds \frac{D}{2}(\varphi'', m_s^0) + (\varphi, z_t) \tag{3.18b}$$

Q-almost surely.

Recalling Definition 2.1, we have that (φ, z_t) is a martingale and $(\varphi, z_t)^2 - 2t(\varphi, \varphi)$ is also a martingale. Then the free process solves the problem FMP and, by the uniqueness of the solutions of the problem, its law is $\mathcal{P}^0$ and it coincides with the limiting law for the voter model. All that, of course, is based on the assumption that there is a free process in the sense of Definition 3.5. This is essentially contained in [15], see also [16], [33], [34]. We give below a proof not only for completeness, but also because several points will be used in the sequel. We start from a "canonical realization" of the Wiener process of Definition 2.1.

By Lévy's characterization theorem, from (2.2) it follows that (z_t, φ) is a Brownian motion with diffusion coefficient $2(\varphi, \varphi)$ and that (z_t, φ) and (z_t, ψ) are two independent Brownian motions if φ and ψ are orthogonal in $L^2(\mathcal{T}, dr)$.

DEFINITION 3.6. (The Canonical realization of the Wiener process.) Let Q be the product probability on $C(\mathbf{R}_+, \mathbf{R}^{\mathbf{Z}})$, whose marginals $b_t^{(n)}$, $n \in \mathbf{Z}$, $t \geqq 0$, are independent Brownian motions with variance $2t$. Then the variables

$$(z_t, \varphi) := \sum_n b_t^{(n)} \widehat{\varphi}_n, \quad \text{where} \quad \varphi = \sum_n \widehat{\varphi}_n e_n, \quad e_n(r) = e^{i2\pi n r}, \tag{3.18c}$$

have the same law as the canonical coordinates of the Wiener process.

We next need a basic notion, that of the stochastic integral with respect to the Wiener process.

DEFINITION 3.7 (Stochastic integrals.) Let $(\lambda_t)_{t \leqq T}$, $T > 0$, denote a $L_2(\mathcal{T}, dr)$-valued process adapted to the Wiener process and such that

$$|||\lambda|||^2 := \int dQ \int_0^T dt\, (\lambda_t, \lambda_t) < \infty. \tag{3.19}$$

Denote by K the set of all such processes and by H the space of martingales, which are in $L^2(dQ)$, the stochastic integral is then the following isometric map from K to a subset of H:

$$\lambda \to \int_0^T (dz_s, \lambda_s), \qquad |||\lambda|||^2 = \int dQ \left(\int_0^T (dz_s, \lambda_s)\right)^2, \tag{3.20}$$

which extends the map defined on the processes that, in the canonical realization of Q, depend on finitely many Fourier components. In that case it reduces the usual stochastic integral with respect to a Brownian motion in $\mathbf{R}^n$.

A reference for the notion of stochastic integrals in infinite-dimensional spaces is [9] and [23]. The definition extends the classical one in $\mathbf{R}^n$, for which we refer to [28], a book that will be frequently quoted in the sequel. The following is also a classical result, which can be easily proved by density, after showing its validity for the processes λ_t, which are linear combination of finitely many Fourier coordinates. We use the notation $\langle M, M\rangle_t$ for the quadratic variation of the martingale M, namely the process such that $M_t^2 - \langle M, M\rangle_t$ is again a martingale. We sometimes write

$$Z_t(\lambda) := \int_0^t (dz_s, \lambda_s). \tag{3.21a}$$

If λ is constantly equal to φ we have

$$Z_t(\varphi) = \int_0^t (dz_s, \varphi) = (z_t, \varphi) \tag{3.21b}$$

as defined in (3.18c).

LEMMA 3.8. *The quadratic variation of the stochastic integral $Z_t(\lambda)$ is equal to*

$$\langle Z(\lambda), Z(\lambda)\rangle_t = \int_0^t ds\, (\lambda_s, \lambda_s). \tag{3.22}$$

We will realize the free process as a stochastic integral. We first need a simple property of the stochastic integral of nonrandom functions.

LEMMA 3.9. *Let $\chi_{s,t}(r)$, $s, t \geqq 0$ and $r \in \mathcal{T}$, be a continuous bounded function. Then Q-a.s.*

$$\int_0^t ds \int_0^s (dz_{s'}, \chi_{s,s'}) = \int_0^t \left(dz_{s'}, \int_{s'}^t ds \chi_{s,s'}\right). \tag{3.23}$$

Proof. We split $[0, t]$ into n equal intervals of length $\delta = t/n$. We then call

$$I_k = \int_{(k+1)\delta}^t ds \int_{k\delta}^{(k+1)\delta} (dz_{s'}, \chi_{s,s'}). \tag{3.24}$$

The left-hand side of (3.23) differs from the sum of the I_k by the term

$$R_\delta := \sum_k \int_{k\delta}^{(k+1)\delta} ds \int_{k\delta}^s ds' (dz_{s'}, \chi_{s,s'}).$$

We have

$$\int dQ\, R_\delta^2 = \sum_k \int_{k\delta}^{(k+1)\delta} ds_1 \int_{k\delta}^{(k+1)\delta} ds_2 \int_{k\delta}^{s_1 \wedge s_2} ds' (\chi_{s_1,s'}, \chi_{s_2,s'}) \leqq c\delta^2$$

for a suitable constant c. By using the linearity of the stochastic integrals we rewrite the sum of the I_k as the right-hand side of (3.23) with an error vanishing, as $n \to \infty$. We thus obtain (3.23).

In a similar fashion we can prove the following statement: let $a \in \mathcal{S}(\mathcal{T})$ and let $b_r(r')$ be in $\mathcal{S}(\mathcal{T}^2)$, then

$$\int_0^t \left(dz_s, \int_{\mathcal{T}} dr\, a(r) b_r\right) = \int_{\mathcal{T}} dr\, a(r) \int_0^t \left(dz_s, b_r\right). \tag{3.25a}$$

We next introduce the Green functions for the problem (3.18a) without noise:

$$\begin{aligned} g_{t,r}(r') &= g_t^p(r'-r) = \sum_n g_t(r'-r+n), \\ g_t(r) &= \frac{1}{\sqrt{2\pi D t}} \exp\{-r^2/(2Dt)\}, \end{aligned} \tag{3.25b}$$

and define for $\varepsilon > 0$, $r \in \mathcal{T}$ and $t \geqq 0$:

$$w_t^\varepsilon(r) = \int_0^t (dz_s, g_{t+\varepsilon-s,r}). \tag{3.26}$$

In [15] it is proven that the limit, as $\varepsilon \to 0$, of $w_t^\varepsilon(r)$ exists with probability 1 and it defines a continuous adapted process $w_t(r)$. We will next check that it solves (3.18b). Given $t > 0$ and φ, define

$$\varphi_{t-s}(r) = \int dr' g_{t-s,r} \varphi(r')$$

so that, by (3.25a),

$$(\varphi, w_t) = \int_0^t (dz_s, \varphi_{t-s}). \tag{3.27}$$

Then

$$\begin{aligned} \int_0^t ds \frac{D}{2} (\varphi'', w_s) &= \int_0^t ds \frac{D}{2} \int_0^s (dz_{s'}, \varphi''_{s-s'}) \\ &= \int_0^t ds \int_0^s \left(dz_{s'}, \frac{d}{ds}\varphi_{s-s'}\right) = \int_0^t \left(dz_{s'}, \int_{s'}^t ds \frac{d}{ds} \varphi_{s-s'}\right) \\ &= \int_0^t (dz_{s'}, \varphi_{t-s'} - \varphi) = (\varphi, w_t) - (\varphi, z_t), \end{aligned} \tag{3.28}$$

hence (3.18b) is valid. We have thus completed the proof that the solution $\mathcal{P}^0$ of the problem FMP is the law of the free process m_t^0 from Definition 3.5.

We will use in the sequel the following bound: for any $T > 0$ there exist c and $\delta > 0$ such that for all $x > 0$

$$\mathcal{P}^0\Big(\sup_{r \in \mathcal{T}, t \leqq T} |m_t^0(r)| > x\Big) \leqq c e^{-\delta x^2}. \tag{3.29}$$

This holds because m_t^0 is a Gaussian process; for details see [15] and [16].

We conclude the section with some martingale relations. Recall that the martingale $Z_t(\varphi)$ in (c_1) of Definition 3.2, with no stopping times, is identified in law to the stochastic integral of the Wiener process via (3.21b). By Itô calculus in **R**, it follows (see, for instance, [28]) that the martingale term in (c_3) is in law equal to

$$\int_0^t (dz_s, \varphi) f'((m_s^0, \varphi)). \tag{3.30}$$

We need a generalization of this type formulas. To this end we rewrite the expression in (c_3) as follows. We think of $f((m^0, \varphi))$ as a function h on $L_2(\mathcal{T}, dr)$. We denote by dh its Fréchet derivative, identifying it to an element of L_2. We denote by d^2h the second derivative, which is an operator on L_2 with values in L_2. Then setting $h_s = h(m_s^0)$,

$$h_t - h_0 - \int_0^t ds\, \mathcal{L}_0 h_s = \int_0^t (dz_s, dh_s), \tag{3.31}$$

$$\mathcal{L}_0 h = \frac{D}{2}\left(m^0, \frac{d^2}{dr^2} dh\right) + 2 \operatorname{tr}(d^2 h), \tag{3.32}$$

where, in this case,

$$dh = f'\varphi, \qquad d^2 h = f'' \pi_\varphi, \tag{3.33}$$

π_φ being the orthogonal projection on φ.

The relations (3.31)–(3.32) hold as well for

$$h = f_1((m^0, \varphi_1)) f_2((m^0, \varphi_2)) \tag{3.34}$$

as it can be seen by a density argument starting from $\exp\{\theta_1(m^0, \varphi_1) + \theta_2(m^0, \varphi_2)\}$. By iteration, (3.31) and (3.32) hold for any finite product of functions like in (3.34), as well as for their linear combination. Then, by a density argument, we have (3.31) and (3.32) valid for all functions h which can be approximated in the norm:

$$N(h)^2 = \sup_{s \leqq t} \int d\mathcal{P}^0 \left\{ h_s^2 + \left(\frac{d^2}{dr^2} dh_s, \frac{d^2}{dr^2} dh_s\right) + (dh_s, dh_s) + [\operatorname{tr} d^2 h_s]^2 \right\}. \tag{3.35}$$

In particular, in the next section we will need (3.31) for the following choice of h:

$$h = V^l(s) = -\tfrac{1}{8}(m_s^0, v_s^l), \qquad v_s^l := -\tfrac{1}{3}\psi_l \star (m_s^0)^3, \tag{3.36a}$$

where ψ_ℓ is a symmetric function in $\mathcal{T}^4$ and

$$\psi_l \star (m_s^0)^3(r) := \int_{\mathcal{T}^3} dr_1 dr_2 dr_3 \psi_l(r, r_1, r_2, r_3)\, m_s^0(r_1)\, m_s^0(r_2)\, m_s^0(r_3). \tag{3.36b}$$

Then (3.31) becomes

$$-V^l(t) + \int_0^t ds \left(G^l(s) + H^l(s)\right) = \tfrac{1}{2} Z_t(v^l). \tag{3.37}$$

where

$$Z_t(v^l) := \int_0^t (dz_s, v_s^l) \tag{3.38}$$

and

$$G^l(s) := \frac{1}{3!}\frac{D}{2}\left(m_s^0, \partial^2 \psi_l \star (m_s^0)^3\right), \tag{3.39a}$$

$$H^l(s) := \frac{1}{3!} 3 \left(1, \psi_l(0) \star (m_s^0)^2\right), \tag{3.39b}$$

with

$$\text{(3.40a)}\quad \partial^2\psi_l * (m_s^0)^3(r) := \int_{\mathcal{T}^3} dr_1 dr_2 dr_3 \partial_{r_1}^2 \psi_l(r, r_1, r_2, r_3)\, m_s^0(r_1)\, m_s^0(r_2)\, m_s^0(r_3),$$

$$\text{(3.40b)}\quad \psi_l(0) * (m_s^0)^2(r) := \int_{\mathcal{T}^2} dr_2 dr_3 \psi_l(r, r, r_2, r_3) m_s^0(r_2) m_s^0(r_3).$$

4. Convergence of the interacting process. We fix $t > 0$ and consider the Glauber process (σ_s) for $0 \leqq s \leqq \gamma^{-2/3}t$. We denote by $\mathbf{P}^\gamma$ its law and by $\mathbf{P}_0^\gamma$ the law of the voter model restricted to $D([0,\gamma^{-2/3}t],\mathcal{H}_\gamma)$. $\mathcal{P}^\gamma$ and, respectively, $\mathcal{P}_0^\gamma$ are the images of $\mathbf{P}^\gamma$ and of $\mathbf{P}_0^\gamma$ on $D([0,t],\mathcal{S}'(\mathcal{T}))$, as introduced in Definition 2.5.

The Radon–Nikodym derivative of $\mathbf{P}^\gamma$ with respect to $\mathbf{P}_0^\gamma$ is (see, for instance, [4])

$$\text{(4.1)}\qquad \frac{d\mathbf{P}^\gamma}{d\mathbf{P}_0^\gamma} = \exp\{\Gamma_t^\gamma\},$$

where

$$\text{(4.2)}\quad \Gamma_t^\gamma := \sum_{x\in\mathcal{T}_\gamma}\left\{\int_0^{\gamma^{-2/3}t} N_x(ds)\log\frac{c_\gamma(x,\sigma_s)}{c_\gamma^0(x,\sigma_s)} - \int_0^{\gamma^{-2/3}t}[c_\gamma(x,\sigma_s) - c_\gamma^0(x,\sigma_s)]\,ds\right\};$$

$N_x(ds)$ is the counting measure of the spin flips at x. As usual, the integrand has to be evaluated at s^- if the flip occurs at s.

As we shall see in § 5 the Radon–Nikodym derivative of the interacting process of Theorem 2.2 with respect to the free process of Definition 3.5 exists and is given by an exponential martingale, i.e., it has the form $\exp\{M_t - \frac{1}{2}\langle M,M\rangle_t\}$, where M_t is a martingale. The first step in the proof of Theorem 2.6 is to recognize this structure in (4.1).

4.1. Taylor expansion of the Radon–Nikodym derivative. In this section we denote by $h_\gamma(x,s)$ the same function, as in (2.7), evaluated at $\sigma = \sigma_s$.

THEOREM 4.1. *For any $\delta > 0$,*

$$\text{(4.3)}\qquad \lim_{\gamma\to 0}\mathbf{P}_0^\gamma\left(\left|\Gamma_t^\gamma - \left(M_t^\gamma - \tfrac{1}{2}\langle M^\gamma, M^\gamma\rangle_t\right)\right| > \delta\right) = 0,$$

where M_t^γ is the following $\mathbf{P}_0^\gamma$-martingale

$$\text{(4.4a)}\qquad M_t^\gamma := \frac{1}{3}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\Lambda_x(ds)\,\sigma_s(x)\,h_\gamma^3(x,s),$$

$$\text{(4.4b)}\qquad \Lambda_x(ds) := N_x(ds) - c_\gamma^0(x,\sigma_s)\,ds,$$

and $\langle M^\gamma, M^\gamma\rangle_t$ is its quadratic variation given by

$$\text{(4.5)}\qquad \langle M^\gamma, M^\gamma\rangle_t = \tfrac{1}{9}\textstyle\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} c_\gamma^0(x,\sigma_s)\,ds\, h_\gamma^6(x,s).$$

The proof of this theorem is based on the Taylor expansion in (4.2) up to order 6. We first perform the Taylor expansion in the second integral in (4.2).

LEMMA 4.2. *For any $\delta > 0$,*

$$\lim_{\gamma\to 0}\mathbf{P}_0^\gamma\Bigg(\bigg|\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} ds\Big[\big(c_\gamma(x,\sigma_s) - c_\gamma^0(x,\sigma_s)\big)$$

$$\text{(4.6)}\qquad - \tfrac{1}{2}\left(\tfrac{1}{3}\sigma(x)h_\gamma^3(x,s) - \tfrac{2}{15}\sigma_s(x)h_\gamma^5(x,s)\right)\Big]\bigg| > \delta\Bigg) = 0.$$

Proof. After some algebraic manipulations we can rewrite the intensity of the Glauber process $c_\gamma(x,\sigma)$ in (2.6) as

$$c_\gamma(x,\sigma) = \tfrac{1}{2}\left(1 - \sigma(x)\tanh h_\gamma(x)\right).$$

Since the derivatives of $\tanh(\cdot)$ are uniformly bounded we have

$$\text{(4.7)}\qquad \left|\left(c(x,\sigma) - c_\gamma^0(x,\sigma)\right) - \tfrac{1}{2}\left(\tfrac{1}{3}\sigma(x)h_\gamma^3(x) - \tfrac{2}{15}\sigma(x)h_\gamma^5(x)\right)\right| \leqq c_1|h_\gamma(x)|^7.$$

Using the Chebyshev inequality and then the Hölder inequality, we obtain

$$\mathbf{P}_0^\gamma\Bigg(\bigg|\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} ds\Big[\big(c(x,\sigma_s) - c_\gamma^0(x,\sigma_s)\big) - \tfrac{1}{2}\left(\tfrac{1}{3}\sigma_s(x)h_\gamma^3(x,s) - \tfrac{2}{15}\sigma_s(x)h_\gamma^5(x,s)\right)\Big]\bigg| > \delta\Bigg)$$

$$\leqq c_1\delta^{-1}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\mathbf{E}_0^\gamma|h_\gamma(x,s)|^7 ds$$

$$\text{(4.8)}\qquad \leqq c_2\delta^{-1}\gamma^{-4/3}\gamma^{-2/3}\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}}\left(\mathbf{E}_0^\gamma h_\gamma^8(x,s)\right)^{7/8}.$$

To estimate the supremum in (4.8) we use the following bound proved in Appendix A: for any $t > 0$ and any $n \geqq 1$ there exists C such that

$$\text{(4.9)}\qquad \left|\mathbf{E}_0^\gamma\left(\prod_{i=1}^{2n}\sigma_s(x_i)\right)\right| \leqq C\gamma^{n2/3}$$

for all $s \leqq \gamma^{-2/3}t$ and all $2n$-tuple of distinct sites x_i in $\mathcal{T}_\gamma$. We then have

$$\text{(4.10)}\qquad \mathbf{E}_0^\gamma\left(h_\gamma^8(x,s)\right) = \sum_{x_1,\dots,x_8}\prod_{i=1}^{8}J_\gamma(x,x_i)\cdot\mathbf{E}_0^\gamma\left(\prod_{i=1}^{8}\sigma_s(x_i)\right)$$

and we can use (4.9) when $x_i \neq x_j$. If $x_i = x_j$, as in (3.13), we get an extra factor γ from $J_\gamma(x_i,x_j)^2$. The leading term in (4.10) is then bounded by $c_3\gamma^{8/3}$. Thus (4.8) can be bounded by $c_4\delta^{-1}\gamma^{1/3}$.

In order to justify the Taylor expansion also in the counting integral in (4.2) we need two auxiliary lemmas. First we estimate the number of flips of the voter model.

LEMMA 4.3. *Let $\nu(\tau)$ be the total number of spin flips up to time τ. Then for any $b > 2$,*

$$\lim_{\gamma\to 0} \mathbf{P}_0^\gamma\big(\nu(\gamma^{-2/3}t) > \gamma^{-b}\big) = 0. \tag{4.11}$$

Proof. Let $\nu_x(\tau)$ the number of flips of the spin at x up to time τ. We then have $\nu(\tau) = \sum_x \nu_x(\tau)$. Using the Chebyshev inequality and recalling that by (3.1) the flip intensity is bounded by $\frac{1}{2}$, we get

$$\mathbf{P}_0^\gamma\left(\nu(\gamma^{-2/3}t) > \gamma^{-b}\right) \leqq \gamma^b \sum_{x\in\mathcal{T}_\gamma} \mathbf{E}_0^\gamma\left(\nu_x(\gamma^{-2/3}t)\right) \leqq \tfrac{1}{2}\gamma^b\gamma^{-4/3}\gamma^{-2/3}t.$$

We now prove a uniform bound on $h_\gamma(x,s)$.

LEMMA 4.4. *For any $\alpha < \frac{1}{3}$,*

$$\lim_{\gamma\to 0} \mathbf{P}_0^\gamma\Bigg(\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}} |h_\gamma(x,s)| > \gamma^\alpha\Bigg) = 0. \tag{4.12}$$

Proof. The supremum over the lattice sites can be bounded considering the inclusion

$$\Bigg\{\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}} |h_\gamma(x,s)| > \gamma^\alpha\Bigg\} \subset \bigcup_{x\in\mathcal{T}_\gamma}\Bigg\{\sup_{s\leqq\gamma^{-2/3}t} |h_\gamma(x,s)| > \gamma^\alpha\Bigg\},$$

hence

$$\mathbf{P}_0^\gamma\Bigg(\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}} |h_\gamma(x,s)| > \gamma^\alpha\Bigg) \leqq \sum_{x\in\mathcal{T}_\gamma}\mathbf{P}_0^\gamma\Bigg(\sup_{s\leqq\gamma^{-2/3}t} |h_\gamma(x,s)| > \gamma^\alpha\Bigg). \tag{4.13}$$

We now estimate the supremum over time. We first localize the problem in a "small" interval. Let $a > \frac{10}{3}$ and introduce $\tau_{i+1} = \tau_i + \gamma^a$, $\tau_0 = 0$; we can bound (4.13) by

$$\sum_{x\in\mathcal{T}_\gamma}\sum_{i=0}^{[\gamma^{-a}\gamma^{-2/3}t]} \mathbf{P}_0^\gamma\Bigg(\sup_{s\in[\tau_i,\tau_{i+1}]} |h_\gamma(x,s)| > \gamma^\alpha\Bigg). \tag{4.14}$$

The probability to have at least two spin flips in the interval $[\tau_i,\tau_{i+1}]$ is bounded by $c_1\gamma^{-4/3}\gamma^{2a}$. On the other hand, for $s\in[\tau_i,\tau_{i+1}]$, if the number of flips is one or zero the value of $h_\gamma(x,s)$ differs from $h_\gamma(x,\tau_i)$ at most by $c_2\gamma$. Since $\alpha < \frac{1}{3}$ for γ small enough $\gamma^\alpha - \gamma > \frac{1}{2}\gamma^\alpha$ and we can bound (4.14) by

$$\sum_{x\in\mathcal{T}_\gamma}\sum_{i=0}^{[\gamma^{-a}\gamma^{-2/3}t]} \left(\mathbf{P}_0^\gamma\left(c_3|h_\gamma(x,\tau_i)| > \gamma^\alpha\right) + c_1\gamma^{-4/3}\gamma^{2a}\right). \tag{4.15}$$

Using the Chebyshev inequality of order $2n$ and repeating the same argument of equation (4.10) we can finally bound (4.15) with

$$c_4\gamma^{-4/3}\gamma^{-a}\gamma^{-2/3}\left(\gamma^{-2\alpha n}\gamma^{n2/3} + \gamma^{-4/3}\gamma^{2a}\right),$$

which converges to 0 for $\alpha < \frac{1}{3}$ if n is taken large enough.

We are now ready to prove the Taylor expansion also for the first integral in (4.2).

LEMMA 4.5. *For any $\delta > 0$,*

$$\lim_{\gamma\to 0}\mathbf{P}_0^\gamma\Bigg(\Bigg|\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} N_x(ds) \times\Bigg[\log\frac{c_\gamma(x,\sigma)}{c_\gamma^0(x,\sigma)} - \Big(\tfrac{1}{3}\sigma(x)h_\gamma^3(x,s) + \tfrac{1}{3}h_\gamma^4(x,s) + \Big(\tfrac{1}{3}-\tfrac{2}{15}\Big)\sigma(x)h_\gamma^5(x,s) + \Big(\tfrac{1}{3}-\tfrac{2}{15}-\tfrac{1}{18}\Big)h_\gamma^6(x,s)\Big)\Bigg]\Bigg| > \delta\Bigg) = 0. \tag{4.16}$$

Proof. As in Lemma 4.2 we expand $[\log c_\gamma(x,\sigma_s) - \log c_\gamma^0(x,\sigma_s)]$ up to the sixth order. Since $h_\gamma(x,s) \leqq 1$, the remainder $R_7(x,\sigma_s)$ is bounded:

$$|R_7(x,\sigma_s)| \leqq c_1\Bigg[1 - \sup_{s\leqq\gamma^{-2/3}t}|h_\gamma(x,s)|\Bigg]^{-7}|h_\gamma(x,s)|^7. \tag{4.17}$$

Let $b > 2$; Lemmas 4.3 and 4.4 being valid, it suffices then to show that

$$\lim_{\gamma\to 0}\mathbf{P}_0^\gamma\Bigg(\Bigg|\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} N_x(ds)\,R_7(x,\sigma_s)\Bigg| > \delta, \sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}}|h_\gamma(x,\gamma^{-2/3}s)| < \frac{1}{2}, \nu(\gamma^{-2/3}t) < \gamma^{-b}\Bigg) = 0. \tag{4.18}$$

From the definition of the measure $N_x(ds)$, denoting by s_i the flips times for the spin at x and using (4.17), we have

$$\begin{aligned}\Bigg|\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t} N_x(ds)\,R_7(x,\sigma_s)\Bigg| &= \Bigg|\sum_{x\in\mathcal{T}_\gamma}\sum_{i=1}^{\nu_x(\gamma^{-2/3}t)} R_7(x,\sigma_{s_i^-})\Bigg|\\ &\leqq c_1\nu(\gamma^{-2/3}t)\Bigg[1 - \sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}}|h_\gamma(x,s)|\Bigg]^{-7}\sup_{x\in\mathcal{T}_\gamma, s\leqq\gamma^{-2/3}t}|h_\gamma(x,s)|^7\end{aligned} \tag{4.19}$$

recalling that we denote by $\nu_x(\tau)$ the total number of flips for the spin at x up to time τ.

We thus can bound the probability in (4.18) by

$$\mathbf{P}_0^\gamma\Bigg(c_2\gamma^{-b}\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}}|h_\gamma(x,s)|^7 > \delta\Bigg) = \mathbf{P}_0^\gamma\Bigg(c_3\sup_{\substack{x\in\mathcal{T}_\gamma\\ s\leqq\gamma^{-2/3}t}}|h_\gamma(x,s)| > \delta^{1/7}\gamma^{b/7}\Bigg),$$

which converges to 0 by Lemma 4.4 if b is chosen such that $2 < b < \frac{7}{3}$.

Proof of Theorem 4.1. Lemmas 4.2 and 4.4 allow us to perform the Taylor expansion of Γ_t^γ in (4.2). After a suitable permutation of the terms we can write

$$\Gamma_t^\gamma = M_t^\gamma - \tfrac{1}{2}\langle M^\gamma, M^\gamma\rangle_t + N_t^{1,\gamma} + N_t^{2,\gamma} + N_t^{3,\gamma} + R_t(\gamma) + \eta(\gamma),$$

where $\eta(\gamma)$ is the remainder in the Taylor expansion, $N_t^{i,\gamma}$ are the following martingales:

$$N_t^{1,\gamma} := \sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\Lambda_x(ds)\,\tfrac{1}{3}\,h_\gamma^4(x,s),$$

$$N_t^{2,\gamma} := \sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\Lambda_x(ds)\left(\frac{1}{3}-\frac{2}{15}\right)\sigma_s(x)\,h_\gamma^5(x,s),$$

$$N_t^{3,\gamma} := \sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\Lambda_x(ds)\left(\frac{1}{3}-\frac{2}{15}-\frac{1}{18}\right)h_\gamma^6(x,s),$$

and

$$R_t(\gamma) := -\frac{1}{2}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}ds\left(\frac{1}{3}-\frac{2}{15}\right)\sigma_s(x)\,h_\gamma^7(x,s).$$

We can now repeat the argument of Lemma 4.2 to check that $R_t(\gamma)$ converges to 0 in probability. In order to conclude that the same holds for the martingales $N_t^{i,\gamma}$ it is enough to compute their second moments. For instance, consider $N_t^{1,\gamma}$. We have

$$\mathbf{E}_0^\gamma\left(N_t^{1,\gamma}\right)^2 = \frac{1}{9}\mathbf{E}_0^\gamma\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}c_\gamma^0(x,\sigma_s)\,ds\,h_\gamma^8(x,s),$$

which converges to 0 as in the proof of Lemma 4.2.

4.2. Modification. It would be tempting, after Theorem 4.1, to conclude the proof of Theorem 2.6 by saying that M_t^γ converges to a martingale M_t given by the stochastic integral

$$-\frac{1}{3!}\int\left(dz_s,\,(m_s^0)^3\right).$$

A posteriori this is indeed the correct result. Unfortunately there does not seem to exist good criteria for extending the convergence of a process to the convergence of its stochastic integrals, see [31] for a discussion on this point. As a matter of fact the expression $M_t^\gamma - \frac{1}{2}\langle M^\gamma, M^\gamma\rangle_t$ is not a continuous function on $D([0,T],\mathcal{S}'(T))$ or even on $C([0,T],C(T))$, so that we cannot use, at least directly, the convergence of $\mathcal{P}_0^\gamma$ to the free field, also in the improved version of Proposition 3.4.

Our strategy is the following. We "replace" the martingale M_t^γ by a smoother expression $M_t^{\gamma,\ell}$, something analogous to the two block estimates in the analysis of the hydrodynamical limits, see [14]. The stochastic integral $M_t^{\gamma,\ell}$ enters in a martingale relation, which involves regular terms, so that we can use it to express $M_t^{\gamma,l}$ as a linear combination of regular functions; for these functions we are then allowed to take the macroscopic limit, as $\gamma\to 0$. Repeating the previous procedure in the reverse order, the limit will reconstruct a stochastic integral. As it will become clear in the sequel, this strategy is successful because the limit equation is of gradient type.

DEFINITION 4.6. Let $\varphi\in C_0^\infty(T^3)$, $\varphi\geqq 0$, such that

$$\int_{T^3}dr_1dr_2dr_3\varphi(r_1,r_2,r_3)=1.$$

Define the following functions:

$$\psi_\ell(r,r_1,r_2,r_3)$$

$$:= \frac{1}{4!}l^{-3}\left(\varphi(l^{-1}(r-r_1),l^{-1}(r-r_2),l^{-1}(r-r_3))+\text{perm.}\right), \tag{4.20a}$$

$$\psi_{\ell,\gamma}(x,x_1,x_2,x_3) := (\gamma^{4/3})^3\psi_\ell(\gamma^{4/3}x,\gamma^{4/3}x_1,\gamma^{4/3}x_2,\gamma^{4/3}x_3). \tag{4.20b}$$

By construction ψ_ℓ and $\psi_{\ell,\gamma}$ are symmetric function and

$$\int_{T^3}dr_1dr_2dr_3\psi_\ell(r,r_1,r_2,r_3)=1. \tag{4.20c}$$

We use the following notation, which is the discrete approximation of (3.36b):

$$\psi_{\ell,\gamma}\circ\sigma^3(x) := \sum_{x_1,x_2,x_3}^{*}\psi_{\ell,\gamma}(x,x_1,x_2,x_3)\sigma(x_1)\,\sigma(x_2)\,\sigma(x_3), \tag{4.21}$$

where the star means that the sum is only for $x_i\neq x_j$, $x_i\neq x$.

Introducing this modification in the martingale M_t^γ (4.4), we obtain

$$M_t^{\gamma,l} := \frac{1}{3}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}\Lambda_x(ds)\,\sigma_s(x)\,\psi_{\ell,\gamma}\circ\sigma_s^3(x) \tag{4.22}$$

and

$$\langle M^{\gamma,\ell},M^{\gamma,\ell}\rangle_t = \frac{1}{9}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}c_\gamma^0(x,\sigma_s)\,ds\,\left(\psi_{\ell,\gamma}\circ\sigma_s^3(x)\right)^2. \tag{4.23}$$

We will now show that, when l is small, M_t^γ can be replaced by $M_t^{\gamma,l}$ with a negligible error.

THEOREM 4.7. *Let M_t^γ and $M_t^{\gamma,l}$ as in (4.4) and (4.22); then*

$$\lim_{l\to 0}\lim_{\gamma\to 0}\mathbf{E}_0^\gamma\left(M_t^{\gamma,l}-M_t^\gamma\right)^2 = 0 \tag{4.24}$$

and

$$\lim_{l\to 0}\lim_{\gamma\to 0}\mathbf{E}_0^\gamma\left|\langle M^{\gamma,\ell},M^{\gamma,\ell}\rangle_t-\langle M^\gamma,M^\gamma\rangle_t\right| = 0. \tag{4.25}$$

Proof. We make the following decomposition:

$$\mathbf{E}_0^\gamma(M_t^\gamma-M_t^{\gamma,l})^2 = \mathbf{E}_0^\gamma\left[\langle M^\gamma,M^\gamma\rangle_t - 2\langle M^\gamma,M^{\gamma,l}\rangle_t + \langle M^{\gamma,\ell},M^{\gamma,\ell}\rangle_t\right] \tag{4.26}$$

and prove that all the term converge to the same constant. Consider the first of them, which is given by (4.5). For the same argument of Lemma 4.1 the leading term is obtained taking $c_\gamma^0=\frac{1}{2}$. In fact we have

$$\left|\mathbf{E}_0^\gamma\langle M^\gamma,M^\gamma\rangle_t - \frac{1}{18}\sum_{x\in\mathcal{T}_\gamma}\int_0^{\gamma^{-2/3}t}ds\sum_{x_1,\dots,x_6}\prod_{i=1}^6 J_\gamma(x,x_i)\right.$$

$$\left.\times\,\mathbf{E}_0^\gamma\left(\prod_{i=1}^6\sigma_s(x_i)\right)\right| \leqq c_1\gamma^{1/3}. \tag{4.27}$$

The key point in the proof of the theorem is that the correlation function of the voter model in (4.27) is smooth on the scale $\gamma^{-4/3}$, i.e., it does not change much if the spins $\sigma(x_i)$ are translated by the "small" macroscopic quantity $l\gamma^{-4/3}$. In the limit $\gamma \to 0$ and then $l \to 0$ this will prove that all the quadratic variations on the right-hand side of (4.26) converge to the same value. This is the real point where the "two block estimates" problem previously mentioned enters in our analysis; as is well known this is related to the achievement of the "local equilibrium." We refer to Appendix A for the technical estimates and continue the proof of the theorem.

Using the results in Corollary A.2, it can be checked that the only term in (4.27), which does not vanish, when γ goes to zero, is given by the sum over $x_i \neq x_j$, $i,j \in (1,\dots,6)$. As in (4.21) we indicate this sum with a star. Let now

$$B^\gamma := \gamma^{4/3} \sum_{x\in\mathcal{T}_\gamma} \sum_{x_1,\dots,x_6}^{*} \prod_{i=1}^{6} J_\gamma(x,x_i) \tag{4.28}$$

from Theorem A.1 we have the following bound from below and above: there exist $\xi_0, C > 0$ such that

$$\begin{aligned} &\sum_{x\in\mathcal{T}_\gamma} \int_0^{\gamma^{-2/3}t} ds \sum_{x_1,\dots,x_6}^{*} \prod_{i=1}^{6} J_\gamma(x,x_i)\cdot \mathbf{E}_0^\gamma\left(\prod_{i=1}^{6}\sigma_s(x_i)\right) \\ &\qquad \geq B^\gamma \int_0^t ds\, a_3(\gamma^{1/3}, s) - C\gamma^{\xi_0}, \\ &\sum_{x\in\mathcal{T}_\gamma} \int_0^{\gamma^{-2/3}t} ds \sum_{x_1,\dots,x_6}^{*} \prod_{i=1}^{6} J_\gamma(x,x_i)\cdot \mathbf{E}_0^\gamma\left(\prod_{i=1}^{6}\sigma_s(x_i)\right) \\ &\qquad \leq \int_0^t ds\, a_3(0,s) + C\gamma^{\xi_0}, \end{aligned} \tag{4.29a}$$

where $a_3(r,s)$ is the continuous function given by

$$a_3(r,s) = 15\left(\int_0^s du\, g_u^P(r)\right)^3 \tag{4.29b}$$

($g_u^P(r)$ is the heat kernel on the torus $\mathcal{T}$ defined in (3.25b)).

Since $\lim_{\gamma\to 0} B^\gamma = 1$ and $a(r,s)$ is uniformly integrable in s, from (4.29a) we thus have

$$\lim_{\gamma\to 0} \mathbf{E}_0^\gamma \langle M^\gamma, M^\gamma\rangle_t = \frac{1}{18}\int_0^t ds\, a(0,s). \tag{4.30}$$

By the same arguments

$$\begin{aligned} \mathbf{E}_0^\gamma \langle M^\gamma, M^{\gamma,l}\rangle_t = \frac{1}{2} \sum_{x\in\mathcal{T}_\gamma} \sum_{x_1,\dots,x_6}^{*} \int_0^{\gamma^{-2/3}t} ds\, \psi_{\ell,\gamma}(x,x_4,x_5,x_6) \\ \times \mathbf{E}_0^\gamma\left(\prod_{i=1}^{3} J_\gamma(x,x_i) \prod_{i=1}^{6}\sigma_s(x_i)\right) + \eta(\gamma), \end{aligned}$$

where $\eta(\gamma) \leq c_2\gamma^{1/3}$. Now let

$$B^{\gamma,l} := \gamma^{4/3} \sum_{x\in\mathcal{T}_\gamma} \sum_{x_1,\dots,x_6}^{*} \psi_{\ell,\gamma}(x,x_4,x_5,x_6) \prod_{i=1}^{3} J_\gamma(x,x_i), \tag{4.31}$$

from Theorem A.1 we have the following bounds

$$\begin{aligned} &\sum_{x\in\mathcal{T}_\gamma} \sum_{x_1,\dots,x_6}^{*} \int_0^{\gamma^{-2/3}t} ds\, \psi_{l,\gamma}(x,x_4,x_5,x_6) \\ &\qquad \times \mathbf{E}_0^\gamma \prod_{i=1}^{3} J_\gamma(x,x_i)\sigma_s(x_i) \geq B^{\gamma,l}\int_0^t a_3(l,s)\,ds - C\gamma^{\xi_0}, \\ &\sum_{x\in\mathcal{T}_\gamma} \sum_{x_1,\dots,x_6}^{*} \int_0^{\gamma^{-2/3}t} ds\, \psi_{\ell,\gamma}(x,x_4,x_5,x_6) \\ &\qquad \times \mathbf{E}_0^\gamma \prod_{i=1}^{3} J_\gamma(x,x_i)\sigma_s(x_i) \leq \int_0^t a_3(0,s)\,ds + C\gamma^{\xi_0}, \end{aligned}$$

where $a_3(r,s)$ is always given by (4.29b). Recalling (4.20c) we have

$$\lim_{l\to 0}\lim_{\gamma\to 0} B^{\gamma,l} = 1.$$

Taking first the limit $\gamma \to 0$, then $l \to 0$, we obtain that $\mathbf{E}_0^\gamma\langle M^\gamma, M^{\gamma,l}\rangle_t$ converges to the same constant as in (4.30). The other term in (4.26), i.e., $\mathbf{E}_0^\gamma\langle M^{\gamma,\ell}, M^{\gamma,\ell}\rangle_t$, can be handled in the same way; this proves (4.24). Equation (4.25) is then a straightforward consequence of (4.24).

4.3. Approximation of $M_t^{\gamma,l}$ by continuous functions. We will show that the martingale $M_t^{\gamma,l}$ in (4.22) may be approximated by a function in the class of those considered in Proposition 3.4, for which we can take the limit $\gamma \to 0$.

Analogously to (4.21), we introduce the following notation, which is the discrete approximation to (3.40):

$$\partial^2\psi_{\ell,\gamma}\circ\sigma^3(x) := \gamma^{-2}\sum_{x_1,x_2,x_3}^{*} \partial_{x_1}^2\psi_{\ell,\gamma}(x,x_1,x_2,x_3)\,\sigma(x_1)\,\sigma(x_2)\,\sigma(x_3), \tag{4.31a}$$

$$\psi_{\ell,\gamma}(0)\circ\sigma^2(x) := \sum_{x_2,x_3}^{*} \psi_{\ell,\gamma}(x,x,x_2,x_3)\,\sigma(x_2)\,\sigma(x_3), \tag{4.31b}$$

where as usual the star means that the sum is only for $x_i \neq x_j$, $x_i \neq x$.

Finally we introduce the following analogues of (3.38) and (3.39):

$$V^{\gamma,l}(s) := \frac{1}{4!}\sum_{x\in\mathcal{T}_\gamma}\sigma_s(x)\,\psi_{\ell,\gamma}\circ\sigma_s^3(x), \tag{4.32a}$$

$$G^{\gamma,l}(s) := \frac{1}{3!}\frac{D}{2}\sum_{x\in\mathcal{T}_\gamma}\sigma_s(x)\,\partial^2\psi_{\ell,\gamma}\circ\sigma_s^3(x), \tag{4.32b}$$

$$H^{\gamma,l}(s) := \frac{1}{3!}3\sum_{x\in\mathcal{T}_\gamma}\sigma_s(x)\,\psi_{\ell,\gamma}(0)\circ\sigma_s^2(x). \tag{4.32c}$$

We next show that the martingale $M_t^{\gamma,l}$ (4.22) may be rewritten in terms of the functions (4.32) plus a small error.

THEOREM 4.8. *With the above notation, for all $l > 0$, the following identity holds $d\mathbf{P}_0^\gamma$-a.s.:*

$$M_t^{\gamma,l} = -(V^{\gamma,l}(\gamma^{-2/3}t) - V^{\gamma,l}(0)) + \int_0^{\gamma^{-2/3}t} (G^{\gamma,l}(s) + H^{\gamma,l}(s))ds + \eta(\gamma), \tag{4.33}$$

where $\eta(\gamma) = \eta(\gamma, l)$ converges to 0 in probability, as $\gamma \to 0$.

Proof. Given any f on $\{-1,1\}^{\mathcal{T}_\gamma}$ we define

$$(\delta_x f)(\sigma) := f(\sigma^{(x)}) - f(\sigma); \tag{4.34}$$

we then have

$$f(\sigma_t) - f(\sigma_0) - \int_0^{\gamma^{-2/3}t} ds\, \mathcal{L}_0^\gamma f(\sigma_s) = \sum_{x\in\mathcal{T}_\gamma} \int_0^{\gamma^{-2/3}t} \Lambda_x(ds)\, \delta_x f(\sigma_s), \tag{4.35}$$

where $\mathcal{L}_0^\gamma$ is defined in (3.2b).

The key observation in the proof of the theorem is that, due to the symmetry of $\psi_{\ell,\gamma}$, we have

$$\delta_x V^{\gamma,l}(s) = -\tfrac{1}{3}\sigma_s(x)\, \psi_{\ell,\gamma} \circ \sigma_s^3(x), \tag{4.36}$$

using (4.35) this allows us to identify the martingale $M_t^{\gamma,l}$ in terms of $V^{\gamma,l}$ and $\mathcal{L}_0^\gamma V^{\gamma,l}$. It remains to compute $\mathcal{L}_0^\gamma V^{\gamma,l}$; we have

$$\begin{aligned} \mathcal{L}_0^\gamma V^{\gamma,l}(s) &= \frac{1}{3}\sum_{x\in\mathcal{T}_\gamma} \frac{1}{2}\Big(\sigma_s(x) - \sum_y J_\gamma(x,y)\sigma_s(y)\Big) \\ &\quad\times \sum_{x_1,x_2,x_3}^{*} \psi_{\ell,\gamma}(x,x_1,x_2,x_3)\,\sigma_s(x_1)\,\sigma_s(x_2)\,\sigma_s(x_3), \end{aligned} \tag{4.37}$$

whereas $x \neq x_i$ by definition, it is possible that $x_i = y$; these terms have to be handled differently.

We consider first the case when $x_i \neq y$, $i = 1,2,3$; denoting this constraint by another star in the sum, we have

$$\begin{aligned} &\frac{1}{3!}\Big[\sum_{x,x_1,x_2,x_3}^{*} \psi_{\ell,\gamma}(x,x_1,x_2,x_3)\sigma_s(x)\sigma_s(x_1)\sigma_s(x_2)\sigma_s(x_3) \\ &\quad - \sum_{x,y,x_1,x_2,x_3}^{**} J_\gamma(x,y)\psi_{\ell,\gamma}(x,x_1,x_2,x_3)\,\sigma_s(y)\sigma_s(x_1)\sigma_s(x_2)\sigma_s(x_3)\Big], \end{aligned} \tag{4.38}$$

permutating x, y in the second sum, expanding $\psi_{\ell,\gamma}$ up to the second order with respect to the first argument and noticing that the first order gives no contribute, by symmetry we can write (4.38) as

$$-\frac{1}{3!}\sum_{x,x_1,x_2,x_3}^{*} \frac{D_\gamma}{2}\gamma^{-2}\partial_x^2\psi_{\ell,\gamma}(x,x_1,x_2,x_3)\,\sigma_s(x)\sigma_s(x_1)\sigma_s(x_2)\sigma_s(x_3) + \eta_\gamma^1(s), \tag{4.39}$$

where

$$D_\gamma := \sum_{y\in\mathcal{T}_\gamma} J_\gamma(0,y)\gamma^2 y^2. \tag{4.40}$$

The remainder $\eta_\gamma^1(s)$ in (4.39) contains the error in the Taylor expansion and the terms added to reconstruct the sum over x, x_1, x_2, x_3 without the constraint $x_i \neq y$, i.e.,

$$\begin{aligned} \eta_\gamma^1(s) := -\frac{1}{3!}\sum_{x,x_1,x_2,x_3}^{*} &\sigma_s(x)\sigma_s(x_1)\sigma_s(x_2)\sigma_s(x_3)\Big(-3J_\gamma(x_1,x)\psi_{\ell,\gamma}(x,x_1,x_2,x_3) \\ &- 3(x_1 - x)J_\gamma(x_1,x)\partial_y\psi_{\ell,\gamma}(y,x_1,x_2,x_3)\,|_{y=x} \\ &- \frac{3}{2}(x_1-x)^2 J_\gamma(x_1,x)\partial_y^2\psi_{\ell,\gamma}(y,x_1,x_2,x_3)\,|_{y=x} \\ &+ \frac{1}{3!}\sum_y^{*} J_\gamma(x,y)(y-x)^3\partial_y^3\,\psi_{\ell,\gamma}(y,x_1,x_2,x_3)\big|_{y=\bar{y}}\Big). \end{aligned} \tag{4.41}$$

We now consider the case when in (4.37) exists i such that $x_i = y$. We remark that, since $x_i \neq x_j$, this can occur just for one i. By symmetry this term is given by

$$-\frac{1}{3!}3\sum_{x,y,x_2,x_3}^{*} \psi_{\ell,\gamma}(x,y,x_2,x_3)\,J_\gamma(x,y)\,\sigma_s(x_2)\,\sigma_s(x_3). \tag{4.42}$$

We now expand $\psi_{\ell,\gamma}$ with respect to the second argument up to the first order obtaining

$$-\frac{1}{3!}3\sum_{x,x_2,x_3}^{*} \psi_{\ell,\gamma}(x,x,x_2,x_3)\,\sigma_s(x_2)\,\sigma_s(x_3) + \eta_\gamma^2(s), \tag{4.43}$$

where $\eta_\gamma^2(s)$ reflects the error in the Taylor expansion, i.e.,

$$\eta_\gamma^2(s) := \frac{1}{2}\sum_{x,y,x_2,x_3}^{*} J_\gamma(x,y)\,(y-x)\,\partial_y\psi_{\ell,\gamma}(x,y,x_2,x_3)|_{y=\bar{y}}\sigma_s(x_2)\,\sigma_s(x_3).$$

The equations (4.37), (4.39), and (4.42) give the statement in (4.34); in fact as in (3.10a) $|D_\gamma - D| \leq c_1\gamma$.

To complete the proof we have only to show that

$$\eta(\gamma) := \int_0^{\gamma^{-2/3}t} \left(\eta_\gamma^1(s) + \eta_\gamma^2(s)\right) ds$$

is small in probability.

It is enough to compute the second moment. Considering for instance, the last term in (4.41), using the Cauchy–Swartz inequality, we have

$$\begin{aligned} \mathbf{E}_0^\gamma\Bigg(\int_0^{\gamma^{-2/3}t} &\sum_{x,x_1,x_2,x_3,y}^{*} J_\gamma(x,y)(y-x)^3 \\ &\times \partial_y^3\psi_{\ell,\gamma}(y,x_1,x_2,x_3)|_{y=\bar{y}}\sigma_s(x)\,\sigma_s(x_1)\,\sigma_s(x_2)\,\sigma_s(x_3)\Bigg)^2 ds \end{aligned}$$

$$\le \gamma^{-2/3}t \int_0^{\gamma^{-2/3}t} \mathbf{E}_0^\gamma \Bigg(\sum_{x,x_1,x_2,x_3,y}^{\star} J_\gamma(x,y)\,(y-x)^3$$

$$\times\, \partial_y^3 \psi_{\ell,\gamma}(y,x_1,x_2,x_3)|_{y=\bar y}\sigma_s(x)\,\sigma_s(x_1)\,\sigma_s(x_2)\,\sigma_s(x_3) \Bigg)^2 ds. \tag{4.44}$$

From the definition (4.20b) of $\psi_{\ell,\gamma}$ there exists $c_2 = c_2(l)$ such that

$$\left| \partial_y^3\, \psi_{\ell,\gamma}(y,x_1,x_2,x_3)\big|_{y=\bar y} \right| \le c_2(\gamma^{4/3})^3(\gamma^{4/3})^3.$$

As in (4.10) the leading term in the correlation function with 8 spin is given by $c_3\gamma^{8/3}$. It is thus possible to bound (4.44) by

$$c_4(\gamma^{-2/3})^2(\gamma^{-4/3})^8 \Bigg(\sum_{y\in\mathcal{T}_\gamma} J_\gamma(x,y)(\gamma|x-y|)^3(\gamma^{4/3})^3(\gamma^{1/3})^3 \Bigg)^2 \gamma^{8/3},$$

which converges to 0 since

$$\lim_{\gamma\to 0} \sum_{y\in\mathcal{T}_\gamma} J_\gamma(x,y)(\gamma|y-x|)^3 = \hat J(0)^{-1} \int_{\mathcal{T}} dr\,|r|^3 J(|r|).$$

The other terms are analogous.

4.4. Conclusions. To prove Theorem 2.6 we will show that for all function f bounded and measurable on $D([0,t],\ \mathcal{S}'(\mathcal{T}))$ and continuous on $C([0,t],\ C(\mathcal{T}))$

$$\lim_{\gamma\to 0} \int d\mathcal{P}^\gamma f = \int d\mathcal{P} f, \tag{4.45}$$

where $\mathcal{P}$ is the law of the process in Theorem 2.2.

Let

$$\Gamma_t := \tfrac{1}{2}\, Z_t(v) - \tfrac{1}{8}\, \langle Z(v), Z(v)\rangle_t, \qquad v_s = -\tfrac{1}{3}\,(m^0)^3(s), \tag{4.46}$$

where, as in (3.21), Z_t denotes the stochastic integral with respect to the Wiener process.

We prove (4.45) through the convergence of the Radon–Nikodym derivative:

$$\lim_{\gamma\to 0} \mathbf{E}_0^\gamma(f \exp\{\Gamma_t^\gamma\}) = \int d\mathcal{P}_0 f \exp\{\Gamma_t\}, \tag{4.47}$$

where, with an abuse of notations, we have denoted by f both the function on $D([0,t],\mathcal{H}_\gamma)$ and its image on $D([0,t],\mathcal{S}'(\mathcal{T}))$ via the correspondence in Definition 2.5.

After the identification $d\mathcal{P} = d\mathcal{P}_0 \exp\{\Gamma_t\}$ this proves Theorem 2.6. However, in order to take the limit $\gamma \to 0$ we have also to treat the "large field" problem, i.e., the configurations, such that the exponent in the Radon–Nikodym derivative is large. In a first step we exclude these configurations, introducing a large field regulator which will be eventually removed. Let $\mathbf{1}_K \in C^\infty(\mathbf{R})$, $0 \le \mathbf{1}_K \le 1$, such that

$$\mathbf{1}_K(r) = \begin{cases} 1 & \text{if} \quad |r| \le K/2, \\ 0 & \text{if} \quad |r| \ge K. \end{cases} \tag{4.48}$$

The macroscopic limit in the "small field" region is proved in the following Theorem.

THEOREM 4.9. *For any $K > 0$ and for all f bounded and measurable on $D([0,t],$ $\mathcal{S}'(\mathcal{T}))$ and continuous on $C([0,t], C(\mathcal{T}))$*

$$\left| \lim_{\gamma\to 0} \mathbf{E}_0^\gamma(f \exp\{\Gamma_t^\gamma\}) - \int d\mathcal{P}_0 f \exp\{\Gamma_t\}\mathbf{1}_K(\Gamma_t) \right|$$

$$\le \|f\|_\infty \left| 1 - \int d\mathcal{P}_0 \exp\{\Gamma_t\}\mathbf{1}_K(\Gamma_t) \right|. \tag{4.49}$$

Let us define, with the notation (4.21) and (4.32), the "nice version" of Γ_t^γ in (4.2)

$$\Gamma_t^{\gamma,l} := \left(V^{\gamma,l}(\gamma^{-2/3}t) - V^{\gamma,l}(0) \right) + \int_0^{\gamma^{-2/3}t} ds \left(G^{\gamma,l}(s) + H^{\gamma,l}(s) \right)$$

$$- \frac{1}{4} \sum_{x\in\mathcal{T}_\gamma} \int_0^{\gamma^{-2/3}t} ds \left(\frac{1}{3}\psi_{\ell,\gamma} \circ \sigma_s^3(x) \right)^2 \tag{4.50}$$

and its "small" macroscopic version, with the notations introduced in (3.36) and (3.39),

$$\Gamma_t^l := -V^l(t) + \int_0^t ds \left(G^l(s) + H^l(s) \right)$$

$$- \frac{1}{(3!)^2} \int_0^t ds \left(\psi_\ell * (m^0)^3(s), \psi_\ell * (m^0)^3(s) \right). \tag{4.51}$$

We may now take advantage of all our probability bounds: let us introduce the "nice" set

$$G_\delta^{\gamma,l} := \{ |\Gamma_t^{\gamma,l} - \Gamma_t^\gamma| \le \delta \}. \tag{4.52}$$

From Theorems 4.1, 4.7, and 4.8 we can get the following statement: for any $\delta > 0$

$$\lim_{l\to 0} \lim_{\gamma\to 0} \mathbf{P}_0^\gamma \left(G_\delta^{\gamma,l} \right) = 1. \tag{4.53}$$

In this set all the approximations in the Radon–Nikodym derivative are correct.

We first take the macroscopic limit in this set.

LEMMA 4.10. *For any $K > 0$ and for all f bounded and measurable on $D([0,t],$ $\mathcal{S}'(\mathcal{T}))$ and continuous on $C([0,t], C(\mathcal{T}))$ there exist $\delta_0 > 0$, $C = C(K,\delta_0)$ such that for all $l > 0$ and for any $\delta \in (0,\delta_0)$*

$$\Bigg| \lim_{\gamma\to 0} \mathbf{E}_0^\gamma \left(f \exp\{\Gamma_t^\gamma\}\, \mathbf{1}_K(\Gamma_t^{\gamma,l})\, \mathbf{1}_{G_\delta^{\gamma,l}} \right)$$

$$- \int d\mathcal{P}_0 f \exp\{\Gamma_t^l\} \mathbf{1}_K(\Gamma_t^l) \Bigg| \le C\,\delta\, \|f\|_\infty + R_l(K,\delta), \tag{4.54}$$

where $R_l(K,\delta)$ converges to 0 as $l \to \infty$.

Proof. We have

$$\mathbf{E}_0^\gamma \left(f \exp\{\Gamma_t^\gamma\} \mathbf{1}_K(\Gamma_t^{\gamma,l}) \mathbf{1}_{G_\delta^{\gamma,l}} \right) = \mathbf{E}_0^\gamma \left(f \exp\{\Gamma_t^{\gamma,l}\} \mathbf{1}_K(\Gamma_t^{\gamma,l}) \mathbf{1}_{G_\delta^{\gamma,l}} \right) + R_{\gamma,l}^1(K,\delta), \tag{4.55}$$

where

$$|R^1_{\gamma,l}(K,\delta)| = \left|\mathbf{E}^\gamma_0\left(f\left(e^{\Gamma^\gamma_t} - e^{\Gamma^{\gamma,l}_t}\right)1_K(\Gamma^{\gamma,l}_t)1_{G^{\gamma,l}_\delta}\right)\right| \leqq \|f\|_\infty C\delta \tag{4.56}$$

and C may be taken independent of l and δ, if $\delta \leqq \delta_0$. We can now write the expectation on the right-hand side of (4.55) as

$$\mathbf{E}^\gamma_0\left(f\exp\{\Gamma^{\gamma,l}_t\}1_K(\Gamma^{\gamma,l}_t)\right) + R^2_{\gamma,l}(K,\delta), \tag{4.57}$$

where

$$|R^2_{\gamma,l}(K,\delta)| = \left|-\mathbf{E}^\gamma_0\left(f\exp\{\Gamma^{\gamma,l}_t\}1_K(\Gamma^{\gamma,l}_t)1_{(G^{\gamma,l}_\delta)^c}\right)\right| \leqq \|f\|_\infty e^K \mathbf{P}^\gamma_0\left((G^{\gamma,l}_\delta)^c\right). \tag{4.58}$$

Since Γ^l_t is a measurable function on $D([0,t],\mathcal{S}'(\mathcal{T}))$ and continuous on $C([0,t], C(\mathcal{T}))$, by the Proposition 3.4 we have

$$\lim_{\gamma\to 0}\mathbf{E}^\gamma_0\left(f\exp\{\Gamma^{\gamma,l}_t\}, 1_K(\Gamma^{\gamma,l}_t)\right) = \int d\mathcal{P}_0 f\exp\{\Gamma^l_t\}1_K(\Gamma^l_t), \tag{4.59}$$

the remainder $R_l(K,\delta)$ is given by $\lim_{\gamma\to 0} R^2_{\gamma,l}(K,\delta)$; this converges to 0, as $l \to 0$, by the bounds (4.58) and (4.53).

LEMMA 4.11. *For any $K > 0$ and for all f bounded and measurable on $D([0,t], \mathcal{S}'(\mathcal{T}))$ and continuous on $C([0,t],\ C(\mathcal{T}))$ there exist $\delta_0 > 0$, $C = C(K,\delta_0)$ such that for all $\delta \in (0,\delta_0)$*

$$\left|\lim_{l\to 0}\lim_{\gamma\to 0}\mathbf{E}^\gamma_0\left(f\exp\{\Gamma^\gamma_t\}1_K(\Gamma^{\gamma,l}_t)1_{G^{\gamma,l}_\delta}\right) - \int d\mathcal{P}_0 f\exp\{\Gamma_t\}1_K(\Gamma_t)\right| \leqq C\delta. \tag{4.60}$$

Proof. We have to study the limit, as $l \to 0$, of the second integral on the left-hand side of (4.54); before doing so it is convenient to identify $\exp\{\Gamma^l_t\}$ as the exponential martingale of a stochastic integral. Let $Z_t(v^l)$ as in (3.38); we have

$$\langle Z(v^l),\ Z(v^l)\rangle_t = 2\frac{1}{3^2}\int_0^t ds\,(\psi_\ell * (m^0)^3(s),\ \psi_\ell * (m^0)^3(s)),$$

hence, by (4.51) and (3.37)

$$\exp\{\Gamma^l_t\} = \exp\left\{\tfrac{1}{2}Z_t(v^l) - \tfrac{1}{8}\langle Z(v^l),\ Z(v^l)\rangle_t\right\}. \tag{4.61a}$$

We remark that neither G^l nor H^l has a limit for $l \to 0$, however, their linear combination in $Z_t(v^l)$ has a well-defined limit. In fact we have

$$\lim_{l\to 0}\int d\mathcal{P}_0\left(Z_t(v^l) - Z_t(v)\right)^2 = 0, \tag{4.61b}$$

which follows from the convergence of v^l to v in the $|||\cdot|||$ norm defined in (3.19). We remark that from (4.61) we have also the $L^1(d\mathcal{P}_0)$-convergence of the quadratic variation.

We are now ready to take the limit $l \to 0$ of the interacting process. Since for all fixed K the function $\exp\{r\}1_K(r)$ satisfies the uniform Lipschitz condition, from the $L^1(d\mathcal{P}_0)$-convergence of Γ^l_t to Γ_t we have

$$\lim_{l\to 0}\int d\mathcal{P}_0 f\exp\{\Gamma^l_t\}1_K(\Gamma^l_t) = \int d\mathcal{P}_0 f\exp\{\Gamma_t\}1_K(\Gamma_t). \tag{4.62}$$

Equation (4.60) is then a straightforward consequence of Lemma 4.10.

Proof of Theorem 4.9. Lemma 4.11 proves the convergence in the "nice" and "small field" set $G^{\gamma,l}_\delta \cap \{\Gamma^{\gamma,l}_t < K\}$. We can write the expectation on its complement as

$$\begin{aligned}&\mathbf{E}^\gamma_0\left(f\exp\{\Gamma^\gamma_t\}\left[1 - 1_K(\Gamma^{\gamma,l}_t)1_{G^{\gamma,l}_\delta}\right]\right)\\ &\quad\leqq \|f\|_\infty\left|1 - \mathbf{E}^\gamma_0\left(\exp\{\Gamma^\gamma_t\}1_K(\Gamma^{\gamma,l}_t)1_{G^{\gamma,l}_\delta}\right)\right|,\end{aligned} \tag{4.63}$$

since $\exp\{\Gamma^\gamma_t\}$ is the Radon–Nikodym density. We are now in a situation, where Lemma 4.11 holds again (with $f \equiv 1$). Taking first $\gamma \to 0$, then $l \to 0$, and finally $\delta \to 0$ we prove (4.49).

Proof of Theorem 2.6 (Girsanov characterization). In [20] a solution of the Cauchy problem (2.1) is constructed and it is shown that the measure of the interacting process $\mathcal{P}$ may be expressed as a density with respect to the free measure $\mathcal{P}_0$; furthermore, the density is given by the Girsanov formula, i.e.,

$$d\mathcal{P} = \exp\{\Gamma_t\}\,d\mathcal{P}_0, \tag{4.64}$$

where Γ_t is given by (4.48).

This result allows us to take the limit, as $K \to \infty$, in (4.49); in fact, recalling (4.47), by dominated convergence we have

$$\lim_{K\to\infty}\int d\mathcal{P}_0 f\exp\{\Gamma_t\}1_K(\Gamma_t) = \int d\mathcal{P}_0 f\exp\{\Gamma_t\} \tag{4.65}$$

and by [20]

$$\int d\mathcal{P}_0\exp\{\Gamma_t\} = 1. \tag{4.66}$$

This concludes the proof of the convergence of the Glauber process to the [20] solution of (2.1). In Theorem 2.6, however, we refer to the solution of the problem (2.1) in the strong sense of Theorem 2.2; it is not clear to us whether there is in [20] a characterization of the process also in this sense, i.e., as a process adapted to z_t which satisfies (2.3). In the next section we will complete the proof of Theorem 2.6 by showing the equivalence of the Girsanov characterization with the strong one of Theorem 2.2 and with a martingale formulation of (2.1). In the course of the proof we will also derive the result in [20] quoted above.

5. Characterization of the interacting process and identification of the limit.

In this section we prove the equivalence of three different definitions of a process solution of (2.1), see Propositions 5.1, 5.2, 5.3 and Theorem 5.4 below. As a corollary we will deduce Theorem 2.2 and we will complete the proof of Theorem 2.6, as discussed at the end of the previous section. Even though the content of this section may not be completely new, it makes the whole paper self-contained and, hopefully, it provides a unified view of a collection of results on (2.1) existing in the literature and not all easily accessible.

PROPOSITION 5.1 (The Girsanov characterization). *Let Γ_t be as in (4.46), then, for all t, $e^{\Gamma_t} \in L_1(\mathcal{P}_0)$ and $(e^{\Gamma_t})_{t\geqq 0}$ is a $\mathcal{P}_0$-martingale. Therefore, there is a unique probability denoted by $\mathcal{P}_1$ which, for any $t \geqq 0$, coincides with $e^{\Gamma_t}d\mathcal{P}_0$ on $C([0,t], C(\mathcal{T}))$.*

PROPOSITION 5.2 (The martingale characterization). *There is a unique law $\mathcal{P}_2$, which is solution of the martingale problem MP stated below.*

We denote by $(m_t) = (m(r,t), r \in \mathcal{T}, t \in \mathbf{R}_+)$ the canonical coordinates in $C(\mathbf{R}_+, C(\mathcal{T}))$.

DEFINITION (The martingale problem MP). The probability $\mathcal{P}_2$ on $C(\mathbf{R}_+, C(\mathcal{T}))$ solves the martingale problem MP if $m(r,0) = 0$ almost surely and the condition (a) below is satisfied.

Let τ be any stopping time such that, for some K,

$$\sup_{r,t} \left| m(r, t \wedge \tau) \right| \leqq K \quad \mathcal{P}_2\text{-a.s.}$$

(a) For any stopping time τ and any $\varphi \in \mathcal{S}(\mathcal{T})$ the processes $Z_t(\varphi,\tau)$ and $\Lambda_t(\varphi,\tau)$ defined below are $\mathcal{P}_2$-martingales:

$$Z_t(\varphi,\tau) := (m_{t\wedge\tau}, \varphi) - \int_0^{t\wedge\tau} ds\, \frac{D}{2}\, (m_s, \varphi'') + \int_0^{t\wedge\tau} ds \left(\frac{1}{3} m_s^3, \varphi \right),$$

$$\Lambda_t(\varphi,\tau) := Z_t(\varphi,\tau)^2 - 2(t \wedge \tau)(\varphi,\varphi).$$

PROPOSITION 5.3 (Strong solutions of (2.1)). *Given any (continuous) realization $(m_t^0)_{t\geqq 0}$ of the free process (see Definition 3.5) there is a unique continuous function $m(r,t)$, such that for all $r \in \mathcal{T}$ and all $t \geqq 0$*

$$m(r,t) - \int_0^t ds\, (g_{t-s,r}, v_s) = m_0(r,t), \tag{5.1a}$$

$$v_t(r) = -\tfrac{1}{3} m(r,t)^3. \tag{5.1b}$$

Therefore, (5.1) *defines a process $(m_t)_{t\geqq 0}$, $(m_t) = (m(r,t),\ r \in \mathcal{T},\ t \in \mathbf{R}_+)$, adapted to m_t^0 and supported by $C(\mathbf{R}_+, C(\mathcal{T}))$, whose law is denoted by $\mathcal{P}_3$.*

The proof of Proposition 5.3 may be found in [15] to which we refer. Propositions 5.1 and 5.2 will be proved jointly with the following theorem.

THEOREM 5.4. *The laws $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ defined in Propositions 5.1–5.3 are all equal to each other and to the law $\mathcal{P}$ of Theorem 2.2.*

We will first prove that the problem MP has a unique solution $\mathcal{P}_2$ and that $\mathcal{P}_2 = \mathcal{P}_3$. The proof extends to the infinite-dimensional case the following well-known property. Existence and uniqueness of strong solutions of a stochastic ODE implies existence and uniqueness of weak solutions to the same equation. The extension to infinite dimensions uses in several parts arguments introduced by Holley and Stroock ([16], [33], [34]) for the characterization of the generalized Ornstein–Uhlenbeck processes.

5.1. Proof that $\mathcal{P}_3$ solves the problem MP. We know that m_t^0 hence m_t are adapted to the Wiener process z_t of Definition 2.1, and in this subsection we consider m_t as a process realized in the space of z_t. We will prove the martingale condition (a) of the problem MP by showing that

$$(m_t, \varphi) - \frac{D}{2} \int_0^t ds\, (m_s, \varphi'') - \int_0^t ds\, (v_s, \varphi) = (z_t, \varphi). \tag{5.2}$$

This identifies $Z_t(\varphi,\tau)$ in MP as the martingale (z_t,φ) in (5.2) stopped at τ. By (2.2) it then follows that also $\Lambda_t(\varphi,\tau)$ is a martingale, hence condition (a) is proved, once we show (5.2).

We define $\varphi_{t-s}(r) = \int dr' g_{t-s,r}\varphi(r')$, then from (5.1) we have

$$(m_t, \varphi) = \int_0^t ds\, (\varphi_{t-s}, v_s) + (m_t^0, \varphi). \tag{5.3}$$

We use this relation to rewrite the second term in (5.2) as

$$\begin{aligned} \frac{D}{2} \int_0^t ds\, (m_s, \varphi'') &= \frac{D}{2} \int_0^t ds \int_0^s ds_1 (\varphi''_{s-s_1}, v_{s_1}) + \frac{D}{2} \int_0^t ds\, (m_s^0, \varphi'') \\ &= \int_0^t ds_1 \int_{s_1}^t ds \left(\frac{\partial}{\partial s} \varphi_{s-s_1}, v_{s_1} \right) + \frac{D}{2} \int_0^t ds\, (m_s^0, \varphi'') \\ &= -\int_0^t ds\, (v_s, \varphi) + \int_0^t ds\, (v_s, \varphi_{t-s}) + (m_t^0, \varphi) - (\varphi, z_t), \end{aligned} \tag{5.4}$$

having used (3.18b) in the last equality. We then obtain (5.2) as a consequence of (5.4) and (5.3).

Proof of Theorem 2.2. The process m_t of Proposition 5.3 realized as in § 5.1 solves (2.3). On the other hand, if a process adapted to z_t solves (2.3) then it is a solution of MP, thus uniqueness in Theorem 2.2 will follow from Proposition 5.2.

5.2. Proof that any law which solves MP is equal to $\mathcal{P}_3$. We denote by E_i the expectation with respect to $\mathcal{P}_i$, $i = 1,2,3$. We define, for any $t \geqq 0$ and $r \in \mathcal{T}$,

$$N_t(r) := m_t(r) - \int_0^t ds\, (g_{t-s,r}, v_s), \qquad N_t(\varphi) := (N_t, \varphi) \tag{5.5}$$

with $g_{t,r}$ as in (3.25b). Since by assumption $\mathcal{P}_2$ is supported by $C(\mathbf{R}_+, C(\mathcal{T}))$, it follows that $N_t(r)$ is continuous $\mathcal{P}_2$-almost surely. We will prove that

$$m_t^0(r) \overset{\text{in law}}{=} N_t(r) \tag{5.6}$$

for all r and $t \geqq 0$. At this point we can use Proposition 5.3 to solve (5.5), expressing m_t in terms of N_s, $s \leqq t$, and by (5.6), we then conclude that $\mathcal{P}_2 = \mathcal{P}_3$. We are, therefore, left with the proof of (5.6). By the continuity of the process $N_t(r)$, its law is determined by the characteristic functions

$$E_2 \left(\prod_k e^{i\theta_k N_{t_k}(\varphi^{(k)})} \right), \tag{5.7}$$

where $i^2 = 1$, $k = 1, \ldots, K$, $K \geqq 1$, θ_k are real numbers, $\varphi^{(k)} \in \mathcal{S}(\mathcal{T})$ and $0 \leqq t_1 \leqq \cdots \leqq t_K$. We call $\mathcal{F}_t$ the σ-algebra generated by the process m_s for $0 \leqq s \leqq t$. To simplify notation we call $T = t_K$, $S = t_{K-1}$, $\varphi = \theta_K \varphi_K$. For $t \leqq T$ we define

$$\widehat{N}_t(\varphi) := N_t(\varphi_{T-t}), \tag{5.8a}$$

where φ_t is defined in § 5.1. Since

$$\widehat{N}_T(\varphi) = N_T(\varphi), \tag{5.8b}$$

we may compute

$$E_2 \Big(\exp\{ i\widehat{N}_T(\varphi) \} \big| \mathcal{F}_S \Big). \tag{5.9}$$

We will prove that $\exp\{\lambda_t(\varphi)\}_{t\leqq T}$ is a $\mathcal{P}_2$-martingale, where

$$\lambda_t(\varphi) = i\widehat{N}_t(\varphi) + \int_0^t ds\,(\varphi_{T-s}, \varphi_{T-s}). \tag{5.10}$$

We postpone the proof of this property, and use it to deduce that

$$E_2\Big(\exp\{i\widehat{N}_T(\varphi)\}\big|\mathcal{F}_S\Big) = \exp\{iN_S(\varphi_{T-S}) + C_{T,S}(\varphi)\}, \tag{5.11a}$$

$$C_{T,S}(\varphi) := -\int_S^T ds\,(\varphi_{T-s}, \varphi_{T-s}). \tag{5.11b}$$

By iteration we then get

$$E_2\left(\prod_k e^{i\theta_k N_{t_k}(\varphi^{(k)})}\right) = \prod_k e^{C_{t_k,t_{k-1}}(\psi^{(k)})}, \tag{5.12a}$$

where

$$\psi^{(k)} = \sum_{h\geq k} \theta_h \varphi^{(h)}_{t_h - t_k}. \tag{5.12b}$$

We next prove that

$$E_0\left(\prod_k e^{i\theta_k (m^0_{t_k}, \varphi^{(k)})}\right) = \prod_k e^{C_{t_k,t_{k-1}}(\psi^{(k)})}. \tag{5.13}$$

The proof is just the same as the previous one, after defining for $t \leqq T$:

$$\widehat{M}^0_t(\varphi) = (m^0_t, \varphi_{T-t}). \tag{5.14}$$

Then by (3.27) and (3.22)

$$\exp\left\{i\widehat{M}^0_t(\varphi) + \int_0^t ds\,(\varphi_{T-s}, \varphi_{T-s})\right\} \tag{5.15}$$

is a $\mathcal{P}_0$-martingale. From this we derive (5.13) in just the same way as we did for (5.12a).

The proof that $\mathcal{P}_2 = \mathcal{P}_3$ is thus reduced to the proof that $\exp\{\lambda_t(\varphi)\}$ is a $\mathcal{P}_2$-martingale. Let $T \geqq t_1 > t_0 \geqq 0$. We denote by $Z_t(\varphi)$ the expression appearing in condition (a) of the problem MP without the stopping time τ. We have

$$\begin{aligned}\lambda_{t_1}(\varphi) - \lambda_{t_0}(\varphi) &= i\{Z_{t_1}(\varphi_{T-t_0}) - Z_{t_0}(\varphi_{T-t_0})\} \\ &\quad + (t_1 - t_0)(\varphi_{T-t_0}, \varphi_{T-t_0}) + e_1 + e_2 + e_3,\end{aligned} \tag{5.16}$$

where

$$\begin{aligned} e_1 &= i\int_{t_0}^{t_1} ds\,\frac{D}{2}\left[(m_s, \varphi''_{T-t_0}) - (m_{t_1}, \varphi''_{T-s})\right], \\ e_2 &= -i\int_{t_0}^{t_1} ds\,\left[(v_s, \varphi_{T-t_0}) - (v_s, \varphi_{T-s})\right], \\ e_3 &= -(t_1 - t_0)(\varphi_{T-t_0}, \varphi_{T-t_0}) + \int_{t_0}^{t_1} ds\,(\varphi_{T-s}, \varphi_{T-s}). \end{aligned} \tag{5.17}$$

We denote the continuity modulus of a function $f = f(t,r)$ by $\omega_f(\delta)$:

$$\omega_f(\delta) = \sup_{r\in\mathcal{T}} \sup_{\substack{|s-t|\leqq\delta \\ s,t\in[0,T]}} \big|f(t,r) - f(s,r)\big|. \tag{5.18}$$

Since $\varphi_{T-s}(r) \in C^\infty([0,T], C(\mathcal{T}))$ and $m_s \in C([0,T], C(\mathcal{T}))$ $\mathcal{P}_2$-almost surely, their moduli of continuity are finite. Hence, for $j = 1, 2$,

$$|e_j| \leqq c|t_1 - t_0|\left[\omega_m(|t_1 - t_0|) + \omega_{\varphi_s}(|t_1 - t_0|)\right] \tag{5.19}$$

and c is almost surely finite. The same bound holds for $|e_3|$, but with a true constant c and with ω_m missing. We fix $t > s$ in $[0,T]$ and consider a set $A \in \mathcal{F}_s$. We partition $[s,t]$ into n intervals of equal length denoted by δ. The endpoints of the intervals are called t_k. By (5.17) we have

$$\begin{aligned} &E_2\Big(\mathbf{1}_A \exp\big(\lambda_t(\varphi) - \lambda_s(\varphi) - \eta_\delta\big)\Big) \\ &= E_2\Big(\mathbf{1}_A \prod_k \exp\Big(i\{Z_{t_{k+1}}(\varphi_{T-t_k}) - Z_{t_k}(\varphi_{T-t_k})\} + \delta(\varphi_{T-t_k}, \varphi_{T-t_k})\Big)\Big), \end{aligned} \tag{5.20}$$

where η_δ is the sum of all errors e_j relative to each of the intervals in the decomposition.

We will first prove that the right-hand side of (5.12) is equal to $\mathcal{P}_2(A)$. To see this we define the stopping times

$$\tau_n = \inf\Big\{s \in \mathbf{R}_+ \colon \sup_r |m(r,s)| > n\Big\}. \tag{5.21}$$

As for each $\varphi \in C(\mathcal{T})$, $iZ_t(\varphi, \tau_n)$ is a martingale with bounded variation, it follows from [28, Chap. VIII, Prop. 1.15] that

$$\exp\big(iZ_s(\varphi, \tau_n) + (s \wedge \tau_n)(\varphi, \varphi)\big) \text{ is a } \mathcal{P}_2\text{-martingale.} \tag{5.22}$$

By the assumptions on $\mathcal{P}_2$ and the boundedness of the martingale in (5.22), we obtain that

$$\exp\big(iZ_s(\varphi) + s(\varphi, \varphi)\big) \text{ is a } \mathcal{P}_2\text{-martingale} \tag{5.23}$$

having used the Lebesgue theorem. We then perform the expectation on the right hand side of (5.20) by successively conditioning on $\mathcal{F}_{t_k}$. We thus obtain $\mathcal{P}_2(A)$.

We next take the limit, as $\delta \to 0$, on the left-hand side of (5.20). By (5.17) $\exp\{\eta_\delta\}$ is bounded by a nonstochastic constant. Moreover, by (5.19) η_δ converges almost surely to 0. We have thus proved that

$$E_2\Big(\mathbf{1}_A \exp\big(\lambda_t(\varphi) - \lambda_s(\varphi)\big)\Big) = \lim_{\delta\to 0} E_2\big(\mathbf{1}_A \exp\big(\lambda_t(\varphi) - \lambda_s(\varphi) - \eta_\delta\big)\big) = \mathcal{P}_2(A),$$

hence that $\exp\{\lambda_t(\varphi)\}$ is a $\mathcal{P}_2$-martingale.

5.3. Proof of Proposition 5.1 and that $\mathcal{P}_1 = \mathcal{P}_3$. In order to prove Proposition 5.1, since $\exp(\Gamma_t)$ is the (local) exponential of the martingale $\frac{1}{2}\int_0^t (dz_s, v_s)$, by [28, Chap. VIII, Proposition 1.13], it is enough to show that $\exp(\Gamma_t) \in L_1(\mathcal{P}_0)$ and

$$E_1(\exp(\Gamma_t)) = 1. \tag{5.24}$$

At the same time we will also prove that $\mathcal{P}_1 = \mathcal{P}_3$.

We introduce a cutoff on v_t defining

$$v_t^\varepsilon(r) := v_t(r) \wedge \pm\varepsilon^{-1} = \begin{cases} v_t(r) & \text{if} \quad |v_t(r)| \leqq \varepsilon^{-1}, \\ \pm\varepsilon^{-1} & \text{if} \quad v_t(r) \gtrless \pm\varepsilon^{-1}, \end{cases}$$

$$\Gamma_t^\varepsilon := \frac{1}{2}\int_0^t (dz_s, v_s^\varepsilon) - \frac{1}{4}\int_0^t ds\,(v_s^\varepsilon, v_s^\varepsilon).$$

As $\exp\{\Gamma_t^\varepsilon\}$ is the (local) exponential martingale of

$$\frac{1}{2}Z_t(v^\varepsilon) = \frac{1}{2}\int_0^t (dz_s, v_s^\varepsilon),$$

whose quadratic variation is bounded, it follows from [28, Chap. VIII, Prop. 1.15] that $\exp(\Gamma_t^\varepsilon)$ is a martingale and $d\mathcal{P}_1^\varepsilon := \exp(\Gamma_t^\varepsilon)d\mathcal{P}_0$ defines a unique probability measure on $C(\mathbf{R}_+, C(\mathcal{T}))$.

Let

$$T_\varepsilon = \inf\left\{t \in \mathbf{R}_+ \colon \sup_r |m(r,t)^3| > \varepsilon^{-1}\right\}$$

be a stopping time on $C(\mathbf{R}_+, C(\mathcal{T}))$. We denote by $\mathcal{P}_3^\varepsilon$ the law of $m_{T_\varepsilon \wedge t}$.

LEMMA 5.5. *With reference to the definitions introduced above*

$$\mathcal{P}_1^\varepsilon = \mathcal{P}_3^\varepsilon. \tag{5.25}$$

Proof. For $T > t$, let φ_{T-t} be as in § 5.1, then, since

$$(m_t^0, \varphi_{T-t}) = \int_0^t (dz_s, \varphi_{T-s}) \tag{5.26}$$

by [28, Chap. VIII, Thm 1.7],

$$(m_t, \varphi_{T-t}) - \int_0^t ds\,(v_s^\varepsilon, \varphi_{T-s}) =: \chi_t \quad \text{is a } \mathcal{P}_1^\varepsilon\text{-martingale}, \tag{5.27a}$$

$$\chi_t^2 - \int_0^t (\varphi_{T-s}, \varphi_{T-s}) \quad \text{is a } \mathcal{P}_1^\varepsilon\text{-martingale}. \tag{5.27b}$$

These martingale relationships characterize uniquely the law $\mathcal{P}_1^\varepsilon$, as can be verified by repeating the argument in § 5.2. From (5.1a) with ε and (5.26), it follows that $\mathcal{P}_3^\varepsilon$ solves the martingale problem (5.27).

We now prove (5.24). Let $\mathbf{1}_K$ be as in (4.48). Denoting by E_0 the expectation with respect to $\mathcal{P}_0$, we have by Lemma 5.5

$$E_0(e^{\Gamma_t}\mathbf{1}_K(\Gamma_t)) = E_3^\varepsilon(\mathbf{1}_K(\Gamma_t^\varepsilon)) + R_{\varepsilon,K}, \tag{5.28a}$$

$$R_{\varepsilon,K} := E_0(e^{\Gamma_t}\mathbf{1}_K(\Gamma_t)) - E_0(e^{\Gamma_t^\varepsilon}\mathbf{1}_K(\Gamma_t^\varepsilon)). \tag{5.28b}$$

For any fixed K, the value $R_{\varepsilon,K}$ converges to zero when $\varepsilon \to 0$. In fact, we note that in the set $\{T_\varepsilon > t\}$ the Girsanov exponent Γ_t^ε equals Γ_t, hence

$$|R_{\varepsilon,K}| \leqq E_0\left(\left|e^{\Gamma_t}\mathbf{1}_K(\Gamma_t) - e^{\Gamma_t^\varepsilon}\mathbf{1}_K(\Gamma_t^\varepsilon)\right|\mathbf{1}_{\{T_\varepsilon \leqq t\}}\right) \leqq 2e^K\mathcal{P}_0(\{T_\varepsilon \leqq t\}), \tag{5.29}$$

which converges to zero, as $\varepsilon \to 0$, by (3.29).

We will prove that

$$\lim_{K\to\infty}\lim_{\varepsilon\to 0}\mathcal{P}_3^\varepsilon\left(|\Gamma_t^\varepsilon| > K\right) = 0. \tag{5.30}$$

By (5.29) and (5.30) we derive (5.24) from (5.28a).

By [28, Chap VIII, Thm 1.7],

$$\int_0^T (dz_t, v_t^\varepsilon) = \tilde{L}_T^\varepsilon + \int_0^T dt\,(v_t^\varepsilon, v_t^\varepsilon), \tag{5.31}$$

where $\tilde{L}_t^\varepsilon$ is a $\mathcal{P}_3^\varepsilon$-martingale and

$$[\tilde{L}_T^\varepsilon]^2 - \int_0^T dt\,(v_t^\varepsilon, v_t^\varepsilon) \quad \text{is a } \mathcal{P}_3^\varepsilon\text{-martingale}.$$

Using the Chebyshev inequality we get

$$\mathcal{P}_3^\varepsilon\left(\left|\tilde{L}_t^\varepsilon + \frac{1}{2}\int_0^t ds\,(v_s^\varepsilon, v_s^\varepsilon)\right| > K\right) \leqq K^{-1}E_3^\varepsilon\left(|\tilde{L}_t^\varepsilon| + \frac{1}{2}\int_0^t ds\,(v_s^\varepsilon, v_s^\varepsilon)\right),$$

$$E_3^\varepsilon(|\tilde{L}_t^\varepsilon|) \leqq \left[E_3^\varepsilon\left(\int_0^t ds\,(v_s^\varepsilon, v_s^\varepsilon)\right)\right]^{1/2},$$

$$E_3^\varepsilon\left(\int_0^t ds\,(v_s^\varepsilon, v_s^\varepsilon)\right) \leqq \varepsilon^{-2}\mathcal{P}_3^\varepsilon(T_\varepsilon \leqq t) + \int_0^t ds\,E_3^\varepsilon(v_s^\varepsilon, v_s^\varepsilon)$$

$$\leqq \varepsilon^{-2}ce^{-\delta\varepsilon^{-2}} + ct,$$

the last inequality follows from (3.29) and [15], where it is proved that there exists a constant C such that

$$\sup_{s\in[0,t]}\|m_s\|_\infty \leqq C \sup_{s\in[0,t]}\|m_s^0\|_\infty.$$

We now prove that $\mathcal{P}_3 = \mathcal{P}_1$. For any bounded measurable function f

$$\left|E_0\left(fe^{\Gamma_t}\mathbf{1}_K(\Gamma_t)\right) - E_3^\varepsilon\left(f\mathbf{1}_K(\Gamma_t^\varepsilon)\right)\right| \leqq \|f\|_\infty R_{\varepsilon,K}.$$

Recall that $\lim_{\varepsilon\to 0} R_{\varepsilon,K} = 0$; it follows from (5.30) that $\mathcal{P}_1^\varepsilon \to \mathcal{P}_1$. As $\mathcal{P}_3^\varepsilon \to \mathcal{P}_3$, then $\mathcal{P}_3 = \mathcal{P}_1$.

Appendix A. Correlation functions for the voter model. In this appendix we prove the sharp estimates on the equal time correlation functions for the voter model used in §§ 3 and 4.

We use the following notation.

Given $\gamma > 0$ and $n \geqq 1$ we set $\underline{x}_n := \{x_i\}_{i=1}^n \in \mathcal{T}_\gamma^n$, where $\{x_i\}_{i=1}^n$ is such that $x_i \neq x_j$ for $i \neq j$. For $k \geqq 1$

$$a_k(r,t) := (2k-1)!!\left(\int_0^t ds\,g_{2s}^p(r)\right)^k, \tag{A.1}$$

where $g_s^p(r)$ is the heat kernel on $\mathcal{T}$, as defined in (3.25b).

We notice that by symmetry, for any $\gamma > 0$ and for any $\underline{x}_{2k+1}$, $k \geqq 0$,

$$\mathbf{E}_0^\gamma\left(\prod_{i=1}^{2k+1}\sigma_{\gamma^{-2/3}t}(x_i)\right) = 0. \tag{A.2}$$

The main result in this appendix is the following statement.

THEOREM A.1. *For any $t>0$ and $k \geqq 1$ there are C and $\xi_0>0$ such that for all $\underline{x}_{2k}$*

$$a_k(d\gamma^{4/3},t) - C\gamma^{\xi_0} \leqq \gamma^{-k2/3}\mathbf{E}_0^\gamma\left(\prod_{i=1}^{2k}\sigma_{\gamma^{-2/3}t}(x_i)\right) \leqq a_k(0,t) + C\gamma^{\xi_0}, \tag{A.3}$$

where $d=\sup_{i,j}|x_i-x_j|$. Moreover,

$$\mathbf{E}_0^\gamma\left(\prod_{i=1}^{2k}\sigma_{\gamma^{-2/3}t}(x_i)\right) \geqq 0. \tag{A.4}$$

From the upper bound in (A.3) we deduce the following corollary.

COROLLARY A.2. *For any $t>0$ and $k\geqq 1$ there is $C>0$ such that for all $\underline{x}_{2k}$, $s \leqq \gamma^{-2/3}t$ and $\gamma \leqq \gamma_0$*

$$\left|\mathbf{E}_0^\gamma\left(\prod_{i=1}^{2k}\sigma_s(x_i)\right)\right| \leqq C\gamma^{k2/3}. \tag{A.5}$$

Proof of Theorem A.1. We use the duality between the voter model of definition (3.2b) and the annihilating random walks [21]. Given $\underline{x}_n$ we denote by $\underline{x}_n(t)$ the process on $\mathcal{T}_\gamma$ defined as follows: it starts at $\underline{x}_n$, each x_i moves at independent Poisson time of mean 1 and jumps from x to y with probability $J_\gamma(x,y)$, and if two x_i became equal, they both die. Since the initial measure for the voter model is Bernoulli with mean zero, denoting with an hat the law of the annihilating random walkers, we obtain the duality relation

$$\mathbf{E}_0^\gamma\left(\prod_{i=1}^{n}\sigma_t(x_i)\right) = \widehat{\mathbf{P}}_{\underline{x}}\left\{\text{all the particles are dead at time } t\right\}. \tag{A.6}$$

Equation (A.4) is then obvious. We prove (A.3) for the case $k=3$; the case $k=2$ is easier, while all other cases are proved in the same way. Since particles in the dual process die two by two, from (A.6) we get

$$\mathbf{E}_0^\gamma\left(\prod_{i=1}^{6}\sigma_{\gamma^{-2/3}t}(x_i)\right) = 3!\sum_{\{\imath(h)\}}\int_0^{\gamma^{-2/3}t} ds_3\int_0^{s_3} ds_2\int_0^{s_2} ds_1 \times \widehat{\mathbf{E}}^\gamma\left(\prod_{h=1}^{3}\mathbf{1}_{\{x_{\imath(2h-1)}(s_h)=x_{\imath(2h)}(s_h)\}}\right), \tag{A.7}$$

where $\{\imath(h)\}$ is the set of pairings of $\{1,\dots,6\}$, $3!$ is due to the time ordering and $\widehat{\mathbf{E}}^\gamma$ denotes the expectation with respect to annihilating random walkers starting from $\underline{x}_6$.

To avoid cumbersome notations, in the sequel we just consider the pairing x_1x_2, x_3x_4, x_5x_6. We couple this dual process with independent random walks, denoted by a tilde, for which we introduce the following measurable sets:

$$\begin{aligned}
A &= \left\{\widetilde{x}_1(s_1)=\widetilde{x}_2(s_1)\right\}\cap\left\{\widetilde{x}_3(s_2)=\widetilde{x}_4(s_2)\right\}\cap\left\{\widetilde{x}_5(s_3)=\widetilde{x}_6(s_3)\right\},\\
B_1 &= \left\{\forall u\in(0,s_1), \widetilde{x}_i(u)\neq\widetilde{x}_j(u), \quad \forall i,j\in\{1,\dots,6\}\right\},\\
B_2 &= \left\{\forall u\in[s_1,s_2), \widetilde{x}_i(u)\neq\widetilde{x}_j(u), \quad \forall i,j\in\{3,\dots,6\}\right\},\\
B_3 &= \left\{\forall u\in[s_2,s_3), \widetilde{x}_5(u)\neq\widetilde{x}_6(u)\right\}.
\end{aligned}$$

We thus have

$$\begin{aligned}
\widehat{\mathbf{E}}^\gamma\left(\prod_{h=1}^{3}\mathbf{1}_{\{x_{2h-1}(s_h)=x_{2h}(s_h)\}}\right) &= \widetilde{\mathbf{E}}^\gamma\left(\mathbf{1}_A(1-\mathbf{1}_{B_1^c})(1-\mathbf{1}_{B_2^c})(1-\mathbf{1}_{B_3^c})\right)\\
&= \widetilde{\mathbf{E}}^\gamma\left(\mathbf{1}_A-\sum_{i=1}^{3}\mathbf{1}_A\mathbf{1}_{B_i^c}+\sum_{1\leqq i<j\leqq 3}\mathbf{1}_A\mathbf{1}_{B_i^c}\mathbf{1}_{B_j^c}-\mathbf{1}_A\mathbf{1}_{B_1^c}\mathbf{1}_{B_2^c}\mathbf{1}_{B_3^c}\right),
\end{aligned} \tag{A.8}$$

where $\widetilde{\mathbf{E}}^\gamma$ denotes the expectation with respect to the coupled process of independent random walks.

Let us consider the first term in (A.8). We have

$$\int_0^{\gamma^{-2/3}t} d\underline{s}_3\widetilde{\mathbf{E}}^\gamma(\mathbf{1}_A) = (\gamma^{-2/3})^3\int_0^t d\underline{s}_3\prod_{h=1}^{3}p^\gamma_{\gamma^{-2/3}s_h}(x_{2h},x_{2h-1}), \tag{A.9}$$

where $\int_0^t d\underline{s}_3$ are the three-iterated integrals in (A.7) and $p^\gamma_{\gamma^{-2/3}s}(x,y)$ is the probability that two independent random walks, starting from x,y, meet at time $\gamma^{-2/3}s$. By (2.8)

$$p^\gamma_{\gamma^{-2/3}s}(x,y) = p^\gamma_{2\gamma^{-2/3}s}(x-y), \tag{A.10}$$

where $p_t^\gamma(x)$ is the probability that a random walk, starting from 0, is at x at time t. In Appendix B we prove a local central limit theorem for this random walk; because of Theorem B.1 there are C and $\xi_0>0$ such that

$$\left|\int_0^{\gamma^{-2/3}t} d\underline{s}_3\widetilde{\mathbf{E}}^\gamma(\mathbf{1}_A) - \gamma^2\int_0^t d\underline{s}_3\prod_{h=1}^{3}g^P_{2s_h}(\gamma^{4/3}(x_{2h}-x_{2h-1}))\right| \leqq C\gamma^{2+\xi_0}, \tag{A.11}$$

where $g_s^P(r)$ is the heat kernel in the torus $\mathcal{T}$. The inequalities (A.3) are now consequence of the following bounds on the heat kernel

$$g_s^P(r) \leqq g_s^P(0) = (2\pi Ds)^{-1/2}\sum_{m\in\mathbb{Z}}\exp\left\{-\frac{m^2}{2Ds}\right\}, \tag{A.12a}$$

$$g_s^P(r) \geqq g_s^P(r') = (2\pi Ds)^{-1/2}\sum_{m\in\mathbb{Z}}\exp\left\{-\frac{(m+r')^2}{2Ds}\right\}, \tag{A.12b}$$

where (A.12b) holds for $|r|\leqq|r'|\leqq\frac{1}{2}$.

Considering all the pairings in (A.7) the function $a_3(r,t)$ in (A.3) is thus given by

$$a_3(r,t) = 3!\,5!!\int_0^t d\underline{s}_3\prod_{h=1}^{3}g^P_{2s_h}(r),$$

which is exactly (A.1).

To complete the proof we have to show that the other terms in (A.8) are negligible. We discuss in some detail only the second one; the others can be reduced to this case. We have

$$\int_0^{\gamma^{-2/3}t} d\underline{s}_3\widetilde{\mathbf{E}}^\gamma(\mathbf{1}_A\mathbf{1}_{B_1^c}) = \sum_{1\leqq i<j\leqq 3}\int_0^{\gamma^{-2/3}t} d\underline{s}_4\widetilde{\mathbf{E}}^\gamma \times\left(\prod_{h=2}^{4}\mathbf{1}_{\{\widetilde{x}_{2h-1}(s_h)=\widetilde{x}_{2h}(s_h)\}}\cdot\mathbf{1}_{\{\widetilde{x}_i(s_1)=\widetilde{x}_j(s_1)\}}\right). \tag{A.13}$$

Introducing successively conditional expectations with respect to the times s_h, $h = 1, \dots, 4$ and using Theorem B.1, we can bound (A.13) by

$$c_1 \gamma^{8/3} \int_0^t d\underline{s}_4 \left(\prod_{h=1}^4 g^p_{2(s_h - s_{h-1})}(0) \right) + c_2 \gamma^{8/3+\xi_0} \leqq c_3 \gamma^{8/3},$$

where we understood $s_0 = 0$.

Appendix B. Local central limit theorem for random walk with intensity J_γ. In this appendix we study a single random walk in $\mathcal{T}_\gamma$ which moves following a mean 1 Poisson process and jumps from x to y with probability $J_\gamma(x,y)$. In particular, we prove a local central limit theorem for its transition probability at time $\gamma^{-2/3}t$.

In our discussion we will follow closely the analysis in [12] with some slight modifications necessary since we are working in the torus and we are considering time which scales as $\gamma^{-2/3}$ and not as $\log \gamma^{-1}$; in the latter point this appendix may be considered as an improvement of the result in [12].

Let

$$p^\gamma_{\gamma^{-2/3}t}(x) := p^\gamma_{\gamma^{-2/3}t}(0,x) \tag{B.1}$$

be the probability that a single random walk in $\mathcal{T}_\gamma$, which jumps with intensity $J_\gamma(x,y)$ and starts from 0, is at x at time $\gamma^{-2/3}t$. The main result of this appendix is the following bound we used in Appendix A.

THEOREM B.1. *For any $\alpha \in [0, \frac{1}{2})$ there are C and $\zeta > 0$ such that for all $t \in (0,T]$, $x \neq 0$ in $\mathcal{T}_\gamma$*

$$\left| p^\gamma_{\gamma^{-2/3}t}(x) - \gamma^{4/3} g^p_t(\gamma^{4/3}x) \right| \leqq C \gamma^{4/3+\zeta}(1 + t^{-1+\alpha}), \tag{B.2}$$

where $g^p_t(r)$ is the Gaussian kernel in the torus defined in (3.25b).

This theorem is proved as follows. First we show that the random walk in $\mathcal{T}_\gamma$ with intensity $J_\gamma(x,y)$ behaves essentially as a jump process with intensity $J(|r - r'|)\, dr'$ on the torus of length $\gamma^{-1/3}$ provided one makes the correspondence $x \mapsto r = \gamma x$. Then from usual arguments in central limit theorems we will deduce (B.1).

We start with an elementary lemma for a particular approximation of the Riemann integral.

LEMMA B.2. *Let $f \in C^2([-a,a])$ be an even function. For $N \in \mathbb{N}$ let $\delta = a/N$ and set $x_i = i\delta$, $i = -N, \dots, N$. Define the left Riemann approximation of the integral of f as*

$$S_\delta = \delta \sum_{i=-N}^{N-1} f(x_i), \tag{B.3a}$$

then there is a $C > 0$ such that for all δ as above

$$\left| \int_{-a}^a dx f(x) - S_\delta \right| \leqq C\delta^2 \sup_{x \in [-a,a]} \left(|f''(x)| + |f'(x)| \right). \tag{B.3b}$$

Proof. We have

$$\int_{-a}^a dx f(x) = \sum_{i=-(N-1)}^{-1} \int_{x_i}^{x_{i+1}} dx f(x) + \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} dx f(x)$$

$$+ \int_{x_0}^{x_1} dx f(x) + \int_{x_{-N}}^{x_{-(N-1)}} dx f(x).$$

In the two summations we perform the Taylor expansion of f up to the second order with respect to the left point of the integration interval. It is easy to check that the zero order gives the left Riemann approximation and the first order cancels out since f' is an odd function. For the last two terms we expand only to the first order.

We have thus shown that in this case it is possible to bound the difference between the Riemann integral and its left approximation with a δ^2 instead of the usual δ.

LEMMA B.3. *There are $C > 0$ and $\zeta > 0$ such that for all $t \in (0,T]$, $x \neq 0$, in $\mathcal{T}_\gamma$*

$$\Bigg| p^\gamma_{\gamma^{-2/3}t}(x) - \gamma^{4/3} \sum_{h \in \mathbb{Z}} e^{-i2\pi\gamma^{4/3}hx} e^{-\gamma^{-2/3}t} \times \sum_{n=1}^\infty \frac{(\gamma^{-2/3}t\hat{J}(\gamma^{1/3}h))^n}{n!} \Bigg| \leqq C\gamma^{4/3+\zeta}, \tag{B.4}$$

$$\hat{J}(k) = \int_{\mathcal{T}} dr e^{-i2\pi kr} J(|r|), \qquad \hat{J}(0) = \int_{\mathcal{T}} dr J(|r|) = 1. \tag{B.5}$$

Proof. Recalling Definition 2.3, and, in particular, that L_γ is the integer part of $\gamma^{-4/3}/2$, set

$$\widehat{\mathcal{T}}_\gamma = \left\{ 0, \pm\frac{L_\gamma^{-1}}{2}, \pm\frac{2L_\gamma^{-1}}{2}, \dots, \pm\frac{1}{2} \right\},$$

where $-\frac{1}{2}$ is identified with $\frac{1}{2}$. From the usual Fourier analysis we get

$$\widehat{p}^\gamma_{\gamma^{-2/3}t}(k) := \sum_{x \in \mathcal{T}_\gamma} e^{i2\pi kx} p^\gamma_{\gamma^{-2/3}t}(x), \qquad k \in \widehat{\mathcal{T}}_\gamma, \tag{B.6a}$$

$$p^\gamma_{\gamma^{-2/3}t}(x) = (2L_\gamma)^{-1} \sum_{k \in \widehat{\mathcal{T}}_\gamma} e^{-i2\pi kx} \widehat{p}^\gamma_{\gamma^{-2/3}t}(k). \tag{B.6b}$$

Thus, recalling (2.8) we have

$$p^\gamma_{\gamma^{-2/3}t}(x) = (2L_\gamma)^{-1} \sum_{k \in \widehat{\mathcal{T}}_\gamma} e^{-i2\pi kx} e^{-\gamma^{-2/3}t} \sum_{n=0}^\infty \frac{(\gamma^{-2/3}t\,\hat{J}_{\gamma,k})^n}{n!}, \tag{B.7}$$

where

$$\hat{J}_{\gamma,k} := \sum_{y \in \mathcal{T}_\gamma} e^{i2\pi ky} J_\gamma(0,y). \tag{B.8}$$

We now distinguish different regions of values of k. We start from $|k| > \gamma^{1-b}$. Let us define

$$p^{\gamma,>}_{\gamma^{-2/3}t}(x) := (2L_\gamma)^{-1} \sum_{k \in \widehat{\mathcal{T}}_\gamma,\ |k| > \gamma^{1-b}} e^{-i2\pi kx} e^{-\gamma^{-2/3}t} \sum_{n=1}^\infty \frac{(\gamma^{-2/3}t\,\hat{J}_{\gamma,k})^n}{n!}. \tag{B.9}$$

From Lemma 5.2 in [12], which may be applied also in our situation, as the reader can check, we get the following estimate. Given any $b > 0$ for all m there exists $C = C(m)$, such that, for any $t \in (0,T]$,

$$|p^{\gamma,>}_{\gamma^{-2/3}t}(x)| \leqq C\gamma^{1+m}\gamma^{-2/3}t \leqq c_1\gamma^{5/3}, \tag{B.10}$$

where the second inequality is obtained with an appropriate choice of m. The contributions from "large" k are, therefore, negligible.

We now consider the "small" values of k. Define

$$p^{\gamma,<}_{\gamma^{-2/3}t}(x) := (2L_\gamma)^{-1} \sum_{k\in\widehat{T}_\gamma,\ |k|\le\gamma^{1-b}} e^{-i2\pi kx} e^{-\gamma^{-2/3}t} \sum_{n=1}^{\infty} \frac{(\gamma^{-2/3}t\,\hat{J}_{\gamma,k})^n}{n!}. \tag{B.11}$$

Performing the change of variable $k \mapsto \gamma^{-1}k$, we get

$$p^{\gamma,<}_{\gamma^{-2/3}t}(x) = (2L_\gamma)^{-1} \sum_{\gamma k\in\widehat{T}_\gamma,\ |k|\le\gamma^{-b}} e^{-i2\pi\gamma kx} e^{-\gamma^{-2/3}t} \sum_{n=1}^{\infty} \frac{(\gamma^{-2/3}t\,\hat{J}_{\gamma,\gamma k})^n}{n!}. \tag{B.12}$$

Recalling (2.8) and (B.5), we note that n_γ^{-1} is the left Riemann approximation, as defined in (B.3a), of $\hat{J}(0)^{-1}$; from Lemma B.2 we get

$$|n_\gamma - \hat{J}(0)| \le c_2\gamma^2,$$

since n_γ is uniformly bounded. Comparing (B.5) and (B.8), since J is an even function, we have, for $|k|\le\gamma^{-b}$,

$$\begin{aligned}|\hat{J}_{\gamma,\gamma k} - \hat{J}(k)| &\le c_3\gamma^2 + \left|\gamma\sum_{y\in T_\gamma}\cos(2\pi k\gamma y)J(|\gamma y|) - \int_T dr\cos(2\pi kr)J(|r|)\right| \\ &\le c_4\gamma^{2-2b},\end{aligned} \tag{B.13}$$

where we used again Lemma B.2.

We now can proceed, as in Lemma 5.3 of [12], and obtain that

$$\begin{aligned}&\Bigg|p_t^{\gamma,<}(x) - (2L_\gamma)^{-1}\sum_{\gamma k\in\widehat{T}_\gamma,|k|\le\gamma^{-b}} e^{-i2\pi\gamma kx}e^{-\gamma^{-2/3}t} \\ &\quad\times\sum_{n=1}^{\infty}\frac{(\gamma^{-2/3}t\,\hat{J}(k))^n}{n!}\Bigg| \le (2L_\gamma)^{-1}\sum_{\gamma k\in\widehat{T}_\gamma,|k|\le\gamma^{-b}} c_4\gamma^{-2/3}t\gamma^{2-2b} \\ &\quad\times e^{c_4\gamma^{-2/3}t\gamma^{2-2b}} \le c_5T\gamma^{5/3-3b}e^{c_4T\gamma^{-2/3}\gamma^{2-2b}},\end{aligned} \tag{B.14}$$

if b is chosen small enough.

As

$$|(2L_\gamma)^{-1} - \gamma^{4/3}| \le c_6\gamma^{8/3},$$

and $\hat{J}(k)$ is uniformly bounded, there exists $\zeta > 0$ such that

$$\begin{aligned}&\Bigg|(2L_\gamma)^{-1}\sum_{\gamma k\in\widehat{T}_\gamma,\ |k|\le\gamma^{-b}} e^{-i2\pi\gamma kx}e^{-\gamma^{-2/3}t}\sum_{n=1}^{\infty}\frac{(\gamma^{-2/3}t\hat{J}(k))^n}{n!} \\ &\quad-\gamma^{4/3}\sum_{h\in\mathbf{Z},|h|\le 2L_\gamma\gamma^{1-b}} e^{-i2\pi\gamma^{4/3}hx}e^{-\gamma^{-2/3}t} \\ &\quad\times\sum_{n\ge1}\frac{(\gamma^{-2/3}t\hat{J}(\gamma^{1/3}h))^n}{n!}\Bigg| \le c_7\gamma^{4/3+\zeta},\end{aligned} \tag{B.15}$$

in fact, for $|h|\le 2L_\gamma\gamma^{1-b}$, we have $|\hat{J}(L_\gamma^{-1}\gamma^{-1}h/2) - \hat{J}(\gamma^{1/3}h)| \le c_8\gamma^{5/3-b}$, and we can repeat the argument of (B.14). This completes the analysis of the "small" values of k.

To complete the proof we have show that in (B.4) the sum for $|h| > 2L_\gamma\gamma^{1-b}$ is negligible. For the assumptions on the regularity of J, $|\hat{J}(k)| \le c_9|k|^{-2}$; we thus have

$$\begin{aligned}&\left|\gamma^{4/3}\sum_{h\in\mathbf{Z},|h|>2L_\gamma\gamma^{1-b}} e^{-i2\pi\gamma^{4/3}hx}e^{-\gamma^{-2/3}t}\sum_{n=1}^{\infty}\frac{(\gamma^{-2/3}t\hat{J}(k))^n}{n!}\right| \\ &\quad\le c_{10}\gamma^{4/3}\gamma^{-2/3}e^{-\gamma^{-2/3}t}\sum_{h\in\mathbf{Z},|h|>2L_\gamma\gamma^{1-b}}|k|^{-2}e^{c_9\gamma^{-2/3}t\gamma^{2b}} \le c_{11}\gamma^{5/3}.\end{aligned} \tag{B.16}$$

As

$$p^\gamma_{\gamma^{-2/3}t}(x) = e^{-\gamma^{-2/3}t}\delta_{x,0} + p^{\gamma,>}_{\gamma^{-2/3}t}(x) + p^{\gamma,<}_{\gamma^{-2/3}t}(x)$$

and we are assuming $x\ne0$, the result stated in the lemma follows from (B.10), (B.14), (B.15), (B.16).

Proof of Theorem B.1. We have

$$\gamma^{1/3}g_t^p(\gamma^{4/3}x) = \sum_{m\in\mathbf{Z}} g_{\gamma^{-2/3}t}(\gamma x + m\gamma^{-1/3}), \tag{B.17}$$

where $g_t(r)$ is the heat kernel on $\mathbf{R}$ (3.25b). Using the result in Lemma B.3, it is then enough to show that for any $\alpha\in[0,\frac12)$ there is C such that for all $t\in(0,T]$ and for all $x\ne0$ in T_γ

$$\begin{aligned}&\Bigg|\gamma^{1/3}\sum_{h\in\mathbf{Z}} e^{-i2\pi\gamma^{1/3}h\gamma x}e^{-\gamma^{-2/3}t}\sum_{n=1}^{\infty}\frac{(\gamma^{-2/3}t\hat{J}(\gamma^{1/3}h))^n}{n!} \\ &\quad-\sum_{m\in\mathbf{Z}} g_{\gamma^{-2/3}t}(\gamma x+\gamma^{-1/3}m)\Bigg| \le C(\gamma^{-2/3}t)^{-1+\alpha}.\end{aligned} \tag{B.18}$$

Let $\mathcal{T}_{\gamma^{-1/3}}$ be the torus of length $\gamma^{-1/3}$, $\tau=\gamma^{-2/3}t$; and, for $r\in\mathcal{T}_{\gamma^{-1/3}}$, define

$$\begin{aligned}q^\gamma_\tau(r) &:= e^{-\tau}\sum_{n\ge1}\frac{\tau^n}{n!}\int_{\mathcal{T}_{\gamma^{-1/3}}}dr_1\cdots\int_{\mathcal{T}_{\gamma^{-1/3}}}dr_{n-1} \\ &\quad\times\prod_{i=1}^{n}J(|r_i-r_{i-1}|),\qquad r_n=r,\quad r_0=0.\end{aligned} \tag{B.19}$$

Consider a jump process on $\mathcal{T}_{\gamma^{-1/3}}$ with intensity $J(|r-r'|)\,dr'$; we can interpret $q^\gamma_\tau(r)$ as the probability density that this process starting from 0 arrives in r at time τ with at least one jump.

We note that the h summation in (B.17) is just the Fourier expansion of $q^\gamma_t(r)$, in fact,

$$q^\gamma_\tau(r) = \gamma^{1/3}\sum_{h\in\mathbf{Z}} e^{-i2\pi\gamma^{1/3}hr}e^{-\tau}\sum_{n\ge1}\frac{(\tau\hat{J}(\gamma^{1/3}h))^n}{n!}. \tag{B.20}$$

Let us introduce, for $r\in\mathbf{R}$,

$$\begin{aligned}q^\infty_\tau(r) &:= e^{-\tau}\sum_{n\ge1}\frac{\tau^n}{n!}\int dr_1\cdots dr_{n-1}\prod_{i=1}^{n}J(|r_i-r_{i-1}|), \\ r_n &= r,\quad r_0=0,\end{aligned} \tag{B.21}$$

the same probability density as in (B.18), but for a jump process in $\mathbf{R}$. We thus have

$$q_\tau^\gamma(r) = \sum_{m\in\mathbf{Z}} q_\tau^\infty(r + m\gamma^{-1/3}). \tag{B.22}$$

We can then write the left-hand side of (B.18) as (with $r = \gamma x$)

$$\Big|q_\tau^\gamma(r) - \sum_{m\in\mathbf{Z}} g_\tau(r + m\gamma^{-1/3})\Big|$$
$$\le \sum_{m\in\mathbf{Z}} \left|\frac{1}{2\pi}\int dk e^{-ik(r+m\gamma^{-1/3})}\left(e^{-\tau}\sum_{n\ge1}\frac{(\tau\hat J(k))^n}{n!} - \exp\Big\{-\frac{D\tau k^2}{2}\Big\}\right)\right|, \tag{B.23}$$

where we used the Fourier transform expression for q_τ^∞ and $g_\tau(r)$.

To complete the proof of Theorem B.1 we show that (B.23) may be bounded by $C\tau^{-1+\alpha}$.

We analyze first the term for $m = 0$. Introduce $\delta_\tau = \tau^{-1/4}$ and consider the values of k smaller than δ_τ. We have

$$\int_{|k|\le\delta_\tau} dk \left|e^{-\tau}\sum_{n\ge1}\frac{(\tau\hat J(k))^n}{n!} - e^{\tau(\hat J(k)-1)}\right| \le \int_{|k|\le\delta_\tau} dk\, e^{-\tau} = 2\tau^{-1/4}e^{-\tau}. \tag{B.24}$$

Since $J \in C_0^4(\mathbf{R})$, recalling the definition of D (2.10), we have

$$\left|\hat J(k) - 1 + \tfrac12 Dk^2\right| \le c_1 k^4;$$

thus

$$\int_{|k|\le\delta_\tau} dk \left|e^{\tau(\hat J(k)-1)} - e^{-D\tau k^2/2}\right| \le \tau^{-1/2}\int_{|k|\le\tau^{1/4}} dk\, e^{-Dk^2/2} c_1\tau^{-1}k^4 e^{c_1\tau^{-1}k^4}$$
$$\le c_2\tau^{-1+\alpha}\int dk\,|k|^{4(1/2-\alpha)}e^{-Dk^2/2}. \tag{B.25}$$

We now consider $|k| > \delta_\tau$ and estimate separately the difference in (B.23). For the Gaussian kernel we have

$$\int_{|k|>\delta_\tau} dk\, e^{-\tau Dk^2/2} = \tau^{-1/2}\int_{|k|>\tau^{1/4}} dk\, e^{-Dk^2/2}$$
$$\le \tau^{-1+\alpha}\int dk\,|k|^{4(1/2-\alpha)}e^{-Dk^2/2}. \tag{B.26}$$

From the regularity conditions on J (see (5.24b) in [12] for the details)

$$\exists\, \delta_0,\ \varepsilon_0 > 0\colon\ \forall \delta \le \delta_0 \qquad \sup_{|k|\ge\delta}\left|\hat J(k)\right| \le 1 - \varepsilon_0\delta^2,$$

if $\delta_\tau \le \delta_0$ we thus have

$$\int_{|k|>\delta_\tau} dk\,\tau\left|\hat J(k)\right|e^{\tau(\hat J(k)-1)} \le \tau e^{-\varepsilon_0\tau^{1/2}}\int dk\,\left|\hat J(k)\right|. \tag{B.27}$$

If $\delta_\tau > \delta_0$ we simply note that there exists $c_3 = c_3(\delta_0)$ such that $\tau \le c_3\tau^{-1+\alpha}$ and use $|\hat J(k)| \le 1$ to get

$$\int_{|k|>\delta_\tau} dk\,\tau\left|\hat J(k)\right|e^{\tau(\hat J(k)-1)} \le c_3\tau^{-1+\alpha}\int dk\left|\hat J(k)\right|. \tag{B.28}$$

From (B.24)–(B.28) it follows that for any $\alpha \in (0,\frac12)$ the term for $m = 0$ in (B.23) may be bounded by $C\tau^{-1+\alpha}$.

We now consider the other terms in the m summation: by integrating twice by parts they may be bounded by

$$\sum_{m\ne0}\frac{1}{2\pi}\int dk\,|r + m\gamma^{-1/3}|^{-2}\left|\frac{d^2}{dk^2}\left(e^{-\tau(1-\hat J(k))} - e^{-\tau} - e^{-\frac{D\tau k^2}{2}}\right)\right|$$
$$\le c_4\gamma^{2/3}\int dk\left|e^{-\tau(1-\hat J(k))}\left(\tau\hat J''(k) + (\tau\hat J'(k))^2\right)\right.$$
$$\left. - e^{-\frac{D\tau k^2}{2}}\left(-\tau D + (\tau Dk)^2\right)\right|, \tag{B.29}$$

since $|r| \le \frac12$.

We now show that the integral may be bounded by $C\tau^{-1+\alpha}$.

As before, we consider first the case $|k| \le \delta_t$. Since

$$\left|\tau\hat J''(k) + (\tau\hat J'(k))^2 + \tau D - (\tau Dk)^2\right| \le c_5\tau(k^2 + \tau k^4),$$

we have

$$\int_{|k|\le\delta_t} dk\, e^{-D\tau k^2/2}\left|\tau\hat J''(k) + (\tau\hat J'(k))^2 + \tau D - (\tau Dk)^2\right|$$
$$\le c_5\tau^{-1+\alpha}\int dk\, e^{-k^2/2}\left(|k|^{4(1-\alpha)} + |k|^{4(1+\alpha)}\right). \tag{B.30}$$

On the other hand,

$$\left|\tau^2\hat J'(k)^2 + \tau\hat J''(k)\right| \le D\tau(D\tau k^2 + 1),$$

thus, as in (B.25)

$$\int_{|k|\le\delta_\tau} dk\left|\left(e^{\tau(\hat J(k)-1)} - e^{-D\tau k^2/2}\right)(\tau^2\hat J'(k)^2 + \tau\hat J''(k))\right|$$
$$\le c_6\tau^{-1+\alpha}\int dk\, e^{-Dk^2/2}(Dk^2 + 1)|k|^{4(1/2+\alpha)}. \tag{B.31}$$

We now consider the integral for $|k| > \delta_t$ and bound separately the difference in (B.29). For the Gaussian kernel, as in (B.26) we have

$$\int_{|k|>\delta_\tau} dk\, e^{-\tau Dk^2/2}D\tau(1 + D\tau k^2)$$
$$\le D\tau^{-1+\alpha}\int dk\, e^{-Dk^2/2}|k|^{4(3/2-\alpha)}(1 + D\tau k^2). \tag{B.32}$$

For the other term, if $\delta_\tau \leq \delta_0$, we have, as in (B.27)

$$\text{(B.33)} \quad \int_{|k|>\delta_\tau} dk\, e^{\tau(\hat{J}(k)-1)}\tau\left|\tau\hat{J}'(k)^2 + \hat{J}''(k)\right| \leq \tau e^{-\epsilon_0\tau^{1/2}} \int dk\left|\tau\hat{J}'(k)^2 + \hat{J}''(k)\right|.$$

We remark that in our regularity hypotheses on J the above integral is finite. If $\delta_\tau > \delta_0$ we proceed, as in (B.28). This ends the proof of (B.18), hence of the Theorem B.1.

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