Quantum framed knot invariants

Marta Catalano

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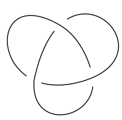
Abstract

In this lecture notes we report a connection between quantum groups and knot theory, due and inspired by works of Turaev, Reshetikhin, Lustzig, Rosso, Jimbo, Drinfel'd and Kauffman, among the others. We derive framed knot invariants from ribbon categories and ribbon categories from (topological) ribbon Hopf algebras. We state that quantum groups have a topological ribbon Hopf algebra structure, which engenders interesting framed knot invariants. In particular, we show how to derive the Kauffman bracket from the quantum group $U_h(\mathfrak{sl}_2)$.

1 Brief introduction to knot theory

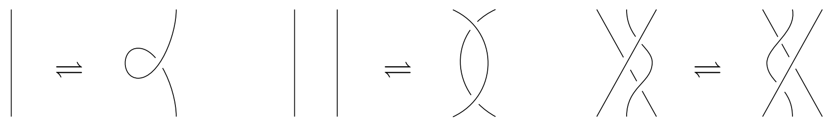
Knot theory is based on assumptions which strictly follow the intuitive idea of a knot. First of all we say that a *knot* is an embedding of S^1 in $\mathbb{R}^3$. We want for two knots to be equivalent when they have the same entanglement, that is when they can be transformed one into the other without breaking the thread. We translate this idea in mathematics by requiring *equivalent knots* to be ambient isotopic.
We could also want to consider *oriented knots* and ask for the (ambient) isotopy to preserve the orientation. Another notion one frequently uses is the one of *link*, that is a disjoint and entangled union of knots in the space. In the rest of the paper, every time we write knot, we are actually referring to links.
In general, one could say that knot theory aims at highlighting and distinguishing equivalence classes of knots and links, both in the oriented and unoriented case.

Knots can be represented by knot diagrams, which have the advantage of living in $\mathbb{R}^2$.



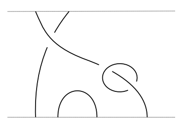
A fundamental result from Reidemeister characterises the isotopy classes of knots through some moves on their diagrams, called the *Reidemeister moves.*

Theorem 1.1. *Two knot diagrams represent equivalent knots if and only if they can be obtained from one another through a finite number of the following moves:*



R1 R2 R3

By looking at the Reidemeister moves, we realise that they do not occur between knot diagrams, but between portions of knots diagrams with boundary, the so called *tangles.* We define an (n, m)-tangle to be a tangle with n upper boundary points and m lower ones. For example, the following picture shows a (2,4)-tangle.



Reidemeister moves can be generalised to oriented knots, by considering all possible orientations of their strings. In this case, we will refer to *oriented* (m, n)*-tangle.*

The notion of *framed knot* will also be useful in what follows. The inspiring idea consists in thickening the knot to obtain a ribbon. Formally, we require the existence of an homotopy class of a normal non singular vector field.
As in the previuos case, we define (m, n)*-ribbon tangles.*

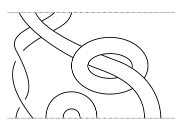
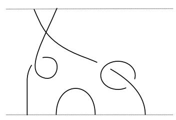


Figure 1: (2,4)-ribbon tangle

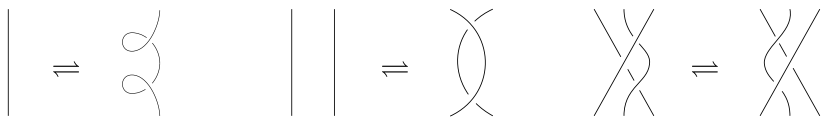
We can consider an equivalence relation on ribbon tangles, which consists in boundary preserving ambient isotopy. One can prove that [1]

This turns out to be very important because it allows us to represent ribbon tangles through standard tangles, up to isotopy. In fact, it suffices to suppose that the normal vector is parallel to the plane. For example, the ribbon tangle in Figure 1 can be represented by



We call *ribbon tangle diagram* a representation of a ribbon tangle through tangles. The next theorem [2] characterises ribbon tangle isotopy through some moves on diagrams.

Theorem 1.2. *Two tangles represent equivalent framed knots if and only if they can be obtained from one another through a finite number of the following moves:*



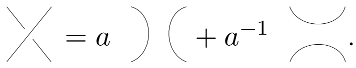
We point out that the second and third move coincide with the second and third Reidemeister move. In particular, every knot invariant will be a framed knot invariant, while the inverse is not true in general. Nonetheless, every time we have a framed knot invariant, it suffices to check the first Reidemeister move to prove that it is a knot invariant as well.
The previous theorem can be easily adapted to oriented ribbon tangles.

2 Kauffman bracket and Jones polynomial

In this section we introduce a framed knot invariant, known as the Kauffman bracket.

[1] [Tur] Chapter I, Section 2.1.
[2] [Tur] Chapter XII Section 1.2

Definition 2.1. If K is a commutative ring with unity and $a \in K$ is an invertible element, we consider the K-module $E_{k,l} = E_{k,l}(a)$ generated by (k,l)-tangles and quotiened by the following relations:

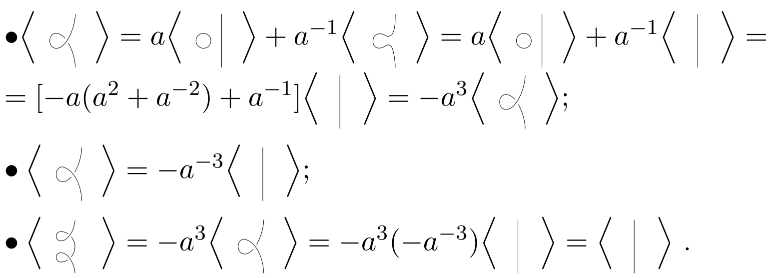
1. Boundary preserving planar isotopy;
2. $D \sqcup \bigcirc = -(a^2 + a^{-2})D \qquad \forall D$ tangle;
3. The skein relations: 

$E_{k,l}$ is called the **(k,l)-skein module** correspondent to a.

Every (k,l)-tangle D represents an element of a class in $E_{k,l}$, denoted by $\langle D \rangle$. By applying the skein relation to all crossings in D, we get $\langle D \rangle$ in the form of a linear combination of classes of diagrams with no crossings; applying the second relation, we can write $\langle D \rangle$ as a linear combination of classes of diagrams with $\frac{k+l}{2}$ simple disjoint arc diagrams, the so called *simple diagrams.* We can therefore conclude that $E_{k,l}$ is a free K-module which has the simple diagrams as basis.
In particular, $E_{0,0} \simeq K$ is generated by the skein class of $\langle \emptyset \rangle = 1$.

Theorem 2.1. *The skein class of a (k,l)-tangle is a ribbon tangle invariant.*

Proof. The proof consists in manually verifying that tangles from the same move are sent in the same skein class. Let's check for example the first move.



□

In particular, if L is a framed knot, its skein class $\langle L \rangle \in E_{0,0} \simeq K$ is an invariant . We can also see $\langle L \rangle \in \mathbb{Z}[a, a^{-1}]$ as a Laurent polynomial, which we call **Kauffman bracket**.
The Kauffman bracket plays an important role in knot theory because under a slight modification we get a very famous oriented knot invariant, the *Jones polynomial.* Though this is not its original definition, we can introduce the Jones polynomial [3] through the Kauffman bracket by setting

$$V_L(a) = (-a)^{w(D)} \langle D \rangle, \tag{1}$$

[3] Actually, this is the Jones Polynomial up to normalisation. In order to get the Jones polynomial one should substitute $a = t^{\frac{1}{4}}$ and divide by $-(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$.

where L is an oriented knot, D is a diagram of L and $W(D)$ is its writhe, i.e. the algebraic sum of the signs of its crossings, where $sign($  $) = 1$ and $sign($  $) = -1$.
Proving that the Jones polynomial is a knot invariant is very easy through this definition. Again, we prove it manually by analysing the three Reidemeister moves.

In the following paragraphs we will describe a method for building framed knot invariants, connect quantum groups to this method and finally show how a specific quantum group determines the Kauffman bracket.

3 Ribbon categories

In order to describe our family of framed knot invariants, we need the notion of ribbon category, which is a monoidal category with some extra structure. Giving complete definitions would result more formal than useful. We therefore refer to [Tur] for a complete review on the subject.

The definition of monoidal category encodes the existence of an associative product with unity. When reading the definition, one should always think of the properties of the tensor product for modules, in order for it not to become too abstract.

Definition 3.1. A **monoidal category** $(\mathcal{C},\ \otimes,\ 1_{\mathcal{C}},\ a,\ r,\ l)$ consists of

- A category $\mathcal{C}$;
- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor product*;
- An object $1_{\mathcal{C}} \in Ob(\mathcal{C})$ called *unit object*;
- Three natural isomorphisms $a,\ l,\ r\ \in Mor(\mathcal{C})$, such that $\forall U, V, Z \in Ob(\mathcal{C})$
  $\cdot\ a_{U,W,Z} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, called *associativity isomorphism*;
  $\cdot\ l_V : 1_{\mathcal{C}} \otimes V \to V$;
  $\cdot\ r_V : V \otimes 1_{\mathcal{C}} \to V$.

We also require the commutativity of three diagrams, which encode the idea that a, l and r behave well with respect to the tensor product.

A monoidal category is said to be *strict* if $a = id$, $r = id$ and $l = id$. Maclane showed that every monoidal category is equivalent to a strict one. We will therefore use an abuse of notation and every time we write monoidal category we will implicitly assume it to be strict.

For modules, we have a canonic isomorphism $U \otimes V \cong V \otimes U$. The existence of a braiding in a monoidal category translates this fact.

Definition 3.2. A **braided category** $(\mathcal{C},\ \otimes,\ 1_{\mathcal{C}}, c)$ is a monoidal category with a natural isomorphism

$$c_{U,V} : U \otimes V \to V \otimes U \tag{2}$$

together with two diagrams which ensure that it behaves well with respect to the tensor product.

The braiding for modules is involutive. The translation of this feature brings to *symmetric categories*, where $c_{U,V}c_{V,U} = id_{W\otimes V}$. In many cases this is too strong a condition and we therefore encode a weaker one in the existence of a twist.

Definition 3.3. A **twist** θ on a braided category $(\mathcal{C},\ \otimes,\ 1_{\mathcal{C}}, c)$ is a natural isomorphism

$$\theta_V : V \to V \tag{3}$$

which behaves well with respect to the braiding in the following sense:

$$\begin{tikzcd}
U \otimes V \arrow[r, "\theta_U \otimes \theta_V"] \arrow[d, "\theta_{U\otimes V}"'] & V \otimes U \arrow[d, "c_{V,U}"] \\
V \otimes U & U \otimes V. \arrow[l, "c_{U,V}"]
\end{tikzcd} \tag{4}$$

Duality in a monoidal category generalises the concept of duality for a module, from the point of view of evaluation and coevaluation pairings.

Definition 3.4. A **duality** $(*, b, d)$ in a monoidal category $(\mathcal{C},\ \otimes,\ 1_{\mathcal{C}})$ consists of

- An object V^* for every $V \in Ob(\mathcal{C})$;
- A morphism $b_v : 1_{\mathcal{C}} \to V \otimes V^*$ for every $V \in Ob(\mathcal{C})$;
- A morphism $d_V : V \otimes V^* \to 1_{\mathcal{C}}$ for every $V \in Ob(\mathcal{C})$;

such that the following diagram commutes for every $V \in Ob(\mathcal{C})$ and a similar one commutes for every V^*:

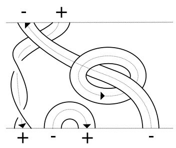
$$\begin{tikzcd}
V \arrow[rr, "id"] \arrow[dr, "b_V \otimes id"'] & & V \arrow[dl, "id \otimes d_V"] \\
& V \otimes V^* \otimes V &
\end{tikzcd} \tag{5}$$

Finally if a braided category has both a twist and a duality and if b_V behaves well with respect to the twist, we call it a **Ribbon category**.

4 *Rib*

In order to understand the role that is played by ribbon categories in knot theory, we now describe a specific ribbon category, which will have a universal property with respect to all other ribbon categories. Moreover, the reader will understand the reason why the characterising morphisms of a ribbon category are called twist and braiding.

Consider an oriented (k,l)-ribbon tangle.



The boundary components of the oriented ribbon tangle inherit a sign in $\{+1, -1\}$ from the orientation of the ribbon tangle: $+1$ if the direction is from the top down, -1 otherwise.
We can then build a ribbon category *Rib* by considering:
- Objects given by the words in the alphabet $\{+1, -1\}$ and the empty set ;
- Morphisms consisting in ribbon tangles modulo isotopy;
- Composition rule obtained by placing one ribbon tangle under the other;
- Tensor product given by juxtaposition of tangles;
- Duality: $(\epsilon_1, .., \epsilon_n)^* := (-\epsilon_1, .., -\epsilon_n)$, $\epsilon_i \in \{+1, -1\}$.
- Braiding:

c_{++} c_{+-} c_{-+} c_{--}

c^{-1}_{++} c^{-1}_{+-} c^{-1}_{-+} c^{-1}_{--}

- Twist:

θ_+ θ_+^{-1} θ_- θ_-^{-1}

- Coevaluation and Evaluation:

b_+ b_- d_+ d_-

The proof of *Rib* being a ribbon category can be found both in [Tur] and in [Kas].
Rib is often called the *free ribbon category* because of the following property [4].

Theorem 4.1. *Let $\mathcal{C}$ be a ribbon category and V be an object of $\mathcal{C}$. There exists one and only one functor*

$$F_V : Rib \to \mathcal{C} \tag{6}$$

such that it preserves tensor products, duals, braiding and twist and $F_V(+) = V$. F_V is called the Reshetikhin-Turaev functor *for V.*

We can consider the full subcategory of *Rib* which has only the empty set as object and framed knots modulo isotopy as morphisms. Clearly, the image of a framed knot through a Reshetikhin-Turaev functor is a framed knot invariant.
As the reader may expect, we will describe a Reshetikhin-Turaev functor which sends a framed knot in its Kauffman bracket.

5 Ribbon hopf algebras

In the last section we have shown how ribbon categories determine framed knot invariants. In the present one, we will show a procedure for deriving ribbon categories, by considering the category of representations of particular Hopf algebras, which we will call ribbon Hopf algebras.

[4] [Kas] Chapter XIV, Section 5.1

Proposition 5.1. [5] *Let (A, Δ, ϵ, s) be a Hopf algebra on K commutative ring with unity 1. The category $Rep(A)$ of representations, i.e. finite rank A-modules, is a monoidal category with duality, through the following definitions: if V, W are A-modules*

- $V \otimes W := V \otimes_K W$ *with A-module operation* $a \cdot v \otimes w = \Delta a(v \otimes w)$;
- $1_{Rep(A)} := K$ *with A-module operation* $a \cdot k = \epsilon(a)k$;
- $V^* := Hom_K(V, K)$ *with A-module operation* $(a \cdot y)(x) = y(s(a)x)$;
- $d_V : V^* \otimes V \to K$ *standard evaluation pairing;*
- $b_V = (d_V)^*$.

The notion of quasi triangular Hopf algebra involves the requirement of a specific element $R \in A \otimes A$, called *universal R-matrix.* This element will ensure for $Rep(A)$ to be a monoidal category with duality and braiding. Before introducing it formally, we need some simple definitions and notation conventions.

Definition 5.1. The **opposite comultiplication** is a linear map $\Delta' : A \to A \otimes A$ defined by $\Delta'(a) = \tau_{1,2}\Delta(a)$, where τ_{12} is the flip.

We use the following notation. If $R \in A \otimes A$, then
- $R_{12} := R \otimes 1_A \in A \otimes A \otimes A$;
- $R_{23} := 1_A \otimes R \in A \otimes A \otimes A$;
- $R_{13} := (id_A \otimes \tau_{12})R_{12} = (\tau_{12} \otimes id_A)R_{23} \in A \otimes A \otimes A$.

Definition 5.2. Let (A, Δ, ϵ, s) be a Hopf algebra and $R \in A \otimes A$ an invertible element. (A, R) is a **quasi triangular Hopf algebra** if

1. $\Delta'(a) = R\Delta(a)R^{-1}$;
2. $(id_A \otimes \Delta)R = R_{13}R_{12}$;
3. $(\Delta \otimes id_A)R = R_{13}R_{23}$.

R is called a **universal R matrix**.

Proposition 5.2. [6] *If (A, R) is a quasi triangular Hopf algebra, $Rep(A)$ is a monoidal category with duality and braiding $c_{UV} : U \otimes V \to V \otimes U$ defined by:*

$$c_{U,V}(x \otimes y) = \tau_{12}(R \cdot (x \otimes y)). \tag{7}$$

We finally introduce some extra structure on A which will ensure that $Rep(A)$ is a ribbon Hopf category.

[5] [Tur] Chatper XI, Section 1.3.1
[6] [Tur] Chapter XI, Section 2.3.1

Definition 5.3. A **ribbon Hopf algebra** consists of (A, R, ν), where (A, R) is a quasi triangular Hopf algebra and ν is an invertible element in the centre of A such that:

1. $\Delta(\nu) = \tau_{12}(R)R(\nu \otimes \nu)$;
2. $s(\nu) = \nu$.

The element ν is called **universal twist** of A.

Proposition 5.3. [7] *If (A, R, ν) is a ribbon Hopf algebra, Rep(A) is a ribbon category with twist $\theta_V : V \to V$ defined by*

$$\theta_V(w) = \nu \cdot w \tag{8}$$

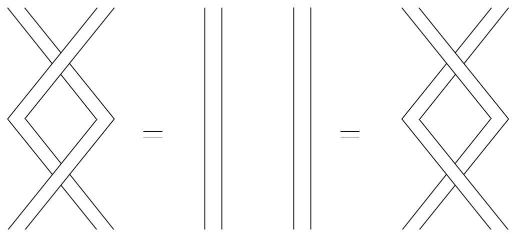
6 Universal quantum enveloping algebras

In the previous sections we saw how every ribbon Hopf algebra determines a framed knot invariant. Our aim consists in finding ribbon Hopf algebras which engender interesting invariants.
The first example of ribbon Hopf algebra is given by the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In fact, this can be considered a ribbon Hopf algebra in a trivial way by setting:

- $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$
  which extends $\Delta(x) = x \otimes 1 + 1 \otimes x$, for every $x \in \mathfrak{g}$;
- $\epsilon : U(\mathfrak{g}) \to K$
  which extends $\epsilon(x) = 0$ for every $x \in \mathfrak{g}$;
- $s : U(\mathfrak{g}) \to U(\mathfrak{g})$
  which extends $s(x) = -x$ for every $x \in \mathfrak{g}$;
- $R = 1 \otimes 1$;
- $\nu = 1$.

In fact, every cocommutative Hopf algebra can be considered a ribbon Hopf algebra: one can prove that Δ cocommutative implies that s is involutive. This implies that the choices $R = 1 \otimes 1$ and $\nu = 1$ are always possible. Unfortunately, the invariant which is engendered by these trivial ribbon Hopf algebras is not interesting at all. In fact, $R = 1 \otimes 1$ implies that the braiding is involutive, that is $c_{U,V} c_{V,U} = id_{V \otimes U}$. This tells us that the knot invariant locally verifies

[7] [Tur] Chapter XI, Section 3.2



If every crossing can be untied, it is clear that the framed knot invariant does not distinguish any class and is therefore useless.
This is where quantum groups come at help, by deforming universal enveloping algebras in such a way that the resulting "Hopf algebra" is not cocommutative and engenders interesting knot invariants. The reason we wrote "Hopf algebra" is because we do not get a standard Hopf algebra, but a topological one. We now give a brief account on this definition.

Definition 6.1. A **topological algebra** A is an algebra on the ring $\mathbb{C}[[h]]$ of formal power series in one variable, whose tensor product is not the standard one, but its completion under the *h-adic topology.* We call it *topological tensor product.*

The reader can find a brief but complete account on the h-adic topology in [Kas]. The definition of *topological Hopf algebra* and *topological ribbon Hopf algebra* can be intuitively recovered from the classical case.

Definition 6.2. A **quantization** or **deformation** of a Hopf algebra A on $\mathbb{C}$ is a topological Hopf algebra A_h on $\mathbb{C}[[h]]$ such that:

1. $A_h \cong A[[h]]$ as modules;

2. $A_h/hA_h \cong A$ as Hopf algebras.

In particular, if we consider the definition of quantum group $U_q(\mathfrak{g})$ by Jimbo and Drinfeld, where $\mathfrak{g}$ is a semisimple Lie algebra, one can show that $U_q(\mathfrak{g})$ is a quantization of the universal enveloping algebra $U(\mathfrak{g})$. We usually call $U_q(\mathfrak{g})$ the *universal quantum enveloping algebra* of $\mathfrak{g}$.
The topological Hopf algebra described by Jimbo and Drinfeld can be canonically considered as a topological ribbon Hopf algebra. One can extend what we have seen for classical ribbon Hopf algebras to topological ribbon Hopf algebras: the representations of a topological ribbon Hopf algebra form a ribbon category in the same way. We must underline though that a representation in the topological context is an A-module which is topologically free and of finite rank, i.e. isomorphic to $V[[h]]$ as module, where V is a finite dimensional vector space.
Quantum groups turn out to be non-cocommutative and non-involutive topological Hopf algebras, and they will therefore hopefully engender interesting

framed knot invariants. In fact, they do. In the next section we will see an example and show how $U_h(\mathfrak{sl}_2)$ brings to the construction of the Kauffman bracket.

7 $U_h(\mathfrak{sl}_2)$

Jimbo and Drinfeld gave a definition of quantum group by generators and relations. Applying it in the case $\mathfrak{g} = \mathfrak{sl}_2$ we get the following definition.

Definition 7.1. $\boldsymbol{U_h(\mathfrak{sl}_2)}$ is the $\mathbb{C}[[h]]$-algebra with $\{E, F, H\}$ as generators and the following relations:

- $[H, E] = 2E$;
- $[H, F] = -2F$;
- $[E, F] = \frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}}$

We observe that for $h \to 0$ we find the relations which define $U(\mathfrak{sl}_2)$:

$$\frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}} = \frac{e^{hH/2} - e^{-hH/2}}{hH/2} \cdot \frac{h/2}{e^{h/2} - e^{-h/2}} \cdot H \xrightarrow{h \to 0} H \tag{9}$$

$U_h(\mathfrak{sl}_2)$ has a structure of a topological hopf algebra by setting:

- $\Delta(H) = H \otimes 1 + 1 \otimes H$;
- $\Delta(E) = E \otimes e^{hH/2} + 1 \otimes E$;
- $\Delta(F) = F \otimes 1 + e^{-hH/2} \otimes F$;
- $\epsilon(H) = \epsilon(E) = \epsilon(F) = 0$;
- $s(H) = -H$;
- $s(E) = -e^{hH/2}E$;
- $s(F) = e^{-hH/2}F$.

We immediately realise that it is not cocommutative nor involutive.
This topological Hopf algebra can be canonically extended to a ribbon one. We do not write down the explicit expression of the universal twist, since we won't use it, but we write down the universal R matrix:

$$R := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(q - q^{-1})^n}{[n]_q!} e^{h(H \otimes H)/2} E^n \otimes F^n \tag{10}$$

where $q := e^{h/2}$.
In general, the representation theory of the universal quantum enveloping algebra of a semisimple Lie algebra is strictly related to the representation theory of the Lie algebra, as shown from the following theorem.

Theorem 7.1. *For every dominant weight λ of $\mathfrak{g}$, there exists one and only one finite and topologically free $U_h(\mathfrak{g})$-module $\tilde{V}_\lambda$ such that:*

1. $\tilde{V}_\lambda / h\tilde{V}_\lambda \cong V_\lambda$

2. $\tilde{V}_\lambda$ *is generated by an element $\tilde{v}_\lambda$, called* heighest weight vector*, such that:*

$$H_i \tilde{v}_\lambda = \lambda(H_i)\tilde{v}_\lambda \qquad E_i \tilde{v}_\lambda = 0 \tag{11}$$

When dealing with $U_h(\mathfrak{sl}_2)$ one can show that every finite and topologically free module is direct sum of modules of the previous form, so that the representation theory of $U_h(\mathfrak{sl}_2)$ is completely similar to the one of $U(\mathfrak{sl}_2)$. The following theorem holds.

Theorem 7.2. *The indecomposable representations of $U_h(\mathfrak{sl}_2)$ are indexed by $\mathbb{N}$. Explicitly, the n-th representation V_n is a free module on $\mathbb{C}[[h]]$ of rank $n+1$, whose basis $\{v_0, .., v_r\}$ verifies:*

- $Hv_r = (n-2r)v_r$;
- $Ev_r = [n-r+1]_q v_{r+1}$;
- $Fv_r = [r+1]_q v_{r+1}$;

where once again $q = e^{h/2}$.

We observe that for $k \in \mathbb{N}$ and $q = e^{h/2}$ we have:

$$[k]_q = \frac{e^{kh/2} - e^{-kh/2}}{e^{h/2} - e^{-h/2}} \xrightarrow{h \to 0} k \tag{12}$$

so that we find the standard representations of $U(\mathfrak{sl}_2)$.

Let $\rho_n : U_h(\mathfrak{sl}_2) \to \mathfrak{gl}(n+1)$ be the representation which corresponds to

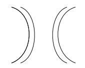
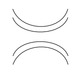
V_n. By considering its matrix form, it will behave in the following way:

$$\rho_n(E) = \begin{pmatrix} 0 & [n]_q & 0 & \cdots & 0 \\ 0 & 0 & [n-1]_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{13}$$

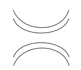
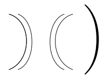
$$\rho_n(F) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & [2]_q & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n]_q & 0 \end{pmatrix} \tag{14}$$

$$\rho_n(H) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -n \end{pmatrix} \tag{15}$$

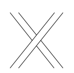
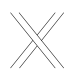
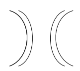
We will show that the Kauffman bracket is the invariant we derive from the 2-dimensional representation of $U_h(\mathfrak{sl}_2)$, once we set $a = q^{1/2} = e^{h/4}$. We indicate with F_1 the Reshetikhin-Turaev functor which corresponds to such representation. We want to prove that

$F_1\Big($  $\Big) = q^{1/2} F_1\Big($  $\Big) + q^{-1/2} F_1\Big($  $\Big)$ (16)

To avoid the calculation of $F_1($  $)$, we observe that equation (16) applied to  tells us that

$F_1\Big($  $\Big) = q^{1/2} F_1\Big($  $\Big) + q^{-1/2} F_1\Big($  $\Big).$ (17)

With the appropriate substitution of $F_1($  $)$, it turns out that it suffices to prove that

$q^{1/2} F_1\Big($  $\Big) - q^{-1/2} F_1\Big($  $\Big) = (q - q^{-1}) F_1\Big($  $\Big)$ (18)

. Translating the last equation in terms of the ribbon category, we must prove that

$$q^{1/2} c_{V_1,V_1} - q^{-1/2}(c_{V_1,V_1})^{-1} = (q - q^{-1}) id_{V_1 \tilde{\otimes} V_1} \tag{19}$$

We have said that $c_{V_1,V_1}(x \otimes y) = \tau_{12}(R \cdot (x \otimes y)$. In terms of the representation map, this tells us that $c_{V_1,V_1} = \tau_{12}\rho_1(R) =: \hat{R}$.

In [Kas] there is the explicit matrix expression of $\hat{R}$, on the basis $\{v_0 \otimes v_0, v_1 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_1\}$, which can be calculated from (10).

$$\hat{R} = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} - q^{-3/2} & q^{-1/2} & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix} \tag{20}$$

An explicit calculation tells us that

$$\hat{R}^{-1} = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & q^{1/2} & q^{-1/2} - q^{3/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}, \tag{21}$$

so that we can finally state that:

$$q^{1/2}\hat{R} - q^{-1/2}\hat{R}^{-1} = \begin{pmatrix} q - q^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & 0 & 0 \\ 0 & 0 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q - q^{-1} \end{pmatrix} = (q - q^{-1})id. \tag{22}$$

References

[Tur] Vladimir G. Turaev, *Quantum invariants of knots and 3-manifolds.* Studies in Mathematics, Walter de Gruyter and Co., 1994.

[Kas] Christian Kassel, *Quantum groups.* Volume 155 of GTM, Springer-Verlag, 1994.