Sapienza Università di Roma
Seminar for the Course "Quantum Groups and C* Categories"

Quantum Groups at Root of Unity and Tilting Modules

Laura Fedele

Among the most important examples of quantum groups, there are those which are obtained from a Lie algebra $\mathfrak{g}$ as deformations of the associated universal enveloping algebra $U(\mathfrak{g})$, the so-called *quantized universal enveloping algebras*. Here, we will review some theory of quantized universal enveloping algebras and their representations, focusing on their rational and integral (restricted) forms.

Throughout the notes the (not so) basic example of $\mathfrak{sl}_2(\mathbb{C})$ will be examined. For all proofs see [CP].

1 Hopf Algebras and Deformations

A Hopf algebra over a commutative ring $\mathbf{K}$ is a quintuple $(A, \iota, \mu, \epsilon, \Delta, S)$, where (A, ι, μ) is a $\mathbf{K}$-algebra, (A, ϵ, Δ) is a $\mathbf{K}$-coalgebra, and S is an antiautomorphism of A called *antipode* which satisfies some compatibility conditions with morphisms ι, μ, ϵ, Δ. Moreover, Δ and ϵ are $\mathbf{K}$-algebras morphisms, while μ and ι are $\mathbf{K}$-coalgebras morphisms.

Given a complex Lie algebra $\mathfrak{g}$, its associated universal enveloping algebra $U(\mathfrak{g})$ has a Hopf algebra structure, where coproduct, counit and antipode are defined on *primitive elements* $x \in \mathfrak{g}$ as follows:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x. \tag{1.1}$$

It is easily checked that Δ satisfies $\Delta([x, y]) = [\Delta(x), \Delta(y)]$, hence it extends to a map $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The morphisms ϵ and S extend to maps $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ and $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ respectively, endowing $U(\mathfrak{g})$ of a Hopf algebra structure. $U(\mathfrak{g})$ actually is a cocommutative Hopf algebra, but it only is commutative if the Lie algebra $\mathfrak{g}$ is commutative.

A representation of a Hopf A over $\mathbf{K}$ just is a left A-module (i.e. we are just focusing on the algebra structure of A) Given two left A-modules V and W there is a natural way to define their tensor product as representations: the left action of $a \in A$ on $V \otimes W$ is given by $a(v \otimes w) = \Delta(a)(v \otimes w)$ for all $v \in V$, $w \in W$. We therefore say that the category of representations of the Hopf algebra A is a *tensor* (or *monoidal*) *category*.

A deformation of a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over $\mathbf{K}$ is abstractly defined as a topological Hopf algebra $(A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over the ring of formal series $\mathbf{K}[[h]]$ such that $A_h \cong A[[h]]$ as $\mathbf{K}[[h]]$-modules and both the product μ_h and the coproduct Δ_h coincide with the "original" ones modulo h. We say that two such deformations are equivalent if there exists a Hopf algebra isomorphism over $\mathbf{K}[[h]]$ which is the identity modulo h. It turns out that every definition of the morphisms ι_h, ϵ_h and S_h is equivalent to the *trivial* one, that is the one obtained simply extending $\mathbf{K}[[h]]$-linearly the original morphisms.

Given a Lie algebra $\mathfrak{g}$, a Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ is called a **quantum universal enveloping algebra**, briefly a **QUE algebra**, and it is denoted by $U_h(\mathfrak{g})$.

We are mainly interested in deformations of $U(\mathfrak{g})$ which are non-cocommutative, since it can be proved that when $\mathbf{K}$ is a field of characteristic zero, any cocommutative QUE algebra A_h over $\mathbf{K}[[h]]$ is isomorphic as a topological Hopf algebra over $\mathbf{K}[[h]]$ to the universal enveloping algebra of the some Lie algebra deformation $\mathfrak{g}_h$ of $\mathfrak{g}$. In particular, for every complex simple Lie algebra $\mathfrak{g}$ every deformation would be trivial, since every deformation of $\mathfrak{g}$ as a Lie algebra is trivial.

To get an idea of the non-trivial deformation we are interested in, let us examine the case of $\mathfrak{sl}_2(\mathbb{C})$, discovered by Kulish and Sklyanin (1982).

Example 1.1. $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is the topological Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by elements X^+, X^-, H with the following relations:

$$X^+X^- - X^-X^+ = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \quad HX^\pm - X^\pm H = \pm 2X^\pm. \tag{1.2}$$

Coproduct, counit and antipode are defined on generators as:

- $\Delta_h(H) = H \otimes 1 + 1 \otimes H, \quad S_h(H) = -H, \quad \epsilon_h(H) = 0;$
- $\Delta_h(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+, \quad S_h(X^+) = -X^+e^{-hH}, \quad \epsilon_h(X^+) = 0;$
- $\Delta_h(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-, \quad S_h(X^-) = -e^{hH}X^-, \quad \epsilon_h(X^-) = 0.$

It is clear that the antiautomorphism S_h is not involutive; by a result of Abe it implies that $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is not cocommutative. However, it is "almost" cocommutative, since we can relate the coproduct Δ_h and its opposite Δ_h^{op} (obtained composing Δ_h with the flip map) by means of a special invertible element $\mathcal{R}_h$ in the completion of the tensor product $U_h(\mathfrak{sl}_2(\mathbb{C})) \otimes U_h(\mathfrak{sl}_2(\mathbb{C}))$, called the *universal R-matrix*. Indeed, among other compatibility properties, the universal R-matrix satisfies $\Delta_h^{\mathrm{op}}(x) = \mathcal{R}_h \Delta_h \mathcal{R}_h^{-1}(x)$ for every $x \in U_h(\mathfrak{sl}_2(\mathbb{C}))$. We therefore say that $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is a *quasitriangular Hopf algebra*.

It is remarkable that an analogue of the PBW Theorem holds for $U_h(\mathfrak{sl}_2(\mathbb{C}))$. Indeed, the monomials $(X^-)^r H^s (X^+)^t$, for r, s, $t \in \mathbb{N}$ form a topological basis of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. Finally, note that in the limit $h \to 0$ we retrieve the "classical" Hopf algebra structure of $U(\mathfrak{sl}_2(\mathbb{C}))$.

This construction can be generalized to arbitrary finite dimensional complex Lie algebras (actually, to Kac-Moody algebras associated with symmetrizable generalized Cartan matrices).

It is useful to introduce the following elements of $\mathbb{Z}[q, q^{-1}]$, for $m \geq n \in \mathbb{N}$ and an indeterminate q.

(i) $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$;

(ii) $[n]_q! = \displaystyle\prod_{j=1}^{n} [j]_q$;

(iii) $\begin{bmatrix} m \\ n \end{bmatrix}_q = \dfrac{[m]_q!}{[n]_q![m-n]_q!} = \dfrac{\prod_{j=1}^{n}[m-j+1]_q}{[n]_q!}$.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, with Cartan matrix $A = (a_{ij})_{i,j=1}^{n}$. Let Φ be the root system of $\mathfrak{g}$, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots and Φ^+ the set of positive roots with respect to Π. Recall that $a_{ij} = \langle \alpha_i, \alpha_j \rangle = 2\dfrac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, for any α_i, $\alpha_j \in \Pi$,. Let d_i be positive integers such that the matrix $(d_i a_{ij})_{i,j=1}^{n}$ is symmetric: it is a result of Kac that $d_i \in \{1, 2, 3\}$, and actually $d_i = (\alpha_i, \alpha_i)/2$ for any simple root α_i.

The QUE algebra $U_h(\mathfrak{g})$, first appeared in Drinfeld (1987) and Kimbo (1985), is the topological Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by elements H_i, $X_i^{\pm}$, $i = 1, \dots, n$ and the following relations:

- $[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}$;
- $X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \dfrac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}}$;
- $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{e^{d_i h}} (X_i^{\pm})^r X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-r} = 0, \quad \text{if } i \neq j.$

Coproduct, counit and antipode are defined on generators as follows:

- $\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i$;
- $\Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+, \quad \Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-$;
- $S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ e^{-d_i h H_i}, \quad S_h(X_i^-) = -e^{d_i h H_i} X_i^-$;
- $\varepsilon_h(H_i) = 1, \quad \varepsilon_h(X_i^{\pm}) = 0$.

We can regard the elements $X_i^{\pm}$ as the "root vectors" associated with the simple roots $\alpha_i \in \Pi$ (the elements H_i play here the role of the elements of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$); however to have root vectors associated with any root β we have to consider an action of the braid group of $\mathfrak{g}$ instead of that of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ as in the classical case.

The braid group $\mathcal{B}_{\mathfrak{g}}$ has generators T_i, $i = 1, \dots, n$ and defining relations

$$T_i T_j T_i T_j \cdots = T_j T_i T_j T_i \cdots \tag{1.3}$$

for all $i \neq j$, and there are m_{ij} elements on both sides of the equation, and $m_{ij} = 2$, 3, 4 if $a_{ij} a_{ji} = 0$, 1, 2 respectively. $\mathcal{B}_{\mathfrak{g}}$ acts on $U_h(\mathfrak{g})$ by algebra automorphisms, and the action on the generators H_j coincides with the action of the simple reflections $s_i \in \mathcal{W}$ on the corresponding generators of $\mathfrak{g}$. In the classical case, choosing a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ of the longest element w_0 of $\mathcal{W}$, we get that N is exactly the number of positive roots of $\mathfrak{g}$ and that every positive root occurs exactly once in the set $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N})$. In the same setting, we define positive and negative root vectors $X^{\pm}_{\beta_r}$ for $U_h(\mathfrak{g})$ as follows:

$$X^{\pm}_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(X^{\pm}_{i_r}). \tag{1.4}$$

Another difference with the classical case is that root vectors may now be very different from one another depending on the choice of the reduced expression for w_0 (while in the classical case they coincide up to a sign). However, they allow us to formulate a quantum analogue of the PBW Theorem for $U_h(\mathfrak{g})$:

Theorem 1.2. *The set of products*

$$(X^-_{\beta 1})^{r_1} \cdots (X^-_{\beta_N})^{r_N} H_1^{s_1} \cdots H_n^{s_n} (X^+_{\beta_N})^{t_N} \cdots (X^+_{\beta_1})^{t_1} \tag{1.5}$$

for $r_1, \dots, r_N, t_1, \dots, t_N, s_1, \dots, s_n \in \mathbb{N}$ *forms a topological basis of* $U_h(\mathfrak{g})$.

This was first proved by Rosso (1989) in the $\mathfrak{sl}_{n+1}(\mathbb{C})$ case, using a different definition of root vectors given by Jimbo, and then by Lusztig (1990) in the simply-laced case first and then in the general case.

As for $U_h(\mathfrak{sl}(\mathbb{C}))$, we can define a universal R-matrix for $U_h(\mathfrak{g})$, making it a quasitriangular Hopf algebra; the category of finite-dimensional representations of $U_h(\mathfrak{g})$ is therefore a *braided category.*

2 QUE Algebras: The Rational Form

Intuitively, we think about $U_h(\mathfrak{g})$ as a family of Hopf algebras parametrized by h; however it does not actually make sense, since being $U_h(\mathfrak{g})$ defined over a ring of formal series, we can not specialize h at any elements of $\mathbf{K}$ except $h = 0$. Moreover we have to take care of its topological structure: as a topological Hopf algebra over $\mathbf{K}[[h]]$ it is endowed with the h-adic topology, and we need, for example, to work with the completion of the usual tensor product with respect to this topology. Therefore we will rather work with a "rational" counterpart of $U_h(\mathfrak{g})$ introduced by Jimbo (1985), its so-called *rational form* $U_q(\mathfrak{g})$.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, with Cartan matrix $A = (a_{ij})^n_{i,j=1}$ and integers d_i, $i = 1, \dots, n$ as above. Let Φ, Π and Φ^+ denote the roots system, the set of simple roots and the set of positive roots of $\mathfrak{g}$ respectively. Denote by $\check{\alpha}_i = 2\alpha_i/(\alpha_i, \alpha_i)$ the coroot associated to a simple root α_i (we therefore have $\langle \beta, \alpha_i \rangle = (\beta, \check{\alpha}_i)$ for any $\beta \in Phi$) and by ρ the semi-sum of all positive roots. Let P be the set of weights and $P^+ \subset P$ the set of dominant weights of $\mathfrak{g}$. Let q be an indeterminate, and let $q_i = q^{d_i}$.

$U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{Q}(q)$ with generators X^+_i, X^-_i, K_i, K_i^{-1} for $i = 1, \dots, n$, and the following relations:

- $K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1;$
- $K_i X^+_j K_i^{-1} = q_i^{a_{ij}} X^+_j, \quad K_i X^-_j K_i^{-1} = q_i^{-a_{ij}} X^-_j;$

- $X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$;
- $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (X_i^{\pm})^{1-a_{ij}-r} X_j^{\pm} (X_i^{\pm})^r = 0$ for $i \neq j$.

Moreover, $U_q(\mathfrak{g})$ has a Hopf algebra structure with coproduct, counit and antipode defined on generators as:

- $\Delta_q(K_i) = K_i \otimes K_i, \quad S_q(K_i) = K_i^{-1}, \quad \varepsilon_q(K_i) = 1$;
- $\Delta_q(X^+)_i = X_i^+ \otimes K_i + 1 \otimes X_i^+, \quad S_q(X_i^+) = -X_i^+ K_i^{-1}, \quad \varepsilon_q(X_i^+) = 0$;
- $\Delta_q(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-, \quad S_q(X_i^-) = -K_i X_i^-, \quad \varepsilon_q(X_i^-) = 0$.

The defining relations can be deduced from the corresponding relations for $U_h(\mathfrak{g})$ if we let $q_i = e^{q_i h}$ and $K_i = e^{d_i h H_i}$. The same holds for the maps Δ_q, S_q and ϵ_q on $X_i^{\pm}$. Actually, many of the results established for $U_h(\mathfrak{g})$ have counterparts for $U_q(\mathfrak{g})$. For example, the braid group $\mathcal{B}_{\mathfrak{g}}$ acts on $U_q(\mathfrak{g})$ as $\mathbb{Q}(q)$-algebra automorphisms in a similar way, and we can analogously define root vectors $X_\beta^{\pm}$ for any root β. We therefore have a quantum analogue of the PBW Theorem for $U_q(\mathfrak{g})$ as well.

Theorem 2.1. *The set of products*

$$(X_{\beta_1}^-)^{r_1} \cdots (X_{\beta_N}^-)^{r_N} K_1^{s_1} \cdots K_n^{s_n} (X_{\beta_N}^+)^{t_N} \cdots (X_{\beta_1}^+)^{t_1} \tag{2.1}$$

for $r_1, \ldots, r_N, t_1, \ldots, t_N \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in \mathbb{Z}$ *forms a* $\mathbb{Q}(q)$ *basis of* $U_q(\mathfrak{g})$.

However, not every property of $U_h(\mathfrak{g})$ translates to $U_q(\mathfrak{g})$. For example, it is not true that $U_q(\mathfrak{g})$ possesses a universal R-matrix; it is indeed clear by an explicit description of the universal R-matrix for $U_h(\mathfrak{g})$ that it does not correspond to any element of $U_q(\mathfrak{g})$.

Also remind that this rational form $U_h(\mathfrak{g})$ only allows us to specialize q up to transcendental numbers. If we want to specialize q to an arbitrary non-zero complex number $\epsilon \in \mathbb{C}^*$ we need to consider a so-called *integral form* of $U_q(\mathfrak{g})$. Namely, it is an $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$-subalgebra $U_{\mathcal{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ such that the natural map $U_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}(q) \to U_q(\mathfrak{g})$ is an isomorphism of $\mathbb{Q}(q)$-algebras.

There are two possible integral forms associated with the QUE algebra $U_h(\mathfrak{g})$, namely the "restricted" and the "non-restricted" integral form. Their corresponding specializations coincide when ϵ is not a root of unity, but behave very differently when ϵ is a root of unity. We will not analyse the non-restricted integral form (which, when specialized at a root of unity, has properties closely resembling those of the classical universal enveloping algebra), but we will focus on the restricted integral form instead. This one, when specialized at a root of unity, provides a characteristic zero analogue to the theory of classical Lie algebras in characteristic p.

3 QUE Algebras: The Restricted Integral Form

For any integer $r \geq 0$ and $1 \leq i \leq n$ define the **divided powers**

$$(X_i^+)^{(r)} = \frac{(X_i^+)^r}{[r]_{q_i}!}, \quad (X_i^-)^{(r)} = \frac{(X_i^-)^r}{[r]_{q_i}!}. \tag{3.1}$$

The restricted integral form $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by elements $(X_i^+)^{(r)}$, $(X_i^-)^{(r)}$, K_i, K_i^{-1} for $i = 1, \dots, n$ and $r \geq 1$. It is a Hopf algebra over $\mathcal{A}$ with coproduct, counit and antipode given by

- $\Delta_q(K_i) = K_i \otimes K_i, \quad S_q(K_i) = K_i^{-1}, \quad \epsilon_q(K_i) = 1;$
- $\Delta_q((X_i^+)^{(r)}) = \sum_{k=0}^{r} q_i^{-k(r-k)} (X_i^+)^{(k)} \otimes K_i^k (X_i^+)^{(r-k)}, \quad \epsilon_q((X_i^+)^{(r)}) = 0;$
- $\Delta_q((X_i^-)^{(r)}) = \sum_{k=0}^{r} q_i^{k(r-k)} (X_i^-)^{(k)} \otimes (X_i^-)^{(r-k)} K_i^{-k}, \quad \epsilon_q((X_i^-)^{(r)}) = 0;$
- $S_q((X_i^+)^{(r)}) = (-1)^r q_i^{r(r+1)} K_i^{-r} (X_i^+)^{(r)}, \quad S_q((X_i^-)^{(r)}) = (-1)^r q_i^{-r(r+1)} (X_i^-)^{(r)} K_i^r.$

The definitions of the maps Δ_q, S_q and ϵ_q on the divided powers $(X_i^\pm)^{(r)}$ is a consequence of their definitions of generators $X_i^\pm$ of $U_q(\mathfrak{g})$, as it can be easily proved by induction on r. Moreover we have

$$K_i (X_j^\pm)^{(r)} K_i^{-1} = q^{\pm a_{ij} r} (X_j^\pm)^{(r)} \tag{3.2}$$

and we can rewrite the "quantum Serre relation" as

$$\sum_{r=0}^{1-a_{ij}} (-1)^r (X_i^\pm)^{(1-a_{ij}-r)} X_j^\pm (X_i^\pm)^{(r)} = 0, \quad \text{for } i \neq j. \tag{3.3}$$

We also have the following immediate identity, which will later be useful.

$$(X_i^\pm)^{(r)} (X_i^\pm)^{(s)} = \begin{bmatrix} r+s \\ r \end{bmatrix}_{q_i} (X_i^\pm)^{(r+s)}. \tag{3.4}$$

Let us introduce the following remarkable elements of $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$:

$$\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i} = \prod_{s=1}^{r} \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}} \tag{3.5}$$

for all $i = 1, \dots, n$, $c \in \mathbb{Z}$ and $r, \in \mathbb{N}$.

When $c = 0$ the elements $\begin{bmatrix} K_i; 0 \\ r \end{bmatrix}_{q_i}$ clearly commute with themselves and with the $K_i^{\pm 1}$, and for any $c \in \mathbb{Z}$ the elements $\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i}$ lie in $U_{\mathcal{A}}^{\mathrm{res}\,0}(\mathfrak{g}) := \mathrm{Span}_{\mathcal{A}} \langle K_i^{\pm 1}, \begin{bmatrix} K_i; 0 \\ r \end{bmatrix}_{q_i} \rangle$, a maximal abelian subalgebra of $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$. Moreover, we have the following relations:

$$\begin{bmatrix} K_i; c \\ r \end{bmatrix}_{q_i} (X_j^\pm)^{(s)} = (X_j^\pm)^{(s)} \begin{bmatrix} K_i; c \pm a_{ij} s \\ r \end{bmatrix}_{q_i}. \tag{3.6}$$

The braid group action of $\mathcal{B}_{\mathfrak{g}}$ on $U_q(\mathfrak{g})$ restricts to $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$. Hence, for $r \in \mathbb{N}$, we can define root vectors

$$(X_{\beta_k}^\pm)^{(r)} = T_{i_1} \cdots T_{i_{k-1}} ((X_{i_k}^\pm)^{(r)}) \tag{3.7}$$

and we can formulate a quantum analogue of the PBW Theorem for $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$ as well.

Theorem 3.1. *The set of products*

$$(X^-_{\beta_1})^{(r_1)} \cdots (X^-_{\beta_N})^{(r_N)} \prod_{i=1}^{n} K_i^{\sigma_i} \begin{bmatrix} K_i; 0 \\ s_i \end{bmatrix}_{q_i} (X^+_{\beta_N})^{(t_N)} \cdots (X^+_{\beta_1})^{(t_1)} \tag{3.8}$$

for $r_1, \ldots, r_N, t_1, \ldots, t_N, s_1, \ldots, s_n \in \mathbb{N}$ and $\sigma_i \in \{0,1\}$ forms an $\mathcal{A}$-basis of $U^{res}_{\mathcal{A}}(\mathfrak{g})$.

This basis in analogue to the basis for $U_q(\mathfrak{g})$ given by Theorem 2.1, therefore ensuring that $U^{\mathrm{res}}_{\mathcal{A}}(\mathfrak{g})$ actually is an integral form of $U_q(\mathfrak{g})$.

For any $\epsilon \in \mathbb{C}^*$ the corresponding *restricted specialization* is

$$U^{\mathrm{res}}_{\epsilon}(\mathfrak{g}) := U^{\mathrm{res}}_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}(\epsilon). \tag{3.9}$$

We will now examine what happens when ϵ is a roots of unity: let ϵ be a primitive ℓ-th root of unity, where ℓ is odd and greater than d_i, for all i (and not divisible by 3 if $\mathfrak{g}$ is of type G_2). Let $\epsilon_i = \epsilon^{d_i}$.

It has remarkable implications for the representation theory of $U^{\mathrm{res}}_{\epsilon}(\mathfrak{g})$ that it can be somehow factorized into a "product" of the classical universal enveloping algebra of $\mathfrak{g}$ and a finite-dimensional Hopf algebra. The origin of this factorization can be found in the following factorization of the Gaussian binomial coefficient through the classical binomial coefficient for a decomposition of $s \leq r \in \mathbb{N}$ as given by the Eucliden division by ℓ, $r = r_0 + \ell r_1$, $s = s_0 + \ell s_1$:

$$\begin{bmatrix} r \\ s \end{bmatrix}_{\epsilon} = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}_{\epsilon} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}. \tag{3.10}$$

Hereafter, we will use the notation x_0, x_1 to denote remainder and quotient respectively of the Euclidean division of a number $x \in \mathbb{N}$ by ℓ; clearly $0 \leq r_0 < \ell$.

Combining this factorization property with Equation (3.4), for $r = r_0 + \ell r_1$ we deduce that in $U^{\mathrm{res}}_{\epsilon}(\mathfrak{g})$

$$(X^{\pm}_i)^{(r)} = (X^{\pm}_i)^{(r_0)} \frac{((X^{\pm}_i)^{(\ell)})^{r_1}}{r_1!}. \tag{3.11}$$

It follows that $U^{\mathrm{res}}_{\epsilon}(\mathfrak{g})$ is generated as a $\mathbb{Q}(\epsilon)$-algebra by elements $X^{\pm}_i$, $(X^{\pm}_i)^{(\ell)}$, $K_i^{\pm 1}$ and $\begin{bmatrix} K_i; 0 \\ r_i \end{bmatrix}_{\epsilon_i}$ with $0 \leq r_i < \ell$, $i = 1, \ldots, n$. Indeed, for $r < \ell$ we can get $(X^{\pm}_i)^{(r)}$ directly as $(X^{\pm}_i)^r / [r]_{\epsilon}!$, and ijn the case of $r \geq \ell$, when the same procedure is not possible since $(X^{\pm}_i)^{\ell} = 0$, we can use the decomposition above.

The factorization of $U^{\mathrm{res}}_{\epsilon}(\mathfrak{g})$ will therefore essentially be into the part generated by the $X^{\pm}_i$ and the remainder consisting of the divided powers $(X^{\pm}_i)^{(\ell)}$. More precisely, we define $U^{\mathrm{fin}}_{\epsilon}(\mathfrak{g})$ to be the $\mathbb{Q}(\epsilon)$-subalgebra of $U^{\mathrm{res}}_{\epsilon}(\mathfrak{g})$ generated by the $X^{\pm}_i$, $K_i^{\pm 1}$ and $\begin{bmatrix} K_i; 0 \\ r_i \end{bmatrix}_{\epsilon_i}$, $i = 1, \ldots, n$. It is a Hopf algebra over $\mathbb{Q}(\epsilon)$ of finite dimension $2^n \ell^{2N+n}$. Rosso (1992) proved that $U^{\mathrm{fin}}_{\epsilon}(\mathfrak{g})$ actually is quasitriangular Hopf algebra, using the quantum double method to give an explicit formula of its universal R-matrix $\mathcal{R}_{\epsilon}$, which cannot be directly deduced from the formula of the universal R-matrix of $U_h(\mathfrak{g})$.

It is immediate that the definition of the restricted integral form mimics that of the Chevalley-Kostant $\mathbb{Z}$-form for the classical universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$, which we call $U_{\mathbb{Z}}(\mathfrak{g})$. It is indeed the subring of $U(\mathfrak{g})$ generated by the divided powers $(x^{\pm})^{(r)} = (x^{\pm})^r/r!$, for $r \in \mathbb{N}$, and $x^{\pm}$ the Chevalley generators of $\mathfrak{g}$. Therefore, it should not come as a surprise that these two objects are related. Indeed, if we consider $U_1^{\mathrm{res}}(\mathfrak{g}) = U_\epsilon^{\mathrm{res}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}$, the specialization of $U_q(\mathfrak{g})$ at 1, and we let $\tilde{U}_1^{\mathrm{res}}(\mathfrak{g})$ be its quotient by the two-sided ideal generated by $K_i - 1$, $i = 1, \ldots, n$, we have the following isomorphism of Hopf algebras over $\mathbb{Q}$:

$$U_{\mathbb{Q}}(\mathfrak{g}) := U_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \tilde{U}_1^{\mathrm{res}}(\mathfrak{g}), \quad x_i^{\pm} \mapsto X_i^{\pm}. \tag{3.12}$$

Let now $p \in \mathbb{N}$ be an odd prime, and consider the hyperalgebra of the algebraic group $G_{\mathbb{F}_p}$ over $\mathbb{F}_p$ associated to $\mathfrak{g}$: $U_{\mathbb{F}_p} = U_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{F}_p$. It is known that there exists a Hopf algebra homomorphism over $\mathbb{F}_p$, called the *Frobenius map*

$$\mathrm{Fr} : U_{\mathbb{F}_p} \longrightarrow U_{\mathbb{F}_p} \tag{3.13}$$

such that

$$\mathrm{Fr}(x_i^{\pm})^{(r)} = \begin{cases} (x_i^{\pm})^{(r/p)}, & \text{if } p \text{ divides } r \\ 0, & \text{otherwise} \end{cases} \tag{3.14}$$

and the kernel of Fr is the two-sided ideal of $U_{\mathbb{F}_p}$ generated by the augmentation ideal of $U_{\mathbb{F}_p}^{\mathrm{fin}}$ (i.e. the kernel of the counit morphism), the latter being the *restricted enveloping algebra* of $\mathfrak{g}$ (that is, the subalgebra of $U_{\mathbb{F}_p}$ generated by the $x^{\pm}$).

It is a result of Lusztig the existence of an analogous map for $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$ in characteristic zero. It is the unique Hopf algebra homomorphism over $\mathbb{Q}(\epsilon)$

$$\mathrm{Fr}_\epsilon : U_\epsilon^{\mathrm{res}}(\mathfrak{g}) \longrightarrow U_{\mathbb{Q}}(\mathfrak{g}) \otimes_{\mathbb{Q}} \mathbb{Q}(\epsilon) \tag{3.15}$$

such that

$$\begin{aligned} &\mathrm{Fr}_\epsilon(K_i) = 1 \\ &\mathrm{Fr}_\epsilon((X_i^{\pm})^{(r)}) = \begin{cases} (x^{\pm})^{(r/l)}, & \text{if } \ell \text{ divides } r \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{3.16}$$

and its kernel is the two-sided ideal of $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$ generated by the augmentation ideal of $U_\epsilon^{\mathrm{fin}}(\mathfrak{g})$ (i.e. the kernel of the counit morphism). This also explains how to think the "factorization" of $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$ mentioned above.

Note however that despite the analogies between $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$ and $U_{\mathbb{F}_p}(\mathfrak{g})$ there are some differences. They reflect the differences between the decomposition (3.10) and its characteristic p analogue:

$$\binom{r}{s} = \binom{r_0}{s_0}\binom{r_1}{s_1} \cdots \pmod{p} \tag{3.17}$$

for $s \leq r \in \mathbb{N}$ and $r = \sum_{k=0}^{\infty} r_k p^k$, $s = \sum_{k=0}^{\infty} s_k p^k$ the corresponding p-adic decompositions. Differently from the Gaussian binomial case, there is no bound (independent of r and s) on the number of factors appearing in the RHS of Equation (3.17), and all the terms have the same form. This reflects in the fact that $U_{\mathbb{F}_p}(\mathfrak{g})$ is not finitely generated, while $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$ is, and that the map Fr is an endomorphism of $U_{\mathbb{F}_p}(\mathfrak{g})$ while Fr_ϵ is a map between different algebras.

4 Representation theory of the rational forms

The theory of finite-dimensional representations of $U_q(\mathfrak{g})$ is very similar to that of $U(\mathfrak{g})$; the main result is an analogue of the classical Weyl Theorem about complete reducibility. Moreover, the irreducible $U_q(\mathfrak{g})$-modules are still parametrized by their highest weight, but in general we will have 2^n irreducible modules corresponding to each irreducible $U(\mathfrak{g})$-module, arising from the choice of $n = |\,\Pi\,|$ signs. However, if we restrict ourselves to the case of $\sigma = (1, \ldots, 1)$ the theory proceeds more or less as for $U(\mathfrak{g})$. All the results still hold we specialize q at an arbitrary complex number $\epsilon \in \mathbb{C}^*$ which is not a root of unity. We will later see what happens for the roots of unity case. Unless otherwise stated, all representations will be on complex vector spaces.

Let us introduce some definitions, which mimic those for the classical case.

- A **weight** is an n-tuple $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in (\mathbb{Q}(q)^*)^n$. We write $\boldsymbol{\omega'} \leq \boldsymbol{\omega}$ if $\omega_i'^{-1}\omega_i = q^{(\alpha_i,\beta)}$ for some $\beta \in \Phi^+$ and for all $i = 1, \ldots, n$ (*Bruhat order*);

- For a $U_q(\mathfrak{g})$-module V, the weight space of weight $\boldsymbol{\omega}$ is (if not empty) the $\mathbb{Q}(q)$-subspace

$$V_{\boldsymbol{\omega}} = \{v \in V | K_i \cdot v = \omega_i v,\ i = 1, \ldots, n\}. \tag{4.1}$$

- A **primitive vector** in V is a non-zero vector $v \in V$ such that

$$X_i^+ \cdot v = 0, \quad i = 1, \ldots, n \quad \text{and} \quad v \in V_{\boldsymbol{\omega}} \text{ for some weight } \boldsymbol{\omega}; \tag{4.2}$$

- A **highest weight $U_q(\mathfrak{g})$-module** is a $U_q(\mathfrak{g})$-module V which contains a primitive vector $v \in V$ so that $V = U_q \cdot v$. Such v is called a **highest weight vector**. In particular, if $v \in V_{\boldsymbol{\omega}}$ ($\boldsymbol{\omega}$ is uniquely determined by V) we have

  (i) $V = \bigoplus_{\boldsymbol{\omega'} \leq \boldsymbol{\omega}} V_{\boldsymbol{\omega'}}$;

  (ii) $\dim_{\mathbb{Q}(q)}(V_{\boldsymbol{\omega}}) = 1$.

  So, V is direct sum of all its weight spaces and the weights occurring in the decomposition necessarily are less or equal than $\boldsymbol{\omega}$ in the Bruhat ordering.

- Given a weight $\boldsymbol{\omega}$ we can construct the corresponding **Verma module**:

$$M_q(\boldsymbol{\omega}) = U_q(\mathfrak{g})/\langle X_i^+, K_i - \omega_i 1\rangle_{i=1,\ldots,n}. \tag{4.3}$$

  It is a highest weight $U_q(\mathfrak{g})$-module with highest weight $\boldsymbol{\omega}$ and canonical highest weight vector $v_{\boldsymbol{\omega}}$ given by the image of $1 \in U_q(\mathfrak{g})$. As in the classical case, every highest weight $U_q(\mathfrak{g})$-module with highest weight $\boldsymbol{\omega}$ is isomorphic to a quotient of $M_q(\boldsymbol{\omega})$, and since $\dim(M_q(\boldsymbol{\omega})_{\boldsymbol{\omega}}) = 1$, $M_q(\boldsymbol{\omega})$ admits a unique irreducible quotient $V_q(\boldsymbol{\omega})$ (hence every irreducible highest weight module is isomorphic to some $V_q(\boldsymbol{\omega})$).

We are mainly interested in highest weight $U_q(\mathfrak{g})$-modules with highest weight $\boldsymbol{\omega}_{\sigma,\lambda}$ where $\omega_i = \sigma(\alpha_i)q^{(\alpha_i,\lambda)}$, for $\lambda \in P$ and $\sigma \in \mathrm{Hom}(\Delta, \{\pm 1\})$ a sign. Indeed it can be proved that every finite-dimensional irreducible $U_q(\mathfrak{g})$-module V is a highest weight module of highest weight $\boldsymbol{\omega} = \boldsymbol{\omega}_{\sigma,\lambda}$ for some sign σ and $\lambda \in P^+$. Moreover, $X_i^\pm$ acts locally nilpotently on V (i.e. V is *integrable*).

For $\lambda = 0$ we have $\omega_i = \sigma(\alpha_i)$ for all $i = 1, \dots, n$, and $\dim(V_q(\boldsymbol{\omega}_{\sigma,0})) = 1$. Moreover, the following decomposition of $U_q(\mathfrak{g})$-modules holds

$$V_q(\boldsymbol{\omega}_{\sigma,\lambda}) \cong V_q(\boldsymbol{\omega}_{\sigma,0}) \otimes_{\mathbb{Q}(q)} V_q(\boldsymbol{\omega}_{1,\lambda}) \tag{4.4}$$

where we denote by $\mathbf{1}$ the sign $(1, \dots, 1)$ (equivalently, $\mathbf{1}(\alpha_i) = 1$ for all $i = 1, \dots, n$). Up to tensoring with a one-dimensional module, we can therefore reduce ourselves to consider the so-called modules of **type 1**: $V_q(\boldsymbol{\omega}_{1,\lambda})$. From now on we will denote by λ the weight $\boldsymbol{\omega}_{\mathbf{1},\lambda}$. In this case, K_i acts as $q^{(\alpha_i,\lambda)}$ and the Bruhat order induces the classical order on weights.

Let us describe what happens in the $\mathfrak{sl}_2(\mathbb{C})$ case.

Example 4.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. There are exactly 2 irreducible $U_q(\mathfrak{sl}_2(\mathbb{C}))$-modules for any (finite) dimension $\lambda + 1 \geq 1$: $V_q(\boldsymbol{\omega}_{\sigma,\lambda})$ for $\sigma = \pm 1$. They have basis $\{v_0^{(\lambda)}, \dots, v_\lambda^{(\lambda)}\}$, and $v_0^{(\lambda)}$ is the highest weight vector. The generators of $U_q(\mathfrak{sl}_2(\mathbb{C})$ act on the basis as follows:

- $K_1 v_r^{(\lambda)} = \sigma q^{\lambda - 2r} v_r^{(\lambda)}$;
- $X_1^+ v_r^{(\lambda)} = \sigma[\lambda - r + 1]_q v_{r-1}^{(\lambda)}$;
- $X_i^- v_r^{(\lambda)} = [r+1]_q v_{r+1}^{(\lambda)}$.

For $V_q(\boldsymbol{\omega}_{1,\lambda})$ the above formulas clearly resemble those describing the irreducible representation of $U(\mathfrak{sl}_2(\mathbb{C}))$ of dimension $\lambda + 1$ in the limit $q \to 1$ (we have to think of K_1 as 'q^H' though): $h \cdot v_i = (\lambda - 2i)v_i$, $x \cdot v_i = (\lambda - i + 1)v_{i-1}$, $y \cdot v_i = (i+1)v_{i+1}$.

In the general case, it can be proved that the structure of the irreducible $U_q(\mathfrak{g})$-module $V_q(\lambda)$, is exactly parallel to that of the corresponding highest weight module in the classical case. Indeed, given a highest weight $U_q(\mathfrak{g})$-module V with highest weight vector v we can use the restricted integral form to construct a $U_1^{\mathrm{res}}(\mathfrak{g})$-module $\overline{V}$ on which K_i acts as the identity, hence a $\tilde{U}_1^{\mathrm{res}}(\mathfrak{g})$-module. By Equation (3.12) $\overline{V}$ is a module for the universal enveloping algebra of $\mathfrak{g}$ (over $\mathbb{Q}$) and it is highest weight as a $U_{\mathbb{Q}}(\mathfrak{g})$-module as well, with the same highest weight of V and highest weight vector the image $\overline{v}$ of v. As a consequence, to any irreducible highest weight $U_q(\mathfrak{g})$-module $V_q(\lambda)$, $\lambda \in P^+$ corresponds an irreducible highest weight $U_{\mathbb{Q}}(\mathfrak{g})$-module $\overline{V_q(\lambda)}$ with the same highest weight, and the dimensions of their weight spaces coincide. Hence, the character of $V_q(\lambda)$ is given by the classical Weyl character formula. By analogy with the classical case we get that every finite dimensional highest weight $U_q(\mathfrak{g})$-module is irreducible and has highest weight in P^+. As a consequence, every finite-dimensional $U_q(\mathfrak{g})$-module is completely reducible. We say that the category of finite-dimensional $U_q(\mathfrak{g})$-modules of type $\mathbf{1}$ is semisimple.

Actually, we know more. Indeed it is possible to define an invertible operator $\tilde{\mathcal{R}}$ that, although not an element of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, acts on any tensor product of two finite-dimensional $U_q(\mathfrak{g})$-modules playing the role of the universal R-matrix of $U_h(\mathfrak{g})$. Therefore, the category of finite-dimensional representations of $U_q(\mathfrak{g})$ is a (semisimple) quasitensor category.

5 Representation theory of the restricted integral form

As already mentioned, the representation theory of $U_q(\mathfrak{g})$ turns out to be very different from the one of $U_q(\mathfrak{g})$ (hence of $U(\mathfrak{g})$) when q is specialized at a root of unity. It is also remarkable that the two integral forms lead to representation theories which are very different from one another. In the restricted case, the finite-dimensional irreducible modules are still parametrized by dominant weights, but their structure is in general different from that of the corresponding $U(\mathfrak{g})$-module parametrized by the same weight (e.g. their dimension is not the same). The main difference it that for finite-dimensional $U_\epsilon^{\text{res}}(\mathfrak{g})$-modules complete reducibility does not hold anymore. However, we will use tilting modules to recover a a suitable semisimple category of representations.

As for the $U_q(\mathfrak{g})$ case, up to tensoring with a one-dimensional representation, we will only consider representations of type $\mathbf{1}$, that is on which $K_i^\ell = 1$ for all $i = 1, \dots, n$.

Let V be a finite-dimensional $U_\epsilon^{\text{res}}(\mathfrak{g})$-module. The action of the commuting elements $K_1, \dots, K_n$ on V is simultaneously diagonalizable with eigenvalues in the set $\{\pm\epsilon^r\}_{r=0}^{\ell-1}$. Let $\lambda \in P$ be a weight of V: we would like to define weight spaces V_λ for $U_\epsilon^{\text{res}}(\mathfrak{g})$ analogously to those for $U_q(\mathfrak{g})$:

$$V_\lambda = \{v \in V \mid K_i \cdot v = \epsilon_i^{(\lambda, \check{\alpha}_i)} v, \, \forall i\}. \tag{5.1}$$

However, it is clear that we would have $V_\lambda = V_\mu$ as soon as $\lambda - \mu \in \ell P$, hence the weight space decomposition would not be direct. The reason why this definition fails is that to get a maximal abelian subalgebra of $U_\epsilon^{\text{res}}(\mathfrak{g})$ we have to add to the K_i all the elements $\begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix}_{\epsilon_i}$. The correct definition of the weight space associated to $\lambda \in P$ is therefore

$$V_\lambda = \{v \in V \mid K_i \cdot v = \epsilon_i^{(\lambda, \check{\alpha}_i)} v, \, \begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix}_{\epsilon_i} \cdot v = \begin{bmatrix} (\lambda, \check{\alpha}_i) \\ \ell \end{bmatrix}_{\epsilon_i} v, \, \forall i = 1, \dots, n\}. \tag{5.2}$$

The elements $\begin{bmatrix} K_i; c \\ \ell \end{bmatrix}_{\epsilon_i}$ act on V_λ as $\begin{bmatrix} (\lambda, \check{\alpha}_i) + c \\ \ell \end{bmatrix}_{\epsilon_i}$, and it follows from Equations (3.2) and (3.6) that $(X_i^\pm)^{(r)}(V_\lambda) \subset V_{\lambda \pm r\alpha_i}$.

The definitions of weight, primitive vector, highest weight module, etc... can be carried out as in the $U_q(\mathfrak{g})$-case; we just have to remind that for a primitive vector v we also require $(X_i^+)^{(\ell)} \cdot v = 0$ for $i = 1, \dots, n$. As for the $U_q(\mathfrak{g})$ case, it can be proved that a finite-dimensional irreducible $U_\epsilon^{\text{res}}(\mathfrak{g})$-module V of type $\mathbf{1}$ is the direct sum of its weight spaces.

We can construct a highest weight $U_\epsilon^{\text{res}}(\mathfrak{g})$-module of highest weight $\lambda \in P$ from the correspondent highest weight $U_q(\mathfrak{g})$-module $V_q(\lambda)$ as follows:

$$W_\epsilon^{\text{res}}(\lambda) := V_{\mathcal{A}}^{\text{res}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C} \tag{5.3}$$

where $V_{\mathcal{A}}^{\text{res}}(\lambda)$ is the $U_{\mathcal{A}}^{\text{res}}$-submodule of $V_q(\lambda)$ generated by its highest weight vector v ($\mathbb{C}$ is an $\mathcal{A}$-module via the homomorphism sending q to ϵ). This so-called **Weyl module** clearly is a highest weight module, but it is in general not irreducible, neither finite-dimensional. However, for a dominant weight $\lambda \in P^+$ we have a remarkable result.

Proposition 5.1. *Let $\lambda \in P^+$. Then $dim(W_\epsilon^{res}(\lambda)) < \infty$, and it is irreducible if one of the following conditions holds:*

(i) $(\lambda + \rho, \check{\alpha}) < \ell$*, for all positive roots* α*;*

(ii) $\lambda = (\ell - 1)\rho + \ell\mu$*, for some* $\mu \in P^+$.

Moreover, as in the $U_q(\mathfrak{g})$-case, the character of $W_\epsilon^{res}(\lambda)$ is given by the classical Weyl character formula.

Being a highest weight module, $W_\epsilon^{\mathrm{res}}(\lambda)$ possesses a unique irreducible quotient, which we denote by $V_\epsilon^{\mathrm{res}}(\lambda)$. Actually, it is possible to prove that every finite-dimensional irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-module V of type $\mathbf{1}$ is isomorphic to a module $V_\epsilon^{\mathrm{res}}(\lambda)$ for some $\lambda \in P^+$. It is equivalently possible to construct the irreducible modules $V_\epsilon^{\mathrm{res}}(\lambda)$ directly via a usual Verma module construction; however, Weyl modules are very useful tool in the study of the representation theory of $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$, as it will later be clear with the introduction of the so-called *tilting modules*.

Example 5.2. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. For $m \in \mathbb{Z}$ we can consider the Weyl module $W_\epsilon^{\mathrm{res}}(m)$ of highest weight m. It has a basis $\{v_0^{(m)}, \ldots, v_m^{(m)}\}$ on which the action of $U_\epsilon^{\mathrm{res}}(\mathfrak{sl}_2(\mathbb{C}))$ is given by

- $X_1^+ \cdot v_r^{(m)} = [m-r+1]_\epsilon v_{r-1}^{(m)}, \quad (X_1^+)^{(\ell)} \cdot v_r^{(m)} = ((m-r)_1+1)v_{r-\ell}^{(m)};$
- $X_1^- \cdot v_r^{(m)} = [r+1]_\epsilon v_{r+1}^{(m)}, \quad (X_1^-)^{(\ell)} \cdot v_r^{(m)} = (r_1+1)v_{r+\ell}^{(m)};$
- $K_1 \cdot v_r^{(m)} = \epsilon^{m-2r} v_r^{(m)}.$

$W_\epsilon^{\mathrm{res}}(m)$ has a unique maximal submodule, spanned by those $v_r^{(m)}$ for which $m_0 < r_0 < \ell$ and $r_1 < m_1$. It is isomorphic to the irreducible module $V_\epsilon^{\mathrm{res}}(\ell m_1 - m_0 - 2)$ and the quotient $W_\epsilon^{\mathrm{res}}(m)/V_\epsilon^{\mathrm{res}}(\ell m_1 - m_0 - 2)$ is the unique quotient module $V_\epsilon^{\mathrm{res}}(m)$. Clearly, $W_\epsilon^{\mathrm{res}}(m)$ is irreducible if and only if $m < \ell$ or $m_0 = \ell - 1$: in these cases it is in fact not possible to define a unique maximal submodule as above. For $\ell = 5$, $m = 17$ the weight structure of $V_\epsilon^{\mathrm{res}}(17)$ is described by the following picture

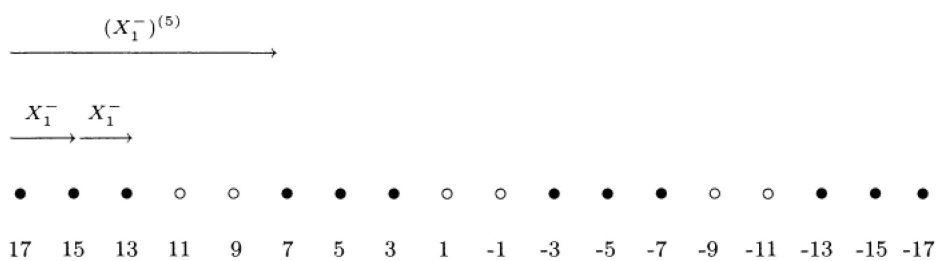

Figure 1: Weight structure of $V_\epsilon^{\mathrm{res}}(17)$, for $\ell = 5$

where black and white dots indicate basis vectors for $V_\epsilon^{\mathrm{res}}(17)$ and for the unique maximal submodule (isomorphic to $V_\epsilon^{\mathrm{res}}(11)$) respectively. The actions of X_1^+ and $(X_1^+)^{(5)}$ are given by analogous arrows pointing from the right side to the left side of the diagram. As a difference with the $\mathfrak{sl}_2(\mathbb{C})$ case, note that $\dim V_\epsilon^{\mathrm{res}}(17) = 12$. In general we have $\dim(V_\epsilon^{\mathrm{res}}(m)) = (m_0+1)(m_1+1) \neq m+1$.

We can actually reduce the study of the irreducible highest weight $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules of highest weight λ to two cases: those for which all components of λ are strictly less than ℓ, and those for which all components of λ are divisible by ℓ. Indeed, every irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-module can be decomposed as a tensor product of two such irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules.

Theorem 5.3. *Let $\lambda \in P^+$ and let $\lambda = \lambda_0 + l\lambda_1$ be its unique decomposition with $\lambda_0, \lambda_1 \in P^+$, $0 \le (\lambda_0, \check{\alpha}_i) < \ell$. Then there is an isomorphism of $U_\epsilon^{res}(\mathfrak{g})$-modules*

$$V_\epsilon^{res}(\lambda) \cong V_\epsilon^{res}(\lambda_0) \otimes V_\epsilon^{res}(l\lambda_1). \tag{5.4}$$

This is an analogue to Steinberg's Tensor Product Theorem in characteristic p. It occurs in the representation theory of the hyperalgebra $U_{\mathbb{F}_p}(\mathfrak{g})$, where finite-dimensional irreducible modules are again parametrized by dominant weights. For a dominant weight $\lambda \in P^+$ and an irreducible $U_{\mathbb{F}_p}(\mathfrak{g})$-module $V_{\mathbb{F}_p}(\lambda)$, Steinberg's Theorem describes how it decomposes as the tensor product of irreducible $U_{\mathbb{F}_p}(\mathfrak{g})$-modules with respect to the p-adic expansion of λ. With all the due differences (e.g. as for the decomposition of the binomial coefficient in characteristic p, the number of factors in Steinberg's decomposition is not a priori bounded), it remarks the analogy with the theory of classical Lie algebras in characteristic p.

However, how should we interpret the modules $V_\epsilon^{\mathrm{res}}(\lambda_0)$ and $V_\epsilon^{\mathrm{res}}(\ell\lambda_1)$?

- The modules $V_\epsilon^{\mathrm{res}}(\lambda_0)$ essentially are the irreducible modules for the finite-dimensional Hopf algebra $U_\epsilon^{\mathrm{fin}}(\mathfrak{g})$ (they bijectively correspond to irreducible $U_\epsilon^{\mathrm{fin}}(\mathfrak{g})$-modules on which K_i^l acts as 1);
- The modules $V_\epsilon^{\mathrm{res}}(\ell\lambda_1)$ are isomorphic as $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules to the pull-back of the corresponding irreducible highest weight $U(\mathfrak{g})$-modules $V(\lambda)$ by the Frobenius map Fr_ϵ.

5.1 Tilting Modules and Decompositions of Tensor Products

From now on we assume moreover that $\ell > h$, where h is the Coxeter number of $\mathfrak{g}$.

We want to analyse tensor products of irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules: they are not in general completely reducible, as it appears from Example 5.2. To find some kind of complete reducibility we need to introduce tilting modules.

A finite-dimensional $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-module V of type $\mathbf{1}$ is a **tilting module** if both V and V^* possess a **Weyl filtration**, that is if there exists a sequence of submodules

$$0 = V_0 \subset V_1 \subset \ldots \subset V_p = V \tag{5.5}$$

with $V_r/V_{r-1} \cong W_\epsilon^{\mathrm{res}}(\lambda_r)$ for some $\lambda \in P^+$, $r = 1, \ldots, p$, and an analogous one for V^*. We equivalently say that V is a tilting module if it possesses both a Weyl and a *dual Weyl filtration*, that is a filtration whose consecutive quotients are isomorphic to the dual of some Weyl module $W_\epsilon^{\mathrm{res}}(\lambda_r)^*$ for some $\lambda \in P^+$. Indeed, V^* possesses a Weyl filtration if and only if V possesses a dual Weyl filtration.

Example 5.4. Let $\mathcal{C}_\ell$ be the principal alcove with respect to the action of the affine Weyl group of $\mathfrak{g}$:

$$\mathcal{C}_\ell = \{\lambda \in P^+ \,|\, (\lambda + \rho, \check{\alpha}) < \ell, \forall \alpha \in \mathbf{\Phi}^+\}. \tag{5.6}$$

This is not empty since we have required $\ell > h$. By Proposition 5.1 we have $W_\epsilon^{\mathrm{res}}(\lambda) = V_\epsilon^{\mathrm{res}}(\lambda)$ for $\lambda \in \mathcal{C}_\ell$; hence $V_\epsilon^{\mathrm{res}}(\lambda)$ admits a Weyl filtration. Moreover $V_\epsilon^{\mathrm{res}}(\lambda)^* \cong V_\epsilon^{\mathrm{res}}(-w_0(\lambda))$, because the same holds for $U(\mathfrak{g})$-modules, and $w_0(\lambda)$ is in $\mathcal{C}_\ell$ as well. Therefore, $V_\epsilon^{\mathrm{res}}(\lambda)$ is a tilting module.

Some basic properties of tilting modules:

(i) The dual of a tilting module is a tilting;

(ii) Any (finite) direct sum of tilting modules is tilting;

(iii) Any direct summand of a tilting module is tilting;

(iv) Any (finite) tensor product of tilting modules is tilting.

We can therefore restrict our attention to indecomposable tilting modules., for which we can give a nice classification via dominant weights. Indeed, for any $\lambda \in P^+$ there exists a unique (up to isomorphism) indecomposable tilting module $T_\epsilon(\lambda)$ such that:

(i) The set of weights of $T_\epsilon(\lambda)$ is contained in the convex hull of $\mathcal{W} \cdot \lambda$;

(ii) λ is the unique maximal weight of $T_\epsilon(\lambda)$;

(iii) $\dim T_\epsilon(\lambda)_\lambda = 1$;

(iv) $T_\epsilon(\lambda)^* \cong T_\epsilon(-w_0(\lambda))$;

Conversely, every indecomposable tilting module is isomorphic to a (unique) $T_\epsilon(\lambda)$, for some $\lambda \in P^+$. Therefore $\{T_\epsilon(\lambda) \,|\, \lambda \in P^+\}$ is a complete set of isomorphism classes of indecomposable tilting modules for $U_\epsilon^{\text{res}}(\mathfrak{g})$. As a consequence, any tilting module T decomposes (non uniquely) as

$$T \cong \bigoplus_{\lambda \in P^+} T_\epsilon(\lambda)^{n_\lambda(T)} \tag{5.7}$$

and the multiplicities $n_\lambda(T)$ are uniquely determined by T.

We can prove that if T is a tilting module then $\mathrm{Ext}^1(T,T) = 0$ and T has projective dimension 1. Hence our tilting modules actually correspond to the partial tilting modules, in the original definition by Ringel (1991) extended by Donkin for algebraic groups (1993).

It is in general not easy to describe the structures of indecomposable Tilting modules, even in the simplest cases. However, for some special values of $\lambda \in P^+$ we have the following result.

Example 5.5. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Identifying $\lambda \in P^+$ with the integer (λ, α) (α being the simple root of $\mathfrak{sl}_2(\mathbb{C})$), we have the following description of the indecomposable tilting modules $T_\epsilon(\lambda)$:

(i) If $0 \le \lambda < \ell - 1$, then $T_\epsilon(\lambda)$ is irreducible because of Example 5.2;

(ii) If $\lambda = \ell - 1$, then $T_\epsilon(\lambda)$ is irreducible because of Proposition 5.1: $T_\epsilon(\ell-1) \cong V_\epsilon^{\text{res}}(\ell-1)$;

(iii) If $\ell \le \lambda \le 2\ell - 2$, then $T_\epsilon(\lambda)$ is the 2ℓ-dimensional module with basis $\{t_r\}_{r=0}^{\lambda} \cup \{t'_r\}_{r=0}^{2l-2-\lambda}$ and the following action of the generators:

- $K_1 \cdot t_r = \epsilon^{\lambda - 2r} t_r$, $\begin{bmatrix} K_1; 0 \\ \ell \end{bmatrix}_\epsilon \cdot t_r = r_1 t_r$, $\quad K_1 \cdot t'_r = \epsilon^{2\ell - 2r - 2 - \lambda} t'_r$, $\begin{bmatrix} K_1; 0 \\ \ell \end{bmatrix}_\epsilon \cdot t'_r = 0$;
- $X_1^+ \cdot t_r = [\lambda - r + 1]_\epsilon t_{r-1}$, $\quad X_1^- \cdot t_r = [r+1]_\epsilon t_{r+1}$;

- $X_1^+ \cdot t'_r = [2\ell - 1 - \lambda - r]_\epsilon t'_{r-1} + \begin{bmatrix} \lambda + r - \ell \\ r \end{bmatrix}_\epsilon t_{\lambda+r-\ell}, \quad \text{if } 0 < r \leq 2\ell - 2 - \lambda;$
- $X_1^- \cdot t'_r = [r+1]_\epsilon t'_{r+1}, \quad \text{if } 0 \leq r < 2\ell - 2 - \lambda;$
- $X_1^+ \cdot t'_0 = [\lambda - \ell + 1]_\epsilon t_{\lambda-\ell}, \quad X_1^- \cdot t'_{2l-2-\lambda} = \begin{bmatrix} \ell - 1 \\ \lambda - \ell + 1 \end{bmatrix}_\epsilon t_l;$
- $(X_1^+)^{(\ell)} \cdot t_r = ((\lambda - r)_1 + 1) t_{r-\ell}, \quad (X_1^-)^{(\ell)} \cdot t_r = (r_1 + 1) t_{r+l}, \quad (X_1^\pm)^{(\ell)} \cdot t'_r = 0.$

Hence $T_\epsilon(\lambda)$ is indecomposable, and it contains a submodule spanned by the t_r which is isomorphic to $W_\epsilon^{\mathrm{res}}(\lambda)$; the corresponding quotient, spanned by the t'_r, is isomorphic to $W_\epsilon^{\mathrm{res}}(2\ell - 2 - \lambda)$. Thus, $T_\epsilon(\lambda)$ admits a Weyl filtration. Actually, $T_\epsilon(\lambda)$ is a tilting module since $T_\epsilon(\lambda)^*$ admits a Weyl filtration as well, given by the same Weyl modules of $T_\epsilon(\lambda)$.

Note that although λ is the unique maximal weight of $T_\epsilon(\lambda)$, it is not a highest weight module, since it is not generated by the highest weight vector t_0. For $\ell = 7$, $\lambda = 8$ we have the following picture, where the action of X_1^+ is given by both the upward and the right-to-left arrows, and the downward arrows describe the action of X_1^-. The structure of $T_\epsilon(\lambda)^*$ is simply given by turning the diagram upside down.

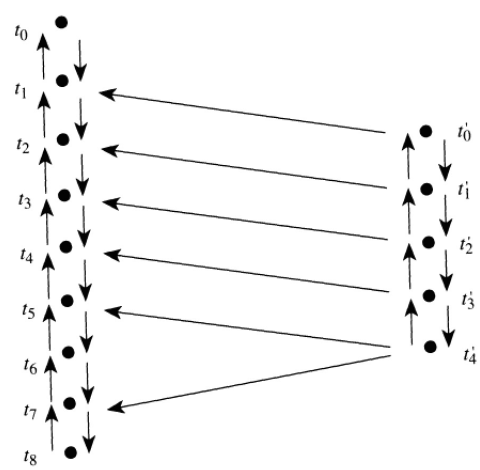

Figure 2: Structure of the tilting module $T_\epsilon(8)$, for $\ell = 7$

The question now is how to decompose the tensor product of irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules $V_\epsilon^{\mathrm{res}}(\lambda) \otimes V_\epsilon^{\mathrm{res}}(\mu)$ for $\lambda, \mu \in P^+$. Although it does not decompose in general into the direct sum of irreducible $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules, for $\lambda, \mu \in \mathcal{C}_\ell$ we have $V_\epsilon^{\mathrm{res}}(\lambda) \cong T_\epsilon(\lambda)$ and $V_\epsilon^{\mathrm{res}}(\mu) \cong T_\epsilon(\mu)$ hence the tensor product decomposes as the direct sum of indecomposable tilting modules.

The first attempt to retrieve some kind of complete reducibility would hence be to define a new tensor product of $U_\epsilon^{\mathrm{res}}(\mathfrak{g})$-modules discarding all the non-irreducible tilting components. However, we would only get a non-associative tensor product, as it is clear from the following example.

Example 5.6. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\ell = 5$ we have the following decompositions into indecomposable tilting modules:

$$\begin{aligned}
&V_\epsilon^{\text{res}}(0) \otimes V_\epsilon^{\text{res}}(4) \cong V_\epsilon^{\text{res}}(4)\\
&V_\epsilon^{\text{res}}(2) \otimes V_\epsilon^{\text{res}}(2) \cong V_\epsilon^{\text{res}}(0) \oplus V_\epsilon^{\text{res}}(2) \oplus V_\epsilon^{\text{res}}(4)\\
&V_\epsilon^{\text{res}}(2) \otimes V_\epsilon^{\text{res}}(4) \cong V_\epsilon^{\text{res}}(4) \oplus T_\epsilon(6)\\
&V_\epsilon^{\text{res}}(4) \otimes V_\epsilon^{\text{res}}(4) \cong V_\epsilon^{\text{res}}(4) \oplus T_\epsilon(6) \oplus T_\epsilon(8)
\end{aligned}$$

If we denote by $\tilde{\otimes}$ the tensor product obtained by discarding the indecomposable non-irreducible tilting components (e.g. all those of the form $T_\epsilon(\lambda)$) we get

$$V_\epsilon^{\text{res}}(2)\tilde{\otimes}(V_\epsilon^{\text{res}}(2)\tilde{\otimes}V_\epsilon^{\text{res}}(4)) \cong V_\epsilon^{\text{res}}(2)\tilde{\otimes}V_\epsilon^{\text{res}}(4) \cong V_\epsilon^{\text{res}}(4) \tag{5.8}$$

while

$$\begin{aligned}
(V_\epsilon^{\text{res}}(2)\tilde{\otimes}V_\epsilon^{\text{res}}(2))\tilde{\otimes}V_\epsilon^{\text{res}}(4) &\cong (V_\epsilon^{\text{res}}(0) \oplus V_\epsilon^{\text{res}}(2) \oplus V_\epsilon^{\text{res}}(4))\tilde{\otimes}V_\epsilon^{\text{res}}(4) \cong\\
&\cong V_\epsilon^{\text{res}}(4) \oplus V_\epsilon^{\text{res}}(4) \oplus V_\epsilon^{\text{res}}(4).
\end{aligned} \tag{5.9}$$

We therefore need a different approach. For a finite dimensional $U_\epsilon^{\text{res}}(\mathfrak{g})$-module V we define its quantum dimension as the quantum trace of the identity morphism id_V:

$$\operatorname{qdim} V = \operatorname{qtr}(\operatorname{id}_V) = \operatorname{trace}(K_{\rho^*}) \tag{5.10}$$

where $K_{\rho^*} = \prod_i K_i^{r_i}$ and $2\rho = \sum_{\alpha_i} r_i\alpha_i$ is the sum of all the positive roots of $\mathfrak{g}$ with coefficients $r_i \in \mathbb{Z}$. For an indecomposable tilting module $T_\epsilon(\lambda)$, it is true that $\operatorname{qdim} T_\epsilon(\lambda) \neq 0$ if and only if $\lambda \in \mathcal{C}_\ell$. Moreover, for $\lambda \in P^+ \setminus \mathcal{C}_\ell$ every direct summand of the tensor product $T_\epsilon(\lambda) \otimes V$, where V is any $U_\epsilon^{\text{res}}(\mathfrak{g})$-module of type $\mathbf{1}$, has quantum dimension zero (it is *negligible*).

Proposition 5.7. *Let T_1, T_2 be tilting $U_\epsilon^{res}(\mathfrak{g})$-modules. Then their tensor product decomposes as*

$$T_1 \otimes T_2 \cong \left(\bigoplus_{\lambda\in\mathcal{C}_l} V_\epsilon^{res}(\lambda)^{n_\lambda}\right) \oplus Z \tag{5.11}$$

where Z is a $U_\epsilon^{res}(\mathfrak{g})$-module with $\operatorname{qdim} Z = 0$.

This is clear since we can decompose a tilting module into the direct sum of indecomposable tilting modules, parametrized by $\lambda \in P^+$. Hence, if we denote by Z the sum of all the indecomposable tilting modules parametrized by $\lambda \in P^+ \setminus \mathcal{C}_\ell$, it obviously has quantum dimension zero, and it provides the above decomposition.

The idea is therefore to define a new tensor product of T_1 and T_2 discarding the component Z. More precisely, if T is any tilting module, we define $\overline{T}$ to be the sum of all the indecomposable summands in the decomposition of T whose maximal weights lie in $\mathcal{C}_\ell$. Hence we define the **truncated tensor product** $\overline{\otimes}$ of tilting modules T_1, T_2 as

$$T_1\overline{\otimes}T_2 := \overline{T_1 \otimes T_2}. \tag{5.12}$$

This tensor product is endowed with nice properties. Indeed, for tilting modules T_1, T_2 and T_3 we have the following isomorphisms of $U_\epsilon^{\text{res}}(\mathfrak{g})$-modules:

(i) $T_1 \overline{\otimes} T_2 \cong T_2 \overline{\otimes} T_1$, (because the same holds for the "old" tensor product);

(ii) $T_1 \overline{\otimes} (T_2 \overline{\otimes} T_3) \cong (T_1 \overline{\otimes} T_2) \overline{\otimes} T_3$.

In the setting of Example 5.6 above, we have instead

$$V_\epsilon^{\text{res}}(2) \overline{\otimes} (V_\epsilon^{\text{res}}(2) \overline{\otimes} V_\epsilon^{\text{res}}(4)) = 0 = (V_\epsilon^{\text{res}}(2) \overline{\otimes} V_\epsilon^{\text{res}}(2)) \overline{\otimes} V_\epsilon^{\text{res}}(4).$$

Since the decomposition of a tilting module T into indecomposable components is not unique in general, the choice of a module $\overline{T}$ is not unique, but it is well defined up to isomorphism. Hence it is important to make a "canonical choice" for this module: in what follows, assume that $\overline{T}$ is such a canonical choice. We can now summarize our result in a categorical language. Let us introduce the following categories

$\mathbf{rep}\,\mathbf{U}_\epsilon^{\mathbf{res}}(\mathfrak{g})$ category of finite-dimensional $U_\epsilon^{\text{res}}(\mathfrak{g})$-modules of type $\mathbf{1}$;

$\mathbf{tilt}_l$ full subcategory of $\text{rep}\,\mathbf{U}_\epsilon^{\text{res}}(\mathfrak{g})$ whose objects are tilting modules;

$\overline{\mathbf{tilt}}_l$ full subcategory of $\mathbf{tilt}_l$ whose objects are finite-dimensional $U_\epsilon^{\text{res}}(\mathfrak{g})$-modules with weights belonging to

$$\{\lambda \in P \mid \langle \lambda + \rho, \check{\alpha} \rangle < l, \; \forall \text{ roots } \alpha\}. \tag{5.13}$$

$\mathbf{tilt}'_l$ full subcategory of $\mathbf{tilt}_l$ whose objects are those tilting modules T having $\bar{T} = 0$.

Then $\mathbf{tilt}_l = \overline{\mathbf{tilt}}_l \oplus \mathbf{tilt}'_l$ (to be precise, $\overline{\mathbf{tilt}}_l$ is a "quotient category" of $\mathbf{tilt}_l$), and we have the following results:

- With the usual operations of direct sum, dual and tensor product, $\mathbf{rep}\,\mathbf{U}_\epsilon^{\text{res}}(\mathfrak{g})$ is a rigid, $\mathbb{C}$-linear, braided category.
- With the usual operations of direct sum and dual, $\mathbf{tilt}_l$ is a rigid, $\mathbb{C}$-linear, braided category;
- With the usual operations of direct sum and dual, and with the truncated tensor product, $\overline{\mathbf{tilt}}_l$ is a rigid, semisimple, $\mathbb{C}$-linear braided category.

References

[APW] H. H. Andersen, P. Polo, Wen K. *Representations of quantum algebras*, Invent. Math. **104** (1991) 1-59.

[CP] V. Chari, A. Pressley *A guide to quantum groups*, Cambridge University Press (1997).

[Fe] J. Feldvoss *Tilting modules for quantum groups*, Notes of two Talks at the Workshop on Quantum Groups in Hesselberg, Germany (1994).

[Hu] J. E. Humphreys *Introduction to Lie algebras and representation theory*, Springer, New York (1972).

[Ja] J. C. Jantzen *Lectures on quantum groups*, American Mathematical Society **6**, (1996).

[Lu1] G. Lusztig *Modular representations and quantum groups*, Contemp. Math. **82** (1989), 59-77.

[Lu2] G. Lusztig *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math Soc. **3 (1)** (1990), 257-296.

[Lu3] G. Lusztig *Quantum Groups at Root of 1*, Geom. Ded. **35** (1990), 89-114.

[Sa] S. F. Sawin *Quantum Groups at Roots of Unity and Modularity*, J. Knot Theory Ramifications **15 (10)** (2006), 1245-1277.