Takesaki's proof of a Tannaka–Stinespring–Tatsuuma-type pre-duality for Commutative Measured Involutive Hopf–von Neumann algebras

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Abstract

We give here a short review of Takesaki's seminal paper [5], in which a characterization of a commutative measured involutive Hopf–von Neumann algebra is given, as the algebra $L^\infty(G)$ of a locally compact Hausdorff topological group G.

While referring the reader to the original paper for the majority of proofs and for the details of all others, we focus on the measure theoretical tools and implications underlying the general ideas.

1 Preliminaries

Definition 1.1 (Hopf–von Neumann Algebra)**.** Let $\mathfrak{H}$ be a Hilbert $\mathbb{C}$-space and $\mathcal{L}_b(\mathfrak{H})$ be the $\mathbb{C}^*$-algebra of the bounded linear operators on $\mathfrak{H}$. A *trace-class operator* on $\mathfrak{H}$ is any operator $t \in \mathcal{L}_b(\mathfrak{H})$ such that $\operatorname{tr}_{\mathfrak{H}} |t| < \infty$. Let now $\mathcal{L}_b(\mathfrak{H})_*$ be the Banach space of all trace-class operators. The *σ-weak* (read: *ultra-weak*) *topology* on $\mathcal{L}_b(\mathfrak{H})$ is defined as the $\sigma(\mathcal{L}_b(\mathfrak{H}), \mathcal{L}_b(\mathfrak{H})_*)$-topology. A *von Neumann algebra* M is any *unital σ-weakly closed* $*$-subalgebra of $\mathcal{L}_b(\mathfrak{H})$. The *tensor product* $\overline{\otimes}_i M_i$ of von Neumann algebras M_i acting on Hilbert spaces $\mathfrak{H}_i$ $(i = 1, \dots, n)$ is defined as the weak closure of the algebraic tensor product $\otimes_i M_i$ acting on $\otimes_i \mathfrak{H}_i$.

A *co-multiplication* δ on M is a *unital σ-weakly continuous co-associative *-homomorphism* $\delta\colon M \to M \overline{\otimes} M := M^{\overline{\otimes}2}$, where co-associativity is to be understood as the commutativity of the diagram

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & M^{\overline{\otimes}2} \arrow[d, "\iota\otimes\delta"] \\ M^{\overline{\otimes}2} \arrow[r, "\delta\otimes\iota"'] & M^{\overline{\otimes}3} \end{tikzcd}$$

(with $\iota := \mathrm{id}_M$ the identity morphism of M).

A *Hopf–von Neumann algebra* (M, δ) is any von Neumann algebra endowed with a co-multiplication δ and is said to be *commutative* whenever M is. The *twist* $\tau\colon M \otimes M$ is defined by $\tau\colon (x \otimes y) \mapsto (y \otimes x)$. The co-multiplication δ is said to be *symmetric* or *co-commutative* if it absorbs τ (i.e. $\tau \circ \delta = \delta$). If so, (M, δ) is said to be *symmetric*.

Everywhere in the following the algebraic tensor product of algebras is denoted by $\otimes$, while the topological one by $\overline{\otimes}$.

Theorem 1.2 (Sakai)**.** Every von Neumann algebra M admits a unique *predual* M_*, i.e. a Banach space such that $(M_*)^* = M$ (see [6, I.3.9]). Furthermore, M_* may be regarded as the Banach space of σ-weakly continuous functionals on M (see [4, §1.1.1]).

Lemma 1.3 (Pre-duality for Hopf–von Neumann algebras)**.** Let M be a Hopf–von Neumann algebra and $(M_*, *)$ be the predual of M endowed with the *convolution* $*$ defined by

$$\forall f, g \in M_*, \forall x \in M \qquad \langle f * g \,|\, x \rangle \coloneqq \langle f \otimes g \,|\, \delta(x) \rangle \,.$$

Then $(M_*, *)$ is a Banach algebra.

Proof. Straightforward. ■

Definition 1.4 (Involutive Hopf-von Neumann algebras)**.** An *involution* (or *antipode*) $j \colon M \to M$ is any 2-involutive anti-automorphism (i.e. σ-weakly continuous unital morphism) such that

$$\tau \circ j \circ \delta = (j \otimes j)\delta. \tag{1.1}$$

Lemma 1.5. Setting $j_* \coloneqq j^*\big|_{M_*}$, (M_*, j_*) is an involutive Banach algebra.

Proof. Straightforward. ■

Everywhere in the following let $j(a) \coloneqq a\check{}$ and $j_*(f) \coloneqq f\hat{}$ for $a \in M$, $f \in M_*$.

Definition 1.6 (Trace)**.** A *trace* $\boldsymbol{\mu}$ on M is any $\overline{\mathbb{R}}^{\geq}$*-valued* (*positively*) *additive positively homogeneous* function on M which is *conjugation-invariant* w.r.t. the unitary operators of M. A trace $\boldsymbol{\mu}$ is said to be *faithful* if it is strictly positive on $M^+ \setminus \{0\}$, *finite* if $\mathbb{R}^{\geq}$-valued and *semi-finite* if for all $a \in M^+$ it holds that $\boldsymbol{\mu}(a) = \sup_{M^+ \ni b \leq a} \boldsymbol{\mu}(b)$. A σ-weakly continuous semi-finite faithful trace is said to be a *measure* on M.

Definition 1.7 (Measured involutive Hopf–von Neumann (MIHvN) algebra)**.** An involutive Hopf–von Neumann algebra is said to be *right-* (resp. *left-*)*measured* whenever endowed with a *measure* which is also *right* (resp. *left*) *invariant*, i.e.

$$\begin{gathered}(\boldsymbol{\mu} \otimes \boldsymbol{\mu})\,[(a \otimes b)\delta(c)] = (\boldsymbol{\mu} \otimes \boldsymbol{\mu})\,[(a\hat{} \otimes c)\delta(b)] \\ (\text{resp. } (\boldsymbol{\mu} \otimes \boldsymbol{\mu})\,[(a \otimes b)\delta(c)] = (\boldsymbol{\mu} \otimes \boldsymbol{\mu})\,[(c \otimes b\hat{})\delta(a)]).\end{gathered}$$

A right-left invariant measure (resp. algebra) is said to be *unimodular.*

A *morphism* θ *of measured involutive Hopf–von Neumann algebras* $(M_i, \delta_i, j_i, \boldsymbol{\mu}_i)$ is any morphism of von Neumann algebras preserving δ, j and $\boldsymbol{\mu}$ in the sense that

$$\theta \circ j_1 = j_2 \circ \theta \qquad (\theta \otimes \theta) \circ \delta_1 = \delta_2 \circ \theta \qquad \boldsymbol{\mu}_2 = \theta_\sharp \boldsymbol{\mu}_1.$$

2 MIHvN algebra of a locally compact group

Everywhere in the following, a topological measurable space is always endowed with its Baire algebra $\mathfrak{A}$, that is the σ-algebra generated by *functionally* closed subsets of X.

Theorem 2.1 (Canonical MIHvN algebra of a locally compact group)**.** To any locally compact Hausdorff topological group G it may be naturally associated a MIHvN algebra.

Proof. Recall that G has a (unique, locally finite, regular) left-invariant *Haar measure* μ. Letting $M_G \coloneqq L^\infty(G, \mu)$, set

$$\begin{aligned} &\forall s, t \in G, \forall f \in M_G & \delta_G(f)(s,t) &\coloneqq f(st) \\ &j_G(f)(s) \doteq f\hat{}(s) \coloneqq f(s^{-1}) & \boldsymbol{\mu}_G(f) &\coloneqq \int_G f \,\mathrm{d}\mu. \end{aligned}$$

Let $\mu^{\otimes 2}$ denote the product measure on the σ-algebra generated by the squared σ-algebra of G. By Stone–Weierstraß theorem it holds $C(K) \overline{\otimes} C(K) \cong C(K^2)$ for every compact subset of G. Since μ is locally finite and inner regular w.r.t. closed sets and G is locally compact, μ is also Radon, whence the inclusion $C(K) \subseteq L^\infty(K, \mu\big|_K)$ is a weak* dense one. Varying K among the compact subsets of G thus yields $M^{\overline{\otimes}2} \cong L^\infty(G^2, \mu^{\otimes 2})$.

Thanks to the properties of Haar measure, it can easily be verified that δ_G is a co-multiplication; also, j_G is trivially an involution and $\boldsymbol{\mu}_G$ is a left-invariant measure on (M_G, δ_G, j_G) (a proof of the latter statement may be easily deduced by the one of Corollary 2.4 below).

Thus $(M_G, \delta_G, j_G, \boldsymbol{\mu}_G)$ is a left-measured involutive Hopf–von Neumann algebras. ∎

Notice that $L^\infty(G, \mu)$ is a natural realization of the dual space $(L^1(G, \mu))^*$ and it is not difficult to verify that it holds $(M_G)_* \cong L^1(G, \mu)$ and the convolution defined via δ_G (see Lemma 1.3) is the usual one.

¿Question? When does the converse of Theorem 2.1 hold? Read: when given a MIHvN algebra (M, δ, j, μ) there exists a locally compact Hausdorff group such that $(M_G, \delta_G, j_G, \boldsymbol{\mu}_G) \cong (M, \delta, j, \boldsymbol{\mu})$ in the sense of Definition 1.7.

Since $(M_G, \delta_G, j_G, \boldsymbol{\mu}_G)$ is a commutative algebra, we can restrict ourselves to the case when (M, δ, j, μ) is a commutative MIHvN algebra (cMIHvN). Then, the statement we want to prove is the following.

Theorem 2.2 (Duality realization for cMIHvN algebras)**.** Let $(M, \delta, j, \boldsymbol{\mu})$ be any commutative measured involutive Hopf–von Neumann algebra. There exists a locally compact Hausdorff topological group G such that $(M, \delta, j, \boldsymbol{\mu}) = (M_G, \delta_G, j_G, \boldsymbol{\mu}_G)$ (see Theorem 2.1 for the definitions).

As a first step, we want to realize M as the space $L^\infty(X, \mu)$ where X is some locally compact Hausdorff topological space endowed with its Baire σ-algebra $\mathfrak{A}$ and μ is some measure on $\mathfrak{A}$. In fact, this will not be our final realization of M, for we need to get rid of the dependence on the space X. To achieve this, we shall need to realize M in some "more canonical" way, using $L^\infty(X, \mu)$ as a starting point to construct some representations also involving $L^1(X, \mu)$ and $L^2(X, \mu)$.

Theorem 2.3 (Realization of a commutative von Neumann algebra)**.** Let M be a *commutative* von Neumann algebra. Then there exists a locally compact Hausdorff topological measurable space $(X, \mathfrak{T}, \mathfrak{A})$ endowed with a σ-ideal $\mathfrak{N}$ of $\mathfrak{A}$ such that

$$M \cong L^\infty(X, \mathfrak{A}, \mathfrak{N}).$$

Proof. See [3, I.7.3.1]. ∎

At this time, the dependence of this realization on μ is misleading, since no measure is in fact required in defining a L^∞ space, for the latter is the datum of a measurable space $(X, \mathfrak{A})$ endowed with a σ-ideal $\mathfrak{N}$ of $\mathfrak{A}$. Nonetheless, the measure is relevant when we want to identify any such space $L^\infty(X, \mathfrak{A}, \mathfrak{N})$ as a $*$-subalgebra of $\mathcal{L}_b(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. Indeed, letting μ_i $(i = 1, 2)$ be different measures on $\mathfrak{A}$ which generate the same ideal $\mathfrak{N} = \mathfrak{N}_{\mu_i}$ of null-measured sets, the spaces $L^\infty(X, \mathfrak{A}, \mu_i)$ are $*$-isomorphic to each other and they may be both regarded as a $*$-subalgebra of $\mathcal{L}_b(L^2(X, \mathfrak{A}, \mu_i))$ via the (left) multiplication action

$$L \colon L^\infty(X, \mathfrak{A}, \mathfrak{N}) \ni f \mapsto (L_f \colon L^2(X, \mathfrak{A}, \mu_i) \ni h \mapsto fh),$$

whereas the spaces $\mathfrak{H}_i \coloneqq L^2(X, \mathfrak{A}, \mu_i)$ need not to be the same.

The absence of a specific measure is not an issue if we confine ourselves to the study of the Banach algebra structure of M, since, taking e.g. $L^1(X, \mu)$ to be a realization of its predual M_*, this must be isomorphic to any other by Theorem 1.2. Nonetheless, since our main goal is to realize M as induced by a Haar measure (rather than simply by a σ-ideal $\mathfrak{N}$ of null-measured sets), we need to be more careful in the construction of all spaces involved, for the measure actually plays a key role.

At this point we ask ourselves wether the choice of the measure μ generating the null-sets ideal $\mathfrak{N}_\mu$ involved in the definition of $L^\infty(X, \mathfrak{A}, \mathfrak{N}_\mu)$ may be chosen in some canonical way. We can answer in the affirmative as soon as we justify calling a σ-weakly continuous semi-finite faithful trace on M a *measure* on it.

Corollary 2.4 (Canonical realization of a commutative measured von Neumann algebra)**.** Let $(M, \boldsymbol{\mu})$ be a *commutative measured von Neumann algebra*. Then there exists a (unique) topological measured space $(X, \mathfrak{T}, \mathfrak{A}, \mu)$ such that

$$M \cong L^\infty(X, \mathfrak{A}, \mathfrak{N}_\mu)$$

and μ represents $\boldsymbol{\mu}$ in the sense that

$$\boldsymbol{\mu}(f) = \langle \mu \,|\, f \rangle := \int_X f \, \mathrm{d}\mu.$$

Proof. The existence of the space $(X, \mathfrak{T}, \mathfrak{A}, \mathfrak{N})$ is granted by Theorem 2.3 so that it suffices to construct μ representing $\boldsymbol{\mu}$ as above and verify that $\mathfrak{N} = \mathfrak{N}_\mu$.

Firstly, for any $E \in \mathfrak{A}$ set $\mu(E) := \boldsymbol{\mu}(\mathbf{1}_E)$; since $\boldsymbol{\mu}$ is faithful and a trace, it is straightforward that $\mu(\varnothing) = \boldsymbol{\mu}(\mathbf{1}_\varnothing) = \boldsymbol{\mu}(0) = 0$, thus it suffices to show that μ is countably additive on disjoint sets. Let $(E_n)_n$ be a countable family of $\mathfrak{A}$-measurable mutually disjoint sets and set $E^k := \cup_n^k E_n$. Then

$$\begin{aligned}
\mu(E^\infty) &= \boldsymbol{\mu}(\mathbf{1}_{E^\infty}) && \text{(semi-finiteness of } \boldsymbol{\mu}) \\
&= \sup_{0 \le f \le \mathbf{1}_{E^\infty}} \boldsymbol{\mu}(f) \\
&\ge \limsup_k \boldsymbol{\mu}(\mathbf{1}_{E^k}) && \left(\text{positive additivity of } \boldsymbol{\mu}, \mathbf{1}_{E^k} = \sum_n^k \mathbf{1}_{E_n}\right) \\
&= \limsup_k \sum_n^k \boldsymbol{\mu}(\mathbf{1}_{E_n}) \\
&= \sum_n^\infty \boldsymbol{\mu}(\mathbf{1}_{E_n}) = \sum_n^\infty \mu(E_n).
\end{aligned}$$

For the opposite inequality it is sufficient to take the limit in k in

$$\mu(E^k) = \boldsymbol{\mu}(\mathbf{1}_{E^k}) = \sum_n^k \boldsymbol{\mu}(E_n) = \mu(E_n),$$

which follows again by positive additivity.

Therefore μ is a measure and, since $\boldsymbol{\mu}$ is faithful it is straightforward that $\mathfrak{N}_\mu \subseteq \mathfrak{N}$. On the other hand, if $E \in \mathfrak{N}$, then $\mu(E) = \boldsymbol{\mu}(\mathbf{1}_E) = \boldsymbol{\mu}(0) = 0$, hence $\mathfrak{N} = \mathfrak{N}_\mu$. ■

Corollary 2.5 (Modular function of a cMIHvN algebra)**.** Let $(M, \delta, j, \boldsymbol{\mu})$ be a cMIHvN algebra and notice that $\hat{\boldsymbol{\mu}} := \boldsymbol{\mu} \circ j^{-1}$ is a measure on μ. Let then $(X, \mathfrak{A}, \mu)$ and $(X, \mathfrak{A}, \hat{\mu})$ be canonical realizations of $(M, \delta, j, \boldsymbol{\mu})$ and $(M, \delta, j, \hat{\boldsymbol{\mu}})$ respectively.

Then there exists a strictly positive (non necessarily bounded) continuous function $\Delta \colon X \to X$ such that

$$\forall f \in \mathrm{C}_c(X) \qquad \int_X f \, \mathrm{d}\mu = \int_X f \Delta \, \mathrm{d}\hat{\mu}$$

Proof. Since μ and $\hat{\mu}$ are induced by semi-finite traces, they are easily seen to be τ-additive (i.e. for every net (E_α) of increasing measurable sets it holds $\lim_\alpha \mu(E_\alpha) = \mu(\cup_\alpha E_\alpha)$). It follows by local compactness of X (see Theorem 2.3) that μ is Radon whenever restricted to a compact subset of X (see [1, 7.2.2.(ii)]). As a consequence, $\langle \mu \,|\, f \rangle < \infty$ for every continuous compactly supported f.

Since $\hat{\boldsymbol{\mu}}$ is also a measure, by Corollary 2.4 it holds $\mathfrak{N}_\mu = \mathfrak{N}_{\hat{\mu}}$, thus μ and $\hat{\mu}$ are mutually absolutely continuous w.r.t. each other, whence the Radon–Nikodým derivative $\Delta := \mathrm{d}_{\hat{\mu}} \mu$ is strictly positive.

The continuity of Δ is nontrivial consequence of the σ-weak continuity of j and μ, which may be deduced by comparing the σ-weak topology restricted to $\mathrm{C}_c(X)$ with the $\sigma(\mathrm{C}_c(X), \mathscr{M}_{bR})$-topology ($\mathscr{M}_{bR}$ denoting the Banach space of totally finite Radon measures on X). ■

Finally, as a result of some calculations (see [5, 2.1]), it can be proved that $\mathrm{C}_c \coloneqq \mathrm{C}_c(X)$ is j-invariant when regarded as a subspace of M and also j_*-invariant when regarded as a subspace of $L^1(X,\mu) \cong M_*$, thus the $(\mathrm{C}_c(X), \mathfrak{A}, \mu)$ may be chosen as a canonical realization of $(M, \delta, j, \boldsymbol{\mu})$. To see the latter invariance property notice that, for every $f \in \mathrm{C}_c$

$$f\hat{\ } = (\Delta f)\check{\ } = \Delta^{-1} f\check{\ } \qquad\qquad \Delta f = f\hat{\ }\check{\ } \tag{2.1}$$

Indeed, for every continuous compactly supported f, g one has

$$\begin{aligned}\int_X f\hat{\ } g \,\mathrm{d}\mu = \Big(\int_X f g\check{\ } \,\mathrm{d}\mu = \int_X f\check{\ } g \,\mathrm{d}\hat{\mu} = \int_X \Delta^{-1} f\check{\ } g \,\mathrm{d}\mu\Big) \\ = \int_X \Delta f g\check{\ } \,\mathrm{d}\hat{\mu} = \int_X (\Delta f)\check{\ } g \,\mathrm{d}\mu\end{aligned}$$

and so $f\hat{\ } = (\Delta f)\check{\ }$ for every $f \in \mathrm{C}_c$ by σ-weak density of $\mathrm{C}_c(X)$ in $L^\infty(X, \mathfrak{A}, \mathfrak{N}_\mu)$. Now since Δ was continuous by Corollary 2.5, it follows that $\Delta\mathrm{C}_c = \mathrm{C}_c$ and thus $\mathrm{C}_c\check{\ } = \mathrm{C}_c$. The invariance now follows, the inclusion $\mathrm{C}_c(X) \subseteq L^1(X,\mu)$ being σ-weakly dense.

Reassuringly, it is straightforward that whenever $(M, \delta, j, \boldsymbol{\mu}) = (M_G, \delta_G, j_G, \boldsymbol{\mu}_G)$, then $\hat{\boldsymbol{\mu}}_G$ induces the right-invariant Haar measure on G and Δ is its modular function.

3 Interpolation between multiplication and convolution

The main goal is now the construction of some suitable representation of $L^1(X,\mu)$. To further this task we need some explicit automorphism of the space $L^2(X^{\overline{\otimes}2}, \mu^{\otimes 2})$. In view of the previous results, we drop from now on the $\boldsymbol{\mu}$ (bold) notation.

Since we deal with a product space, the following verifications are required (see [5, 3.1&3.2])

$$\forall f,g,h \in \mathrm{C}_c \qquad (\mu\otimes\mu)\,[(f\otimes g)\delta(h)] = (\mu\otimes\mu)\,[(h\otimes g\check{\ })\delta(f)] \tag{3.1}$$

$$(\mu\otimes\mu)\,[(f\otimes 1)\delta(g)] = \mu(f)\mu(g) \tag{3.2}$$

which follows by the commutation properties of j, δ and τ (cfr. (1.1)).

Via the identification of the measure with its concrete counterpart, equation (3.2) implies that

$$\|(f\otimes 1)\delta(g)\|_{(L^1)^2} = \|f\|_{L^1}\,\|g\|_{L^1} \tag{3.3}$$

so that the $*$-morphism $\omega_0 \colon {\mathrm{C}_c}^{\otimes 2} \to M^{\otimes 2}$ defined as the linear extension of

$$(f\otimes g) \mapsto (f\otimes 1)\delta(g)$$

has, by (3.3) a unique continuous extension

$$\omega\colon \Big({\mathrm{C}_c}^{\otimes 2}, \|\cdot\|_{L^1(X,\mu)}\Big) \to \Big(M^{\otimes 2} \cap L^1(X^{\overline{\otimes}2}, \mu^{\overline{\otimes}2})\Big).$$

It also holds

$$\left\|\omega\left(\sum_i f_i\otimes g_i\right)\right\|^2_{(L^2)^2} = (\mu\otimes\mu)\left[\sum_{i,j}^n (f_i\bar{f}_j\otimes 1)\delta(g_i\bar{g}_j)\right] \qquad ((3.2))$$
$$= \sum_{i,j}^n \mu(f_i\bar{f}_j)\mu(g_i\bar{g}_j) = \left\|\sum_i f_i\otimes g_i\right\|^2_{(L^2)^2}$$

so that ω uniquely extends to an isometry $W\colon L^2(X^{\overline{\otimes}2},\mu^{\otimes 2})\to L^2(X^{\overline{\otimes}2},\mu^{\otimes 2})$. It is readily seen (see [5, 3.3&3.4]) that W is in fact unitary, since it has an isometric adjoint W^*, for which it also holds

$$(j\otimes\iota)\omega_0(j\otimes\iota) = W^*\big|_{\mathrm{C}_c{}^{\otimes 2}}.$$

Thanks to its reflexivity and Riesz Representation Theorem for Hilbert spaces, $L^2(X,\mu)$ is now the perfect environment to study the convolution. By (3.1) and Lemma 1.3, we can equivalently define the convolution $f*g := \delta_*(f\otimes g)$ on M_* as the unique element satisfying

$$\mu\left[(f*g)h\right] = \mu\left[g(f^{\hat{}}*h)\right] = \mu\left[f(h*g^{\check{}})\right] \tag{3.4}$$

for all $h\in L^1(X,\mu)$.

When we restrict ourselves to functions in $L^1(X,\mu)\cap L^\infty(X,\mathfrak{N}_\mu)$, it is possible to prove a suitable Young inequality.

Lemma 3.1 (Young inequality)**.** For $f,g\in L^1(X,\mu)\cap L^\infty(X,\mathfrak{N}_\mu)$ it holds

$$\|f*g\|_{L^\infty}\le \|f\|_{L^2}\|g^{\check{}}\|_{L^2},\ \|f*g^{\check{}}\|_{L^\infty}\le\|f\|_{L^2}\|g\|_{L^2}.$$

Proof. First of all notice that the inequality is well-posed and non-vacuous for $f,g\in L^1(X,\mu)\cap L^\infty(X,\mathfrak{N}_\mu)$ implies $f,g\in L^2(X,\mu)$ by Interpolation Inequality (see [2, §4.2, Rem.2, p.93]). By definition of convolution, for all $h\in\mathrm{C}_c$ and $u\in M$ such that $u\,|h|=h$, we have

$$\begin{aligned}
|\mu\left[(f*g^{\check{}})h\right]| = & && ((3.1))\\
&= |(\mu\otimes\mu)\left[(f\otimes g^{\check{}})\delta(h)\right]| && ((3.4))\\
&= |(\mu\otimes\mu)\left[(h\otimes g)\delta(f)\right]| && (u\,|h|=h)\\
&= \left|(\mu\otimes\mu)\left[(u\,|h|^{1/2}\otimes g)(|h|^{1/2}\otimes 1)\delta(f)\right]\right| && (\text{def. of }\omega)\\
&= \left|(\mu\otimes\mu)\left[(u\,|h|^{1/2}\otimes g)\omega(|h|^{1/2}\otimes f)\right]\right| && (W\text{ bounded})\\
&\le \left\|u\,|h|^{1/2}\otimes g\right\|_{(L^2)^2}\|W\|_{\mathcal{L}_b(L^2)}\left\||h|^{1/2}\otimes f\right\|_{(L^2)^2} && (W\text{ unitary})\\
&= \|h\|_{L^1}\|g\|_{L^2}\|f\|_{L^2}. && \blacksquare
\end{aligned}$$

By the previous Lemma 3.1, the convolution operator

$$*\colon\ \left(L^1(X,\mu)\cap L^\infty(X,\mathfrak{N}_\mu)\right)^2\to L^\infty(X,\mathfrak{N}_\mu)$$

is L^2-continuous, and may thus be extended to the whole $(L^2(X,\mu))^2$ by density of its original domain in the latter space. In fact, it is possible to prove that this $L^2(X,\mu)$-convolution coincides with the original one on $(L^1(X,\mu)\cap L^2(X,\mu))^2$ (see [5, 4.2]).

4 Involution and representations of predual algebras

Thanks to the properties of the convolution, we are able to define an involution on $L^1(X,\mu)$ via the following lemma.

Lemma 4.1. Let $f, g \in L^1(X,\mu) \cap L^\infty(X,\mu)$. Then

$$(f * g)\hat{} = g\hat{} * f\hat{} \qquad\qquad (f * g)\check{} = g\check{} * f\check{}. \tag{4.1}$$

Proof. For any $h \in L^\infty(X,\mu)$ we have by (3.1) and (1.1)

$$\begin{aligned}\mu\left[(f*g)\hat{}h\right] =& \mu\left[(f*g)h\check{}\right] = (\mu\otimes\mu)\left[(f\otimes g)\delta(h)\right]\\ =&(\mu\otimes\mu)\left[(f\otimes g)\tau(j\otimes j)\delta(h)\right]\\ =&(\mu\otimes\mu)\left[(g\hat{}\otimes f\hat{})\delta(h)\right]\\ =&\mu\left[(g\hat{} * f\hat{})h\right].\end{aligned}$$

The conclusion follows by faithfulness of μ. Proceeding in almost the same way it is possible to that it also holds $(f*g)\check{} = g\check{} * f\check{}$ (see [5, 5.3]). ■

The previous lemma allows us to define the involution

$$-^*\colon L^1(X,\mu) \ni f \mapsto f^* := \overline{f\hat{}} \in L^1(X,\mu)$$

making $(L^1(X,\mu), -^*)$ an involutive Banach algebra.

Our next goal will be the construction of some representation of the predual algebra $L^1(X,\mu) \cong M_*$ (at this point an involutive Banach algebra) to further our construction.

By Lemma 3.1 and the strong density of the inclusion $\mathrm{C}_c(X) \subseteq L^1(X,\mu)$,

$$(L^2(X,\mu))^2 \ni (g_1, g_2) \mapsto \mu\left[f(\bar{g}_1 * g_2\check{})\right]$$

is a continuous sesquilinear on $(L^2(X,\mu))^2$ and may therefore be represented by $\langle\lambda(f)- \mid -\rangle_{L^2}$ where $\lambda(f)\colon L^2(X,\mu) \to L^2(X,\mu)$ is a bounded operator. Again by Lemma 3.1, taking the supremum for g_1, g_2 in the unit sphere of $L^2(X,\mu)$, it follows that $\|\lambda(f)\|_{\mathcal{L}_b(L^2(X,\mu))} \le \|f\|_{L^1}$. Furthermore, by (3.4) it holds

$$\lambda(f)^* = \lambda(f^*)$$

for every $f \in L^1(X,\mu)$, representing the convolution $*$ in the sense that $\lambda(f)g = f*g$ for all $f, g \in \mathrm{C}_c(X)$ (hence in $L^2(X,\mu)$ by density of the former).

Definition 4.2 (Left regular representation of $L^1(X,\mu)$)**.** At this point we have proved that

$$\lambda\colon L^1(X,\mu) \to \mathcal{L}_b(L^2(X,\mu))$$

is a *-representation of $L^1(X,\mu)$ on $L^2(X,\mu)$. Further calculations (proceeding as for the usual convolution on $\mathbb{R}^n$, see [5, 5.1&5.2]) show that λ is a *faithful* representation.

Now we are ready to introduce a second operator V in $\mathcal{L}_b(L^2(X,\mu))$ which will allow us to define a *right regular representation* of $L^1(X,\mu)$. To this purpose notice that by (4.1) and (2.1) it follows that

$$\Delta(f*g) = (\Delta f) * (\Delta g).$$

Along the same line of reasoning it is possible to see (see [5, 5.4]) that it also holds

$$\Delta^{1/2}(f*g) = (\Delta^{1/2} f) * (\Delta^{1/2} g).$$

Using again $\mathrm{C}_c(X)$ as included in every Lebesgue space on (X,μ), we define

$$Vf := \Delta^{-1/2} f\check{}$$

and notice that

$$\|Vf\|_{L^2}^2 = \mu\left[\Delta^{-1}|f^{\check{}}|^2\right] = \hat{\mu}\left[|f^{\check{}}|^2\right] = \mu(|f|^2) = \|f\|_{L^2}^2$$

whence V uniquely extends to an operator on $L^2(X,\mu)$, which we denote in the same way.

Lemma 4.3 (Extension of the predual involution j_* to $L^2(X,\mu)$)**.** Let $\Delta^{1/2}$ denote the (unbounded) multiplication operator induced by the same function and set

$$J := \Delta^{1/2}V\colon L^2(X,\mu) \to L^2(X,\mu)$$

with $\operatorname{dom} J := \operatorname{dom}\Delta^{1/2}$. Then J is a closed extension of j on $L^2(X,\mu)\cap L^\infty(X,\mu)$ and its adjoint J^* its a closed extension of j_* on $L^2(X,\mu)\cap L^1(X,\mu)$. The following also hold

$$\begin{aligned} J &= \Delta^{1/2}V = V\Delta^{-1/2} & VJV &= J^* & \Delta &= JJ^* \\ J^* &= \Delta^{-1/2}V = V\Delta^{1/2} & VJ^*V &= J & V\Delta V &= \Delta^{-1} \\ \forall f &\in L^1(X,\mu)\cap L^2(X,\mu) & J^*\bar{f} &= f^* \end{aligned}$$

Proof. It is sufficient to notice that $Jf = \Delta^{1/2}Vf = \Delta^{1/2}\Delta^{-1/2}f^{\check{}} = f^{\check{}}$ for all f in $\mathrm{C}_c(X)$ and that j_* is a $\|\cdot\|_{L^2}$-isometry on $\mathrm{C}_c(X)$. All the above equalities are a consequence of both the definition of J and (2.1). ■

Definition 4.4 (Right regular representation)**.** Another straightforward consequence of the previous lemma is that V is also a self-adjoint operator, thus the conjugation of λ via V defined by

$$\lambda'\colon L^1(X,\mu) \ni f \mapsto V\lambda(f)V \in \mathcal{L}_b(L^2(X,\mu))$$

is again a *-representation of $L^1(X,\mu)$ on $L^2(X,\mu)$. Since one has

$$V\lambda(\,\cdot\,) = \lambda(\,\cdot\,)'V$$

this justifies the name of *right* (and left) regular representation. On the other hand, these are said to be *regular* for they coincide, in the case $M = M_G$, with the regular representation of G in the usual sense.

Now that we have defined the morphisms λ and λ', we set

$$M(\lambda) := \left[\lambda(L^1(X,\mu))\right]'' \qquad M(\lambda') := \left[\lambda'(L^1(X,\mu))\right]''$$

to be the von Neumann algebras generated by them (by von Neumann Bicommutant Theorem).

5 Commutants and dual IHvN algebras

The notation λ' might seem confusing w.r.t. the construction of $M(\lambda')$, though it is consistent. It is in fact the major achievement of [5, §6&7] the proof that

$$M(\lambda)' = M(\lambda') \qquad M(\lambda')' = M(\lambda).$$

Thus, thanks to Tomita's result [3, p.29], it also holds

$$\left[M(\lambda)^{\overline{\otimes}2}\right]' = \left[M(\lambda)'\right]^{\overline{\otimes}2} = M(\lambda')^{\overline{\otimes}2}.$$

With the previous results at hand it is possible to define the *dual IHvN algebra* of our MIHvN algebra (M,δ,j,μ) (see [5, §9] for the proofs). This will be von Neumann algebra $M(\lambda)$, endowed with the co-multiplication

$$\pi(-) := W^{-1}(-\otimes\mathbf{1})W$$

and the involution

$$\kappa\colon a \mapsto \bar{a}^*$$

where $-^*$ is the usual adjoint of an operator (recall that $M(\lambda)$ consists of operators acting on $L^2(X,\mu)$) and the conjugate of an operator a is defined by $a(f) := \overline{a(\bar{f})}$, $\bar{f}$ being the complex conjugate.

It turns out that π is indeed co-associative and it commutes with κ, so that $(M(\lambda), \pi, \kappa)$ is truly a symmetric IHvN algebra.

6 The predual group of a MIHvN algebra

The main goal of this final section will be the construction of the group G whose canonical MIHvN algebra coincides with the assigned (M, δ, j, μ). Given the algebra $(M(\lambda), \pi, \kappa)$ constructed above, since λ is a faithful representation (see Definition 4.2), it makes sense to push the measure $\boldsymbol{\mu}$ forward to $M(\lambda)$, by setting

$$\boldsymbol{\mu}(\lambda(f)) := \boldsymbol{\mu}(f)$$

and then extending to the whole algebra by continuity of λ. We will denote this new trace (which is clearly a measure since $\boldsymbol{\mu}$ is) by $\lambda_\sharp\boldsymbol{\mu}$. At this point, $(M(\lambda), \pi, \kappa, \lambda_\sharp\boldsymbol{\mu})$ is a MIHvN algebra, so that we can reason as in §2 and identify it with some L^∞-space, which we denote simply by $L^\infty(\lambda)$, while its predual will be $L^1(\lambda) \cong M_*(\lambda)$ as constructed via the measure $\lambda_\sharp\boldsymbol{\mu}$.

It is now our task to prove that the group G we are looking for from the beginning is in fact the spectrum of $L^1(\lambda)$.

We replicate here the constructions of §3, in order to construct the space $L^2(\lambda)$. Indeed, set for $\phi \in L^1(\lambda)$

$$\|\phi\|_{L^2(\lambda)} := \sup\left\{ |\langle \lambda(f)^* \,|\, \phi\rangle| \,; \quad f \in L^1(X,\mu) \cap L^\infty(X,\mu), \|f\|_{L^2(X,\mu)} \leq 1 \right\}$$

where the coupling is taken to be the one induced by the dual couple $\big(L^1(\lambda), L^\infty(\lambda)\big)$. Letting $L^1_2(\lambda)$ be the set of ϕ's with finite $\|\cdot\|_{L^2(\lambda)}$ norm, the closure of $L^1_2(\lambda)$ w.r.t. $\|\cdot\|_{L^2(\lambda)}$ is a Hilbert space (as realized in its bidual via the canonical embedding), which we denote by $L^2(\lambda)$.

Now, for any fixed $\phi \in L^1(\lambda)$, the map $L^1(\mu) \ni f \mapsto \langle \lambda(f)^* \,|\, \phi\rangle$ is a bounded anti-linear (thanks to taking the adjoint of $\lambda(f)$) functional on $L^1(\mu)$ for every ϕ, thus it is uniquely represented by some $g := g_\phi \in L^\infty(X,\mu)$ satisfying

$$\mu(\bar{f}g) = \langle \lambda(f)^* \,|\, \phi\rangle \,.$$

Thus, we can the dual map γ of λ by setting

$$\gamma\colon L^1(\lambda) \ni \phi \mapsto g_\phi \in L^\infty(\mu).$$

From the definition of λ^* it also follows that $\lambda^*\colon L^2(\lambda) \cap L^1(\lambda) \to L^1(X,\mu) \cap L^2(X,\mu)$ is a $(\|\cdot\|_{L^2(\lambda)}\,; \|\cdot\|_{L^2(X,\mu)})$-isometry, thus it may be extended to an isometry $\Gamma\colon L^2(\lambda) \to L^2(X,\mu)$.

At this point it is possible to replicate the reasoning of §3 and §4, Γ playing the role of W, to construct ρ the left regular representation of $L^1(\lambda)$ as acting on the Hilbert space $L^2(\lambda)$. Namely,

$$\begin{aligned} \rho\colon L^1(\lambda) &\longrightarrow \mathcal{L}_b(L^2(\lambda)) \\ \phi &\longmapsto - * \phi. \end{aligned}$$

Furthermore, notice that γ is itself a representation on $L^1(\lambda)$, this time acting on $L^2(X,\mu)$, when $\gamma(\phi)$ is understood as the multiplication operator induced on $L^2(X,\mu)$ by the $L^\infty(X,\mu)$-function $\gamma(\phi)$). Also, by definition of γ it follows that

$$\gamma(-) = \Gamma\rho(-)\Gamma^{-1}.$$

We therefore have constructed representations

$$\begin{aligned}\lambda\colon L^1(\mu) &\to \mathcal{L}_b(L^2(\mu))\\ \rho\colon L^1(\lambda) &\to \mathcal{L}_b(L^2(\lambda))\\ \gamma\colon L^1(\lambda) &\to \mathcal{L}_b(L^2(\mu)),\end{aligned}$$

where γ may be thought of as ρ' (the role of V being also played by Γ).

This triplet of representation morphisms (which turn out to be all faithful) may be used to show that $L^1(\lambda)$ is a *semi-simple* involutive abelian Banach algebra.

By known results of the theory (see [6, 3.8&3.11]), the semi-simplicity of $L^1(\lambda)$ as a Banach algebra yields that the Gel'fand representation $\widehat{\ }$ is faithful, its kernel being coincident with the Jacobson radical of the algebra. Thus, $L^1(\lambda) \cong M_*(\lambda)$ is faithfully represented by $\widehat{\ }$ on $C_0(G)$, where $G = \Phi_{M_*(\lambda)}$ denotes the spectrum of the algebra.

We are now able to endow G with a group structure. Indeed, letting $s \in G$ and $\widehat{\phi}(s)$ the Gel'fand representation of $\phi \in L^1(\lambda)$ evaluated on s, by faithfulness of the representation there exists a (unique) unitary vector $u \in L^\infty(\lambda)$ such that

$$\langle u^* \mid \phi\rangle = \widehat{\phi}(s);$$

we will denote such vector by $\lambda(s)$. Since λ is injective, we may define, for all $s,t \in G$,

$$st := \lambda^{-1}\left[\lambda(s)\lambda(t)\right].$$

It is in fact possible to show that $\lambda^{-1}(\mathbf{1}) =: e$ is the identity in G and that for every $u \in \lambda(G)$, $\bar{u}^* = \bar{u}^{-1}$ also belongs to $\lambda(G)$ and is the inverse of u, so that G is in fact a group. The continuity of the operations w.r.t. the topology of G follows by the coincidence of the strong and weak operator topologies on the closed unit ball of $L^\infty(\lambda)$, so that G is a locally compact topological group.

Finally, since $\|u\|_{\mathrm{op}} = \|u\|_{L^\infty(\lambda)} = 1$ for all $u \in \lambda(G)$ whenever regarded as their corresponding multiplication operator acting on $L^2(\lambda)$, the morphism λ is itself a *faithful unitary group representation* of G, whose image generates the von Neumann algebra $L^\infty(\lambda) \cong M(\lambda)$.

At this point it is possible to construct (see [5, §11]) a Haar measure μ_G on G, which is induced by the representation ρ. The MIHvN algebra induced by G is then isomorphic to our original algebra $(M,\delta,j,\boldsymbol{\mu})$, so that our conclusive theorem holds.

Theorem 6.1 (Duality realization for cMIHvN algebras)**.** Let $(M,\delta,j,\boldsymbol{\mu})$ be any commutative measured involutive Hopf–von Neumann algebra.

There exists a locally compact Hausdorff topological group G such that $(M,\delta,j,\boldsymbol{\mu}) = (M_G,\delta_G,j_G,\boldsymbol{\mu}_G)$ (see Theorem 2.1 for the definitions).

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