HOPF ALGEBRAS WITH TRACE AND REPRESENTATIONS

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Abstract. We study the restriction of representations of Cayley-Hamilton algebras to subalgebras. This theory is applied to determine tensor products and branching rules for representations of quantum groups at roots of 1.

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Introduction

Irreducible representations of quantized universal enveloping algebra were classified in [DKP1]D2. These algebras are finite dimensional over their center and are Cayley-Hamilton algebras [P2].

Here we study the restriction representations of Cayley-Hamilton algebras to subalgebras. This theory is applied to the tensor product of two generic representations of $U_\epsilon(\mathfrak{g})$, and of $U_\epsilon(\mathfrak{b})$, and to the branching of generic irreducible representations when $U_\epsilon(\mathfrak{b}) \subset U_\epsilon(\mathfrak{g})$. Here ϵ is an primitive root of 1 of an odd degree.

The center Z of $U_\epsilon(\mathfrak{g})$ has the central Hopf subalgebra Z_0 generated by ℓ-th powers of root generators of and by ℓ-th powers of generators of the Cartan subalgebra [DKP1]. This central Hopf subalgebra Z_0 is isomorphic to the algebra of polynomial functions on G^* which is a Poisson Lie dual to G. Let $\pi : Spec(Z) \to Spec(Z_0) = G^*$ be the natural projection induced by the inclusion of Z_0 to the center Z of $U_\epsilon(\mathfrak{g})$. According to the general theory of Cayley-Hamilton algebras there exist a Zariski open subvariety $S \subset Spec(Z_0)$ such that π is a finite covering map and the algebra is semisimple over S. In case of $U_\epsilon(\mathfrak{g})$ this projection has ℓ^r fibers where r is the rank of the Lie algebra $\mathfrak{g}$. Thus, each central character $\chi \in Spec(Z)$ with $\pi(\chi) \in S$ defines an irreducible representation V_χ.

The tensor product $V_\chi \otimes V_{\chi'}$ is completely reducible if the product $\pi(\chi)\pi(\chi')$ (in G^*) is generic, i.e. belongs to S. Our results imply:

$$V_\chi \otimes V_{\chi'} \simeq \oplus_{\chi'' \in Spec(Z), // \pi(\chi'')=\pi(\chi)\pi(\chi')} (V_{\chi''})^{\oplus m} \tag{1}$$

where $m = \ell^{|\Delta_+|-r}$. Here $|\Delta_+|$ is the number of positive roots.

Similar decompositions hold for the restriction of V_χ to $U_\epsilon(\mathfrak{b})$ and for the tensor product of $U_\epsilon(\mathfrak{b})$-modules.

In sections 1 to 4 we recall the general theory of semisimple representations of Cayley-Hamilton algebras (CH-algebras for short). Then in sections 5 to 7 we study how a semisimple representation of a CH-algebra restricts to a CH-subalgebra. Then we apply this theory to the decomposition of tensor product of semisimple representations of the special class of Hopf algebras that we call CH-Hopf algebras. Finally we study examples of such Hopf algebras which are quantum groups at roots of 1. The special form of the multiplicities suggests that for many natural varieties related to the corresponding Poisson Lie groups the Poisson tensor is constant in certain birational coordinate system. Such coordinates are known in many cases.

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1. $n-$dimensional representations.

In this paper by **ring** we mean an associative ring with 1. An algebra over a commutative ring A will be an associative and unital algebra.

In this section we remind some basic facts of universal algebra. We will use the following categories (we denote by A a commutative ring):

- <u>*Set*</u> is the category of sets,
- $\mathcal{C}$ and $C(A)$ are categories of commutative rings and commutative A-algebras respectively,
- $\mathcal{N}$ and $N(A)$ are categories of non commutative rings and of non commutative A-algebras respectively.

1.1. **The universal n-dimensional representation.** Given a ring B one denotes by $M_n(B)$ the full ring of $n \times n$ matrices over B. If $f : B \to C$ is a ring homomorphism we can construct $M_n(f) : M_n(B) \to M_n(C)$ the homomorphism induced on matrices. We will use systematically the following simple well known Lemma.

Lemma 1.1. *If I is a 2-sided ideal in $M_n(B)$ then $I = M_n(J)$ for a (unique) 2-sided ideal J in B (and $M_n(B)/I = M_n(B/J)$).*

Furthermore, if $f : B \to C$ is such that $M_n(f) : M_n(B) \to M_n(C)$ annihilates I, then there is a morphism $\bar{f} : B/J \to C$ such that the following diagram commutes

$$\begin{array}{ccc} M_n(B) & \xrightarrow{f} & M_n(C) \\ \searrow & & \nearrow_{M_n(\bar{f})} \\ & M_n(B/J) & \end{array}$$

Let R be a ring, by an $n-$dimensional representation of R over a commutative ring B one means a homomorphism

$$\phi : R \to M_n(B)$$

We will use notation $\mathcal{R}_R^n(B) := \hom_{\mathcal{C}}(R, M_n(B))$ for the set of all these representations. It is clear that it defines a functor $\mathcal{R}_R^n : \mathcal{C} \to \underline{Set}$. The image of a given ring homomorphism $f : B \to C$ under this functor is the map $\mathcal{R}_R^n(B) \to \mathcal{R}_R^n(C)$ obtained by composing with

$$M_n(B) \xrightarrow{M_n(f)} M_n(C).$$

Lemma 1.2. *The functor $\mathcal{R}_R^n$ is representable.*

Proof. We should prove that there is a commutative ring $A_n(R)$ such that $\mathcal{R}_R^n(B) = hom_{\mathcal{C}}(A_n(R), B)$.

Let a_α be a set of generators for the ring R. This gives a presentation of it as a quotient ring of the free (non commutative) algebra $\mathbb{Z}\langle x_\alpha \rangle$:

$$\pi : \mathbb{Z}\langle x_\alpha \rangle \to R = \mathbb{Z}\langle x_\alpha \rangle / K, \ K := Ker\pi.$$

with generators a_α being images of x_α.

For each α choose a set of n^2 variables $\xi_{i,j}^\alpha$. Let $A := \mathbb{Z}[\xi_{i,j}^\alpha]$ be the polynomial ring in all these variables.

Define the *generic matrices* ξ_α in $M_n(A)$ by setting ξ_α to be the matrix which, in the i,j entry, has coefficient $\xi_{i,j}^\alpha$.

Let $j : \mathbb{Z}\langle x_\alpha \rangle \to M_n(A)$ be the algebra homomorphism defined by $j(x_\alpha) = \xi_\alpha$. Let finally I be the 2-sided ideal in $M_n(A)$ generated by $j(K)$. By the previous Lemma, $I = M_n(J)$ for some ideal J of A and thus we have the mapping $j_R : R \to M_n(A/J)$ and the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\langle x_\alpha \rangle & \xrightarrow{j} & M_n(A) \\ \downarrow_\pi & & \downarrow_{M_n(p)} \\ R & \xrightarrow{j_R} & M_n(A/J) \end{array}$$

Here $p : A \to A/J$ is the quotient map.

Let $\phi : R \to M_n(B)$ be any representation $\phi(a_\alpha) = (f_{i,j}^\alpha)$. It gives the homomorphism of commutative rings $A \to B$, $\xi_{i,j}^\alpha \mapsto f_{i,j}^\alpha$. This homomorphism induces the required map $\overline{\phi}: A/J \to B$ for which the diagram

$$\begin{tikzcd} R \arrow[rr, "j_R"] \arrow[dr, "\phi"'] & & M_n(A/J) \arrow[dl, "M_n(\overline{\phi})"] \\ & M_n(B) & \end{tikzcd}$$

commutes. This proves representability of $\mathcal{R}^n_R$.

□

Notice that in the proof we constructed the commutative ring $A_n(R) := A/J$.

Notice that, for a finitely generated ring R also $A_n(R)$ is finitely generated. This construction is functorial. Indeed, for each ring homomorphism $f : R \to S$ we have a corresponding homomorphism of commutative rings $A_n(f) : A_n(R) \to A_n(S)$ defined naturally. It is clear that the diagram

$$\begin{array}{ccc} R & \xrightarrow{j_R} & M_n(A_n(R)) \\ \downarrow_f & & \downarrow_{M_n(A_n(f))} \\ S & \xrightarrow{j_S} & M_n(A_n(S)) \end{array}$$

is commutative.

Definition 1.3. *We shall refer to the mapping*

$$R \xrightarrow{j_R} M_n(A_n(R)) \tag{2}$$

as the universal $n-$dimensional representation.

Universality of the ring $A_n(R)$ means the commutativity of the diagram above.

Remark 1.4. *The commutative ring $A_n(R)$ may be zero. This means that the ring R does not have any n-dimensional representation.*

Instead of working with rings we can work with algebras over a commutative ring A. Clearly all the discussion from above carries over. Moreover if R is a finitely generated algebra so is the universal ring $A_n(R)$. The universality of $A_n(R)$ implies the following theorem

Theorem 1.5. *The functor $B \to M_n(B)$ from the category $\mathcal{C}(A)$ of commutative $A-$algebras to the category $\mathcal{N}(A)$ of non commutative $A-$algebras has a left adjoint*

$$hom_{\mathcal{N}}(R, M_n(B)) = hom_{\mathcal{C}}(A_n(R), B).$$

for each R in $\mathcal{N}$.

Example 1.6. *Consider the ring U generated by three elements H, X, Y with defining relations*

$$HX - XH = Y \tag{3}$$

In this case the ring $A_n(U)$ is the polynomial ring in the $2n^2$ variables h_{ij}, x_{ij}.

Example 1.7. *Consider the commutative polynomial ring $\mathbb{Z}[x, y]$ generated by two elements x and y.*

In this case the ring $A_n(\mathbb{Z}[x, y])$ is generated by the $2n^2$ variables x_{ij}, y_{ij} modulo the quadratic equations $\sum_s x_{is}y_{sj} - \sum_s y_{is}x_{sj}$. It is not known if these equations generate in general a prime ideal!

Example 1.8. *Consider the $\mathbb{Q}$ algebra U generated by two elements X, Y with defining relations*

$$XY - YX = 1 \tag{4}$$

In this case the ring $A_n(U)$ is 0. If instead we work over $\mathbb{Z}$ we get not trivial rings, since the matrix equation $XY - YX = 1$ can be solved in some characteristic $p > 0$.

Example 1.9. *Let k be a field and $R := M_m(k)$ the algebra of $m \times m$ matrices. Then $A_n(R) = 0$ unless m divides n; in the case $n = mr$, let $\overline{k}$ be an algebraic closure of k. Consider the embedding $GL(r, \overline{k})$ into $GL(mr, \overline{k})$ given by the tensor product $1 \otimes A : \overline{k}^m \otimes \overline{k}^r \to \overline{k}^m \otimes \overline{k}^r$. $A_n(R)$ is the coordinate algebra of the homogeneous space $GL(mr, \overline{k})/GL(r, \overline{k})$ (cf.* [LP1], [LP2]*)*

1.2. **Equivalence between universal n-dimensional representations.** When one studies representations one has a natural equivalence given by *changing the basis.* [1] We shall not describe this theory for general rings (cf. [P]), but assume now that all rings are algebras over a field k. Let B be a k-algebra. An invertible matrix $g \in GL(n, k)$ defines a $B-$automorphism by conjugation:

$$M_n(B) \xrightarrow{C(g)} M_n(B), \qquad C(g)(A) := gAg^{-1}$$

Let R be an algebra over k. From the universal property of the universal n-dimensional representation, every matrix $g \in GL(n, k)$ defines a homomorphism, $\overline{g} : A_n(R) \to A_n(R)$ making the following diagram commutative

$$\begin{array}{lcl} R & \xrightarrow{j_R} & M_n(A_n(R)) \\ \downarrow j_R & & \downarrow M_n(\overline{g}) \\ M_n(A_n(R)) & \xrightarrow{C(g)} & M_n(A_n(R)) \end{array}$$

Notice that such $\overline{g}$ is unique due to the universality of A_n.

Composing with some other $h \in G$ we get:

$$M_n(\overline{hg})) \circ j_R = C(hg) \circ j_R = C(h) \circ C(g) \circ j_R =$$

$$C(h) \circ M_n(\overline{g}) \circ j_R = M_n(\overline{g}) \circ C(h) \circ j_R = M_n(\overline{g}) \circ M_n(\overline{h}) \circ j_R = M_n(\overline{g} \circ \overline{h}) \circ j_R$$

and therefore

$$M_n((\overline{hg})) = M_n(\overline{g} \circ \overline{h})$$

which implies $\overline{hg} = \overline{g} \circ \overline{h}$. In other words we get an action of $GL(n, k)$ on $A_n(R)$, $g : a \mapsto \overline{g}^{-1}(a)$, an action of $GL(n, k)$ on $M_n(k)$ by conjugation and the diagonal action $m \otimes_k a \mapsto C(g)m \otimes_k \overline{g^{-1}}(a)$ on $M_n(A_n(R)) = M_n(k) \otimes_k A_n(R)$.

The identity $C(g) \circ j_R = M_n(\overline{g}) \circ j_R$ means that $M_n(\overline{g})^{-1} C(g) \circ j_R = j_R$ or that j_R maps R into the elements which are $GL(n, k)$ invariant:

$$R \xrightarrow{j_R} M_n(A_n(R))^{GL(n,k)}.$$

One of the aims of the theory is to understand better the previous map; it is clear that in the algebra of invariants we find all the *characters* $Tr(j_R(a))$, $a \in R$ or even all coefficients of characteristic polynomials. This justifies introducing such

[1] In the arithmetic theory one needs a little care, one has to define the automorphism group of matrices as *group scheme*.

characters formally as will be done in the next section. In the meantime let us interpret the construction for the free algebra. So let us assume that k is an infinite field which allows us to identify formal polynomials with functions.

For the free (non commutative) algebra $k\langle x_\alpha\rangle_{\alpha\in I}$, we have seen that for each α we choose a set of n^2 variables $x^\alpha_{i,j}$ and let $A_{n,I} := k[x^\alpha_{i,j}]$ be the polynomial ring in all these variables.

We have seen that the universal map is the map $j(x_\alpha) = \xi_\alpha$ sending each variable x_α to the corresponding generic matrix ξ_α, the matrix which, in the i,j entry, has value $x^\alpha_{i,j}$.

The ring $A_{n,I}$ is best thought of as the ring $k[M_n(k)^I]$ of polynomial functions on $M_n(k)^I$ and the ring $M_n(A_{n,I})$ is best thought of as the ring of polynomial maps $f : M_n(k)^I \to M_n(k)$.

Now assume that I is a set with m elements. Choose it to be $I = \{1,\dots,m\}$. Then

$$M_n(k)^I = M_n(k)^m = \{(\xi_1,\dots,\xi_m),\ \xi_i \in M_n(k)\}$$

Define

$$A_{n,m} = k[M_n(k)^m] = k[\xi_i^{hk}]$$

The generic matrix ξ_i is thus a *coordinate function* mapping $(\xi_1,\dots,\xi_m) \to \xi_i$. The $GL(n,k)$ action is identified to the obvious action on functions:

$$(gf)(\xi_1,\dots,\xi_m) := gf(g^{-1}\xi_1 g,\dots,g^{-1}\xi_m g)g^{-1}$$

Finally the ring $M_n(A_{n,m})^{GL(n,k)}$ can be identified to the *ring of $GL(n,k)$ equivariant maps*:

$$f : M_n(k)^n \to M_n(k) \,|\quad f(g\xi_1 g^{-1},\dots,g\xi_m g^{-1}) = gf(\xi_1,\dots,\xi_m)g^{-1}.$$

Let us denote by $C_n(\xi_1,\dots,\xi_m) := M_n(A_{n,m})^{GL(n,k)}$ the ring of equivariant maps.

2. Cayley-Hamilton algebras.

2.1. **Algebras with trace.** We start with a formal definition which belongs to universal algebra.

Definition 2.1. *An associative algebra with trace, over a commutative ring A is an associative algebra R with a 1-ary operation*

$$t : R \to R$$

which is assumed to satisfy the following axioms:

(1) *t is $A-$linear.*
(2) *$t(a)b = b\,t(a), \quad \forall a,b \in R$.*
(3) *$t(ab) = t(ba), \quad \forall a,b \in R$.*
(4) *$t(t(a)b) = t(a)t(b), \quad \forall a,b \in R$.*

This operation is called a formal trace. We denote $t(R) := \{t(a),\ a \in R\}$ the image of t.

Remark 2.2. *We have the following implications:*
Axiom 1) implies that $t(R)$ is an $A-$submodule.
Axiom 2) implies that $t(R)$ is in the center of R.
Axiom 3) implies that t is 0 on the space of commutators $[R,R]$.
Axiom 4) implies that $t(R)$ is an $A-$subalgebra and that t is also $t(R)-$linear.

The basic example of algebra with trace is of course the algebra of $n \times n$ matrices over a commutative ring B with the usual trace.

Notice that we have made no special requirements on the value of $t(1)$.

Algebras with trace form a category, where objects are algebras with trace and morphisms algebra homomorphisms which commute with trace mappings. An ideal in a trace algebra is a *trace* ideal i.e. one which is stable under the trace. Then the usual homomorphism theorems are valid.

The subalgebra $t(R)$ is called the *trace algebra.*

Example 2.3. *As usual in universal algebra the category of algebras with trace has free algebras. Given a set I it is easily seen that the free algebra with trace in the variables x_i, $i \in I$, is obtained as follows:*

First one constructs the free algebra $A\langle x_i \rangle$ in the variables x_i, $i \in I$ whose basis over A are the free monomials in these variables. Next one defines *cyclic equivalence* of monomials where when we decompose a monomial $M = XY$ we set $XY \cong YX$.

Finally for every equivalence class of monomials we pick a commutative variable $t(M)$ and form finally the polynomial algebra:

$$F\langle x_i \rangle := A\langle x_i \rangle [t(M)], \; i \in I,$$

where M a monomial up to cyclic equivalence.

We set $T\langle x_i \rangle := A[t(M)]$ the commutative polynomial algebra in the variables $t(M)$. The trace is defined as the unique $T\langle x_i \rangle-$linear map for which $t : M \to t(M)$. It is easily verified that this gives the free algebra with trace.

Remark 2.4. *There is an obvious base change construction on algebras with trace. If R is an $A-$ algebra with trace and B a commutative A algebra then $R \otimes_A B$ has the natural trace $t(r \otimes b) := t(r)b$. We have $t(R \otimes_A B) = t(R) \otimes_A B$.*

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2.2. **n-dimensional representations of algebras with trace.** We can now apply the theory of §1 to algebras with trace. The only difference is now that, if R is an algebra with trace, by an $n-$dimensional representation over a commutative ring B one means a homomorphism $\phi : R \to M_n(B)$ which is compatible with traces, where matrices have the standard trace. The discussion of §1 can be repeated verbatim, we have again a representable functor, a universal (trace preserving) map $R \xrightarrow{i_R} M_n(B_n(R))$, an action of $GL(n,k)$ on $B_n(R)$ and a map c

$$R \xrightarrow{i_R} M_n(B_n(R))^{GL(n,k)}$$

We shall call $B_n(R)$ the *coordinate ring of the $n-$dimensional representations of* R and the map i_R the *generic $n-$dimensional representation of R.* We will use the notation $B_n(R)$ to emphesize that now we work with algberas with traces.

We shall see that, under suitable assumptions, this map i_R is an isomorphism. For this we first point out some elements which are always in the kernel of i_R.

2.3. **Cayley Hamilton algebras.** At this point we will restrict the discussion to the case in which A is a field of characteristic 0. The positive characteristic theory can to some extent be developed, provided we start the axioms from the idea of a *norm* and not a trace. Since the theory is still incomplete we will not go into it now.

The basic algebraic restriction which we know for the algebra of $n \times n$ matrices over a commutative ring B is the Cayley Hamilton theorem:

Every matrix M satisfies its characteristic polynomial $\chi_M(t) := \det(t - M)$.

The main remark that allows to pass to the formal theory is that, in characteristic 0, there are universal polynomials $P_i(t_1, \ldots, t_i)$ with rational coefficients, such that:

$$\chi_M(t) = t^n + \sum_{i=1}^{n} P_i(tr(M), \ldots, tr(M^i))t^{n-i}.$$

The polynomials $P_i(t_1, \ldots, t_i)$ are the ones which express the elementary symmetric functions $e_i(x_1, \ldots, x_n)$ defined by $\prod(t - x_i) = t^n + \sum_{i=1}^{n}(-1)^i e_i(x_1, \ldots, x_n)t^{n-i}$ in terms of the Newton functions $\psi_k := \sum_i x_i^k$, i.e. $e_i(x_1, \ldots, x_n) = P_i(\psi_1, \ldots, \psi_i)$.

At this point we can formally define, in an algebra with trace R, for every element a a formal $n-$characteristic polynomial:

$$\chi_a^n(t) := t^n + \sum_{i=1}^{n}(-1)^i P_i(t(a), \ldots, t(a^i))t^{n-i}.$$

With this definition we obviously see that, given any element a, the element $\chi_a^n(a)$ vanishes in every $n-$dimensional representation or equivalently it is in the kernel of the universal map. Thus we are led to make the following.

Definition 2.5. *An algebra with trace R is said to be an $n-$Cayley Hamilton algebra, or to satisfy the n^{th} Cayley Hamilton identity if:*

1) $t(1) = n$.[2]

2) $\chi_a^n(a) = 0, \ \forall a \in R$.

It is clear that $n-$Cayley Hamilton algebras form a category. This category has obviously free algebras. By definition the free n-Cayley-Hamilton algebra $F_n\langle x_i \rangle$ is the algebra generated freely by x_i and by traces of monomials modulo the trace ideal generated by evaluating in all possible ways the n^{th} Cayley Hamilton identity. It is thus a quotient of the free algebra $F\langle x_i \rangle$ with trace. Some remarks are in order, they are all standard from the theory of identities [P],[P3].

One can *polarize* the Cayley-Hamilton identity getting a multilinear identity $CH(x_1, \ldots, x_n)$, since we are in characteristic 0 this identity is equivalent to the 1-variable Cayley-Hamilton identity, $CH(x_1, \ldots, x_n)$ has a nice combinatorial description as follows. given a permutation σ of $n+1$ elements written into cycles as $\sigma = (i_1, i_2, \ldots, i_a)(j_1, j_2, \ldots, j_b) \ldots (v_1, v_2, \ldots, v_r)(u_1, u_2, \ldots, u_s, n+1)$ set

$$\phi_\sigma(x_1, \ldots, x_n) := t(x_{i_1}x_{i_2} \ldots x_{i_a})t(x_{j_1}x_{j_2} \ldots x_{j_b}) \ldots t(x_{v_1}x_{v_2} \ldots x_{v_r})x_{u_1}x_{u_2} \ldots x_{u_s}$$

then

$$CH(x_1, \ldots, x_n) = (-1)^n \sum_{\sigma \in S_{n+1}} sgn(\sigma)\phi_\sigma(x_1, \ldots, x_n).$$

[2]this restriction could be to some extent dropped

Base change If R is a $A-$algebra satisfying the n-Cayley-Hamilton identity and B is a commutative A algebra, then $R \otimes_A B$ acquires naturally a $B-$linear trace for which it is also an n-Cayley-Hamilton algebra.

By construction the free trace algebra satisfying the n^{th} Cayley Hamilton identity in variables $x_i,\ i \in I$ has as universal map to the algebra $C_n(\xi_1, \dots, \xi_m)$ of $GL(n)$-equivariant maps of matrices $M_n(k)^I \to M_n(k)$.

The first main theorem of the theory is the following:

Theorem 2.6. (1) *The universal map*

$$F_n\langle x_i\rangle \xrightarrow{i} C_n(\xi_1, \dots, \xi_m)$$

from the free n^{th} Cayley Hamilton algebra in variables x_i to the ring of equivariant maps on $m-$tuples of matrices is an isomorphism.

(2) *The trace algebra $T_n(\xi_1, \dots, \xi_m)$ of $C_n(\xi_1, \dots, \xi_m)$ is the algebra of invariants of $m-$tuples of matrices. As soon as $m > 1$, it is the center of $C_n(\xi_1, \dots, \xi_m)$.*

(3) *$C_n(\xi_1, \dots, \xi_m)$ is a finite $T_n(\xi_1, \dots, \xi_m)$ module.*

Proof. We sketch the proof, (cf. [P3], [P4], [R]). We take an infinite set of variables $x_i,\ i = 1, \dots, \infty$ we let the linear group $G := GL(\infty, k)$ act by linear transformations on variables on both sides. The map i is clearly G equivariant. By standard representation theory, in order to prove that i is an isomorphism it is enough to check it on multilinear elements. Now the space of multilinear equivariant maps of matrices in $m-$variables $f : M_n(k)^m \to M_n(k)$ can be identified with the space of multilinear invariant functions of matrices in $m + 1-$variables $g : M_n(k)^{m+1} \to k$ by the formula $tr(f((\xi_1, \dots, \xi_m)\xi_{m+1})$. This last space can be identified with the centralizer of $GL(n, k)$ acting on the $m + 1^{th}$ tensor power of k^n. Finally this space is identified with the group algebra of the symmetric group S_{m+1} modulo the ideal generated by the antisymmetrizer on $n + 1$ elements. Finally one has to identify the element of the symmetric group decomposed into cycles $(i_1, \dots, i_k) \dots (s_1, \dots, s_l)(j_1, \dots, j_h, m + 1)$ with covariant map $tr(\xi_{i_1} \dots \xi_{i_k}) \dots tr(\xi_{s_1} \dots \xi_{s_l})\xi_{j_1} \dots \xi_{j_h}$ and finally identify the antisymmetrizer with the Cayley Hamilton identity and the elements in the ideal of the symmetric group with the elements deduced from this identity in the free algebra. ii) follows easily from the previous description. For iii) one has the estimate of Razmyslov that $C_n(\xi_1, \dots, \xi_m)$ is generated as $T_n(\xi_1, \dots, \xi_m)$ module by the monomials in the ξ_i of degree $\leq n^2$. Conjecturally the right estimate is rather $\leq \binom{n+1}{2}$. □

From this theorem, using a suitable method of Reynolds operators, one gets ([P5]):

Theorem 2.7. *If R is a n^{th} Cayley Hamilton algebra, the universal trace preserving maps $i_R, \bar{i}_R$*

$$\begin{array}{ccc} R & \xrightarrow{i_R} & M_n(B_n(R))^{GL(n,k)} \\ t\downarrow & & tr\downarrow \\ T & \xrightarrow{\bar{i}_R} & B_n(R)^{GL(n,k)} \end{array}$$

are isomorphisms.

3. Semisimple representations

In this section and the next ones we will work on finitely generated algebras over an algebraically closed field k. If A is a commutative finitely generated algebra over

k, we set $V(A)$ to be the associated algebraic variety, which can be identified either to the maximal spectrum of A or to the homomorphisms $\phi : A \to k$.Let us look at $F_n\langle x_i\rangle = C_n(\xi_1,\ldots,\xi_m)$ in the case that k is algebraically closed. In this case we can apply geometric invariant theory. First we analyze the case of the free algebra in $m-$variables.

Here we have seen that the coordinate ring of the $n-$dimensional representations of $F\langle x_1, x_2,\ldots,x_m\rangle$ or equivalently of $F_n\langle x_1, x_2,\ldots x_m\rangle$ is the coordinate ring of the space of $m-$tuples of $n \times n$ matrices $M_n(k)^m$. The action of the linear group is by simultaneous conjugation. The ring of invariants, which we have denoted by $T_n(\xi_1,\ldots,\xi_m)$ is the coordinate ring of the *quotient variety*

$$V_{n,m} := M_n(k)^m//GL(n,k)$$

the quotient map $M_n(k)^m \xrightarrow{\pi} V_{n,m} = M_n(k)^m//GL(n,k)$ is surjective and each fiber contains exactly one closed orbit. By the analysis of M. Artin (cf. [A],[P],[P2]) we have that an $n-$tuple $(A_1,\ldots,A_m)$ of matrices is in a closed orbit if and only if it is *semisimple* in the sense that the subalgebra $k[A_1,\ldots,A_m] \subset M_n(k)$ is semisimple. Given any $n-$tuple $(A_1,\ldots,A_m)$ of matrices the unique closed orbit contained in the closure of its orbit is constructed by taking a composition series of k^n thought as a $k[A_1,\ldots,A_m]$ module and constructing the associated graded semisimple representation (whose isomorphism class is uniquely determined).

It is interesting to analyze more closely this picture. Let us use again the notation $A_{n,m} := k[M_n(k)^m]$ the coordinate ring of the space of $m-$tuples of matrices. Given a point $p \in V_{n,m}$ this is given by a maximal ideal m_p of $T_n(\xi_1,\ldots,\xi_m)$, by the previous theory it corresponds to an equivalence class of semisimple representations of $F\langle x_1, x_2,\ldots,x_m\rangle$, the closed orbit in the fiber $\pi^{-1}(p)$. An explicit representation ϕ of $F\langle x_1, x_2,\ldots,x_m\rangle$ in the fiber of p is given by a maximal ideal M_ϕ in the coordinate ring of matrices lying over m_p and the representation is given by the evaluation:

$$\phi : F_n\langle x_i\rangle \xrightarrow{i} M_n[A_{n,m}] \longrightarrow M_n[A_{n,m}/M_\phi] = M_n[k]$$

Take now any finitely generated Cayley Hamilton algebra R, and let T be its trace algebra. $R = F_n\langle x_1, x_2,\ldots,x_m\rangle/I$ is the quotient of the free Cayley Hamilton algebra $F_n\langle x_1, x_2,\ldots,x_m\rangle$ modulo a trace ideal I and correspondingly T is the quotient of $T_n(\xi_1,\ldots,\xi_m)$ modulo $I \cap T_n(\xi_1,\ldots,\xi_m)$.

By Theorem 2.6 and functoriality we have a commutative diagram:

$$\begin{array}{lll}
F_n\langle x_i\rangle \xrightarrow{i\ \cong} & M_n[A_{n,m}]^{GL(n,k)} \longrightarrow & M_n[A_{n,m}] \\
\downarrow & \downarrow & \downarrow \\
R \qquad \xrightarrow{i_R\ \cong} & M_n[B_n(R)]^{GL(n,k)} \longrightarrow & M_n[B_n(R)]
\end{array}$$

The ring $B_n(R) = A_{n,m}/I'$ need not be reduced, nevertheless it defines a $GL(n,k)$ stable subvariety of $M_n(k)^m$ made of the (trace) representations of R, furthermore $T = B_n(R)^{GL(n,k)}$. Again explicitly a homomorphism $\psi : A_{n,m} \to B_n(R) \to k$ gives maximal ideals M_ψ, M'_ψ and the commutative diagram:

$$\begin{array}{lllll}
F_n\langle x_i\rangle & \xrightarrow{i} & M_n[A_{n,m}] & \longrightarrow & M_n[A_{n,m}/M'_\psi] = M_n[k] \\
\downarrow & & \downarrow & & 1\downarrow \\
R & \xrightarrow{i_R} & M_n[B_n(R)] & \longrightarrow & M_n[B_n(R)]/M_\psi] = M_n[k]
\end{array}$$

If $\psi : A_{n,m} \to B_n(R) \to k$ gives a point in a closed orbit then the corresponding representation of R is semisimple. It follows:

Theorem 3.1. *The algebraic variety associated to the ring T parametrizes isomorphism classes of (trace compatible) semisimple representations of R.*

Consider again $\phi : C_n(\xi_1, \dots, \xi_m) = F_n\langle x_i\rangle \xrightarrow{i} M_n[A_{n,m}] \longrightarrow M_n[A_{n,m}/M_\phi] = M_n[k]$ which induces a map $\phi : T_n(\xi_1, \dots, \xi_m) \longrightarrow k$ with kernel some maximal ideal m_p. The representation ϕ factors through

$$C_n(p) := C_n(\xi_1, \dots, \xi_m)/m_p C_n(\xi_1, \dots, \xi_m).$$

The algebra $C_n(p)$ is by construction a finitely generated (over the trace algebra) Cayley Hamilton algebra with trace and the trace takes values in k, so we need to start with analyzing this picture.

Lemma 3.2. *Let a be a matrix with entries in a commutative ring A. If a is nilpotent also $tr(a)$ is nilpotent.*

Proof. Since in a commutative ring A the set of nilpotent elements is the intersection of the prime ideals we are reduced to the case in which A is a domain. Hence we can embed it into a field. For matrices over a field the trace of a nilpotent matrix is 0.

Proposition 3.3. *Let R be a $n^{th}-$Cayley Hamilton algebra and $r \in R$ a nilpotent element, then $tr(r)$ is nilpotent.*

Proof. By 2.7 we can embed R into matrices over a commutative ring so that the trace becomes the ordinary trace. Hence the statement follows from the previous Lemma.

Proposition 3.4. *Let R be a $n^{th}-$Cayley Hamilton algebra over an algebraically closed field k, with trace values in k and finitely generated over k, denote by t the trace, then*[3]*:*

1. *R is finite dimensional over k.*
2. *The Jacobson radical J of R is the Kernel of the trace form $t(ab)$. So t factors through R/J which is also a $n^{th}-$Cayley Hamilton algebra, with non degenerate trace form.*
3. *$R/J = \oplus_{i=1}^s M_{k_i}(k)$ and there exists positive integers h_i with $n = \sum_{i=1}^s h_i k_i$ such that given $a_i \in M_{k_i}(k)$ and $tr(a_i)$ the ordinary trace, we have:*

$$t(a_1, a_2, \dots, a_s) = \sum_{i=1}^s h_i tr(a_i)$$

Proof. 1) The finite dimensionality follows from Theorem 2.6

2) From 3.3 it follows that the trace t vanishes on the radical J of R which is a nilpotent ideal. Conversely let N be the kernel of the trace form. It is clearly an ideal. Given $a \in N$, the (formal) trace $t(a^i)$ is 0 for all i. Therefore the Cayley Hamilton polynomial of a is $\chi_a^n(t) = t^n$ hence $a^n = 0$ and every element $a \in N$ is nilpotent, so $N \subset J$.

---

[3] one could drop the hypothesis that R is finitely generated and still get 2),3) but not 1)

3) We can assume now that R is semisimple, $R = \oplus_{i=1}^{s} M_{k_i}(k)$ and the trace form is non degenerate. By the previous discussion, if we present R as a quotient of $C_n(\xi_1, \ldots, \xi_m) = F_n\langle x_i \rangle$ we have a commutative diagram:

$$\begin{array}{lcl} C_n(\xi_1, \ldots, \xi_m) & \xrightarrow{tr} & T_n(\xi_1, \ldots, \xi_m) \\ \downarrow & & p \downarrow \\ R & \xrightarrow{t} & k \end{array}$$

thus a point p in $V_{n,m}$ which can be lifted to a semisimple representation of $C_n(\xi_1, \ldots, \xi_m)$ factoring through R.

Now $R = \oplus_{i=1}^{s} M_{k_i}(k)$ and an $n-$dimensional representation of this algebra is of the form $\oplus h_i k^{k_i}$ with trace exactly $t(a_1, a_2, \ldots, a_s) = \sum_{i=1}^{s} h_i tr(a_i)$. Since the trace form is non degenerate we must have $h_i > 0$ for all i.

□

The previous proposition applies thus to the algebras

$$C_n(p) := C_n(\xi_1, \ldots, \xi_m)/m_p C_n(\xi_1, \ldots, \xi_m).$$

We need to understand a special case, when ϕ is irreducible.

Proposition 3.5. *If ϕ is irreducible then*

$$C_n(p) := C_n(\xi_1, \ldots, \xi_m)/m_p C_n(\xi_1, \ldots, \xi_m) = M_n(k).$$

Proof. By the previous proposition we have that $C_n(p)/J = M_n(k)$ where J is the radical, we have to show that $J = 0$. If $u_i,\ i = 1, \ldots, n^2$ are elements of $C_n(\xi_1, \ldots, \xi_m)$ whose images in $C_n(p)/J = M_n(k)$ form a basis, we have that the matrix $tr(u_i u_j)$ is invertible in the local ring $T_n(\xi_1, \ldots, \xi_m)_{m_p}$ since the trace form on $M_n(k)$ is non degenerate. Moreover these elements form also a basis for the ring of matrices $M_n(A_{n,m})$ localized at m_p. Thus given any $u \in C_n(p)$ we have $u = \sum_i x_i u_i,\ x_i \in [A_{n,m}]_{m_p}$. The x_i can be computed by the equations $tr(uu_j) = \sum_i x_i tr(u_i u_j)$ which shows that $x_i \in T_n(\xi_1, \ldots, \xi_m)_{m_p}$ specializing modulo m_p we see that the classes of the u_i are a $k-$basis of $C_n(p)$ hence $C_n(p) = C_n(p)/J = M_n(k)$. □

From the previous analysis it follows that the set of points p in which $C_n(p) = M_n(k)$ is the open set defined by the non vanishing of at least one of the *discriminants* $\det(tr(u_i u_j))$ where the elements u_i vary on all n^2-tuples of elements of $C_n(\xi_1, \ldots, \xi_m)$.

This set is empty if and only if $m = 1$. For $m = 2$ one takes a diagonal matrix with distinct entries and the matrix of a cyclic permutation which generate $M_n(k)$.

A more careful analysis of the previous argument shows that the localized algebra $C_n(\xi_1, \ldots, \xi_m)_{m_p}$ is an Azumaya algebra over $T_n(\xi_1, \ldots, \xi_m)_{m_p}$. In the geometric language, the points of the spectrum of $T_n(\xi_1, \ldots, \xi_m)_{m_p}$ where $C_n(\xi_1, \ldots, \xi_m)_{m_p}$ is an Azumaya algebra are exactly the points over which the quotient map

$$M_n(k)^m \xrightarrow{\pi} M_n(k)^m//Gl(n,k)$$

is a *principal bundle* over the projective linear group. In fact this can be viewed as a special case of M. Artin characterization of Azumaya algebras by polynomial identities (cf. [A], [P],[S]).

One easily obtains the following

Corollary 3.6. *Let R be an $n-$Cayley Hamilton algebra with trace values in k and Jacobson radical J. If $R/J = M_n(k)$ then $J = 0$.*

Corollary 3.7. *Let R be an $n-$Cayley Hamilton algebra finitely generated over k. If R is a simple algebra then $R = M_h(k)$ for some $h|n$.*

Proof. From the previous theory there is at least one semisimple representation of R but, since R is simple it follows that this is injective, hence R is finite dimensional and the values of the trace are in k. The statement follows from 3.4.

Let us finish this paragraph by pointing out the relationship, for CH-algebras, between trace representations and irreducible representations.

We assume as before that R is finitely generated over an algebraically closed field k.

Let V be an irreducible representation of R.

Lemma 3.8. *V is finite dimensional over k.*

Proof. Let Z be the center of R, it contains $T(R)$ and so R is a finite module over Z. By Schur's lemma, the centralizer $End_R(V)$ is a division ring with some center F and clearly the representation induces a homomorphism of Z to F. Since V is irreducible $V = R/I$ with I a maximal left ideal, hence it follows that V is finite dimensional over F. Let L be the ideal of R annihilator of V. By the Jacobson density theorem it follows that R/L is a simple algebra, finite dimensional over F and finitely generated over k. We cannot apply directly the previous corollary but can argue as follows. Let us choose a basis u_i of R/L over F and consider the multiplication constants $u_i u_j = \sum_r c^r_{i,j} u_r$, moreover for the generators x_i of R over k write $x_i = \sum_h d_{i,j} u_j$. It follows easily that $F = k[c^r_{i,j}, d_{i,j}]$. From the Hilbert Nullstellensatz it follows then that $F = k$.

From the previous Lemma it follows that, associated to V we have a *central character* $\chi_V : Z \to k$ so that, $zv = \chi_V(z)(v)$, $\forall z \in Z, \forall v \in V$. This central character, restricted to $T(R)$ induces also a maximal ideal m of $T(R)$ and point P of the spectrum of semisimple trace representations. V is also an irreducible module for the trace algebra R/mR which, by 3.4 is finite dimensional. Of course the action of R/mR on V is trivial on its Jacobson radical, hence V is one of the irreducible modules appearing in the semisiple representation associated to the point P.

Thus if $Spec_i(R)$ denotes the set of equivalence classes of irreducible representations of R and $V_{ss}(R)$ the equivalence classes of semisimple trace representations we have:

Theorem 3.9. *There is a map $\pi : Spec_i(R) \to V_{ss}(R)$.*

i) π is surjective.

ii) The fiber $\pi^{-1}(P)$ is the finite set of irreducible representations appearing in the semisimple representation associated to P.

3.1. **Some categorical constructions.** Let R be a $n^{th}-$Cayley-Hamilton algebra finitely generated algebra over an algebraically closed field k, with trace t and $t(R) = A$. A is also a finitely generated algebra over k. By Theorem 3.1 the (closed) points of $V(A)$ parametrize semisimple representations of dimension n of R. Fix a positive integer r and change trace taking the new trace $\tau = rt_{R/A}$.

Proposition 3.10. *Let R be a $n^{th}-$Cayley-Hamilton algebra with trace t and $t(R) = A$. A a finitely generated algebra over an algebraically closed field k and $V(A)$ the reduced variety of $Spec(A)$.*

The algebra R with trace $\tau := rt$ is an $(rn)^{th}-$Cayley-Hamilton algebra. Given a point $p \in V(A)$ it determines an $n-$dimensional semisimple representation M_p^t compatible with the trace t and also an $rn-$dimensional semisimple representation M_p^τ compatible with the trace τ we have:

$$M_p^\tau = rM_p^t = (M_p^t)^{\oplus r}$$

Proof. Let us show first that, if R is a $n^{th}-$Cayley-Hamilton algebra with trace t the algebra R with trace $\tau := rt$ is an $(rn)^{th}-$Cayley-Hamilton algebra. We know that R embeds in a trace compatible way into $n \times n$ matrices over a commutative ring A, then the natural diagonal embedding of $M_n(A)$ into $M_{rn}(A)$ gives the claim.

Now we have that $V(A)$ parametrizes also semisimple representations (compatible with the new trace) of dimension rn. We have an obvious map from the variety of n dimensional representations compatible with the reduced trace to the variety of rn dimensional representations compatible with trace τ it is simply the map that associates to a representation M its direct sum $M^{\oplus r}$.

From this the statement is clear. □

Let now R_1 and R_2 two trace algebras over A which are Cayley-Hamilton for two integers n_1, n_2, and $t(R_1) = t(R_2) = A$, then, a point $p \in V(A)$ determines a semisimple representation M_p^1 of dimension n_1 of R_1 and a semisimple representation M_p^2 of dimension n_2 of R_2:

Proposition 3.11. *The algebra $R := R_1 \oplus R_2$ with trace $t(r_1, r_2) := t(r_1) + t(r_2)$ is an $n^{th}-$Cayley-Hamilton for $n := n_1 + n_2$, and $t(R) = A$, a point $p \in V(A)$ determines also a semisimple representation M_p of dimension $n = n_1 + n_2$ of R and:*

$$M_p = M_p^1 \oplus M_p^2$$

The proof is similar to that of the previous proposition and it is omitted.

Finally let now R_1 and R_2 two trace algebras over k which are Cayley-Hamilton for two integers n_1, n_2, and $t(R_1) = A_1$, $t(R_2) = A_2$, then, a point $p \in V(A_1)$ determines a semisimple representation M_p^1 of dimension n_1 of R_1 and a point $q \in V(A_2)$ determines a semisimple representation M_q^2 of dimension n_2 of R_2:

Proposition 3.12. *The algebra $R := R_1 \otimes R_2$ with trace $t(r_1 \otimes r_2) := t(r_1) \otimes t(r_2)$ is an $n^{th}-$Cayley-Hamilton for $n := n_1 n_2$, and $t(R) = A := A_1 \otimes A_2$, a point $(p, q) \in V(A) = V(A_1) \times V(A_2)$ determines also a semisimple representation M_p of dimension $n = n_1 n_2$ of R and:*

$$M_p = M_p^1 \otimes M_p^2$$

Again the proof is similar.

4. The reduced trace

Let us recall that a prime ring R is a ring in which the product of two non-zero ideals is non-zero. Now let R be a prime algebra over a commutative ring A and assume that $A \subset R$ and R is an $A-$module of finite type. One easily sees that:

(1) A is an integral domain.
(2) R is a torsion free module.

If F is the field of fractions of A then $R \subset R \otimes_A F$ and $S := R \otimes_A F$ is, by a Theorem of Wedderburn a (finite dimensional) simple algebra isomorphic to $M_k(D)$ where D is a finite dimensional division ring.

If Z is the center of S, it is also the center of D and $\dim_Z D = h^2$; moreover, if $\overline{Z}$ is an algebraic closure of Z we have $M_k(D) \otimes_Z \overline{Z} = M_{hk}(\overline{Z})$, if the finite extension $Z \supset F$ is separable, as happens in characteristic 0 and if $p := [Z : F] = \dim_F Z$ we also have

$$S \otimes_F \overline{Z} = M_k(D) \otimes_F \overline{Z} = M_{hk}(\overline{Z})^{\oplus p}$$

We define the number hkp to be the *degree* of S over F and let:

$$hkp := [S : F],$$

If $a \in S$ we have that $a \otimes 1 \in M_k(D) \otimes_F \overline{Z}$ is a $p-$tuple of matrices $(A_1, \dots, A_p)$ one defines the *reduced trace* of a to be the sum

$$t(a) = t_{S/F}(a) := \sum_{i=1}^{p} tr(A_i),$$

A standard argument of Galois theory shows that $t(a) \in F$, in characteristic 0 this can be even more easily seen as follows.

Consider the $F-$linear operator $a^L : S \to S$, $a^L(b) := ab$ let us compute its trace. This can be done in $M_k(D) \otimes_F \overline{Z} = M_{hk}(\overline{Z})^{\oplus p}$ where $a^L = (A_1^L, \dots, A_p^L)$ and so

$$tr(a^L) = \sum_{i=1}^{p} tr(A_i^L) = hk \sum_{i=1}^{p} tr(A_i) \quad \implies t_{S/F}(a) = \frac{1}{hk} tr(a^L)$$

Theorem 4.1. *If $S = R \otimes_A F$ as before and A is integrally closed we have that the reduced trace $t_{S/F}$ maps R into A, so we will denote by $t_{R/A}$ the induced trace.*

The algebras R, S with their reduced trace are $n-$Cayley Hamilton algebras of degree $n = hkp = [S : F] = [R : A]$, (we set $[R : A] := [S : F]$).

Proof. We have a natural representation of $M_{hk}(\overline{Z})^{\oplus p}$ by hkp matrices for which the reduced trace is the trace. If R is a finite $A-$module, it is easy to see that the reduced trace of an element of r is integral over A. If A is integrally closed then the trace takes values in A. □

The importance of the reduced trace comes from the next result. R, A are as before, $n := [R : A]$ the degree and we are assuming characteristic 0:

Theorem 4.2. *If $\tau : R \to A$ is any trace for which R is an $m-$Cayley Hamilton algebra then there is a positive integer r for which:*

$$m = rn, \qquad \tau = r\, t_{R/A}$$

Proof. Clearly τ extends to a trace on S with values in F for which S is an $m-$Cayley Hamilton algebra.

Let G be a finite Galois extension of F for which $S \otimes_F G = M_{hk}(G)^{\oplus p}$, τ extends to a $G-$valued trace on $S \otimes_F G = M_{hk}(G)^{\oplus p}$ which is invariant under the Galois group and for which $S \otimes_F G$ is an $m-$Cayley Hamilton algebra. Passing to the algebraic closure we can now apply statement 3 in Proposition 3.4 where we know that, by invariance under the Galois group, all the integers h_i must be equal to some positive integer r. The formula follows from the definitions. □

5. The unramified locus and restriction maps

5.1. **The unramified locus.** Let us go back to the previous setting. R a prime algebra over A, F the field of fractions of A, $S = R \otimes_A F$ and finally Z the center of S. Let now $B := R \cap Z$ be the center of R. If we further assume that A, B are
c integrally closed we have reduced traces and the formulas:

$$t_{R/A} = t_{B/A} \circ t_{R/B}, \quad [R : A] = [R : B][B : A]$$

Let us assume now that A is a finitely generated algebra over an algebraically closed field k and $V(A)$ the associated affine variety parametrizing semisimple representations (compatible with the reduced trace $t_{R/A}$) of R of dimension $n = [R : A]$. Since we are assuming that R is a finite A module it follows that also B is a finite A module. Then B is a finitely generated algebra over k and its associated affine variety $V(B)$ parameterizes semisimple representations (compatible with the reduced trace $t_{R/B}$) of R of dimension $m = [R : B]$. Moreover we can also use $V(A)$ to parameterize semisimple representations (compatible with the reduced trace $t_{B/A}$) of B of dimension $p = [B : A]$. Finally the inclusion $A \subset B$ defines a morphism of algebraic varieties $\pi : V(B) \to V(A)$ of degree p. We want to put all these things together.

Given a point $Q \in V(A)$ denote by N_Q the corresponding mp dimensional semisimple representation of R. Given a point $P \in V(B)$ denote by M_P the corresponding m dimensional semisimple representation of R .

First of all an irreducible representation of B is 1-dimensional and corresponds to a point $P \in V(B)$, a semisimple representation corresponds to a positive cycle $\sum h_i P_i$ of degree $p = \sum_i h_i$. Proposition 3.4 implies:

Proposition 5.1. *Given a point $Q \in V(A)$ we have for the associated semisimple representation $\sum_{i=1}^s h_i P_i$ of B, that the points P_i are exactly the points in the fiber $\pi^{-1}(Q)$. So we may identify formally $\sum_{i=1}^s h_i P_i$ with the cycle $[\pi^{-1}(P)]$.*

In general a fiber need not have exactly p points but it can have $s \leq p$ points.

In terms of algebras, Q corresponds to a maximal ideal m of A and the points P_i to the maximal ideals of B/mB. This is a finite dimensional commutative algebra and so $\overline{B} := B/mB = \oplus_{i=1}^s B_i$ where B_i is a local ring supported in the point P_i. Let $\overline{n}_i$ be the maximal ideal of B_i and n_i the corresponding maximal ideal of B. We have $B_i/\overline{n}_i = k$ and again Proposition 3.4 implies that the trace $t_{B/A}$
c induces a trace $t_{\overline{B}/k}$ on $\overline{B}$, which decomposes as the sum of *local factors* $h_i t_i$ where
c $t_i : B_i \to B_i/\overline{n}_i = k$ is the projection. In particular, if e_i is the idempotent, unit of B_i we have $t_{\overline{B}/k}(e_i) = h_i$.

Remark 5.2. *If B is a projective $A-$module of rank n the reduced trace $t_{B/A}(b)$ is just the trace of the linear map $x \to bx$. In this case* $\dim_k \overline{B} = n, \quad \dim_k B_i = h_i$

Passing to the algebra R we have a direct sum decomposition.

$$\overline{R} := R/mR = R \otimes_A A/m = R \otimes_B (B \otimes_A A/m) = \oplus_{i=1}^s R \otimes_B B_i = \oplus_{i=1}^s \overline{R}_i$$

We pass to the traces: we know that $t_{R/A} = t_{B/A} \circ t_{R/B}$, and modulo m we get traces $t_{\overline{R}/k} = t_{\overline{B}/k} \circ t_{\overline{R}/\overline{B}}$. If $e_i \in B_i$ is the idempotent identity of B_i we have $\overline{R}_i = \overline{R} e_i$ thus $t_{\overline{R}/\overline{B}}$ restricts to $\overline{R}_i$ to a B_i trace and $t_{\overline{R}/\overline{B}} = \oplus_{i=1}^s t_{\overline{R}_i/B_i}$. Similarly $t_{\overline{B}/k} = \oplus_{i=1}^s t_{B_i/k}$ and

$$t_{\overline{R}/k} = \oplus_{i=1}^s t_{B_i/k} \circ t_{\overline{R}_i/B_i}$$

The algebra R_i with the trace $t_{\overline{R}_i/B_i}$ is an $m = [R : B]$-Cayley Hamilton algebra. If $\overline{n}_i$ is the maximal ideal of B_i, the unique point of $Spec(B_i)$ given by $B_i \to B_i/\overline{n}_i = k$ corresponds to some semisimple representation M_i. As a representation of R, $M = M_{P_i}$ where P_i corresponds to the maximal ideal n_i, since we have that $R/n_iR = \overline{R}_i/\overline{n}_i\overline{R}_i$. If we denote by $\bar{t}_{\overline{R}_i/B_i}$ the image of $t_{\overline{R}_i/B_i}$ modulo $\overline{n}_i$, we have

$$h_i\bar{t}_{\overline{R}_i/B_i} = t_{\overline{B}_i/k} \circ t_{\overline{R}_i/B_i}.$$

It follows that the semisimple representation of $\overline{R}_i$ relative to the trace $t_{\overline{B}_i/k} \circ t_{\overline{R}_i/B_i}$ is h_iM_i. Thus the semisimple representation of $\overline{R}$ relative to the trace $t_{\overline{R}/k} = \oplus_{i=1}^{s} t_{B_i/k} \circ t_{\overline{R}_i/B_i}$ is $\oplus_i h_i M_{P_i}$, (cf. 3.10, 3.11).

We have proved:

Theorem 5.3. *Given a point $Q \in V(A)$ and its cycle $\sum_i h_iP_i$ in $V(B)$ we have*

$$N_Q = \oplus_i h_i M_{P_i}$$

Let R be an algebra with trace finitely generated over an algebraically closed field k satisfying the n^{th} Cayley Hamilton identity with the trace algebra $t(R)$. If $P \in V(t(R))$ with maximal ideal m_P. We ask when $R(P) := R \otimes_{t(R)} k = R/m_PR$ is a semisimple algebra.

The answer is implicit in Proposition 3.4. The trace map for R induces the trace $t : R(P) \to k$ for $R(P)$ and the bilinear trace form $t(ab)$. It follows immediately from 3.4 that:

Proposition 5.4. *The radical J of $R(P)$ is the kernel of the trace form $t(ab)$.*

Let us see what is the meaning of this statement in the case $R \supset A$ is a prime algebra over A, finitely generated algebra over an algebraically closed field k, F the field of fractions of A, $S = R \otimes_A F$, Z the center of S, $B := R \cap Z$ be the center of R with B integrally closed. If $m := [R : B]$ the map $\frac{t_{R/B}}{m}$ is a projection on B so that $R = B \oplus R^0$ and B is a direct summand.

With the previous notations $V(A)$, the associated affine variety, parameterizes semisimple representations (compatible with the reduced trace $t_{R/A}$) of R of dimension $n = [R : A]$. B is a finitely generated algebra over k and its associated affine variety $V(B)$ parameterizes semisimple representations (compatible with the reduced trace $t_{R/B}$) of R of dimension $m = [R : B]$. Denote by $\pi : V(B) \to V(A)$.

If $Q \in V(A)$ corresponds to a maximal ideal m_Q and the algebra $R(Q) := R \otimes_A A/m_Q$ is semisimple we have that $R \otimes_A A/m_Q = B \otimes_A A/m_Q \oplus R^0 \otimes_A A/m_Q$, since $B(Q) := B \otimes_A A/m_Q$ is in the center of $R(Q) := R \otimes_A A/m_Q$ we have that $B(Q)$ is semisimple, in other words the scheme theoretic fiber $\pi^{-1}(Q)$ is reduced, $B \otimes_A A/m_Q = \oplus_i B/n_i$ and $B/n_i = k$ is a point P_i in the fiber of Q. We also have $R(P_i) = R \oplus B/n_i$ and $R \otimes_A A/m_Q = \oplus_i R(P_i)$ hence $R(P_i)$ is semisimple. The converse is also clear.

The commutative algebra $B(Q)$ is semisimple if and only if it is reduced, i.e. the fiber of Q under the map π is reduced, which in our case implies that π is étale in the points of this fiber. Now we know that R is a finite module over B and its generic dimension is m^2. The dimension of $R(P)$ over $k = B(P)$, $P \in V(B)$ is a semicontinuous function and we always have $\dim_{B(P)} R(P) \geq m^2$. If $R(P)$ is not simple of dimension m^2 from 3.4, 3) follows that, if J is the radical of $R(P)$ we have $\dim_k R(P) < m^2$ hence we have a dichotomy, either $R(P) = M_m(k)$ that is to say that P corresponds to an irreducible representation, or $R(P)$ is not semisimple.

Proposition 5.5. *Let W^0 be the open set of $V(B)$ made of points P where $R(P) = M_m(k)$ (the irreducible representations), let V^0 be the maximal open set of $V(A)$ with $\pi^{-1}(V^0) \subset W^0$ and V^1 be the open set of $V(A)$ where $B(Q)$ is reduced, then the set of points of V where $R(Q)$ is semisimple is $V^0 \cap V^1$.*

The set of points $V^0 \cap V^1$ of V where $R(Q)$ is semisimple is called the *unramified locus* of the A algebra R.

From our analysis it follows that, if $m = [R : B],\ p = [B : A]$ over a point Q in the unramified locus

$$N_Q = \oplus_{i=1}^{p} M_{P_i}$$

decomposes as the direct sum of the p irreducible representations M_{P_i} supported at the p distinct points of the fiber $\pi^{-1}(Q)$.

5.2. **Restriction maps.** We come now to the final application of the previous theory. The setting we have in mind appears naturally for quantum groups at roots of 1 and their subgroups.

We need a first Lemma. Given a prime algebra R finite over A with center Z, let F be the quotient field of A and $F \subset G$ an extension field.

Lemma 5.6. *The following are equivalent:*
i) $R \otimes_A G$ is a simple algebra.
ii) The algebra $Z \otimes_A G$ is a field.

Proof. Let F be the quotient field of A we have that $S := R \otimes_A F$ is a simple algebra with center the field $W := Z \otimes_A F$ and that $R \otimes_A G = (R \otimes_A F) \otimes_F G = S \otimes_W (W \otimes_F G)$. Since S is a simple algebra with center W it is well known and easy that $S \otimes_W (W \otimes_F G)$ is simple if and only if $W \otimes_F G$ is a field. Finally $W \otimes_F G = (Z \otimes_A F) \otimes_F G = Z \otimes_A G$. □

Assume that we have two prime algebras $R_1 \subset R_2$ over two commutative rings $A_1 \subset A_2 \subset R_2$. Assume as in the previous paragraph that each R_i is finitely generated as A_i module and that the two rings A_i are integrally closed. We thus have the two reduced traces t_{R_i/A_i}, we want to discuss the compatibility of these traces. In general one can see by simple examples that there is no compatibility. Let us thus make the basic assumption of *compatibility* (with trace). We let F_i be the quotient field of A_i and consider $S_i := R_i \otimes_{A_i} F_i$.

Lemma 5.7. *Given two prime algebras $R_1 \subset R_2$ over the rings $A_1 \subset A_2 \subset R_2$ with Z_1 the center of R_1. Assume R_1 is finite over A_1. The following two conditions are equivalent:*
i) $R_1 \otimes_{A_1} F_2$ is a simple algebra.
ii) The algebra $Z_1 \otimes_{A_1} F_2$ is a field.
In this case the map $i : R_1 \otimes_{A_1} F_2 = S_1 \otimes_{F_1} F_2 \to S_2$ is injective.

These conditions are satisfied if:
iii) The algebra $Z_1 \otimes_{A_1} A_2$ is a domain.

Proof. The equivalence of the first two conditions is the content of the previous Lemma. It is clear that iii) implies ii) since if $Z_1 \otimes_{A_1} A_2$ is a domain, $Z_1 \otimes_{A_1} F_2$ is its quotient field.

Definition 5.8. *We say that the two algebras $R_1 \subset R_2$ are compatible with $A_1 \subset A_2 \subset R_2$ if the previous two equivalent conditions are satisfied.*

In the examples which we will study we will usually verify iii).

Remark 5.9. *If R_2 is a domain then $R_1 \subset R_2$ is compatible with $A_1 \subset A_2 \subset R_2$ if and only if the map $i : R_1 \otimes_{A_1} F_2 = S_1 \otimes_{F_1} F_2 \to S_2$ is injective.*

Proof. If R_2 is a domain so is S_2 and so, if i is injective we must have that $Z_1 \otimes_{A_1} F_2$ is a field. □

Example 5.10. *1) If the extension $F_1 \subset F_2$ is unirational then $Z_1 \otimes_{A_1} F_2$ is a field.*

2) If $A_1 = Z$ is the center of R_1 then $Z_1 \otimes_{A_1} F_2 = F_2$ is a field.

Let us show 1). We have $F_2 \subset F_1(t_1, \dots, t_m)$ so $Z_1 \otimes_{A_1} F_2 \subset Z_1 \otimes_{A_1} F_1(t_1, \dots, t_m) = (Z \otimes_{A_1} F_1)(t_1, \dots, t_m)$ is a field.

Theorem 5.11. *Given two compatible algebras $R_1 \subset R_2$ we have that for a positive integer r:*

$$r[R_1 : A_1] = [R_2 : A_2], \qquad r\, t_{R_1/A_1} = t_{R_2/A_2} \quad \text{on } R_1$$

Proof. By the hypotheses made one can reduce the computation to the two algebras $S_1 \otimes_{F_1} F_2 \subset S_2$ over F_2. In this case we know that the reduced trace t_{S_2/F_2} restricted to $S_1 \otimes_{F_1} F_2$ makes it a Cayley Hamilton trace algebra. Since by assumption $S_1 \otimes_{F_1} F_2$ is simple, one can then apply Theorem 4.2. □

Let us now assume to be in the geometric case in which A_1, A_2 are further assumed to be finitely generated over an algebraically closed field k. If $V(A_1), V(A_2)$ are the two associated affine varieties parameterizing semisimple representations we have an induced map $\pi : V(A_2) \to V(A_1)$. If $Q \in V(A_2)$ and M is a representation of R_2 over Q we then see that M is also a representation of R_1 over $\pi(Q)$ but for r times the reduced trace. If M is semisimple as R_2 module it may well be that it is not semisimple as R_1 module.

Theorem 5.12. *Given two compatible algebras $R_1 \subset R_2$ as before, $Q \in V(A_2)$. M_Q the corresponding semisimple representation of R_2 of dimension $[R_2 : A_2]$, $M_{\pi(Q)}$ the corresponding semisimple representation of R_1 of dimension $[R_1 : A_1]$. We have that the restriction of M_Q to R_1 is a trace representation for $r\, t_{R_1/A_1}$, its associated semisimple representation is $rM_{\pi(Q)} = M_{\pi(Q)}^{\oplus r}$.*

If $\pi(Q)$ lies in the unramified locus of R_1 (as A_1 algebra) we have that the restriction of M_Q to R_1 is the semisimple representation $rM_{\pi(Q)}$.

Proof. Everything follows from the previous discussions except the last point. Let $m \subset A_2$ be the maximal ideal associated to Q and $m' := m \cap A_1$, by definition of the unramified locus the algebra $R_1/m'R_1$ is a semisimple algebra for which every representation is semisimple. □

5.3. **Examples.**

In this section we collect examples from quantum groups.

One class of examples is obtained by taking a prime Hopf algebra R, finite over a central Hopf subalgebra A which is the coordinate ring of an algebraic group G (necessarily connected) over $\mathbb{C}$. In this case $R \otimes_{\mathbb{C}} R$ is also prime and finite over $A \otimes_{\mathbb{C}} A$ and, we need to prove that:

Theorem 5.13. *The comultiplication $\Delta : R \to R \otimes_{\mathbb{C}} R$ is compatible with $\Delta : A \to A \otimes_{\mathbb{C}} A$.*

Proof. By the previous lemma it is enough to show that, setting $\mathcal{Z}$ the center of R, we have that $\mathcal{Z} \otimes_{\Delta(A)} (A \otimes_{\mathbb{C}} A)$ is a domain.

Let us use some geometric language. A is the coordinate ring of a connected algebraic group G and the map $\Delta : A \to A \otimes A$ is the comorphism associated to the multiplication $G \times G \xrightarrow{\mu} G$. Let $\nu : G \times G \to G \times G$ be defined by $\nu(x, y) := (xy, y)$, clearly ν is an isomorphism and μ can be identified to ν composed by the first projection. Thus $\nu^* : A \otimes A \to A \otimes A$ maps $\Delta(A) \to A \otimes 1$. Using this isomorphism we see that:

$$\mathcal{Z} \otimes_{\Delta(A)} (A \otimes_{\mathbb{C}} A) \cong (\mathcal{Z} \otimes_A A) \otimes_{\mathbb{C}} A = \mathcal{Z} \otimes_{\mathbb{C}} A$$

is a domain. □

The second class of examples we have in mind is when, using the notations and assumptions of 5.7, R_2 is a Hopf algebra and R_1, A_1, A_2 Hopf subalgebras finitely generated over $\mathbb{C}$. In this case A_2 is the coordinate ring of an algebraic group G_2 and A_1 that of a quotient group. If the quotient morphism $\pi : G_2 \to G_1$ is unirational, we are in the case of Example 5,10 1).

For instance for $R_2 = U_q(\mathfrak{g})$ the quantized enveloping algebra of a semisimple Lie algebra and $R_1 = U_q(\mathfrak{b}^+)$ we have that G is a semidirect product of G_1 by a nilpotent group, so the quotient morphism π is rational.

5.4. **Cayley-Hamilton Hopf algebras.** We formalize the previous discussion as follows:

Definition 5.14. *A Cayley-Hamilton Hopf algebra is a Hopf algebra such that:*

- *it is a Cayley-Hamilton algebra*
- *the trace subalgebra is a Hopf subalgebra*

In the next section we will see plenty examples of such Hopf algebras that are given by quantized universal enveloping algebras at roots of unity.

Let R be a Cayley-Hamilton Hopf algebra with the trace subalgebra $A := t(R)$. Assume R is prime and a finite A module. Let $Z \supset A$ be the center of R, set $m := [R : A]$, $n = [R : Z]$, $p := [Z : A]$ so that $m = np$. For a point $x \in V(A)$ (resp. $P \in V(Z)$) denote by N_x (resp. M_P) the corresponding semisimple $m-$dimensional representation (resp. $n-$dimensional).

Assume that A is finitely generated over an algebraically closed field k, so that $V(A), V(Z)$ are affine algebraic varieties, and let $\pi : V(Z) \to V(A)$ be the corresponding map of varieties. The comultiplication on A defines an associative binary operation on $V(A)$. The antipode defines the inverse operation for this operation on $V(A)$, so $V(A)$ is an algebraic group.

From 5.12, and 5.13 we see that:

Proposition 5.15. *If $x, y \in V(A)$ and N_x, N_y are the corresponding semisimple representations then the semisimple representation associated to $N_x \otimes N_y$ is mN_{xy}.*

We will say that the pair of points $x, y \in V(A)$ is generic if both points and their product in $V(A)$ lie in the unramified locus. Such pairs of points form a Zariski open subvariety in $V(A) \times V(A)$.

For each point $Q \in V(Z)$ in the fiber of, either x, y, xy the corresponding representation M_Q is irreducible.

$$N_x = \oplus_{P \in \pi^{-1}(x)} M_P, \; N_y = \oplus_{Q \in \pi^{-1}(y)} M_Q, \; N_{xy} = \oplus_{R \in \pi^{-1}(xy)} M_R.$$

From 5.15 we get

$$(5) \qquad N_x \otimes N_y = mN_{xy}, \qquad \bigoplus_{P\in\pi^{-1}(x),\ Q\in\pi^{-1}(y)} M_P \otimes M_Q = \bigoplus_{R\in\pi^{-1}(xy)} mM_R,$$

Let V and W be two irreducible representations of the CH-Hopf algebra R such that the restrictions of V and W to A are given by the action of characters $x, y \in V(A)$ respectively. Thus $V = M_P,\ W = M_Q$ where $P \in \pi^{-1}(x),\ Q \in \pi^{-1}(y)$. The restriction to A of the tensor product $V \otimes W$ has the same property, with central character xy .

If the pair $x, y \in V(A)$ is generic the tensor product $M_P \otimes M_Q$ is semisimple as an R-module and we have the:

Theorem 5.16. *Clebsch-Gordan decomposition (cf. equation 5):*

$$(6) \qquad M_P \otimes M_Q \simeq \oplus_{R\in\pi^{-1}(xy)} M_R^{\oplus h_R^{P,Q}}, \quad \sum_R h_R^{P,Q} = n,\ \sum_{P,Q} h_R^{P,Q} = m$$

For quantized enveloping algebras at roots of 1 we will prove the stronger statement that all the multiplicities $h_R^{P,Q}$ are equal.

6. Quantized universal enveloping algebras at roots of 1

6.1. **The definition $\mathcal{U}_\epsilon$.** Let $\mathfrak{g}$ be a simple Lie algebra of rank n with the root system Δ. Denote by Q, P the root and weight lattice. Fix simple roots $\alpha_1, \dots, \alpha_n \in \Delta_+$ and denote by $(a_{ij})_{i,j=1}^r$ the corresponding Cartan matrix. Denote by d_i the length of the i-th simple root.

For an odd positive integer ℓ denote by ϵ a primitive root of 1 of degree ℓ (in case of components of type G_2 we also need to restrict to ℓ prime with 3).

For any lattice $Q \subset \Lambda \subset P$ we have a quantized universal enveloping algebra $U_\epsilon^\Lambda(\mathfrak{g})$. It is the associative algebra with 1 over $\mathbb{C}$ generated by $K_\mu,\ \mu \in \Lambda$, and $E_i, F_i,\ i = 1, \dots, n$ with defining relations:

$$\begin{aligned}
K_\mu K_\nu &= K_\nu K_\mu \ , K_\mu K_{-\mu} = 1 \quad K_0 = 1 \ , \\
K_\mu E_i &= \epsilon^{\alpha_i(\mu)} E_i K_\mu \\
K_\mu F_i &= \epsilon^{-\alpha_i(\mu)} F_i K_\mu \ , \qquad E_i F_j - F_j E_i = \delta_{ij}(K_{\alpha_i} - K_{\alpha_i}^{-1})/(\epsilon_i - \epsilon_i^{-1}) \ , \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{\epsilon_i} E_i^{1-k-a_{ij}} E_j\, E_k^k = 0 \ , \quad i \neq j \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{\epsilon_i} F_i^{1-k-a_{ij}} F_j\, F_k^k = 0 \ , \quad i \neq j
\end{aligned}$$

Here $\epsilon_i = \epsilon^{d_i}$,

$$\begin{bmatrix} m \\ h \end{bmatrix}_\epsilon = \frac{[m]_\epsilon!}{[m-h]_\epsilon! [h]_\epsilon!} \ , \qquad [h]_\epsilon! = [h]_\epsilon \dots [2]_\epsilon [1]_\epsilon \ , \qquad [h]_\epsilon = \frac{\epsilon^h - \epsilon^{-h}}{\epsilon - \epsilon^{-1}} \ .$$

The map Δ acting on generators as

$$\Delta K_\mu = K_\mu \otimes K_\mu \ , \tag{7}$$

$$\Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i \ , \tag{8}$$

$$\Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i \tag{9}$$

extends to the homomorphism of algebras $\Delta : \mathcal{U}_\epsilon \to \mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$. Denote by $\{\omega_i\}$ the fundamental weights, from now on we will use the notation $K_i = K_{\omega_i}$. The pair $(\mathcal{U}_\epsilon, \Delta)$ is a Hopf algebra with the counit $\eta(K_\mu) = 1$, $\eta(E_i) = \eta(F_i) = 0$.

For $\Lambda = P$ we have the *simply connected* quantized algebra, denoted by $\mathcal{U}_\epsilon^s$ or simply $\mathcal{U}_\epsilon$, for $\Lambda = Q$ we have the *adjoint* form denoted by $\mathcal{U}_\epsilon^a$, the definitions hold also if instead of ϵ we have a q generic.

We will denote by $\mathcal{U}_\epsilon^\pm$ the subalgebras of $\mathcal{U}_\epsilon$ generated by E_i and F_i respectively. The subalgebra generated by the K_i's will be denoted $\mathcal{U}_\epsilon^{s,0} = \mathcal{U}_\epsilon^0$. Similarly the subalgebra generated by the K_{α_i}'s will be denoted $\mathcal{U}_\epsilon^{a,0}$.

6.2. **PBW basis and the structure of the center.** One can introduce a monomial basis in the algebras $\mathcal{U}_\epsilon^\pm$ that is the analog of the Poincaré-Birkhoff-Witt basis. We will call it PBW basis. In order to describe this basis we first should introduce root elements $E_\alpha \in \mathcal{U}_\epsilon^+$, $F_\alpha \in \mathcal{U}_\epsilon^-$. This can be done (see [L] for details) by choosing a convex ordering on positive roots.

If $\beta(1) > \cdots > \beta(N)$ is the convex ordering of positive roots Δ_+ (here $N = |\Delta_+|$) then we choose PBW bases as follows. For $\mathcal{U}_\epsilon^+$ this is a basis of monomials

$$E^{\underline{m}} = E_{\beta(N)}^{m_N} \dots E_{\beta(1)}^{m_1}$$

where $m_i \geq 0$. For $\mathcal{U}_\epsilon^-$ this is a basis of monomials

$$F^{\underline{m}} = F_{\beta(1)}^{m_1} \dots F_{\beta(N)}^{m_N}$$

where $m_i \geq 0$. For $\mathcal{U}_\epsilon^0$ we choose a natural basis of Laurent monomials for $\underline{p} = (p_1, \dots, p_r) \in \mathbb{Z}^r$ the monomial is $K^{\underline{p}} = K_1^{p_1} \dots K_r^{p_r}$.

There is a linear isomorphism $\mathcal{U}_\epsilon \simeq \mathcal{U}_\epsilon^- \otimes \mathcal{U}_\epsilon^0 \otimes \mathcal{U}_\epsilon^+$, the PBW basis in $\mathcal{U}_\epsilon$ is the tensor product of bases described above.

Let moreover consider $\mathcal{B}_\epsilon^\pm := \mathcal{U}_\epsilon^0 \otimes \mathcal{U}_\epsilon^\pm$, $\mathcal{B}_\epsilon^{a,\pm} := \mathcal{U}_\epsilon^{a,0} \otimes \mathcal{U}_\epsilon^\pm$.

By the defining relations it follows that $\mathcal{U}_\epsilon^+, \mathcal{U}_\epsilon^-, \mathcal{B}_\epsilon^+$, $\mathcal{B}_\epsilon^-$ are subalgebras while $\mathcal{B}_\epsilon^+$, $\mathcal{B}_\epsilon^-$ are even sub-Hopf algebras. It is known that the subalgebras

- $Z_0^+ \subset \mathcal{B}_\epsilon^+$ generated by E_α^ℓ, K_i^ℓ
- $Z_0^- \subset \mathcal{B}_\epsilon^-$ generated by F_α^ℓ, K_i^ℓ
- $Z_0 \subset \mathcal{U}_\epsilon$ generated by E_α^ℓ, F_α^ℓ and K_i^ℓ,

are central and are Hopf subalgebras [DC-K], we will recall to which groups these Hopf algebras correspond.

The algebras $\mathcal{B}_\epsilon^+$, $\mathcal{B}_\epsilon^-$ and $\mathcal{U}_\epsilon$ are CH-Hopf algebras with trace subalgebras Z_0^+, Z_0^- and Z_0 respectively, and are free of respective ranks $\ell^{N+r}, \ell^{N+r}, \ell^{2N+r}$ over their trace subalgebras [DC-K].

6.3. **Structure of the center.** Let G be the simply connected group associated to $\mathfrak{g}$, T a maximal torus of G and W its Weyl group. U^+, U^- the unipotent radicals of opposite Borel subgroups B^+, B^-.

Let us recall [DKP1] that, as an Hopf algebra, Z_0 is the coordinate ring of the *dual group* H which is the subgroup of $B^+ \times B^-$ given by the kernel of the composed homomorphisms $B^+ \times B^- \xrightarrow{\mu} T \times T \xrightarrow{m} T$ where m is multiplication and μ the quotient modulo the unipotent radical. As a variety, H is identified to $U^- \times T \times U^+$.

Furthermore Z_0^+, Z_0^- are the coordinate rings of the two quotients $B^- = H/U^+$, $B^+ = H/U^-$.

The center Z_ϵ of $\mathcal{U}_\epsilon$ is described in [DKP1], the center of $\mathcal{U}_\epsilon^+$ will be presented later in this paper.

Let us recall briefly the description of Z_ϵ.

Z_ϵ contains also another subalgebra Z_1 (specialization of the central elements for the generic value of q). Z_1 is identified to the coordinate ring of the quotient T/W (isomorphic to $G//G$ the quotient under adjoint action).

The morphism $B^+ \times B^- \to G,\ (x, y) \to xy^{-1}$, restricted to H induces an étale map $\rho : H \to G$, in coordinates $\rho : U^- \times T \times U^+ \to G$ is given by $\rho(u, t, v) := ut^2v^{-1}$.

Let $Z' := Z_0 \cap Z_1$. In [DKP1] it is proved that $Z_\epsilon = Z_0 \otimes_{Z'} Z_1$, moreover there is the following geometric interpretation of this tensor product.

a) Z' is identified to the coordinate ring of the quotient T/W (isomorphic to $G//G$), under the composite map:

$$H \xrightarrow{\rho} G \xrightarrow{\pi} G//G = T/W$$

b) The ℓ power map $t \to t^\ell$ factors to the quotient giving a map $\ell : T/W \to T/W$ which at the level of coordinate rings induces the inclusion $Z' \subset Z_1$.

c) From this we get that $X := V(Z_\epsilon)$ is the schematic fiber product:

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & T/W \\ \sigma \downarrow & & \ell \downarrow \\ H & \xrightarrow{\rho} \quad G & \xrightarrow{\pi} T/W. \end{array}$$

6.4. **Center of $\mathcal{B}_\epsilon^+$.** In [DP1] (where in fact more general algebras are studied), it is proved that the degree of the algebra $\mathcal{U}_\epsilon^+$ (resp. $\mathcal{B}_\epsilon^+$) over the respective centers, is $\ell^{\frac{|\Delta^+|-s}{2}}$ (resp. $\ell^{\frac{|\Delta^+|+s}{2}}$), where s is the number of orbits of the permutation $-w_0$ on the set Δ of simple roots, and w_0 is the longest element of the Weyl group. In particular, since the rank of $\mathcal{B}_\epsilon^+$ over Z_0^+ is $\ell^{|\Delta^+|+n}$ where n is the rank of the group, we have that the center Z^+ of $\mathcal{B}_\epsilon^+$ has rank ℓ^{n-s} over Z_0^+.

For types different from $A_n,\ D_n,\ E_6$ we have $w_0 = -1$ and so $Z^+ = Z_0^+$.

Otherwise there is a bigger center which we want to describe. For type A_{2m} we have $s = m$ and for A_{2m+1} we have $s = m + 1$, for D_n we have $s = n - 1$ and for E_6 we have $s = 4$.

In order to compute the center of $\mathcal{B}_\epsilon^+$ we need to identify this algebra with the so called *quantized function algebra* $F_\epsilon[B^-]$.

The construction of a function algebra is a general construction on Hopf algebras.

Given a Hopf algebra $\mathcal{H}$ and a class of finite dimensional representations closed under direct sum and tensor products one considers the space $\hat{\mathcal{H}}$, of linear functions on $\mathcal{H}$ spanned by the *matrix coefficients* $c_{\phi,v}$.

Here v is a vector in a representation V and $\phi \in V^*$, the function $c_{\phi,v}$ is defined by:

$$c_{\phi,v}(h) := \langle \phi \mid hv \rangle.$$

$\hat{\mathcal{H}}$ is also a Hopf algebra, dual to $\mathcal{H}$ and called *function algebra*.

In [DL] this theory is developed for the algebras of $\mathcal{B}_q^{a\pm}$ first at q generic where these algebras are defined as in 6.2. We will also abbreviate $\mathcal{B}_q^a = \mathcal{B}^{a,+}$. One obtains the algebras $F_q[G]$ and $F_q[B^-]$ which can be specialized to $F_\epsilon[G]$ and $F_\epsilon[B^-]$ when q is specialized to a primitive ℓ-th root of unity ϵ.

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Remark that, as $\mathcal{B}_\epsilon$ is an Hopf subalgebra of $\mathcal{U}_\epsilon$ so $F_\epsilon[B^-]$ is a quotient Hopf algebra of $F_\epsilon[G]$ (the same holds for q generic).

In particular for every dominant weight λ one has in $F_q[G]$ and in $F_\epsilon[G]$, the matrix coefficients of the Weyl modules V_λ for the Lusztig divided power form of $\mathcal{U}_q^a$.

An important ingredient is Drinfeld's duality which gives the following canonical pairing between $\mathcal{B}_q^{a,-}$ and $\mathcal{B}_q^{s,+}$:

$$(\prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha, \prod_{i=N}^{1} E_{\beta_i}^{h_i} K_\beta) = q^{-(\alpha|\beta)} \prod (h_i)_{q_{\beta_i}^2}!(q_{\beta_i}^{-1} - q_{\beta_i})^{-h_i}$$

(where $q_\beta = q^{\frac{(\beta,\beta)}{2}}$ and $(h)_q = \frac{q^h-1}{q-1}$) and 0 otherwise.

From [DL] we have:

Theorem 6.1. *The algebra $\mathcal{B}_q^{s,+}$ under this pairing is identified to $F_q[B^-]$.*

These isomorphisms specialize at q a root of 1 giving an isomorphism between $\mathcal{B}_\epsilon^+$ and $F_\epsilon[B^-]$.

In view of this theorem we compute the center of $F_\epsilon[B^-]$.

We start from some identities at q generic.

From the theory of the $R-$matrix one has an immediate implication on the commutation rules among the elements $c_{\phi,v}$. Assume that v, w have weights μ_1, μ_2 and that ϕ, ψ have weights ν_1, ν_2 with respect to the action of the elements K_i [LS]. Then:

$$c_{\phi,v} c_{\psi,w} = q^{-(\mu_1|\mu_2)+(\nu_1|\nu_2)} c_{\psi,w} c_{\phi,v} + \sum c_{\psi_i,w_i} c_{\phi_i,v_i}$$

where

$$\psi_i \otimes \phi_i = p_i(q)(M_i(E) \otimes M_i(F))\psi \otimes \phi, \;\; w_i \otimes v_i = p_i'(q)(M_i'(E) \otimes M_i'(F)) w \otimes v$$

where the p_i, p_i' are in $\mathbb{C}(q)$ and M_i, M_i' are monomials of which at least one is not constant.

For each dominant weight λ we have an irreducible representation V_λ, we choose for each λ a highest weight vector v_λ. We make the convention that ϕ_λ denotes a dual vector, so it is a lowest weight vector in the dual space and it has weight $-\lambda$.

Take $\phi_{w_0\lambda}$ dual of a vector $v_{w_0\lambda}$. In the commutation take $c_{\phi_{w_0\lambda}, v_\lambda}$ and a matrix coefficient $c_{\phi,v}$ where ϕ has weight ν and v has weight μ:

$$c_{\phi,v} c_{\phi_{w_0\lambda}, v_\lambda} = q^{-(\mu|\lambda)-(\nu|w_0\lambda)} c_{\phi_{w_0\lambda}, v_\lambda} c_{\phi,v}$$

Set $\Delta_\lambda := d_{\phi_{w_0\lambda}, v_\lambda}$ and notice that, from the previous formula we have that;

$$\Delta_\mu \Delta_\lambda = q^{-(\mu|\lambda)-(-w_0\mu|w_0\lambda)} \Delta_\lambda \Delta_\mu = \Delta_\lambda \Delta_\mu$$

Lemma 6.2. $\Delta_\lambda \Delta_\mu = k \Delta_{\lambda+\mu}$, *$k$ a constant.*

Proof. By definition of multiplication between matrix coefficients:

$$c_{\phi_{w_0\lambda}, v_\lambda} c_{\phi_{w_0\mu}, v_\mu} = c_{\phi_{w_0\lambda} \otimes \phi_{w_0\mu}, v_\lambda \otimes v_\mu}$$

Now in the representation $V_\lambda \otimes V_\mu$ the highest weight vector $v_\lambda \otimes v_\mu$ generates the irreducible module $V_{\lambda+\mu}$ and in the dual we have a similar picture, thus the matrix coefficient is only relative to this submodule. □

We can thus normalize the choices of the v_λ so that

$$\Delta_\lambda \Delta_\mu = \Delta_{\lambda+\mu}$$

Lemma 6.3. *A matrix coefficient $c_{\phi,v}$ where ϕ has weight ν and v has weight μ vanishes on $\mathcal{B}_q^{a,-}$ if $\nu \not\geq -\mu$ (in the dominant order).*

Proof. A form of weight ν vanishes on all vectors which do not have weight $-\nu$, the vectors of $\mathcal{B}_q^{a,-} v$ have weights $\leq \mu$ in the dominant order, so the matrix coefficient is 0 unless $-\nu \leq \mu$ or $\nu \geq -\mu$. □

Let us now denote by $d_{\phi,v}$ the restriction of $c_{\phi,v}$ as function on $\mathcal{B}_q^{a,-}$.

Take now $d_{\phi_\lambda,v_\lambda}$ as function

$$d_{\phi_\lambda,v_\lambda}(\prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha) =< \phi_\lambda, \prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha v_\lambda >= \begin{cases} 0 & \text{if } \sum h_i > 0 \\ q^{<\alpha,\lambda>} & \text{otherwise.} \end{cases}$$

c

So, under the canonical pairing we have the identification

$$d_{\phi_\lambda,v_\lambda} = K_{-\lambda},$$

Lemma 6.4. *A matrix coefficient $d_{\phi,v}$ where ϕ has weight ν and v has weight μ and $\nu \geq -\mu$ (in the dominant order), is identified to a linear combination of elements* c
$\prod_{i=N}^{1} E_{\beta_i}^{h_i} K_\alpha$ where $\sum_i h_i\beta_i = \mu + \nu$.

Proof. We have that $< \phi, \prod_{i=N}^{1} F_{\beta_i}^{h_i} K_\alpha v >$ is 0 unless $\mu - \sum_i h_i\beta_i = -\nu$ therefore in the Drinfeld duality previously described, only the terms described can occur. □ c

We have

$$K_\lambda \prod_{i=N}^{1} E_{\beta_i}^{h_i} K_\alpha = q^{(\lambda, \sum_i h_i\beta_i)} \prod_{i=N}^{1} E_{\beta_i}^{h_i} K_\alpha K_\lambda$$

Therefore, from the previous Lemma, we get:

$$d_{\phi_\lambda,v_\lambda} d_{\phi,v} = q^{-(\lambda,\mu+\nu)} d_{\phi,v} d_{\phi_\lambda,v_\lambda}, \quad K_\lambda d_{\phi,v} = q^{(\lambda,\mu+\nu)} d_{\phi,v} K_\lambda$$

Set $T_\lambda := \Delta_\lambda K_{-\lambda}$, from the previous commutation relations we get:

$$d_{\phi,v} T_\lambda = q^{-(\mu|\lambda)-(\nu|w_0\lambda)+(\lambda,\mu+\nu)} T_\lambda d_{\phi,v} = q^{(\lambda - w_0\lambda,\nu)} T_\lambda d_{\phi,v}$$

From the previous relations, the fact that the Δ_λ's commute with each other and $\Delta_\lambda\Delta_\mu = \Delta_{\lambda+\mu}$, we have c

Proposition 6.5.

$$T_\lambda T_\mu = q^{(\lambda, w_0\mu - \mu)} T_{\lambda+\mu},$$

We can now introduce the elements

$$A_{h,\lambda} := T_\lambda^h T_{-w_0\lambda}^{l-h} = T_{h\lambda+(l-h)(-w_0\lambda)}$$

and compute the commutation relations with a matrix coefficient $d_{\phi,v}$ where ϕ has weight ν and v has weight μ:

$$d_{\phi,v} A_{h,\lambda} = q^{h(\lambda - w_0\lambda,\nu)} q^{(\ell-h)(-w_0\lambda+\lambda,\nu)} A_{h,\lambda} d_{\phi,v} = q^\ell A_{h,\lambda} d_{\phi,v}$$

We thus obtain c

Proposition 6.6. *If we specialize q to an ℓ root of 1, the elements $A_{h,\lambda}$ are in the center.*

Remark that, if $\lambda = -w_0\lambda$ we have $T_{h\lambda+(l-h)(-w_0\lambda)} = T_{l\lambda} \in Z_0$. Notice that, since ℓ is odd $T_{\ell\lambda} = T_\lambda^\ell$.

To understand $\Delta_{\ell\lambda}$, (and also $T_{\ell\lambda} = \Delta_{\ell\lambda}K_{-\ell\lambda}$) we must use the Frobenius isomorphism. We identify this element to the classical matrix coefficient (linear function on $U(\mathfrak{g})$), for λ which we will denote by $\delta_\lambda = C_{\phi_{w_0\lambda},v_\lambda}$. By abuse of notations we denote by the same symbols the vectors and forms in the classical representations.

Now recall that, for an algebraic group G, the function algebra $\mathbb{C}[G]$ has a left and a right G action which in terms of functions or of matrix coefficients are

$$(h,k)f(g) := f(h^{-1}gk), \quad (h,k)c_{\phi,v} = c_{h\phi,kv}.$$

When G is semisimple and simply connected, we can exponentiate the finite dimensional representations of $\mathfrak{g}$ and thus identify the function algebra on $U(\mathfrak{g})$ with the function algebra on G. For every dominant weight λ we have an irreducible representation V_λ and the embedding given by matrix coefficients:

$$i_\lambda : V_\lambda^* \otimes V_\lambda \to \mathbb{C}[G].$$

The element $c_\lambda := C_{\phi_{w_0\lambda},v_\lambda}$, with respect to the left and right actions of $B^+ \times B^+$ is an eigenvector of weight $(-w_0(\lambda),\lambda)$. In particular we can analyze it for the fundamental weights. The following is well known

Proposition 6.7. *The elements c_{ω_i} for the various fundamental weights are irreducible elements whose divisors are the closures of the codimension 1 Bruhat cells of G.*

Proof. Let us recall one possible proof for completeness. The ring $\mathbb{C}[G]$ is a unique factorization domain (cf. [Po]), the elements that are $B^+ \times B^+$ eigenvectors will then factor into irreducible $B^+ \times B^+$ eigenvectors. But these elements coincide up to constant with the elements δ_λ hence the first statement is due to the fact that the fundamental weights are free generators of the monoid of dominant weights.

For the second part we have exactly $n = rk(G)$ codimension 1 Bruhat cells of G which must have equations which are $B^+ \times B^+$ eigenvectors. In fact one can identify more precisely the correspondence (cf. [Ch]).

Another interpretation is with the Borel Weil theorem and identifying the c_λ with sections of line bundles on the flag variety. □

When we restrict to B^- we can exploit the fact that B^-U^+ is open in G. Functions invariant under right U^+ action are identified to functions on B^-. We deduce

Proposition 6.8. *The elements d_{ω_i}, restriction to B^- of the elements c_{ω_i}, $i = 1,\dots,n$, are irreducible elements whose divisors are the closures of the codimension 1 Bruhat cells of G intersected with B^-.*

The restriction $t_\lambda(u,t)$ to $B^- = U^- \times T$ of $T_{\ell\lambda} = \Delta_{\ell\lambda}K_{-\ell\lambda}$ is a function only of $u \in U^-$ and independent of $t \in T$.

Proof. We have already proved the first part, for the second remark that, by definition $\Delta_{\ell\lambda}$ transforms under right action of T through the character χ_λ so $\delta_\lambda(t,u) = g_\lambda(u)\chi_\lambda(t)$, but $K_{\ell\lambda}$ restricts to the character $\chi_\lambda(t)$ hence the claim. □

Example. For $SL(n)$, the fundamental representation $\wedge^i V$, the highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_i$ and the lowest weight vector $e_{n-i+1} \wedge e_{n-i+2} \wedge \cdots \wedge e_n$, the matrix coefficient is the determinant formed in the triangular matrix (x_{ij}), by the determinant of the first i rows and the last i columns:

$$\begin{vmatrix} x_{1,n-i+1} & \cdots & x_{1,n} \\ x_{2,n-i+1} & \cdots & x_{2,n} \\ \cdots & \cdots & \cdots \\ x_{i,n-i+1} & \cdots & x_{i,n} \end{vmatrix}$$

Lemma 6.9. *Let A be a unique factorization domain and a Cohen Macaulay ring of characteristic 0 (or prime to ℓ) and containing the ℓ roots of 1.*

Let $f_1, f_2, \dots, f_k \in A$ be distinct irreducible elements and

$$R := A[t_1, \dots, t_k]/(t_1^\ell - f_1, t_2^\ell - f_2, \dots, t_k^\ell - f_k).$$

Then R is a normal domain, Galois extension of A with Galois group $\mathbb{Z}/(\ell)^k$.

Proof. Clearly R is free over A of rank ℓ^k and $\mathbb{Z}/(\ell)^k$ acts as symmetry group. We need only to show that R is a normal domain. For this we shall use Serre's criterion [Se].

1) First of all R is a complete intersection hence it is Cohen Macaulay.

2) Next we will prove that it is smooth in codimension 1 which will prove that it is a normal ring.

3) Finally we prove that its spectrum is connected which will imply that it is a domain.

Let us show 2). Since A is normal we can restrict our analysis to the smooth locus and choose a regular system of parameters $x_1, \dots, x_m$. Consider the Jacobian matrix (e.g. $k = 3$):

$$\begin{vmatrix} \ell t_1^{\ell-1} & 0 & 0 & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ 0 & \ell t_2^{\ell-1} & 0 & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ 0 & 0 & \ell t_3^{\ell-1} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \cdots & \frac{\partial f_3}{\partial x_m} \end{vmatrix}$$

On the open set where the f_i are non zero, the determinant of the minor formed by the first k columns is not zero and thus R is smooth in codimension 0 and hence reduced. In a smooth point of the subvariety $f_i = 0$ where $\prod_{j \neq i} f_j \neq 0$, we also have a non zero maximal determinant so these points are smooth. Thus the ring R is smooth in codimension at least 2.

Finally we have to prove 3) i.e. connectedness. Let F be the quotient field of A and $\overline{F}$ be an algebraic closure of F. Let us consider the ring $\overline{R} \subset \overline{F}$ obtained from R by adding ℓ roots b_i of f_i. We have clearly a homomorphism of R onto $\overline{R}$. Let E be the quotient field of $\overline{R}$, clearly E is a Galois extension of F with Galois group a subgroup Γ of $\mathbb{Z}/(\ell)^k$. It is clearly enough to show that this subgroup is $\mathbb{Z}/(\ell)^k$ itself.

Let $\mathcal{M} := \{b_1^{h_1} \dots b_k^{h_k}, (h_1, h_2, \dots, h_k) \in \mathbb{Z}^k\}$ and ϵ a primitive ℓ-th root of 1. We identify $\mathbb{Z}/(\ell)$ with the multiplicative group generated by ϵ and have a pairing:

$$\Gamma \times \mathcal{M} \xrightarrow{p} \mathbb{Z}/(\ell), \quad p(\sigma, M) := \sigma(M)M^{-1}$$

This pairing factors through $\mathcal{M}^\ell$ and, if by contradiction $|\Gamma| < \ell^k$, we must have an element $M = b_1^{h_1} \dots b_k^{h_k}$, $0 \le h_i < \ell$ not all the $h_i = 0$ which is in the kernel of c the pairing, hence by Galois theory (and the fact that the element is integral over A) $M \in A$. Then

$$M^\ell = f_1^{h_1} \dots f_k^{h_k}.$$

Factoring M into irreducibles, this implies that $\ell \,|\, h_i$ for all i, a contradiction unless all $h_i = 0$. □

Let $a_i := T_{\ell\omega_i} \in Z_0^+$, $i = 1, \dots, n$. Notice that, under the identification of Z_0^+ with $\mathbb{C}[B^-]$, the elements a_i coincide up to a unit with the elements d_{ω_i} defined in Proposition 6.8, so that their divisors are irreducible and distinct. Consider the algebra $R := Z_0[b_1, \dots, b_n]$ with $b_i^\ell = a_i$. Lemma 6.9 implies that R is a normal domain on which acts the Galois group $\mathbb{Z}/(\ell)^n$. Let $\tau : \omega_i \to -w_0\omega_i$ be the standard involution of fundamental weights. τ induces an involution of the factors of $\mathbb{Z}/(\ell)^n$. Let Γ be the invariant subgroup, it is made of those $r-$tuples which have the same entry in the orbits of τ (made of 1 or 2 elements). The invariants under Γ are spanned by the monomials $b_1^{h_1} b_2^{h_2} \dots b_n^{h_n}$ which when i, j are an orbit of τ have exponents $h_i + h_j \equiv 0, \quad \mod \ell$.

Thus R^Γ is isomorphic to the algebra generated by the elements $T_{h\lambda+(\ell-h)(-w_0\lambda)} =$ c $A_{h,\lambda}$.

Theorem 6.10. *The center of $\mathcal{B}_\epsilon^+$ is the algebra Z_ϵ^+ generated by Z_0^+ and by the elements $T_{h\lambda+(\ell-h)(-w_0\lambda)}$.*

c *Proof.* The algebra Z_ϵ^+, by 6.6 is contained in the center, and being isomorphic to R^Γ, it is normal. Also, since the degree of B_ϵ^+ equals $\ell^{\frac{|\Delta^+|+s}{2}}$, where s is the number of orbits of τ, while B_ϵ^+ has rank $\ell^{|\Delta^+|+n}$ over Z_0^+, we deduce that the rank of Z_ϵ^+ over Z_0^+, equals the rank of the center, so Z_ϵ^+ is the center. □

7. Clebsch-Gordan decompositions for generic representations of quantized universal enveloping algebras at roots of 1

7.1. Compatibility for $\mathcal{U}_\epsilon^+ \subset \mathcal{U}_\epsilon$.

Proposition 7.1. *The natural map $\mathcal{U}_\epsilon^+ \otimes_{Z_0^+} Z_\epsilon \to \mathcal{U}_\epsilon$ is injective.*

Proof. Recall that $\mathcal{U}_\epsilon^+$ is free over Z_0^+. As a linear basis in $\mathcal{U}_\epsilon^+$ over Z_0^+ we can choose PBW elements $b = \prod_{\alpha\in\Delta_+} E_\alpha^{m_\alpha}$ with $0 \le m_\alpha < \ell$. The center Z_ϵ is generated by Z_1 and Z_0 where Z_1 is the "specialization at $q = \epsilon$ of the center for generic q. Any element $z \in Z_1$ is completely determined by its component $\phi_{0,0} \in \mathcal{U}_\epsilon^0$ and it is of the form

$$z = \phi_{0,0} + \sum_{\underline{r},\underline{k}} E^{\underline{r}} \phi_{\underline{r},\underline{k}} F^{\underline{k}}, \quad \phi_{\underline{r},\underline{k}} \in \mathcal{U}_\epsilon^0 \tag{10}$$

Moreover $Z_\epsilon = Z_0 \otimes_{Z_0 \cap Z_1} Z_1$.

We want to prove that if $\sum b z_b = 0$ where b are elements of the PBW basis and $z_b \in Z_\epsilon$, then for any b, $z_b = 0$.

Recall that we have a convex ordering on Δ_+. In the product defining PBW elements we choose the decreasing order of $E_\alpha^{m_\alpha}$. This provides a total ordering on PBW elements defined by the lexicographic ordering of E_α.

Lemma 7.2. *Let $b = \prod_\alpha E_\alpha^{m_\alpha}$ be a PBW element and $\beta \in \Delta_+$; then bE_β is a linear combination of PBW elements which are greater than b.*

Proof. We have $b = E_{\beta(1)}^{m_1} \dots E_{\beta(N)}^{m_N}$ where $\beta(1) > \cdots > \beta(N)$ in the convex ordering of Δ_+. Then

- if $\beta(N) \geq \beta$ the statement is clear,
- if $\beta(N) < \beta$ we use the commutation relation

$$\begin{aligned} E_{\beta(N)} E_\beta &= q^{(\beta(N),\beta)} E_\beta E_{\beta(N)} \\ &+ \sum_{\beta(N) < \gamma_1 < \cdots < \gamma_s < \beta} a_{\gamma_1,\dots,\gamma_s r_1,\dots,r_s} E_{\gamma_s}^{r_s} \dots E_{\gamma_1}^{r_1} \end{aligned} \tag{11}$$

c

It is clear that here $E_\beta E_{\beta(N)} > E_{\beta(N)}$ and $E_{\gamma_s}^{r_s} \dots E_{\gamma_1}^{r_1} > E_{\beta(N)}$. Now we can iterate this process to reorder monomials and the lemma follows. □

Coming back to the proof of the proposition we look at the PBW elements b for which $z_b \neq 0$ and let b_0 be the minimal among them. Then let us apply the coproduct and take the component which belongs to $\mathcal{U}_\epsilon^+ \otimes \mathcal{U}_\epsilon^0 \mathcal{U}_\epsilon^- \subset \mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$.

Due to triangular decomposition in $\mathcal{U}_\epsilon$, each $\Delta(b)$ will contribute only by $b \otimes 1$ to this component. The element (10) will contribute as

$$1 \otimes \phi_{0,0} + \sum E^{\underline{r}} \otimes \phi_{\underline{r},\underline{k}} F^{\underline{k}}$$

and any element in Z_0 which is always a polynomial in $x_\alpha = E_\alpha^\ell$, $y_\alpha = F_\alpha^\ell$, $z_i^\pm = k_i^{\pm\ell}$ will contribute as a polynomial in $x_\alpha \otimes 1,\ 1 \otimes y_\alpha, 1 \otimes z_i^\pm$.

The minimal term in the left side of the tensor product is of the form $b_0 P(x_\alpha)$ for some polynomial P because, from lemma 7.2, the terms coming from Z_1 will contribute by 1 up to bigger terms. The contribution of PBW elements $b > b_0$ to the left component of the tensor product will have their minimal monomial of exactly the same form.

Now the proposition follows from the freeness of $\mathcal{U}_\epsilon^+$ over Z_0^+. □

7.2. **Compatibility of comultiplication.** In this paragraph we will strengthen Theorem 5.13 as follows:

Theorem 7.3. *The comultiplication $\Delta : \mathcal{U}_\epsilon \to \mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$ is compatible (with trace) when we think of $\mathcal{U}_\epsilon$ as Z_0 algebra and $\mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$ as $Z_\epsilon \otimes Z_\epsilon$ algebra.* c

Using 5.8 ii) we need to show:

Proposition 7.4. *$\Delta(Z_\epsilon) \otimes_{\Delta(Z_0)} (Z_\epsilon \otimes Z_\epsilon)$ is a normal domain.*

Proof. We have recalled the analysis of [DP1] in 6.3 and in particular the fiber product diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & T/W \\ \sigma \downarrow & & \ell \downarrow \\ H & \xrightarrow{\rho} & G \xrightarrow{\pi} T/W. \end{array}$$

where X is the spectrum of Z_ϵ.

The analysis shows that one can define a regular locus in all these varieties.

G^{reg} is the usual set of regular elements (i.e. elements with conjugacy class of maximal dimension).

Finally we set

$$H^{reg} := \rho^{-1}G^{reg}, \quad X^{reg} := \sigma^{-1}H^{reg}.$$

In the restricted fiber product diagram

$$\begin{array}{ccccc} X^{reg} & \xrightarrow{\quad p \quad} & & & T/W \\ \sigma\downarrow & & & & \ell\downarrow \\ H^{reg} & \xrightarrow{\rho} & G^{reg} & \xrightarrow{\pi} & T/W. \end{array}$$

by [St], we know that the subset G^{reg} of G of regular elements has a complement of codimension 2 and that the map π restricted to G^{reg} is a smooth map. It follows
c that $\pi \circ \rho$ and also p are smooth and, since T/W is smooth, that all varieties in this regular diagram are smooth. Since the complement of H^{reg} in H has codimension ≥ 2 and σ is finite, the complement of X^{reg} in X has also codimension ≥ 2.

We know that the ring Z_ϵ is presented as a complete intersection over Z_0 by the fiber product diagram and it is free of finite rank. Thus we have that, Z_ϵ and $Z_\epsilon \otimes Z_\epsilon$ are normal Cohen Macaulay domains. For the same reasons $\Delta(Z_\epsilon) \otimes_{\Delta(Z_0)} (Z_\epsilon \otimes Z_\epsilon)$ is a complete intersection, hence a Cohen-Macaulay ring.

$\Delta(Z_\epsilon) \otimes_{\Delta(Z_0)} (Z_\epsilon \otimes Z_\epsilon)$ is the coordinate ring of the schematic fiber product Y:

$$\begin{array}{ccccc} Y & \xrightarrow{\quad p \quad} & & & X \\ \downarrow & & & & \sigma\downarrow \\ X \times X & \xrightarrow{\sigma\times\sigma} & H \times H & \xrightarrow{m} & H. \end{array}$$

where m is multiplication. This can also be presented as the unique fiber product map:

$$\begin{array}{ccccccccc} Y & \xrightarrow{\quad p \quad} & & & X & \xrightarrow{\quad p \quad} & & & T/W \\ q\downarrow & & & & \sigma\downarrow & & & & \ell\downarrow \\ X \times X & \xrightarrow{\sigma\times\sigma} & H \times H & \xrightarrow{m} & H & \xrightarrow{\rho} & G & \xrightarrow{\pi} & T/W. \end{array}$$

We will then apply Serre's criterion [Se], and prove that Y is smooth in codimension 1, which will show that its coordinate ring is a *normal ring.*

By the homeomorphism $\nu : H \times H \to H \times H$, $\nu(x, y) := (x, xy)$ it follows that the open set $\mathcal{A} := \{(x, y) \in H \times H\}$ with x, y, xy regular has a complement of codimension 2. On this set the map $(x, y) \to \pi \circ \rho(xy)$ of $H \times H \to T/W$ is a smooth map.

On the open set $\mathcal{B} := (\sigma \times \sigma)^{-1}\mathcal{A}$ the composite map $\pi \circ \rho \circ (\sigma \times \sigma)$ is smooth,
c and since the map $T/W^{reg} \xrightarrow{\ell} T/W^{reg}$ is also smooth, we deduce that $\mathcal{C} := q^{-1}\mathcal{B}$ is smooth. Since q is finite it follows that the complement of $\mathcal{C}$ in Y has codimension ≥ 2 hence Y is a normal variety and it follows that $\Delta(Z_\epsilon) \otimes_{\Delta(Z_0)} (Z_\epsilon \otimes Z_\epsilon)$ is a normal ring.

It remains (as in Lemma 6.9) to finish the argument and prove that Y is connected which implies that $\Delta(Z_\epsilon) \otimes_{\Delta(Z_0)} (Z_\epsilon \otimes Z_\epsilon)$ is a normal domain.

Since the morphism q is finite it is also proper so it suffices to find some closed subvariety M of $X \times X$ with the property that $q^{-1}(M)$ is connected.

First of all, we claim that we have a natural embedding:

$U^+ \xrightarrow{i^+} X$ so that the diagram:

$$\begin{tikzcd} & X \arrow[d, "\rho"] \\ U^+ \arrow[ur, "i+"] \arrow[r, "v"'] & H \end{tikzcd}$$

is commutative.

In fact we see that the composed map $U^+ \to H \to G \to G//G$ is constant with value the class $\overline{1}$ of 1, so it can be lifted by choosing a point in the fiber of $\ell^{-1}(\overline{1})$.

We now embed $U^+ \times U^- \xrightarrow{j} X \times X$ and see that the composed map

$$U^+ \times U^- \xrightarrow{j} X \times X \xrightarrow{\rho\times\rho} H \times H \xrightarrow{m} H \xrightarrow{\sigma} G$$

induces the natural inclusion by multiplication $U^+U^- \subset G$.

In the next Lemma we show that we have a subvariety $M \subset U^+U^-$ (section of π) for which the composed map $M \xrightarrow{\pi} G//G$ is an isomorphism. Hence $q^{-1}(M)$, i.e. Y restricted to M, is isomorphic to T/W and hence connected. c

Lemma 7.5. *There is a section $M \subset U^+U^-$ for which the composed map $M \xrightarrow{\pi} G//G$ is an isomorphism.*

c

Proof. We will need in our analysis a variation of a result of Steinberg. Let us recall his Theorem. Let G be a semisimple simply connected group.

For β a positive root let us denote by $X_\beta = \exp(\mathbb{C}e_\beta)$ the root subgroup associated to β. If $\alpha_1, \dots, \alpha_n$ is the set of simple positive roots denote by σ_i a representative in the normalizer of the torus T of the simple reflection s_i associated to the root α_i. Finally let $\pi : G \to G//G = T/W$ be the quotient under adjoint action. Define:

$$N = X_{\alpha_1}\sigma_1 X_{\alpha_2}\sigma_2 \dots X_{\alpha_n}\sigma_n.$$

The theorem of Steinberg [St] is that N is a slice of the map π. In other words, c
under π, N is isomorphic to T/W. c

For our purposes we have to slightly change this type of slice, we start remarking that

$$N = X_{\beta_1} X_{\beta_2} \dots X_{\beta_n} \sigma_1\sigma_2 \dots \sigma_n, \quad \beta_i := s_1 s_2 \dots s_{i-1}(\alpha_i).$$

Next we want to show that, provided we possibly change the representative σ_n, we c
can express

$$\sigma_1\sigma_2 \dots \sigma_n = a_+ b_- c_+, \quad a_+, c_+ \in U^+,\ b_- \in U^-.$$

For this consider the flag variety $\mathcal{B}$ and in it the point p^+ with stabilizer B^+, consider $q := \sigma_1\sigma_2 \dots \sigma_n p_+$ and then $U^+ q \cap U^- p_+ \neq \emptyset$ (cf. [Ch]). Thus we can find $a_+ \in U^+,\ b_- \in U^-$ with $a_+^{-1} q = b_- p_+$ hence $b_-^{-1} a_+^{-1} \sigma_1\sigma_2 \dots \sigma_n p_+ = p_+$ hence:

$$\sigma_1\sigma_2 \dots \sigma_n = a_+ b_- c_+ t, \quad c_+ \in U^+,\ t \in T.$$

We change then σ_n with $\sigma_n t^{-1}$ and get

$$\sigma_1\sigma_2 \dots \sigma_n = a_+ b_- c_+.$$

Now we obtain the new slice $M := c_+ N c_+^{-1}$. c

The interest for us is that

$$M = c_+ X_{\beta_1} X_{\beta_2} \dots X_{\beta_n} a_+ b_- \subset U^+U^-.$$

as requested. □

There is actually a rather interesting application of the slice that we found.

We can consider U^+U^- also as subset of H, the canonical covering $\sigma : H \to G$ restricted to U^+U^- is a homeomorphism to the image. Therefore we can also
c consider $M \subset H$. In [DKP1] it is shown that the preimage of a regular orbit of G is a unique symplectic leaf in H while it is the union of ℓ^n leaves in X. We deduce that we have the regular elements in X and H which are unions of maximal Poisson leaves and that:

Theorem 7.6. *The set $M \subset H$ is a cross section of the set of regular Poisson leaves in H.*

The set $\rho^{-1}(M) \subset X$ is homeomorphic to T/W and is a cross section of the set of regular Poisson leaves in X.

7.3. **Clebsch Gordan formula.** We know that $\mathcal{U}_\epsilon$ has no zero divisors, as a Z_0-algebra is a free Z_0-module of rank $\ell^{2|\Delta_+|+n}$. By [DP1] its center Z_ϵ is a free Z_0-module of rank ℓ^n. If $Q(Z_0)$ denotes the quotient field of Z_0 we have that $Q(\mathcal{U}_\epsilon) = \mathcal{U}_\epsilon \otimes_{Z_0} Q(Z_0)$ is a division algebra of dimension $\ell^{2|\Delta_+|}$ over its center $Q(Z_\epsilon) = Z_\epsilon \otimes_{Z_0} Q(Z_0)$.

Therefore:

$$[\mathcal{U}_\epsilon : Z_0] = \ell^{|\Delta_+|+n}, \quad [\mathcal{U}_\epsilon : Z_\epsilon] = \ell^{|\Delta_+|}, \quad [Z_\epsilon : Z_0] = \ell^n.$$

Let V and W be two generic irreducible representations of $\mathcal{U}_\epsilon$ of maximal dimension $m = \ell^{|\Delta_+|}$. We want to decompose the representation $V \otimes W$ of $\mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$ into irreducible representations of the subalgebra $\Delta(\mathcal{U}_\epsilon)$.

We apply the methods of Theorem 5.16, recalling that Z_0 is a Hopf subalgebra of $\mathcal{U}_\epsilon$, but Z_ϵ is only a subalgebra. So, if $V = M_P$, $W = M_Q$ where $P, Q \in V(Z_\epsilon)$ and $\pi(P) = x \in V(Z_0), \pi(Q) = y \in V(Z_0)$ we know by 5.16 that, for generic x, y:

$$M_P \otimes M_Q \simeq \oplus_{R \in \pi^{-1}(xy)} M_R^{\oplus h_R^{P,Q}}$$

we want to prove in our case:

Theorem 7.7. *The multiplicities $h_R^{P,Q}$, $R \in \pi^{-1}(xy)$, are all equal to $\ell^{|\Delta_+|-n}$.*

Proof. In view of Theorem 5.12, in order to prove this Theorem, since by the generic assumption $\dim M_P = \dim M_Q = \dim M_R = \ell^{|\Delta_+|}$, and the degree of π is ℓ^n, it is enough to use the compatibility proven in Theorem 7.3.

7.4. **Compatibility for $\mathcal{B}_\epsilon^+ \subset \mathcal{U}_\epsilon$.** Using the results of 6.4 we prove now:

Theorem 7.8. *i) $Z_\epsilon^+ \otimes_{Z_0} Z_\epsilon$ is a normal domain.*

c *ii) The inclusion $\mathcal{B}_\epsilon^+ \subset \mathcal{U}_\epsilon$ gives compatible algebras, where $\mathcal{B}_\epsilon^+$ is thought of as Z_0 and $\mathcal{U}_\epsilon$ as Z_ϵ algebras.*

Proof. From the analysis leading to 6.10 we know that:

$$Z_\epsilon^+ = Z_0[b_1, \ldots, b_n]^\Gamma, \quad b_i^\ell = a_i.$$

thus also:

$$Z_\epsilon^+ \otimes_{Z_0} Z_\epsilon = Z_\epsilon[b_1, \ldots, b_n]^\Gamma, \quad b_i^\ell = a_i.$$

therefore it is enough to prove that $Z_\epsilon[b_1, \ldots, b_n]$ is a normal domain.

Let us argue geometrically. Let V be the variety of $Z_\epsilon[b_1, \dots, b_n]$, a normal variety. $Z_\epsilon[b_1, \dots, b_n]$ is the coordinate ring of the schematic fiber product S:

$$\begin{array}{ccc} S & \xrightarrow{s} & V \\ t \downarrow & & v \downarrow \\ X & \xrightarrow{\sigma} & H \xrightarrow{p} B \end{array}$$

We have that the map t is finite and flat so S is Cohen Macaulay since, as we have seen, X is Cohen Macaulay. The map p is smooth, so the composite map $p \circ \sigma$ is smooth in the corresponding *regular elements* i.e. outside subvarieties of codimension 2.

From Lemma 6.9 we have that also v is smooth outside codimension 2 and, since t is finite we deduce that S is smooth in codimension 1 hence S is a normal variety.

To prove the irreducibility of S we can, reasoning as in Proposition 7.4, restrict to the section $U^- \subset X$, analyze the schematic fiber product diagram:

$$\begin{array}{ccc} S' & \xrightarrow{s} & V \\ t \downarrow & & v \downarrow \\ U^- & \xrightarrow{\sigma} & H \xrightarrow{p} B \end{array}$$

and show that S' is irreducible. The coordinate ring A of S' is obtained from the coordinate ring $\mathbb{C}[U^-]$ of U^- by adding the restrictions $\bar{b}_i$, of the elements b_i which are of course ℓ-th roots of the restrictions of the a_i. Now identify $B^- = U^- \times T$. By the definitions of the functions a_i we have that these functions are invariant under right action of T. It follows that $a_i(u,t) = g_i(u)$, hence the $g_i(u)$ are irreducible polynomials on U^- defining distinct divisors and the coordinate ring A of S' is:

$$A = \mathbb{C}[U^-][\bar{b}_1, \dots, \bar{b}_n], \quad \bar{b}_i^\ell = g_i(u).$$

We can then again apply Lemma 6.9 and deduce that S' is irreducible, concluding the proof of the theorem. $\square$

We obtain as corollary, using Theorems 5.11 and 5.12, the branching rules from $\mathcal{U}_\epsilon$ to $\mathcal{B}_\epsilon^+$. Recall that the degree of $\mathcal{U}_\epsilon$ (as algebra over its center) is $\ell^{|\Delta_+|}$, while the degree of $\mathcal{B}_\epsilon^+$ (as algebra over Z_0^+) is $\ell^{\frac{|\Delta_+|-s}{2}+n}$, so the factor r of 5.11 is $\ell^{\frac{|\Delta_+|+s}{2}}$. c

Theorem 7.9. *Let M be a semisimple trace representation of $\mathcal{U}_\epsilon$ with central character χ on Z_ϵ. Let p the point in B^- induced by χ.*

i) The restriction of M to $\mathcal{B}_\epsilon^+$ has as associated semisimple representation the one N_p of character p on the coordinate ring Z_0^+ of B^- with multiplicity $\ell^{\frac{|\Delta_+|+s}{2}-n}$.

ii) For generic p, N_p is the direct sum of the irreducible trace representation for the center of $\mathcal{B}_\epsilon^+$ with central characters all the ℓ^{n-s} central characters which restricted to Z_0^+, give p. c

In particular consider a generic point $\chi \in X$. We have a unique irreducible representation of $\mathcal{U}_\epsilon$ with central character χ. When we restrict it to $\mathcal{B}_\epsilon^+$ we have a direct sum of all the ℓ^{n-s} irreducible representations, each of dimension $\ell^{\frac{|\Delta_+|+s}{2}}$, which on Z_0^+ have central character p and each with multiplicity $\ell^{\frac{|\Delta_+|+s}{2}-n}$.

7.5. **Compatibility of comultiplication for $\mathcal{B}_\epsilon^+$.** We can repeat for $\mathcal{B}_\epsilon^+$ the same analysis done in §7.2 for $\mathcal{U}_\epsilon$. Recall that:

$$[\mathcal{B}_\epsilon^+ : Z_0^+] = \ell^{\frac{|\Delta_+|-s}{2}+n}, \quad [\mathcal{B}_\epsilon^+ : Z_\epsilon^+] = \ell^{\frac{|\Delta_+|+s}{2}}, \quad [Z_\epsilon^+ : Z_0^+] = \ell^{n-s}$$

Let V and W be two generic irreducible representations of $\mathcal{B}_\epsilon^+$ of maximal dimension $m = \ell^{\frac{|\Delta_+|+s}{2}}$. We want to decompose the representation $V \otimes W$ of $\mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$ into irreducible representations of the subalgebra $\Delta(\mathcal{B}_\epsilon^+)$.

We apply the methods of Theorem 5.16, recalling that Z_0^+ is a Hopf subalgebra of $\mathcal{B}_\epsilon^+$, but Z_ϵ^+ is only a subalgebra. So, if $V = M_P,\ W = M_Q$ where $P, Q \in V(Z_\epsilon^+)$ and $\pi(P) = x \in V(Z_0), \pi(Q) = y \in V(Z_0^+)$ we know by 5.16 that, for generic x, y:

$$M_P \otimes M_Q \simeq \oplus_{R \in \pi^{-1}(xy)} M_R^{\oplus h_R^{P,Q}}$$

we want to prove in our case:

Theorem 7.10. *The multiplicities $h_R^{P,Q},\ R \in \pi^{-1}(xy)$, are all equal to $\ell^{\frac{|\Delta_+|+s}{2} - n + s}$.*

Proof. In order to prove this Theorem, since by the generic assumption $\dim M_P = \dim M_Q = \dim M_R = \ell^{\frac{|\Delta_+|+s}{2}}$, it is enough, by Proposition 5.12 to show the stronger statement that the inclusion $\Delta(\mathcal{B}_\epsilon^+) \subset \mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$ is compatible, when we consider $\mathcal{B}_\epsilon^+$ as Z_0 algebra but $\mathcal{U}_\epsilon \otimes \mathcal{U}_\epsilon$ as $Z_\epsilon^+ \otimes Z_\epsilon^+$ algebra. Hence we need to show, looking at the following restriction diagram :

$$\begin{array}{ccc} \Delta(\mathcal{B}_\epsilon^+) & \subset & \mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+ \\ \cup & & \cup \\ \Delta(Z_0^+) & \subset & Z_\epsilon^+ \otimes Z_\epsilon^+ \end{array}$$

that $\Delta(\mathcal{B}_\epsilon^+) \otimes_{\Delta(Z_0^+)} (Z_\epsilon^+ \otimes Z_\epsilon^+)$ embeds in $\mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$.

Recalling Theorem 5.12 and definition 5.7, we know that comultiplication is a compatible inclusion when we think of $\mathcal{B}_\epsilon^+$ as Z_0^+ algebra and $\mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$ as $Z_0^+ \otimes Z_0^+$ algebras but we need the stronger fact that

c **Theorem 7.11.** *The comultiplication $\Delta : \mathcal{B}_\epsilon^+ \to \mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$ is compatible (with trace) when we think of $\mathcal{B}_\epsilon^+$ as Z_0^+ algebra and $\mathcal{B}_\epsilon^+ \otimes \mathcal{B}_\epsilon^+$ as $Z_\epsilon^+ \otimes Z_\epsilon^+$ algebra.*

Using 5.8 ii) we need to show:

Proposition 7.12. *$\Delta(Z_\epsilon^+) \otimes_{\Delta(Z_0^+)} (Z_\epsilon^+ \otimes Z_\epsilon^+)$ is a normal domain.*

Proof. The proof is similar to the one of Proposition 7.4. We identify this ring to a ring of invariants of a ring obtained by extracting roots and prove the usual fiber product smoothness condition.

7.6. **The degrees and dimensions of cosets.** Here we will argue that the formulae for degrees that we obtained imply the existence of birational Darboux coordinates on the corresponding cosets.

7.6.1. Let $\mathcal{M}_{2d}$ be a compact symplectic manifold. The geometric quantization produces a sequence of vector spaces $\{V_n\}_n,\ n = 1, 2, \ldots$. The corresponding sequence $\{End(V_n)\}$ of matrix algebras can be regarded as a quantization of the
c Poisson algebra of functions on $\mathcal{M}_{2d}$. For large n the dimension of V_n has the following asymptotic behavior

$$dim(V_n) = Vol(\mathcal{M}_{2d}) n^d (1 + O(1/n)) \ , \tag{12}$$

where $Vol(\mathcal{M}_{2d})$ is the symplectic volume of the symplectic manifold.

Let $\mathcal{M}_{2d} = \mathbb{T}^{2d}$ be the $2d$-dimensional torus with coordinates $t_1, \dots, t_{2d} \in \mathbb{C}$, $|t_i| = 1$. Assume that the symplectic structure on this manifold is constant:

$$\omega = \sum_{a,b=1}^{2d} \omega^{ab} \frac{dt_a}{t_a} \wedge \frac{dt_b}{t_b} \tag{13}$$

Geometric quantization of this manifold produces the sequence of vector spaces V_n with $dim(V_n) = n^d$. Because the symplectic structure is constant the asymptotic formula (12) becomes exact. One can argue that tori are typical manifolds for which this takes place.

The complexification of $(\mathbb{T}^{2d}, \omega)$ is the complex torus $(\mathbb{C}^*)^{2d}$ with complex holomorphic symplectic form (13). The algebra of Laurent polynomials in t_i is a Poisson algebra with the brackets

$$\{t_i, t_j\} = \omega_{ij} t_i t_j$$

where ω_{ij} is the matrix inverse to ω^{ij}.

Let q be a nonzero complex number. Define the algebra $C_q((\mathbb{C}^*)^{2d})$ generated by $t_i^{\pm 1}$ with defining relations

$$t_i t_i^{-1} = 1, t_i t_j = q^{\omega_{ij}} t_j t_i$$

This family of algebras is a deformation quantization of the Poisson algebra of functions on $(\mathbb{C}^*)^{2d}$.

For a primitive root of unity ϵ of degree ℓ consider the specialization of the algebra $C_\epsilon((\mathbb{C}^*)^{2d})$ to $q = \epsilon$. It is clear that Laurent polynomials in t_i^ℓ are in the center of this algebra. Moreover, it is well known that they generate the center and that $C_\epsilon((\mathbb{C}^*)^{2d})$ is a Cayley-Hamilton algebra over its center of rank ℓ^d.

It is remarkable that the rank in this case coincides with the dimension of the space obtained by geometrical quantization with $n = \ell$ and that this dimension coincides with its asymptotic (12).

There are many examples of algebraic symplectic varieties which are birationally equivalent to a complex symplectic torus. It is natural to expect that if a sequence of Cayley-Hamilton algebras quantizes such symplectic variety in a certain regular way then ranks of such algebras will be n^d where d is half the dimension of the complex torus. Conversely, if there is a sequence of such Cayley-Hamilton algebras quantizing a Poisson variety then, one can take it as an indication that the variety is birationally equivalent to a symplectic torus.

7.6.2. Dimension of multiplicity spaces which we studied all have the form ℓ^d for some d. We want to make the following conjectures concerning the multiplicities.

1. Let $\mathcal{O} \subset G$ be a generic conjugation G-orbit in G and $\mathcal{O}_- \subset B_-$ be a generic dressing orbit of B_+ in B_- for the standard Poisson structure on B_-. For generic orbits we have $dim(\mathcal{O}) = 2|\Delta_+|$ and $dim(\mathcal{O}_+) = |\Delta_+| - dim(ker(w_0 - id))$ where $w_0 \in W$ is the longest element of the Weyl group.

Consider the Poisson structure on G which comes from G^* via the factorization map. Then $\mathcal{O}$ and $\mathcal{O}_-$ are symplectic leaves in G and B_- respectively.

The Lie group B_+ acts on $\mathcal{O}$ as a subgroup of G. This action and the dressing action on $\mathcal{O}_-$ are quasi-Hamiltonian, in a sense that there is an appropriate moment map [Lu]. Thus, the Lie group B_+ acts (locally) on $\mathcal{O} \times \mathcal{O}_-$ via the diagonal action and this action is quasi-Hamiltonian. Therefore, we can reduce this product via

Hamiltonian reduction and thus we obtain the symplectic variety

$$X(\mathcal{O}, \mathcal{O}_-) = \mathcal{O} \times \mathcal{O}_-///B_+$$

Here two dashes mean that we take the categorical quotient and one extra dash means that we do Hamiltonian reduction.

It is easy to see that $dim(X(\mathcal{O}, \mathcal{O}_-)) = |\Delta_+| - dim(coker(w_0 - id))$. Therefore the multiplicity of the restriction is $\ell^{\frac{dim(X(\mathcal{O},\mathcal{O}_-))}{2}}$.

c From all these considerations we arrive to a conjecture:

Conjecture 7.1. *There is a birational correspondence between the variety $(\mathcal{O} \times \mathcal{O}_-)///B_+$ described above and a symplectic torus of the same dimension.*

2. The dimension in the tensor product. Consider three conjugation G-orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \subset G$. Consider G as a Poisson variety with the Poisson structure inherited from the dual Poisson Lie group G^* via the factorization map. Then conjugation orbits are symplectic leaves and the natural action of G on them is quasi-Hamiltonian (i.e. there is an appropriate moment map). Therefore, via the diagonal action G acts on the product $\mathcal{O}_1 \times \mathcal{O}_2 \times \bar{\mathcal{O}}_3$ and this action is quasi-Hamiltonian. Here $\bar{\mathcal{O}}$ is the opposite symplectic variety to $\mathcal{O}$.

Consider the Hamiltonian reduction

$$X(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = (\mathcal{O}_1 \times \mathcal{O}_2 \times \bar{\mathcal{O}}_3)///G$$

It is a symplectic variety with $dim(X(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)) = |\Delta_+| - r$.

It is clear that the multiplicity of a generic irreducible module in the tensor product of two generic irreducible representations is $\ell^{dim(X(\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3))/2}$.

Conjecture 7.2. *There is a birational equivalence between the symplectic variety $(\mathcal{O}_1 \times \mathcal{O}_2 \times \bar{\mathcal{O}}_3)///G$ and a complex torus of the same dimension.*

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