

CHAP. 9 TENSOR SYMMETRY

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1 Symmetry in tensor spaces

With all the preliminary work done this will be now a short section, it serves as an introduction to the *first fundamental theorem* of invariant theory, according to the terminology of H.Weyl.

1.1 Intertwiners and invariants We have seen in Chap 1. 2.4 that, given two actions of a group G , an equivariant map is just an invariant under the action of G on maps.

For linear representations the action of G preserves the space of linear maps, so if U, V are 2 linear representations

$$\text{Hom}_G(U, V) = \text{Hom}(U, V)^G.$$

For finite dimensional representations, we have identified, in a G equivariant way

$$\text{Hom}(U, V) = U^* \otimes V = (U \otimes V^*)^*.$$

This last space is the space of bilinear functions on $U \times V^*$.

Explicitly a homomorphism $f : U \rightarrow V$ corresponds to the bilinear form

$$\langle f|u \otimes \varphi \rangle = \langle \varphi|f(u) \rangle.$$

We have thus a correspondence between intertwiners and invariants.

We will find it particularly useful, according to the Aronhold method, to use it when the representations are tensor powers $U = A^{\otimes m}$; $V = B^{\otimes p}$ and $\text{Hom}(U, V) = A^{*\otimes m} \otimes B^{\otimes p}$.

In particular when $A = B$; $m = p$ we have:

$$(1.1.1) \quad \text{End}(A^{\otimes m}) = \text{End}(A)^{\otimes m} = A^{*\otimes m} \otimes A^{\otimes m} = (A^{*\otimes m} \otimes A^{\otimes m})^*.$$

Thus in this case

Proposition. *We can identify, at least as vector spaces, the G endomorphisms of $A^{\otimes m}$ with the multilinear invariant functions on m variables in A and m variables in A^* .*

Let V be an m -dimensional space. On the tensor space $V^{\otimes n}$ we consider 2 group actions, one given by the linear group $GL(V)$ by the formula:

$$(1.1.2) \quad g(v_1 \otimes v_2 \otimes \dots \otimes v_n) := gv_1 \otimes gv_2 \otimes \dots \otimes gv_n,$$

and the other of the symmetric group S_n given by:

$$(1.1.3) \quad \sigma(v_1 \otimes v_2 \otimes \dots \otimes v_n) = v_{\sigma^{-1}1} \otimes v_{\sigma^{-1}2} \otimes \dots \otimes v_{\sigma^{-1}n}.$$

We will refer to this second action as to the *symmetry action* on tensors. By the very definition it is clear that these two actions commute.

Before we make any further analysis of these actions recall that in Chap. 5, §2.3 we have studied symmetric tensors. Let us recall the main points of that analysis. Given a vector $v \in V$ the tensor $v^n = v \otimes v \otimes v \dots \otimes v$ is symmetric.

Fixed a basis e_1, e_2, \dots, e_m in V the basis elements $e_{i_1} \otimes e_{i_2} \dots \otimes e_{i_n}$ are permuted by S_n and the orbits are classified by the multiplicities h_1, h_2, \dots, h_m with which the elements e_1, e_2, \dots, e_m appear in the term $e_{i_1} \otimes e_{i_2} \dots \otimes e_{i_n}$.

The sum of the elements of the corresponding orbit are a basis of the symmetric tensors. The multiplicities h_1, h_2, \dots, h_m are non negative integers subject only to $\sum_i h_i = n$.

If $\underline{h} := h_1, h_2, \dots, h_m$ is such a sequence we denote by $e_{\underline{h}}$ be the sum of elements in the corresponding orbit. The image of the symmetric tensor $e_{\underline{h}}$ in the symmetric algebra is

$$\binom{n}{h_1 h_2 \dots h_m} e_1^{h_1} e_2^{h_2} \dots e_m^{h_m}.$$

If $v = \sum_k x_k e_k$ we have:

$$v^n = \sum_{h_1+h_2+\dots+h_m=n} x_1^{h_1} x_2^{h_2} \dots x_m^{h_m} e_{\underline{h}}.$$

A linear function ϕ on the space of symmetric tensors is defined by $\langle \phi | e_{\underline{h}} \rangle = a_{\underline{h}}$ and computed on the tensor v^n gives:

$$\langle \phi | \left(\sum_k x_k e_k \right)^n \rangle = \sum_{h_1+h_2+\dots+h_m=n} x_1^{h_1} x_2^{h_2} \dots x_m^{h_m} a_{\underline{h}}.$$

We have that the dual of the space of symmetric tensors of degree n is identified with the space of homogeneous polynomials of degree n .

Let us recall that a subset $X \subset V$ is Zariski dense if the only polynomial vanishing on X is 0, a typical example that we will use is, when the base field is infinite, the set of vectors where a given polynomial is non zero (easy to verify).

Lemma. *i) The elements $v^{\otimes n}$, $v \in V$ span the space of symmetric tensors.*

ii) More generally, given a Zariski dense set $X \subset V$ the elements $v^{\otimes n}$, $v \in X$ span the space of symmetric tensors.

Proof. Given a linear form on the space of symmetric tensors we restrict it to the tensors $v^{\otimes n}$, $v \in X$ obtaining the values of a homogeneous polynomial on X . Since X is Zariski dense this polynomial vanishes if and only if the form is 0 hence the tensors $v^{\otimes n}$, $v \in X$ span the space of symmetric tensors. \square

Of course the use of the word symmetric is coherent to the general idea of *invariant under the symmetric group*.

1.2 Schur–Weyl duality We want to apply the general theory of semisimple algebras to the two group actions introduced in the previous section. It is convenient to introduce the two algebras of linear operators spanned by these actions; thus

- (1) We call A the span of the operators induced by $GL(V)$ in $End(V^{\otimes n})$.
- (2) We call B the span of the operators induced by S_n in $End(V^{\otimes n})$.

Our aim is to prove:

Proposition. *If V is a finite dimensional vector space over an infinite field of any characteristic B is the centralizer of A .*

Proof. We start by identifying:

$$End(V^{\otimes n}) = End(V)^{\otimes n}.$$

The decomposable tensor $A_1 \otimes A_2 \otimes \dots \otimes A_n$ corresponds to the operator:

$$A_1 \otimes A_2 \otimes \dots \otimes A_n(v_1 \otimes v_2 \otimes \dots \otimes v_n) = A_1 v_1 \otimes A_2 v_2 \otimes \dots \otimes A_n v_n.$$

Thus, if $g \in GL(V)$, the corresponding operator in $V^{\otimes n}$ is $g \otimes g \otimes \dots \otimes g$. From the Lemma 1.1 it follows that the algebra A coincides with the symmetric tensors in $End(V)^{\otimes n}$, since $GL(V)$ is Zariski dense.

It is thus sufficient to show that, for an operator in $End(V)^{\otimes n}$, the condition to commute with S_n is equivalent to be symmetric as a tensor.

It is sufficient to prove that the conjugation action of the symmetric group on $End(V^{\otimes n})$ coincides with the symmetry action on $End(V)^{\otimes n}$.

It is enough to verify the previous statement on decomposable tensors since they span the tensor space, thus we compute:

$$\begin{aligned} \sigma A_1 \otimes A_2 \otimes \dots \otimes A_n \sigma^{-1}(v_1 \otimes v_2 \otimes \dots \otimes v_n) &= \sigma A_1 \otimes A_2 \otimes \dots \otimes A_n(v_{\sigma 1} \otimes v_{\sigma 2} \otimes \dots \otimes v_{\sigma n}) = \\ \sigma(A_1 v_{\sigma 1} \otimes A_2 v_{\sigma 2} \otimes \dots \otimes A_n v_{\sigma n}) &= A_{\sigma^{-1} 1} v_1 \otimes A_{\sigma^{-1} 2} v_2 \dots A_{\sigma^{-1} n} v_n = \\ &= (A_{\sigma^{-1} 1} \otimes A_{\sigma^{-1} 2} \dots A_{\sigma^{-1} n})(v_1 \otimes v_2 \otimes \dots \otimes v_n). \end{aligned}$$

This computation shows that the conjugation action is in fact the symmetry action and finishes the proof. \square

We draw now a main conclusion:

Theorem. *If the characteristic of F is 0, the algebras A, B are semisimple and each is the centralizer of the other.*

Proof. Since B is the span of the operators of a finite group it is semisimple by Maschke's theorem (Chap 6. 1.5), therefore, by the double centralizer Theorem (Chap. 6, Theorem 2.5) all statements follow from the previous Theorem which states that A is the centralizer of B . \square

REMARK If the characteristic of the field F is not 0, in general the algebras A, B are not semisimple, nevertheless it is still true (at least if F is infinite or big enough) that each is the centralizer of the other (cf. Chap. 13, Theorem 7.1).

1.3 Invariants of vectors We formulate Theorem 1.2 in a different language.

Given two vector spaces V, W we have identified $\text{hom}(V, W)$ with $W \otimes V^*$ and with the space of bilinear functions on $W^* \times V$ by the formulas ($A \in \text{hom}(V, W)$, $\alpha \in W^*$, $v \in V$):

$$(1.3.1) \quad \langle \alpha | Av \rangle .$$

In case V, W are linear representations of a group G , A is in $\text{hom}_G(V, W)$ if and only if the bilinear function $\langle \alpha | Av \rangle$ is G invariant.

In particular we see that for a linear representation V the space of G linear endomorphisms of $V^{\otimes n}$ is identified to the space of multilinear functions of n covector² and n vector variables $f(\alpha_1, \alpha_2, \dots, \alpha_n, v_1, v_2, \dots, v_n)$ which are G invariant.

Let us see the meaning of this for $G = GL(V)$, V an m dimensional vector space. In this case we know that the space of G endomorphisms of $V^{\otimes n}$ is spanned by the symmetric group S_n , we want to see which invariant function f_σ corresponds to a permutation σ . By the formula 1.3.1 evaluated on decomposable tensors we get

$$\begin{aligned} f_\sigma(\alpha_1, \alpha_2, \dots, \alpha_n, v_1, v_2, \dots, v_n) &= \langle \alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n | \sigma(v_1 \otimes v_2 \otimes \dots \otimes v_n) \rangle = \\ &= \langle \alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n | v_{\sigma^{-1}1} \otimes v_{\sigma^{-1}2} \otimes \dots \otimes v_{\sigma^{-1}n} \rangle = \prod_{i=1}^n \langle \alpha_i | v_{\sigma^{-1}i} \rangle = \prod_{i=1}^n \langle \alpha_{\sigma i} | v_i \rangle \end{aligned}$$

We can thus deduce:

Proposition. *The space of $GL(V)$ invariant multilinear functions of n covector and n vector variables is spanned by the functions:*

$$(1.3.2) \quad f_\sigma(\alpha_1, \alpha_2, \dots, \alpha_n, v_1, v_2, \dots, v_n) := \prod_{i=1}^n \langle \alpha_{\sigma i} | v_i \rangle .$$

1.4 First fundamental theorem for the linear group (FFT) Up to now we have made no claim on the linear dependence or independence of the operators in S_n or of the corresponding functions f_σ , this will be analyzed in the chapter 13, §8.

²covector means linear form

We want to drop now the restriction that the invariants be multilinear.

Take the space $(V^*)^p \times V^q$ of p covector and q vector variables as representation of $GL(V)$ ($\dim(V) = m$). A typical element is a sequence $(\alpha_1, \alpha_2, \dots, \alpha_p, v_1, v_2, \dots, v_q)$, $\alpha_i \in V^*$, $v_j \in V$.

On this space consider the pq polynomial functions $\langle \alpha_i | v_j \rangle$ which are clearly $GL(V)$ invariant, we prove:³

FFT First fundamental theorem for the linear group.

*The ring of polynomial functions on $V^{*p} \times V^q$ which are $GL(V)$ invariant is generated by the functions $\langle \alpha_i | v_j \rangle$.*

Before starting to prove this theorem we want to make some remarks on its meaning.

Fix a basis of V and its dual basis in V^* , with these bases V is identified with the set of m dimensional column vectors and V^* to the space of m dimensional row vectors.

The group $GL(V)$ is then identified to the group $Gl(m, \mathbb{C})$ of $m \times m$ invertible matrices.

Its action on column vectors is the product Av , $A \in Gl(m, \mathbb{C})$, $v \in V$ while on the row vectors the action is by αA^{-1} .

The invariant function $\langle \alpha_i | v_j \rangle$ is then identified to the product of the row vector α_i with the column vector v_j .

In other words identify the space $(V^*)^p$ of p -tuples of row vectors to the space of $p \times m$ matrices (in which the p rows are the coordinates of the covectors) and (V^q) with the space of $m \times q$ matrices, thus our representation is identified to the space of pairs:

$$(X, Y) | X \in M_{p,m}, Y \in M_{m,q}.$$

The action of the matrix group is by:

$$A(X, Y) := (XA^{-1}, AY).$$

Consider the multiplication map:

$$(1.4.1) \quad f : M_{p,m} \times M_{m,q} \rightarrow M_{p,q}, \quad f(X, Y) := XY$$

the entries of the matrix XY are the basic invariants $\langle \alpha_i | v_j \rangle$ thus the theorem can also be formulated as:

Theorem. *The ring of polynomial functions on $M_{p,m} \times M_{m,q}$ which are $Gl(m, \mathbb{C})$ invariant is given by the polynomial functions on $M_{p,q}$ composed with the map f .*

Proof. We will now prove the theorem in its first form by the Aronhold method.

Let $g(\alpha_1, \alpha_2, \dots, \alpha_p, v_1, v_2, \dots, v_q)$ be a polynomial invariant, without loss of generality we may assume that it is homogeneous in each of its variables, then we polarize it with respect to each of its variables and obtain a new multilinear invariant of the

³at this moment we are in characteristic 0, but in Chapter 13 we will generalize our results to all characteristics.

form $\bar{g}(\alpha_1, \alpha_2, \dots, \alpha_N, v_1, v_2, \dots, v_M)$ where N, M are the total degrees of g in the α, v respectively.

First we show that $N = M$. In fact, among the elements of the linear group we have scalar matrices, given a scalar λ by definition it transforms v in λv and α in $\lambda^{-1}\alpha$ and thus, by the multilinearity hypothesis, it transforms the function \bar{g} in $\lambda^{M-N}\bar{g}$, the invariance condition implies $M = N$.

We can now apply Proposition 1.3 and deduce that \bar{g} is a linear combination of functions of type $\prod_{i=1}^N \langle \alpha_{\sigma_i} | v_i \rangle$.

We now apply restitution to compute g from \bar{g} . It is clear that g has the desired form. \square

The study of the relations among invariants will be the topic of the **second fundamental theorem, SFT**. Here we only remark that, by elementary linear algebra, the multiplication map f has, as image, the subvariety $D_{p,q}(m)$ of $p \times q$ matrices of rank $\leq m$. This is the whole space if $m \geq \min(p, q)$ otherwise it is a proper subvariety, called a *determinantal variety* defined, at least set theoretically, by the vanishing of the determinants of the $m+1 \times m+1$ minors of the matrix of coordinate functions x_{ij} on $M_{p,q}$.

It will be the content of the second fundamental theorem to prove that these determinants generate a prime ideal which is thus the full ideal of relations among the invariants $\langle \alpha_i | v_j \rangle$.

In fact it is even better to introduce a formal language, suppose that V is an affine algebraic variety with the action of an algebraic group G . Suppose that $p: V \rightarrow W$ is a morphism of affine varieties, inducing the comorphism $p^*: k[W] \rightarrow k[V]$.

Definition. We say that $p: V \rightarrow W$ is a quotient under G and write $W := V//G$ if p^* is an isomorphism of $k[W]$ to the ring of invariants $k[V]^G$.

Thus the FFT says in this geometric language, that the determinantal variety $D_{p,q}(m)$ is the quotient under $GL(m, \mathbb{C})$ of $(V^*)^{\oplus p} \oplus V^{\oplus q}$.

2 Young symmetrizers

2.1 Young diagrams We discuss now the symmetric group. The theory of cycles (cf. Chap. 1, §2.2) implies that the conjugacy classes of S_n are in one to one correspondence with the isomorphism classes of \mathbb{Z} actions on $[1, 2, \dots, n]$ and these are parameterized by partitions of n .

As in Chap. 1, we express that $\mu := k_1, k_2, \dots, k_n$ is a partition of n by $\mu \vdash n$.

We shall denote by $C(\mu)$ the conjugacy class in S_n formed by the permutations decomposed in cycles of length k_1, k_2, \dots, k_n , hence $S_n = \sqcup_{\mu \vdash n} C(\mu)$.

Consider the group algebra $R := \mathbb{Q}[S_n]$ of the symmetric group, we wish to work over \mathbb{Q} since the theory has really this more arithmetic flavor. We will (implicitly) exhibit a

decomposition as a direct sum of matrix algebras over \mathbb{Q} :⁴

$$(2.1.1) \quad R = \mathbb{Q}[S_n] := \bigoplus_{\mu \vdash n} M_{d(\mu)}(\mathbb{Q}),$$

The numbers $d(\mu)$ will be computed in several ways from the partition μ .

Recall that, from the theory of group characters we know at least that

$$R_{\mathbb{C}} := \mathbb{C}[S_n] := \sum_i M_{n_i}(\mathbb{C}),$$

where the number of summands is equal to the number of conjugacy classes, hence the number of partitions of n . For every partition $\lambda \vdash n$ we will construct a primitive idempotent e_λ in R so that $R = \bigoplus_{\lambda \vdash n} R e_\lambda R$ and $\dim_{\mathbb{Q}} e_\lambda R e_\lambda = 1$.

In this way the left ideals $R e_\lambda$ will exhaust all irreducible representations. The description of 2.1.1 then follows from Chap. 6, Theorem 3.1, part 5).

In fact we will construct idempotents e_λ , $\lambda \vdash n$ so that $\dim_{\mathbb{Q}} e_\lambda R e_\lambda = 1$ and $e_\lambda R e_\mu = 0$ if $\lambda \neq \mu$. By the previous results we have that R contains a direct summand of the form $\bigoplus_{\mu \vdash n} M_{n(\mu)}(\mathbb{Q})$, or $R = \bigoplus_{\mu \vdash n} M_{n(\mu)}(\mathbb{Q}) \oplus R'$. We claim that $R' = 0$ in fact otherwise, once we complexify, the algebra $R_{\mathbb{C}} = \bigoplus_{\mu \vdash n} M_{n(\mu)}(\mathbb{C}) \oplus R'_{\mathbb{C}}$ would contain more simple summands than the number of partitions of n , a contradiction.

For a partition $\lambda \vdash n$ let B be the corresponding Young diagram, formed by n boxes which are partitioned in rows or in columns, the intersection between a row and a column is either empty or it reduces to a single box.

In a more formal language consider the set $\mathbb{N}^+ \times \mathbb{N}^+$ of pairs of positive integers. For a pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ set $C_{i,j} := \{(h, k) | 1 \leq h \leq i, 1 \leq k \leq j\}$ (this is a *rectangle*).

These rectangular sets have the following simple but useful properties:

- (1) $C_{i,j} \subset C_{h,k}$ if and only if $(i, j) \in C_{h,k}$.
- (2) If a rectangle is contained in the union of rectangles then it is contained in one of them.

Definition. A *Young diagram* is a subset of $\mathbb{N}^+ \times \mathbb{N}^+$ finite union of rectangles $C_{i,j}$.

In the literature this particular way of representing a Young diagram is also called a Ferrer's diagram, sometimes we will use this expression when we want to stress the formal point of view.

There are two conventional ways to display a Young diagram (sometimes referred to as the French and the English way) either as points in the first quadrant or in the fourth:

Example: partition 4311:

French	$\begin{array}{cccc} & & & \cdot \\ & & & \cdot \\ & \cdot & & \cdot \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{array}$	English	$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & & & \end{array}$
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⁴so in this case all the division algebras coincide with \mathbb{Q} .

Any Young diagram can be written uniquely as a union of sets $C_{i,j}$ so that no rectangle in this union can be removed, the corresponding elements (i, j) will be called the *vertices* of the diagram.

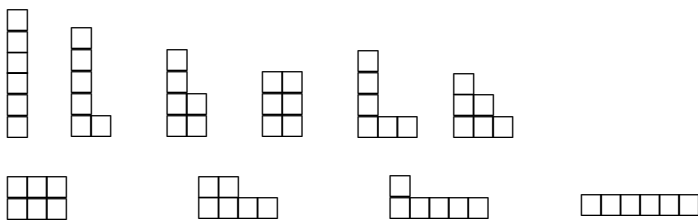
Given a Young diagram D (in French form) the set $C_i := \{(i, j) \in D\}$, i fixed, will be called the i^{th} *column*, the set $R_j := \{(i, j) \in D\}$, j fixed, will be called the j^{th} *row*.

The lengths k_1, k_2, k_3, \dots of the rows are a decreasing sequence of numbers which determine completely the diagrams, thus we can identify the set of diagrams with n boxes with the set of partitions of n , this partition is called **the row shape** of the diagram.

Of course we could also have used the column lengths and the so called *dual* partition which is **the column shape** of the diagram.

The map that to a partition associates its dual is an involutory map which geometrically can be visualized as flipping the Ferrer's diagram around its diagonal.

The elements (h, k) in a diagram will be called *boxes* and displayed more pictorially as (e.g. diagrams with 6 boxes, french display):



2.2 Symmetrizers

Definition 1. A bijective map from the set of boxes to the interval $(1, 2, 3, \dots, n-1, n)$ is called a *tableau*. It can be thought as a filling of the diagram with numbers. The given partition λ is called the *shape* of the tableau.

Example: partition 4311^5 :

$$\text{French} \begin{array}{cccc} & 3 & & 7 \\ 1 & & & 4 \\ 5 & 2 & 7 & \\ 4 & 9 & 6 & 8 \end{array}, \begin{array}{cccc} & & & \\ & & & \\ 3 & 6 & 8 & \\ 1 & 2 & 5 & 9 \end{array}$$

The symmetric group S_n acts on tableaux by composition:

$$\sigma T : B \xrightarrow{T} (1, 2, 3, \dots, n-1, n) \xrightarrow{\sigma} (1, 2, 3, \dots, n-1, n).$$

A tableau induces two partitions on $(1, 2, 3, \dots, n-1, n)$.

The *row partition* defined by:

i, j are in the same part if they appear in the same row of T .

⁵the reader will notice the peculiar properties of the right tableau which we will encounter over and over in the future

Similarly for the *column partition*.

To a partition π of $(1, 2, 3, \dots, n-1, n)$ ⁶ one associates the subgroup S_π of the symmetric group of permutations which preserve the partition. It is isomorphic to the product of the symmetric groups of all the parts of the partition. To a tableau T one associates two subgroups R_T, C_T of S_n .

- (1) R_T is the group preserving the row partition.
- (2) C_T the subgroup preserving the column partition.

It is clear that $R_T \cap C_T = 1$ since each box is an intersection of a row and a column.

Notice that, if $s \in S_n$, the row and column partitions associated to sT are obtained applying s to the corresponding partitions of T thus:

$$(2.2.1) \quad R_{sT} = sR_Ts^{-1}, \quad C_{sT} = sC_Ts^{-1}.$$

We define two elements in $R = \mathbb{Q}[S_n]$:

$$(2.2.2) \quad s_T = \sum_{\sigma \in R_T} \sigma \quad \text{the symmetrizer on the rows}$$

$$a_T = \sum_{\sigma \in C_T} \epsilon_\sigma \sigma \quad \text{the antisymmetrizer on the columns.}$$

Recall that ϵ_σ denotes the sign of the permutation. The two identities:

$$s_T^2 = \prod_i h_i! s_T, \quad a_T^2 = \prod_i k_i! a_T$$

where the h_i are the lengths of the rows and k_i the length of the columns, are clear.

It is better to get acquainted with these two elements from which we will build our main object of interest.

$$(2.2.3) \quad ps_T = s_T p, \quad \forall p \in R_T; \quad qa_T = a_T q = \epsilon_q a_T, \quad \forall q \in C_T.$$

It is then an easy exercise to check

Proposition. *The left ideal $\mathbb{Q}[S_n]s_T$ has as basis the elements gs_T as g runs over a set of representatives of the cosets gR_T and it equals, as representation, the permutation representation on such cosets.*

The left ideal $\mathbb{Q}[S_n]a_T$ has as basis the elements ga_T as g runs over a set representatives of the cosets gC_T and it equals, as representation, the representation induced to S_n by the sign representation of C_T .

Now the remarkable fact comes, consider the product:

$$(2.2.4) \quad c_T := s_T a_T = \sum_{p \in R_T, q \in C_T} \epsilon_q pq$$

we will show that:

⁶there is an ambiguity in the use of the word *partition*. A partition of n is just a non increasing sequence of numbers adding to n , while a partition of a set is in fact a decomposition of the set into disjoint parts.

Theorem. *There exists a positive integer $p(T)$ such that the element $e_T := \frac{c_T}{p(T)}$ is a primitive idempotent.*

Definition 2. *The idempotent $e_T := \frac{c_T}{p(T)}$ is called the Young symmetrizer relative to the given tableau.*

Remark

$$(2.2.5) \quad c_{sT} = sc_T s^{-1}.$$

We have thus, for a given $\lambda \vdash n$ several conjugate idempotents which, we will show to be primitive, associated to tableaux of row shape λ . Each of them will generate an irreducible module associated to λ which will be denoted by M_λ .

For the moment let us remark that, from 2.2.4 follows that the integer $p(T)$ depends only on the shape λ of T and thus we will denote it by $p(T) = p(\lambda)$.

2.3 The main Lemma The main property of the element c_T which we will exploit is the following, clear from its definition and 2.2.2.

$$(2.3.1) \quad pc_T = c_T, \quad \forall p \in R_T; \quad c_T q = \epsilon_q c_T, \quad \forall q \in C_T.$$

We need a fundamental combinatorial lemma, consider the partitions of n as decreasing sequences of integers (including 0) and order them lexicographically.⁷

E.g. the partitions of 6 in increasing lexicographic order:

$$111111, 21111, 2211, 222, 3111, 321, 411, 42, 51, 6.$$

Lemma. *Let S, T be two tableaux of row shapes*

$$\lambda = h_1 \geq h_2 \geq \dots \geq h_n, \quad \mu = k_1 \geq k_2 \geq \dots \geq k_n$$

with $\lambda \geq \mu$, then one and only one of the two following possibilities holds:

- i) Two numbers i, j appear in the same row in S and in the same column in T .*
- ii) $\lambda = \mu$ and $pS = qT$ where $p \in R_S, q \in C_T$.*

Proof. We consider the first row r_1 of S . Since $h_1 \geq k_1$, by the pigeon hole principle either there are two numbers in r_1 which are in the same column in T or $h_1 = k_1$ and we can act on T with a permutation s in C_T so that S and sT have the first row filled with the same elements (possibly in a different order).

Remark that two numbers appear in the same column in T if and only if they appear in the same column in sT or $C_T = C_{sT}$.

We now remove the first row in both S, T and proceed as before. At the end we are either in case i) or $\lambda = \mu$ and we have found a permutation $q \in C_T$ such that S and qT have each row filled with the same elements.

⁷We often drop 0 in the display.

In this case we can find a permutation $p \in R_S$ such that $pS = qT$.

In order to complete our claim we need to show that these two cases are mutually exclusive. Thus we have to remark that, if $pS = qT$ as before then case i) is not verified. In fact two elements are in the same row in S if and only if they are in the same row in pS while they appear in the same column in T if and only if they appear in the same column in qT . Since $pS = qT$ two elements in the same row of pS are in different columns of qT . \square

Corollary. *i) Given $\lambda > \mu$ partitions, S, T tableaux of row shapes λ, μ respectively, and s any permutation, there exists a transposition $u \in R_S$ and a transposition $v \in C_T$ such that $us = sv$.*

ii) If, for a tableau T , s is a permutation not in $R_T C_T$ then there exists a transposition $u \in R_T$ and a transposition $v \in C_T$ such that $us = sv$.

Proof. i) From the previous lemma there are two numbers i, j in the same row for S and in the same column for sT . If $u = (i, j)$ is the corresponding transposition we have $u \in R_S$, $u \in C_{sT}$. We set $v := s^{-1}us$, we have $v \in s^{-1}C_{sT}s = C_T$ by 2.2.1. By definition $sv = uv$.

ii) The proof is similar, we consider again a tableau T , construct $s^{-1}T$ and apply the Lemma to $s^{-1}T, T$.

If there exists a $p' \in R_{s^{-1}T}$, $q \in C_T$ with $p's^{-1}T = qT$ since $p' = s^{-1}ps$, $p \in R_T$ we would have that $s^{-1}p = q$, $s = pq^{-1}$ against the hypothesis. Hence there is a transposition $v \in C_T$ and $v \in R_{s^{-1}T}$ or $v = s^{-1}us$, $u \in R_T$ as required. \square

2.4 Young symmetrizers 2 We draw now the conclusions relative to Young symmetrizers.

Proposition. *i) Let S, T be two tableaux of row shapes $\lambda > \mu$.*

If an element a in the group algebra is such that:

$$pa = a, \forall p \in R_S, \text{ and } aq = \epsilon_q a, \forall q \in C_T$$

then $a = 0$.

ii) Given a tableau T and an element a in the group algebra such that:

$$pa = a, \forall p \in R_T, \text{ and } aq = \epsilon_q a, \forall q \in C_T$$

then a is a scalar multiple of the element c_T .

Proof. i) Let us write $a = \sum_{s \in S_n} a(s)s$, for any given s we can find u, v as in the previous lemma.

By hypothesis $ua = a, av = -a$ then $a(s) = a(us) = a(sv) = -a(s) = 0$ and thus $a = 0$.

ii) Using the same argument as above, we can say that, if $s \notin R_T C_T$ then $a(s) = 0$.

Instead let $s = pq$, $p \in R_T, q \in C_T$ then $a(pq) = \epsilon_q a(1)$ hence $a = a(1)c_T$. \square

Before we conclude let us recall some simple facts on algebras and group algebras.

If R is a finite dimensional algebra over a field F we can consider any element $r \in R$ as a linear operator on R (as vector space) by right or left action. Let us define $tr(r)$ to be the trace of the operator $x \rightarrow xr$,⁸ clearly $tr(1) = \dim_F R$. For a group algebra $F[G]$ of a finite group G an element $g \in G$, $g \neq 1$ gives rise to a permutation $x \rightarrow xg$, $x \in G$ of the basis elements without fixed points, hence: $tr(1) = |G|$, $tr(g) = 0$ if $g \neq 1$.

We are now ready to conclude. For $R = \mathbb{Q}[S_n]$ the theorems which we aim at are:

Theorem 1. *i) $c_T R c_T = c_T R a_T = s_T R c_T = s_T R a_T = \mathbb{Q} c_T$.*

ii) $c_T^2 = p(\lambda)c_T$ with $p(\lambda) \neq 0$ a positive integer.

iii) $\dim_{\mathbb{Q}} R c_T = \frac{n!}{p(\lambda)}$.

iv) If U, V are two tableaux of shapes $\lambda > \mu$ then $s_U R a_V = a_V R s_U = 0$.

v) If U, V are tableaux of different shapes λ, μ we have $c_U R c_V = 0 = s_U R a_V$.

Proof. i) We cannot have $c_T R c_T = 0$ since R is semisimple, hence it is enough to prove $s_T R a_T = \mathbb{Q} c_T$. We apply the previous proposition and get that every element of $s_T R a_T$ satisfies ii) of that proposition, hence $s_T R a_T = \mathbb{Q} c_T$.

ii) In particular we have $c_T^2 = p(\lambda)c_T$. Now compute the trace of c_T , for the right regular representation, from the previous discussion we have $tr(c_T) = n!$ hence $c_T^2 \neq 0$. Since $c_T^2 = p(\lambda)c_T$ we have that $p(\lambda) \neq 0$.

iii) $e_T := \frac{c_T}{p(\lambda)}$ is idempotent and $\frac{n!}{p(\lambda)} = \frac{tr(c_T)}{p(\lambda)} = tr(e_T)$. The trace of an idempotent operator is the dimension of its image. In our case $R e_T = R c_T$, hence $\frac{n!}{p(\lambda)} = \dim_{\mathbb{Q}} R c_T$.

iv) If $\lambda > \mu$ we have, by part i) of the same proposition, $s_U R a_V = 0$.

v) If $\lambda > \mu$ we have, by iv), $c_U R c_V = s_U a_U R s_V a_V \subset s_U R a_V = 0$. Otherwise $c_V R c_U = 0$, which since R has no nilpotent ideals, implies $c_U R c_V = 0$ (Chap. 6, 3.1). \square

From the general discussion performed in 2.1 we finally obtain:

Theorem 2. *i) The elements $e_T := \frac{c_T}{p(\lambda)}$ are primitive idempotents in $R = \mathbb{Q}[S_n]$.*

ii) The left ideals $R e_T$ give all the irreducible representations of S_n explicitly indexed by partitions.

iii) These representations are defined over \mathbb{Q} .

We will indicate by M_λ the irreducible representation associated to a (row-)partition λ .

Remark The Young symmetrizer a priori does not depend only on the partition λ but also on the labeling of the diagram, but to two different labelings we obtain conjugate Young symmetrizers which therefore correspond to isomorphic irreducible representations.

⁸prove in fact that also the operator $x \rightarrow rx$ has the same trace

We could have used, instead of the product $s_T a_T$, the product $a_T s_T$ in reverse order. We claim that also in this way we obtain a primitive idempotent $\frac{a_T s_T}{p(\lambda)}$, relative to the same irreducible representation.

The same proof could be applied but also we can argue applying the antiautomorphism $a \rightarrow \bar{a}$ of the group algebra which sends a permutation σ to σ^{-1} . Clearly:

$$\bar{a}_T = a_T, \bar{s}_T = s_T, \overline{s_T a_T} = a_T s_T.$$

Thus $\frac{1}{p(T)} a_T s_T = \bar{e}_T$ is a primitive idempotent.

Since clearly $c_T a_T s_T = s_T a_T a_T s_T$ is non zero (a_T^2 is a non zero multiple of a_T and so $(c_T a_T s_T) a_T$ is a non zero multiple of c_T^2) we get that e_T and \bar{e}_T are primitive idempotents relative to the same irreducible representation and the claim is proved.

We will need in the computation of the characters of the symmetric group two more remarks.

Consider the two left ideals Rs_T, Ra_T , we have given a first description of their structure as representations in 2.2. They contain respectively $a_T Rs_T, s_T Ra_T$ which are both 1 dimensional. Thus we have

Lemma. M_λ appears in its isotypic component in Rs_T , resp Ra_T with multiplicity 1. If M_μ appears in Rs_T then $\mu \leq \lambda$ and if it appears in Ra_T then $\mu \geq \lambda$.⁹

Proof. To see the multiplicity with which M_μ appears in a representation V it suffices to compute the dimension of $c_T V$ or of $\bar{c}_T V$ where T is a tableau of shape μ . Therefore the statement follows from the previous results. \square

In particular we see that the only irreducible representation which appears in both Rs_T, Ra_T is M_λ .

2.5 Duality There are several deeper informations on the representation theory of the symmetric group of which we will describe some.

A first remark is about an obvious duality between diagrams.

Given a tableau T relative to a partition λ we can exchange its rows and columns obtaining a new tableau \tilde{T} relative to the partition $\tilde{\lambda}$ which in general is different from λ . It is thus natural to ask in which way are the two representations tied.

Let $\mathbb{Q}(\epsilon)$ denote the sign representation.

Proposition. $M_{\tilde{\lambda}} = M_\lambda \otimes \mathbb{Q}(\epsilon)$.

Proof. Consider the automorphism τ of the group algebra defined on the group elements by $\tau(\sigma) := \epsilon_\sigma \sigma$.

⁹we shall prove a more precise theorem later

Clearly, given a representation ϱ the composition $\varrho\tau$ equals to the tensor product with the sign representation; thus, if we apply τ to a primitive idempotent associated to M_λ we obtain a primitive idempotent for $M_{\tilde{\lambda}}$.

Let us thus use a tableau T of shape λ and construct the symmetrizer, we have

$$\tau(c_T) = \sum_{p \in R_T, q \in C_T} \epsilon_p \tau(pq) = \left(\sum_{p \in R_T} \epsilon_p p \right) \left(\sum_{q \in C_T} q \right).$$

We remark now that, since $\tilde{\lambda}$ is obtained from λ by exchanging rows and columns we have:

$$R_T = C_{\tilde{T}}, \quad C_T = R_{\tilde{T}}.$$

thus $\tau(c_T) = a_{\tilde{T}} s_{\tilde{T}} = \bar{c}_{\tilde{T}}$, hence $\tau(e_T) = \bar{e}_{\tilde{T}}$. \square

Remark. From the previous result follows also that $p(\lambda) = p(\tilde{\lambda})$.

3 The irreducible representations of the linear group 1

3.1 Representations of the linear groups We apply now the theory of symmetrizers to the linear group.

Let M be a representation of a semisimple algebra A , B its centralizer. By the structure theorem (Chap. 6) $M = \bigoplus N_i \otimes_{\Delta_i} P_i$ with N_i, P_i irreducible representations respectively of A, B . If $e \in B$ is a primitive idempotent then the subspace $eP_i \neq 0$ for a unique index i_0 and $eM = N_{i_0} \otimes eP_{i_0} \cong N_{i_0}$ is irreducible as representation of A (associated to the irreducible representation of B relative to e).

Thus, from Theorem 2 of 2.4, to get a list of the irreducible representations of the linear group $Gl(V)$ appearing in $V^{\otimes n}$, we may apply to tensor space the Young symmetrizers e_T and see when is $e_T V^{\otimes n} \neq 0$.

Assume we have t columns of length n_1, n_2, \dots, n_t , and decompose the column preserving group C_T as a product $\prod_{i=1}^t S_{n_i}$ of the symmetric groups of all columns.

By definition we get $a_T = \prod a_{n_i}$, the product of the antisymmetrizers relative to the various symmetric groups of the columns.

Let us assume, for simplicity of notations, that the first n_1 indices appear in the first column in increasing order, the next n_2 indices in the second column and so on, so that:

$$V^{\otimes n} = V^{\otimes n_1} \otimes V^{\otimes n_2} \otimes \dots \otimes V^{\otimes n_t},$$

$$a_T V^{\otimes n} = a_{n_1} V^{\otimes n_1} \otimes a_{n_2} V^{\otimes n_2} \otimes \dots \otimes a_{n_t} V^{\otimes n_t} = \wedge^{n_1} V \otimes \wedge^{n_2} V \otimes \dots \otimes \wedge^{n_t} V.$$

Therefore we have that, if there is a column of length $> \dim(V)$, then $a_T V^{\otimes n} = 0$.

Otherwise we have $n_i \leq \dim(V), \forall i$, and we prove the equivalent statement that $a_T s_T V^{\otimes n} \neq 0$. Let e_1, e_2, \dots, e_m be a basis of V and use for $V^{\otimes n}$ the corresponding basis of decomposable tensors; let us consider the tensor:

$$(3.1.1) \quad U = (e_1 \otimes e_2 \otimes \dots \otimes e_{n_1}) \otimes (e_1 \otimes e_2 \otimes \dots \otimes e_{n_2}) \otimes \dots \otimes (e_1 \otimes e_2 \otimes \dots \otimes e_{n_t}).$$

This is the decomposable tensor having e_i in the positions relative to the indices of the i^{th} row. By construction it is symmetric with respect to the group R_T of row preserving permutations hence $s_T U = pU$, $p \neq 0$.

Finally

$$(3.1.2) \quad a_T U = (e_1 \wedge e_2 \wedge \dots \wedge e_{n_1}) \otimes (e_1 \wedge e_2 \wedge \dots \wedge e_{n_2}) \otimes \dots \otimes (e_1 \wedge e_2 \wedge \dots \wedge e_{n_t}) \neq 0.$$

Recall that the length of the first column of a partition λ (equal to the number of its rows) is called the **height** of λ and indicated by $ht(\lambda)$. We have thus proved:

Proposition. *If T is a tableau of shape λ then $e_T V^{\otimes n} = 0$ if and only if $ht(\lambda) > \dim(V)$.*

For a tableau T of shape λ define:

$$(3.1.3) \quad S_\lambda(V) := e_T V^{\otimes n}, \quad \text{the Schur functor associated to } \lambda.$$

We are implicitly using the fact that for two different tableaux T, T' of the same shape we have a unique permutation σ with $\sigma(T) = T'$. Hence we have a canonical isomorphism between the two spaces $e_T V^{\otimes n}, e_{T'} V^{\otimes n}$.

Remark We shall justify the word *functor* in 7.1.

As a consequence, we thus have a description of $V^{\otimes n}$ as representation of $S_n \times GL(V)$.

Theorem.

$$(3.1.4) \quad V^{\otimes n} = \bigoplus_{ht(\lambda) \leq \dim(V)} M_\lambda \otimes S_\lambda(V)$$

Proof. We know that the two algebras A, B spanned by the linear and the symmetric group are semisimple and each the centralizer of the other. By the structure theorem we have thus $V^{\otimes n} = \bigoplus_i M_i \otimes S_i$ where the M_i are the irreducible representations of S_n which appear. We have proved that the ones which appear are the M_λ , $ht(\lambda) \leq \dim(V)$ and that $S_\lambda(V)$ is the corresponding irreducible representation of the linear group.

4 Characters of the symmetric group

As one can easily imagine the character theory of the symmetric and general linear group are intimately tied together. There are basically two approaches, a combinatorial approach due to Frobenius, computes first the characters of the symmetric group and then deduces those of the linear group, and an analytic approach, based on Weyl's character formula which proceeds in the reverse order. It is instructive to see both. There is in fact also a more recent algebraic approach to Weyl's character formula which we will not discuss (cf. [Hu]).

4.1 Character table Up to now we have been able to explicitly parameterize both the conjugacy classes and the irreducible representations of S_n by partitions of n . A way to present a partition is to give the number of times that each number i appears.

If i appears k_i times in a partition μ , the partition is indicated by:

$$(4.1.1) \quad \mu := 1^{k_1} 2^{k_2} 3^{k_3} \dots i^{k_i} \dots$$

Let us indicate by

$$(4.1.2) \quad a(\mu) := k_1! k_2! k_3! \dots k_i! \dots, \quad b(\mu) := 1^{k_1} 2^{k_2} 3^{k_3} \dots i^{k_i} \dots$$

$$(4.1.3) \quad n(\mu) = a(\mu)b(\mu) := k_1! 1^{k_1} k_2! 2^{k_2} k_3! 3^{k_3} \dots k_i! i^{k_i} \dots$$

We need to interpret now the number $n(\mu)$ in terms of the conjugacy class $C(\mu)$:

Proposition. *If $s \in C(\mu)$, $n(\mu)$ is the order of the centralizer G_s of s and $|C(\mu)|n(\mu) = n!$.*

Proof. Let us write the permutation s as a product of a list of cycles c_i . If g centralizes s we have that the cycles $gc_i g^{-1}$ are a permutation of the given list of cycles.

It is clear that in this way we get all possible permutations of the cycles of equal length.

Thus we have a surjective homomorphism of G_s to a product of symmetric groups $\prod S_{k_i}$, its kernel H is formed by permutations which fix each cycle.

A permutation of this type is just a product of permutations, each on the set of indices appearing in the corresponding cycle, and fixing it.

For a full cycle the centralizer is the cyclic group generated by the cycle, so H is a product of cyclic groups of order the length of each cycle. The formula follows. \square

The computation of the character table of S_n consists, given two partitions λ, μ , to compute the value of the character of an element of the conjugacy class $C(\mu)$ on the irreducible representation M_λ . Let us denote by $\chi_\lambda(\mu)$ this value.

The final result of this analysis is expressed in compact form through symmetric functions. Recall that we denote $\psi_k(x) = \sum_{i=1}^n x_i^k$. For a partition $\mu \vdash n := k_1, k_2, \dots, k_n$ set:

$$\psi_\mu(x) := \psi_{k_1}(x) \psi_{k_2}(x) \dots \psi_{k_n}(x).$$

Using the fact that the Schur functions are an integral basis of the symmetric functions there exist (unique) integers $c_\lambda(\mu)$ for which:

$$(4.1.4) \quad \psi_\mu(x) = \sum_{\lambda} c_\lambda(\mu) S_\lambda(x).$$

We interpret these numbers as class functions c_λ on the symmetric group

$$c_\lambda(C(\mu)) := c_\lambda(\mu)$$

and have.

Theorem Frobenius. *For all partitions $\lambda, \mu \vdash n$ we have*

$$(4.1.5) \quad \chi_\lambda(\mu) = c_\lambda(\mu).$$

The proof of this Theorem is quite elaborate, and we divide it in 5 steps.

Step 1 First we transform the Cauchy formula in a new identity.

Step 2 Next we prove that the class functions c_λ are orthonormal.

Step 3 To each partition we associate a permutation character β_λ .

Step 4 We prove that the matrix expressing the functions β_λ in terms of the c_μ is triangular with 1 on the diagonal.

Step 5 We formulate the Theorem of Frobenius in a more precise way and prove it.

Step 1 In order to follow Frobenius approach we go back to symmetric functions in n variables x_1, x_2, \dots, x_n . We shall freely use the Schur functions and the Cauchy formula for symmetric functions:

$$\prod_{i,j=1,n} \frac{1}{1-x_i y_j} = \sum_{\lambda} S_{\lambda}(x) S_{\lambda}(y)$$

proved in Chap. 2, §4.1.

We change its right hand side as follows. Compute:

$$\begin{aligned} \log\left(\prod_{i,j=1}^n \frac{1}{1-x_i y_j}\right) &= \sum_{i,j=1}^n \sum_{h=1}^{\infty} \frac{(x_i y_j)^h}{h} = \\ (4.1.6) \quad \sum_{h=1}^{\infty} \sum_{i,j=1}^n \frac{(x_i y_j)^h}{h} &= \sum_{h=1}^{\infty} \frac{\psi_h(x) \psi_h(y)}{h}. \end{aligned}$$

Taking the exponential we get the following expression:

$$(4.1.7) \quad \exp\left(\sum_{h=1}^{\infty} \frac{\psi_h(x) \psi_h(y)}{h}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{h=1}^{\infty} \frac{\psi_h(x) \psi_h(y)}{h}\right)^k =$$

$$(4.1.8) \quad \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\sum_{i=1}^{\infty} k_i = k} \binom{k}{k_1 \ k_2 \ \dots} \frac{\psi_1(x)^{k_1} \psi_1(y)^{k_1}}{1} \frac{\psi_2(x)^{k_2} \psi_2(y)^{k_2}}{2^{k_2}} \frac{\psi_3(x)^{k_3} \psi_3(y)^{k_3}}{3^{k_3}} \dots$$

then from 4.1.3 we deduce

$$(4.1.9) \quad \sum_{\mu} \frac{1}{n(\mu)} \psi_{\mu}(x) \psi_{\mu}(y) = \sum_{\lambda} S_{\lambda}(x) S_{\lambda}(y).$$

Step 2 Consider two class functions a, b as functions on partitions, their Hermitian product is thus:

$$\sum_{\mu \vdash n} \frac{1}{n!} \sum_{g \in C(\mu)} a(g) \bar{b}(g) = \sum_{\mu \vdash n} \frac{1}{n!} |C(\mu)| a(\mu) \bar{b}(\mu) = \sum_{\mu \vdash n} \frac{1}{n(\mu)} a(\mu) \bar{b}(\mu).$$

Let us now substitute in the identity 4.1.9 the expression $\psi_{\mu} = \sum_{\lambda} c_{\lambda}(\mu) S_{\lambda}$, and get:

$$(4.1.10) \quad \sum_{\mu \vdash n} \frac{1}{n(\mu)} c_{\lambda_1}(\mu) c_{\lambda_2}(\mu) = \begin{cases} 0 & \text{if } \lambda_1 \neq \lambda_2 \\ 1 & \text{if } \lambda_1 = \lambda_2 \end{cases}.$$

We have thus that the class functions c_λ are an orthonormal basis, completing Step 2.

Step 3 We consider now some permutation characters.

Take a partition $\lambda := h_1, h_2, \dots, h_k$ of n .

Consider the subgroup $S_\lambda := S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}$ and the permutation representation on:

$$(4.1.11) \quad S_n / S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}.$$

We will indicate by β_λ the corresponding character.

A permutation character is given by the formula $\chi(g) = \sum_i \frac{|G(g)|}{|H(g_i)|}$ (1.4.3 of Chap. 8).

Let us apply it to the case $G/H = S_n / S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}$ and for a permutation g relative to a partition $\mu := 1^{p_1} 2^{p_2} 3^{p_3} \dots i^{p_i} \dots n^{p_n}$.

A conjugacy class in $S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}$ is given by k partitions $\mu_i \vdash h_i$ of the numbers h_1, h_2, \dots, h_k . The conjugacy class of type μ , intersected with $S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}$, gives all possible k tuples of partitions $\mu_1, \mu_2, \dots, \mu_k$ of type

$$\mu_h := 1^{p_{1h}} 2^{p_{2h}} 3^{p_{3h}} \dots i^{p_{ih}} \dots$$

and:

$$\sum_{h=1}^k p_{ih} = p_i.$$

In a more formal way we may define a direct sum of two partitions $\lambda = 1^{p_1} 2^{p_2} 3^{p_3} \dots i^{p_i} \dots$, $\mu = 1^{q_1} 2^{q_2} 3^{q_3} \dots i^{q_i} \dots$ as the partition:

$$\lambda \oplus \mu := 1^{p_1+q_1} 2^{p_2+q_2} 3^{p_3+q_3} \dots i^{p_i+q_i} \dots$$

and remark that, with the notations of 4.1.2, $b(\lambda + \mu) = b(\lambda)b(\mu)$.

When we decompose $\mu = \bigoplus_{i=1}^k \mu_i$, we have $b(\mu) = \prod b(\mu_i)$.

The cardinality $m_{\mu_1, \mu_2, \dots, \mu_k}$ of the class $\mu_1, \mu_2, \dots, \mu_k$ in $S_{h_1} \times S_{h_2} \times \dots \times S_{h_k}$ is:

$$m_{\mu_1, \mu_2, \dots, \mu_k} = \prod_{j=1}^k \frac{h_j!}{n(\mu_j)} = \prod_{j=1}^k \frac{h_j!}{a(\mu_j) b(\mu)}$$

Now

$$\prod_{j=1}^k a(\mu_j) = \prod_{h=1}^k \left(\prod_{i=1}^n p_{ih}! \right)$$

So we get:

$$m_{\mu_1, \mu_2, \dots, \mu_k} = \frac{1}{n(\mu)} \prod_{j=1}^k h_j! \prod_{i=1}^n \binom{p_i}{p_{i1} p_{i2} \dots p_{ik}}.$$

Finally for the number $\beta_\lambda(\mu)$ we have:

$$\beta_\lambda(\mu) = \frac{n(\mu)}{\prod_{i=1}^k h_i!} \sum_{\mu = \bigoplus_{i=1}^k \mu_i, \mu_i \vdash h_i} m_{\mu_1, \mu_2, \dots, \mu_k} = \sum_{\mu = \bigoplus_{i=1}^k \mu_i, \mu_i \vdash h_i} \prod_{i=1}^n \binom{p_i}{p_{i1} p_{i2} \dots p_{ik}}.$$

This sum is manifestly the coefficient of $x_1^{h_1} x_2^{h_2} \dots x_k^{h_k}$ in the symmetric function $\psi_\mu(x)$.

In fact when we expand

$$\psi_\mu(x) = \psi_1(x)^{p_1} \psi_2(x)^{p_2} \dots \psi_i(x)^{p_i} \dots$$

for each factor $\psi_k(x) = \sum_{i=1}^n x_i^k$ one selects the index of the variable chosen and constructs a corresponding product monomial.

For each such monomial denote by p_{ij} the number of choices of the term x_j^i in the p_i factors $\psi_i(x)$, we have $\prod_i \binom{p_i}{p_{i1} p_{i2} \dots p_{ik}}$ such choices and they contribute to the monomial $x_1^{h_1} x_2^{h_2} \dots x_k^{h_k}$ if and only if $\sum_i i p_{ij} = h_j$.

Step 4 If Σ_λ denotes the sum of all monomials in the orbit of $x_1^{h_1} x_2^{h_2} \dots x_k^{h_k}$ we get the formula:

$$(4.1.12) \quad \psi_\mu(x) = \sum_{\lambda} \beta_{\lambda}(\mu) \Sigma_{\lambda}(x).$$

We wish to expand now the basis $\Sigma_\lambda(x)$ in terms of the basis $S_\lambda(x)$ and conversely:

$$(4.1.13) \quad \Sigma_\lambda(x) = \sum_{\mu} p_{\lambda, \mu} S_{\mu}(x), \quad S_\lambda(x) = \sum_{\mu} k_{\lambda, \mu} \Sigma_{\mu}(x).$$

In order to explicit some information about the matrices:

$$(p_{\lambda, \mu}), (k_{\lambda, \mu})$$

recall that the partitions are totally ordered by lexicographic ordering.

We also order the monomials by the lexicographic ordering of the sequence of exponents h_1, h_2, \dots, h_n of the variables x_1, x_2, \dots, x_n .

We remark that the ordering of monomials has the following immediate property:

If M_1, M_2, N are 3 monomials and $M_1 < M_2$ then $M_1 N < M_2 N$.

For any polynomial $p(x)$ we can thus select the leading monomial $l(p)$ and for two polynomials $p(x), q(x)$ we have:

$$l(pq) = l(p)l(q).$$

For a partition $\mu \vdash n := h_1 \geq h_2 \geq \dots \geq h_n$ the leading monomial of Σ_μ is

$$x^\mu := x_1^{h_1} x_2^{h_2} \dots x_n^{h_n}.$$

Similarly the leading monomial of the alternating function $A_{\mu+\varrho}(x)$ is:

$$x_1^{h_1+n-1} x_2^{h_2+n-2} \dots x_n^{h_n} = x^{\mu+\varrho}.$$

We compute now the leading monomial of the Schur function S_μ :

$$x^{\mu+\varrho} = l(A_{\mu+\varrho}(x)) = l(S_\mu(x)V(x)) = l(S_\mu(x))x^\varrho$$

we deduce that:

$$l(S_\mu(x)) = x^\mu.$$

This computation has the following immediate consequence:

Corollary. *The matrices $P := (p_{\lambda,\mu})$, $Q := (k_{\lambda,\mu})$ are upper triangular with 1 on the diagonal.*

Proof. A symmetric polynomial with leading coefficient x^μ is clearly equal to Σ_μ plus a linear combination of the Σ_λ , $\lambda < \mu$ this proves the claim for the matrix Q . The matrix P is the inverse of Q and the claim follows. \square

Step 5 We can now conclude a refinement of the computation of Frobenius:

Theorem 2. *i) $\beta_\lambda = c_\lambda + \sum_{\phi < \lambda} k_{\phi,\lambda} c_\phi$, $k_{\phi,\lambda} \in \mathbb{N}$. $c_\lambda = \sum_{\mu \geq \lambda} p_{\mu\lambda} b_\mu$.*

ii) The functions $c_\lambda(\mu)$ are a list of the irreducible characters of the symmetric group.

iii) $\chi_\lambda = c_\lambda$.

Proof. From the various definitions we get:

$$(4.1.14) \quad c_\lambda = \sum_{\phi} p_{\phi,\lambda} b_\phi, \quad \beta_\lambda = \sum_{\phi} k_{\phi,\lambda} c_\phi,$$

therefore the functions c_λ are virtual characters. Since they are orthonormal they are \pm the irreducible characters.

From the recursive formulas it follows that $\beta_\lambda = c_\lambda + \sum_{\phi < \lambda} k_{\phi,\lambda} c_\phi$, $m_{\lambda,\phi} \in \mathbb{Z}$.

Since β_λ is a character it is a positive linear combination of the irreducible characters, it follows that each c_λ is an irreducible character and that the coefficients $k_{\phi,\lambda} \in \mathbb{N}$ represent the multiplicities of the decomposition of the permutation representation into irreducible components.¹⁰

iii) Now we prove the equality $\chi_\lambda = c_\lambda$ by decreasing induction. If $\lambda = n$ is one row then the module M_λ is the trivial representation as well as the permutation representation on S_n/S_n .

Assume $\chi_\mu = c_\mu$ for all $\mu > \lambda$. We may use Lemma 2.4 and know that M_λ appears in its isotypic component in Rs_T with multiplicity 1 and does not appear in Rs_U for any tableau of shape $\mu > \lambda$.

We have remarked that Rs_T is the permutation representation of character β_λ in which by assumption the representation M_λ appears for the first time (with respect to the ordering of the λ). Thus the contribution of M_λ to its character must be given by the term c_λ . \square

Remark The basic formula $\psi_\mu(x) = \sum_{\lambda} c_\lambda(\mu) S_\lambda(x)$ can be multiplied by the Vandermonde determinant getting

$$(4.1.15) \quad \psi_\mu(x)V(x) = \sum_{\lambda} c_\lambda(\mu) A_{\lambda+\varrho}(x).$$

Now we may apply the leading monomial theory and deduce that $c_\lambda(\mu)$ is the coefficient in $\psi_\mu(x)V(x)$ belonging to the leading monomial $x^{\lambda+\rho}$ of $A_{\lambda+\varrho}$.

¹⁰The numbers $k_{\phi,\lambda}$ are called Kostka numbers. As we shall see they count some combinatorial objects called semistandard tableaux.

This furnishes a possible algorithm, we will discuss later some features of this formula.

4.2 Frobenius character There is a nice interpretation of the theorem of Frobenius.

Definition. The linear isomorphism between characters of S_n and symmetric functions of degree n which assigns to χ_λ the Schur function S_λ is called the **Frobenius character**. It is denoted by $\chi \rightarrow F(\chi)$.

Lemma. The Frobenius character can be computed by the formula:

$$(4.2.1) \quad F(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) \psi_{\mu(\sigma)}(x) = \sum_{\mu \vdash n} \frac{\chi(\mu)}{n(\mu)} \psi_\mu(x).$$

Proof. By linearity it is enough to prove it for $\chi = \chi_\lambda$. From 4.1.4 and 4.1.10 we have:

$$F(\chi_\lambda) = \sum_{\mu \vdash n} \frac{c_\lambda(\mu)}{n(\mu)} \psi_\mu(x) = \sum_{\mu \vdash n} \frac{c_\lambda(\mu)}{n(\mu)} \sum_{\gamma} c_\gamma(\mu) S_\gamma(x) = \sum_{\gamma} \sum_{\mu} \frac{c_\lambda(\mu) c_\gamma(\mu)}{n(\mu)} S_\gamma(x) = S_\lambda(x).$$

□

Recall that $n(\mu)$ is the order of the centralizer of a permutation with cycle structure μ . This shows the following important multiplicative behavior of the Frobenius character.

Theorem. Given two representations V, W of S_m, S_n respectively we have:

$$(4.2.2) \quad F(\text{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)) = F(V)F(W).$$

Proof. Let us denote by χ the character of $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \otimes W)$. Recall the discussion of induced characters in Chap 8. There we proved (formula 1.4.2), $\chi(g) = \sum_i \frac{|G(g)|}{|H(g_i)|} \chi_V(g_i)$. Where $|G(g)|$ is the order of the centralizer of g in G , the elements g_i run on representatives of the conjugacy classes O_i in H , decomposing the intersection of the conjugacy class of g in G with H .

In our case we deduce that $\chi(\sigma) = 0$ unless σ is conjugate to an element (a, b) of $S_n \times S_m$. In term of partitions, the partitions $\nu \vdash n + m$ which contribute to the characters are the ones of type $\lambda \oplus \mu$. In the language of partitions the previous formula 1.4.2 becomes:

$$\chi(\nu) = \sum_{\nu = \lambda + \mu} \frac{n(\lambda + \mu)}{n(\lambda)n(\mu)} \chi_V(\lambda) \chi_W(\mu)$$

since $\psi_{\lambda \oplus \mu} = \psi_\lambda \psi_\mu$ we obtain for $F(\chi)$:

$$\begin{aligned} F(\chi) &= \sum_{\nu \vdash m+n} \frac{\chi(\nu) \psi_\nu}{n(\nu)} = \sum_{\nu} \frac{\psi_\nu}{n(\nu)} \sum_{\nu = \lambda + \mu} \frac{n(\lambda + \mu)}{n(\lambda)n(\mu)} \chi_V(\lambda) \chi_W(\mu) = \\ &= \sum_{\lambda \vdash m, \mu \vdash n} \frac{\chi_V(\lambda) \chi_W(\mu)}{n(\lambda)n(\mu)} \psi_\lambda \psi_\mu = F(\lambda)F(\mu) \end{aligned}$$

□

4.3 Molien's formula We discuss a complement to the representation theory of S_n .

It will be necessary to work formally with symmetric functions in infinitely many variables, a formalism which has been justified in Chap. 2, 1.1.

With this in mind we think of the identities of §4, as identities in infinitely many variables.

First of all a convention. If we are given a representation of a group on a graded vector space $U := \{U_i\}_{i=0}^{\infty}$ (i.e. a representation on each U_i) its character is usually written as a power series with coefficients in the character ring in a variable q :¹¹

$$(4.3.1) \quad \chi_U(t) := \sum_i \chi_i q^i.$$

Where χ_i is the character of the representation U_i .

Definition. The expression 4.3.1 is called a graded character.

Graded characters have some formal similarities with characters. Given two graded representations $U = \{U_i\}_i, V = \{V_i\}_i$ we have their direct sum, and their tensor product

$$(U \oplus V)_i := U_i \oplus V_i, \quad (U \otimes V)_i := \bigoplus_{h=0}^i U_h \otimes V_{i-h}.$$

For the graded characters we have clearly:

$$(4.3.2) \quad \chi_{U \oplus V}(q) = \chi_U(q) + \chi_V(q), \quad \chi_{U \otimes V}(q) = \chi_U(q)\chi_V(q).$$

Let us consider a simple example.¹²

Lemma Molien's formula. Given a linear operator A on a vector space U its action on the symmetric algebra $S(U)$ has as graded character:

$$(4.3.3) \quad \sum_{i=0}^{\infty} \text{tr}(S^i(A))q^i = \frac{1}{\det(1 - qA)}$$

Its action on the exterior algebra $\wedge U$ has as graded character:

$$(4.3.4) \quad \sum_{i=0}^{\dim U} \text{tr}(\wedge^i(A))q^i = \det(1 + qA)$$

Proof. For every symmetric power $S^k(U)$ the character of the operator induced by A is a polynomial in A . Thus it is enough to prove the formula, by continuity and invariance, when A is diagonal.

¹¹It is now quite usual to use q as variable since it often appears coming from computations on finite fields where $q = p^r$ or as quantum deformation parameter

¹²strictly speaking we are not treating now a group, but the set of all matrices under multiplication, which is only a semigroup, for this set tensor product of representation makes sense but not duality.

Take a basis of eigenvectors u_i , $i = 1, \dots, n$ with eigenvalue λ_i .

Then $S[U] = S[u_1] \otimes S[u_2] \otimes \dots \otimes S[u_n]$ and $S[u_i] = \sum_{h=0}^{\infty} F u_i^h$.

The graded character of $S[u_i]$ is $\sum_{h=0}^{\infty} \lambda_i^h q^h = \frac{1}{1-\lambda_i q}$ hence:

$$\chi_{S[U]}(q) = \prod_{i=1}^n \chi_{S[u_i]}(q) = \frac{1}{\prod_{i=1}^n (1 - \lambda_i q)} = \frac{1}{\det(1 - qA)}.$$

Similarly $\wedge U = \wedge[u_1] \otimes \wedge[u_2] \otimes \dots \otimes \wedge[u_n]$ and $\wedge[u_i] = F \oplus F u_i$ hence

$$\chi_{\wedge[U]}(q) = \prod_{i=1}^n \chi_{\wedge[u_i]}(q) = \prod_{i=1}^n (1 + \lambda_i q) = \det(1 + qA).$$

□

We apply the previous discussion to S_n acting on the space \mathbb{C}^n permuting the coordinates and the representation that it induces on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$.

We denote by $\sum_{i=0}^{\infty} \chi_i q^i$ the corresponding graded character.

If σ is a permutation with cycle decomposition of lengths $\mu(\sigma) = \mu := m_1, m_2, \dots, m_k$ the standard basis of \mathbb{C}^n decomposes into k -cycles each of length m_i . On the subspace relative to a cycle of length m , σ acts with eigenvalues the m -roots of 1 and

$$\det(1 - q\sigma) = \prod_{i=1}^k \prod_{j=1}^{m_i} (1 - e^{\frac{j2\pi\sqrt{-1}}{m_i}} q) = \prod_{i=1}^k (1 - q^{m_i})$$

Thus the graded character of σ acting on the polynomial ring is

$$\begin{aligned} \frac{1}{\det(1 - q\sigma)} &= \prod_i \sum_{j=0}^{\infty} q^{j m_i} = \prod_i \psi_{m_i}(1, q, q^2, \dots, q^k, \dots) = \\ \psi_{\mu}(1, q, q^2, \dots, q^k, \dots) &= \sum_{\lambda \vdash n} \chi_{\lambda}(\sigma) S_{\lambda}(1, q, q^2, \dots, q^k, \dots). \end{aligned}$$

To summarize

Theorem 1. *The graded character of S_n acting on the polynomial ring is*

$$(4.3.5) \quad \sum_{\lambda \vdash n} \chi_{\lambda} S_{\lambda}(1, q, q^2, \dots, q^k, \dots).$$

Exercise Prove this formula directly.

(Hint) $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x]^{\otimes n} = \bigoplus_{\lambda} M_{\lambda} \otimes S_{\lambda}(\mathbb{C}[x])$.

We have a corollary of this formula. If $\lambda = h_1 \geq h_2 \geq \dots \geq h_n$, the term of lowest degree in q in $S_{\lambda}(1, q, q^2, \dots, q^k, \dots)$ is clearly given by the leading term $x_1^{h_1} x_2^{h_2} \dots x_n^{h_n}$ computed in $1, q, q^2, \dots, q^n$ and this gives $q^{h_1 + 2h_2 + 3h_3 + \dots + nh_n}$. We deduce that the representation M_{λ} of S_n appears for the first time in degree $h_1 + 2h_2 + 3h_3 + \dots + nh_n$ and in this degree

it appears with multiplicity 1. This particular submodule of $\mathbb{C}[x_1, x_2, \dots, x_n]$ is called the *Specht module* and it plays an important role.¹³

Now we want to discuss another related representation.

Recall first that $\mathbb{C}[x_1, x_2, \dots, x_n]$ is a free module over the ring of symmetric functions $\mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_n]$ of rank $n!$. It follows that, for every choice of the numbers $\underline{a} := a_1, \dots, a_n$, the ring $R_{\underline{a}} := \mathbb{C}[x_1, x_2, \dots, x_n] / \langle \sigma_i - a_i \rangle$ constructed from $\mathbb{C}[x_1, x_2, \dots, x_n]$ modulo the ideal generated by the elements $\sigma_i - a_i$, is of dimension $n!$ and a representation of S_n .

We claim that it is always the regular representation.

Proof. First we prove it in the case in which the polynomial $t^n - a_1 t^{n-1} + a_2 t^{n-2} - \dots + (-1)^n a_n$ has distinct roots $\alpha_1, \dots, \alpha_n$. This means that the ring $\mathbb{C}[x_1, x_2, \dots, x_n] / \langle \sigma_i - a_i \rangle$ is the coordinate ring of the set of the $n!$ distinct points $\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}$, $\sigma \in S_n$. This is clearly the regular representation.

We know that the condition for a polynomial to have distinct roots is open in the coefficients and given by the non vanishing of the discriminant (Chap. 1).

It is easily seen that the character of $R_{\underline{a}}$ is continuous in \underline{a} and, since the characters of a finite group are a discrete set this implies that the character is constant. \square

It is of particular interest (combinatorial and geometric) to analyze the special case $\underline{a} = 0$ and the ring $R := \mathbb{C}[x_1, x_2, \dots, x_n] / \langle \sigma_i \rangle$ which is a graded algebra affording the regular representation.

Thus the graded character $\chi_R(q)$ of R is a graded form of the regular representation.

To compute it notice that, as a graded representations, we have an isomorphism

$$\mathbb{C}[x_1, x_2, \dots, x_n] = R \otimes \mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_n],$$

and thus an identity of graded characters.

The ring $\mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_n]$ has the trivial representation, by definition, and generators in degree $1, 2, \dots, n$; so its graded character is just $\prod_{i=1}^n (1 - q^i)^{-1}$. We deduce:

Theorem 2.

$$\chi_R(q) = \sum_{\lambda \vdash n} \chi_{\lambda} S_{\lambda}(1, q, q^2, \dots, q^k, \dots) \prod_{i=1}^n (1 - q^i).$$

Notice then that the series $S_{\lambda}(1, q, q^2, \dots, q^k, \dots) \prod_{i=1}^n (1 - q^i)$ represent the multiplicities of χ_{λ} in the various degrees of R and thus are polynomials with positive coefficients with sum the dimension of χ_{λ} .

Exercise Prove that the Specht module has non zero image in the quotient ring $R := \mathbb{C}[x_1, x_2, \dots, x_n] / \langle \sigma_i \rangle$.

The ring $R := \mathbb{Z}[x_1, x_2, \dots, x_n] / \langle \sigma_i \rangle$ has an interesting geometric interpretation as the *cohomology algebra* of the flag variety. This variety can be understood as the space

¹³it appears in the Springer representation, for instance, cf. [DP2].

of all decompositions $\mathbb{C}^n = V_1 \perp V_2 \perp \cdots \perp V_n$ into orthogonal 1-dimensional subspaces. The action of the symmetric group is induced by the topological action permuting the summands of the decomposition (Chap. 10, 6.5).

5 The hook formula

5.1 Dimension of M_λ We want to deduce now a formula, due to Frobenius, for the dimension $d(\lambda)$ of the irreducible representation M_λ of the symmetric group.

From 4.1.15 applied to the partition 1^n , corresponding to the conjugacy class of the identity, we obtain:

$$(5.1.1) \quad \left(\sum_{i=1}^n x_i \right)^n V(x) = \sum_{\lambda} d(\lambda) A_{\lambda+\varrho}(x).$$

Write the development of the Vandermonde determinant as $\sum_{\sigma \in S_n} \epsilon_{\sigma} \prod_{i=1}^n x_i^{\sigma(n-i+1)-1}$. Letting $\lambda + \rho = \ell_1 > \ell_2 > \cdots > \ell_n$, the number $d(\lambda)$ is the coefficient of $\prod_i x_i^{\ell_i}$ in

$$\left(\sum_{i=1}^n x_i \right)^n \sum_{\sigma \in S_n} \epsilon_{\sigma} \prod_{i=1}^n x_i^{\sigma(n-i+1)-1}.$$

Thus a term $\epsilon_{\sigma} \binom{n}{k_1 k_2 \dots k_n} \prod_{i=1}^n x_i^{\sigma(n-i+1)-1+k_i}$ contributes to $\prod_i x_i^{\ell_i}$ if and only if $k_i = \ell_i - \sigma(n-i+1) + 1$. We deduce

$$d(\lambda) = \sum_{\substack{\sigma \in S_n \\ \forall i \\ \ell_i - \sigma(n-i+1) + 1 \geq 0}} \epsilon_{\sigma} \frac{n!}{\prod_{i=1}^n (\ell_i - \sigma(n-i+1) + 1)!}.$$

We change the term

$$n! \prod_{i=1}^n \frac{1}{(\ell_i - \sigma(n-i+1) + 1)!} = \frac{n!}{\prod_{i=1}^n \ell_i!} \prod_{i=1}^n \prod_{\substack{0 \leq k \leq \\ \sigma(n-i+1)-2}} (\ell_i - k)$$

and remark that this formula makes sense, and it is 0, if σ does not satisfy the restriction $\ell_i - \sigma(n-i+1) + 1 \geq 0$.

Thus

$$d(\lambda) = \frac{n!}{\prod_{i=1}^n \ell_i!} \bar{d}(\lambda), \quad \bar{d}(\lambda) = \sum_{\sigma \in S_n} \epsilon_{\sigma} \prod_{i=1}^n \prod_{\substack{0 \leq k \leq \\ \sigma(n-i+1)-2}} (\ell_i - k).$$

$\bar{d}(\lambda)$ is the value of the determinant of a matrix with $\prod_{0 \leq k \leq j-2} (\ell_i - k)$ in the $n - i + 1, j$ position.

$$\begin{vmatrix} 1 & \ell_n & \ell_n(\ell_n - 1) & \cdots & \prod_{0 \leq k \leq n-2} (\ell_n - k) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ell_i & \ell_i(\ell_i - 1) & \cdots & \prod_{0 \leq k \leq n-2} (\ell_i - k) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ell_1 & \ell_1(\ell_1 - 1) & \cdots & \prod_{0 \leq k \leq n-2} (\ell_1 - k) \end{vmatrix}.$$

This determinant, by elementary operations on the columns, reduces to the Vandermonde determinant in the ℓ_i with value $\prod_{i < j} (\ell_i - \ell_j)$. Thus we obtain the formula of Frobenius:

$$(5.1.2) \quad d(\lambda) = \frac{n!}{\prod_{i=1}^n \ell_i!} \prod_{i < j} (\ell_i - \ell_j) = n! \prod_{j=1}^n \frac{\prod_{i, i < j} (\ell_i - \ell_j)}{\ell_j!}.$$

5.2 Hook formula

We want to give a combinatorial interpretation of 5.1.2. Notice that, fixing j , in $\frac{\prod_{i < j} (\ell_i - \ell_j)}{\ell_j!}$ the $j - 1$ factors of the numerator cancel the corresponding factors in the denominator, leaving $\ell_j - j + 1$ factors. In all $\sum_j \ell_j - \sum_{j=1}^n (j - 1) = n$ are left.

These factors can be interpreted as the *hook lengths* of the boxes of the corresponding diagram.

More precisely given a box x of a French diagram its **hook** is the set of elements of the diagram which are either on top or to the right of x , including x . E.g. we mark the hooks of 1, 2; 2, 1; 2, 2 in 4, 3, 1, 1:



The total number of boxes in the hook of x is the *hook length* of x , denoted by h_x .

Frobenius formula for the dimension $d(\lambda)$, can be reformulated into the *hook formula*.

Theorem. Denote by $B(\lambda)$ the set of boxes of a diagram of shape λ :

$$(5.2.1) \quad d(\lambda) = \frac{n!}{\prod_{x \in B(\lambda)} h_x}, \quad \text{hook formula.}$$

Proof. It is enough to show that, the factors in the factorial $\ell_i!$ which are not canceled by the factors of the numerator, are the hook lengths of the boxes in the i^{th} row. This will prove the formula.

In fact let $h_i = \ell_i + i - n$ be the length of the i^{th} row. Given $k > i$ let us consider the $h_{k-1} - h_k$ numbers strictly between $\ell_i - \ell_{k-1} = h_i - h_{k-1} + k - i - 1$ and $\ell_i - \ell_k = h_i - h_k + k - i$.

Remark that $h_{k-1} - h_k$ is the number of cases, in the i^{th} row, for which the hook ends vertically on the $k - 1$ row. It is easily seen, since the vertical *leg* of each such hook has length $k - i$ and the horizontal *arm* length goes from $h_i - h_k$ to $h_i - h_{k-1} + 1$, that the lengths of these hooks vary between $k - i + h_i - h_k - 1$ and $k - i + h_i - h_{k-1}$, the previously considered numbers. \square

6 Characters of the linear group

6.1 Tensor character We plan to deduce from the previous computations the character theory of the linear group.

For this we need to perform another character computation. Given a permutation $s \in S_n$ and a matrix $X \in GL(V)$ consider the product sX as an operator in $V^{\otimes n}$, we want to compute its trace.

Let $\mu = h_1, h_2, \dots, h_k$ denote the cycle partition of s , introduce the obvious notation:

$$(6.1.1) \quad \Psi_\mu(X) = \prod_i tr(X^{h_i}).$$

Clearly $\Psi_\mu(X) = \psi_\mu(x)$ where by x we denote the eigenvalues of X .

Proposition. *The trace of sX as an operator in $V^{\otimes n}$ is $\Psi_\mu(X)$.*

We shall deduce this Proposition as a special case of a more general formula.

Given n matrices X_1, X_2, \dots, X_n and $s \in S_n$ compute the trace of $s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n$ (an operator in $V^{\otimes n}$).

Decompose explicitly s into cycles $s = c_1 c_2 \dots c_k$ and, for a cycle $c := (i_p \ i_{p-1} \ \dots \ i_1)$ define the function of the n matrix variables X_1, X_2, \dots, X_n :

$$(6.1.2) \quad \phi_c(X) = \phi_c(X_1, X_2, \dots, X_n) := tr(X_{i_1} X_{i_2} \dots X_{i_p}).$$

The previous proposition then follows from:

Theorem.

$$(6.1.3) \quad tr(s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n) = \prod_{j=1}^k \phi_{c_j}(X).$$

Proof. We first remark that, for fixed s , both sides of 6.1.3 are multilinear functions of the matrix variables X_i , therefore in order to prove this formula it is enough to do it when $X_i = u_i \otimes \psi_i$ is decomposable.

Let us apply in this case the operator $s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n$ to a decomposable tensor $v_1 \otimes v_2 \dots \otimes v_n$ we have:

$$(6.1.4) \quad s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n(v_1 \otimes v_2 \dots \otimes v_n) = \prod_{i=1}^n \langle \psi_i | v_i \rangle u_{s^{-1}1} \otimes u_{s^{-1}2} \dots \otimes u_{s^{-1}n}.$$

This formula shows that:

$$(6.1.5) \quad s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n = (u_{s^{-1}1} \otimes \psi_1) \otimes (u_{s^{-1}2} \otimes \psi_2) \dots \otimes (u_{s^{-1}n} \otimes \psi_n)$$

so that:

$$(6.1.6) \quad \text{tr}(s \circ X_1 \otimes X_2 \otimes \dots \otimes X_n) = \prod_{i=1}^n \langle \psi_i | u_{s^{-1}i} \rangle = \prod_{i=1}^n \langle \psi_{s(i)} | u_i \rangle .$$

Now let us compute for a cycle $c := (i_p \ i_{p-1} \ \dots \ i_1)$ the function

$$\phi_c(X) = \text{tr}(X_{i_1} X_{i_2} \dots X_{i_p}).$$

We get:

$$(6.1.7) \quad \begin{aligned} \text{tr}(u_{i_1} \otimes \psi_{i_1} u_{i_2} \otimes \psi_{i_2} \dots \otimes u_{i_p} \otimes \psi_{i_p}) &= \text{tr}(u_{i_1} \otimes \langle \psi_{i_1} | u_{i_2} \rangle \langle \psi_{i_2} | u_{i_3} \rangle \dots \langle \psi_{i_{p-1}} | u_{i_p} \rangle \psi_{i_p}) \\ &= \langle \psi_{i_1} | u_{i_2} \rangle \langle \psi_{i_2} | u_{i_3} \rangle \dots \langle \psi_{i_{p-1}} | u_{i_p} \rangle \langle \psi_{i_p} | u_{i_1} \rangle = \prod_{j=1}^p \langle \psi_{c(i_j)} | u_{i_j} \rangle \end{aligned}$$

Formulas 6.1.6 and 6.1.7 imply the claim. \square

6.2 Character of $S_\lambda(V)$ According to Theorem 3.2 of Chapter 2, the formal ring of symmetric functions in infinitely many variables has as basis all Schur functions S_λ . The restriction to symmetric functions in m -variables sets to 0 all S_λ with height $> m$.

We are ready to complete our work, let $m = \dim V$, for a matrix $X \in GL(V)$ and a partition $\lambda \vdash n$ of height $\leq m$ let us denote by $S_\lambda(X) := S_\lambda(x)$ the Schur function evaluated in $x = (x_1, \dots, x_m)$ the eigenvalues of X .

Theorem. Denote $\rho_\lambda(X)$ the character of the representation $S_\lambda(V)$ of $GL(V)$, paired with the representation M_λ of S_n in $V^{\otimes n}$. We have $\rho_\lambda(X) = S_\lambda(X)$.

Proof. If $s \in S_n, X \in GL(V)$ we have seen that the trace of $s \circ X^{\otimes n}$ on $V^{\otimes n}$ is computed by $\psi_\mu(X) = \sum_\lambda c_\lambda(\mu) S_\lambda(X)$ (definition of the c_λ).

If $m = \dim V < n$ only the partitions of height $\leq m$ contribute to the sum.

On the other hand $V^{\otimes n} = \bigoplus_{ht(\lambda) \leq \dim(V)} M_\lambda \otimes S_\lambda(V)$ thus:

$$\psi_\mu(X) = \text{tr}(s \circ X^{\otimes n}) = \sum_{\lambda \vdash n, ht(\lambda) \leq m} \text{tr}(s | M_\lambda) \text{tr}(X^{\otimes n} | S_\lambda(V)) = \sum_{\lambda \vdash n, ht(\lambda) \leq m} \chi_\lambda(\mu) \rho_\lambda(X).$$

If $m < n$ the $\psi_\mu(X)$ with parts of length $\leq m$ (i.e. $ht(\tilde{\mu}) \leq m$) are a basis of symmetric functions in m variables, hence we can invert the system of linear equations and get $S_\lambda(X) = \rho_\lambda(X)$ \square

The eigenvalues of $X^{\otimes n}$ are monomials in the variables x_i and thus we obtain:

Corollary. $S_\lambda(x)$ is a sum of monomials with positive coefficients.

We will see in Chapter 13 that one can index combinatorially the monomials which appear by *semistandard tableaux*.

We can deduce also a dimension formula for the space $S_\lambda(V)$, $\dim V = n$. Of course its value is $S_\lambda(1, 1, \dots, 1)$ which we want to compute from the determinantal formulas giving $S_\lambda(x) = A_{\lambda+\varrho}(x)/V(x)$.

Let as usual $\lambda := h_1, h_2, \dots, h_n$ and $l_i := h_i + n - i$. Of course we cannot substitute directly for the x_i the number 1 else we get $0/0$. Thus we first substitute to $x_i \rightarrow x^{i-1}$ and then take the limit as $x \rightarrow 1$. Under the previous substitution we see that $A_{\lambda+\varrho}$ becomes the Vandermonde determinant of the elements x^{l_i} hence

$$S_\lambda(1, x, x^2, \dots, x^{n-1}) = \prod_{1 \leq i < j \leq n} \frac{(x^{l_i} - x^{l_j})}{(x^{n-i} - x^{n-j})}.$$

If $a > b$ we have $x^a - x^b = x^b(x-1)(x^{a-b-1} + x^{a-b-2} + \dots + 1)$ hence we deduce that:

$$\dim S_\lambda(V) = S_\lambda(1, 1, 1, \dots, 1) = \prod_{1 \leq i < j \leq n} \frac{(l_i - l_j)}{(j - i)} = \prod_{1 \leq i < j \leq n} \frac{(h_i - h_j + j - i)}{(j - i)}.$$

6.3 Cauchy formula as representations We want to give now an interpretation, in the language of representations, of the Cauchy formula.

Suppose we are given a vector space U over which a torus T acts with a basis of weight vectors u_i with weight χ_i .

The graded character of the action of T on the symmetric and exterior algebras, are given by Molien's formula, 4.5 and are respectively:

$$(6.3.1) \quad \frac{1}{\prod (1 - \chi_i q)}, \quad \prod (1 + \chi_i q).$$

As an example consider two vector spaces U, V with basis $u_1, \dots, u_m; v_1, \dots, v_n$ respectively, we may assume $m \leq n$.

The maximal torus of diagonal matrices has eigenvalues $x_1, \dots, x_m; y_1, \dots, y_n$ respectively for each of them. On the tensor product we have the action of the product torus and the basis $u_i \otimes v_j$ has eigenvalues $x_i y_j$.

Therefore the graded character on the symmetric algebra $S[U \otimes V]$ is $\prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j q}$.

By Cauchy's formula we deduce that the character of the n^{th} symmetric power $S^n(U \otimes V)$ equals $\sum_{\lambda \vdash n, ht(\lambda) \leq m} S_\lambda(x) S_\lambda(y)$.

We know that the rational representations of $GL(U) \times GL(V)$ are completely reducible and their characters can be computed by restricting to diagonal matrices. Thus we have the description:

$$(6.3.2) \quad S^n(U \otimes V) = \bigoplus_{\lambda \vdash n, ht(\lambda) \leq m} S_\lambda(U) \otimes S_\lambda(V).$$

This is also referred to as *Cauchy's formula*.

Remark that, if $W \subset V$ is a subspace which we may assume formed by the first k basis vectors, then the intersection of $S_\lambda(U) \otimes S_\lambda(V)$ with $S(U \otimes W)$ has as basis the part of the basis of weight vectors of $S_\lambda(U) \otimes S_\lambda(V)$ relative to weights in which the variables $y_j, j > k$ do not appear. Thus its character is obtained by setting to 0 these variables in $S_\lambda(y_1, y_2, \dots, y_m)$ thus we clearly get that

$$(6.3.3) \quad S_\lambda(U) \otimes S_\lambda(V) \cap S(U \otimes W) = S_\lambda(U) \otimes S_\lambda(W)$$

Similarly it is clear, from the definition 3.1.3: $S_\lambda(V) := e_T V^{\otimes n}$, that:

Proposition. *If $U \subset V$ is a subspace then $S_\lambda(U) = S_\lambda(V) \cap U^{\otimes n}$.*

6.4 Multilinear elements One should stress the strict connection between the two formulas, 6.3.2 and 3.1.4.

$$(6.3.2) \quad S^n(U \otimes V) = \bigoplus_{\lambda \vdash n, ht(\lambda) \leq m} S_\lambda(U) \otimes S_\lambda(V), \quad (3.1.4) \quad V^{\otimes n} = \bigoplus_{ht(\lambda) \leq dim(V)} M_\lambda \otimes S_\lambda(V).$$

This is clearly explained when we assume that $U = \mathbb{C}^n$ with canonical basis e_i and we consider the diagonal torus T acting by matrices $Xe_i = x_i e_i$.

Let us go back to the formula 6.3.2, and apply it when $\dim V = n, W = \mathbb{C}^n$.

Consider the subspace T_n of $S[\mathbb{C}^n \otimes V]$ formed by the elements $\prod_{i=1}^n e_i \otimes v_i, v_i \in V$. T_n is stable under the subgroup $S_n \times GL(V) \subset GL(n, \mathbb{C}) \times GL(V)$, where S_n is the group of permutation matrices. We have a mapping $i : V^{\otimes n} \rightarrow T_n$ defined by:

$$(6.4.1) \quad i : v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \prod_{i=1}^n e_i \otimes v_i.$$

Proposition. *i) T_n is the weight space in $S[\mathbb{C}^n \otimes V]$, of weight $\chi(X) = \prod_i x_i$ for the torus T .*

ii) The map i is an $S_n \times GL(V)$ linear isomorphism between $V^{\otimes n}$ and T_n .

Proof. The verification is immediate and left to the reader. \square

Remark The character $\chi := \prod_{i=1}^n x_i$ is invariant under the symmetric group (and generates the group of these characters) we call it *the multilinear character*.

As usual, when we have a representation W of a torus, we denote by W^χ the weight space of character χ .

Now for every partition λ consider

$$(6.4.2) \quad S_\lambda(\mathbb{C}^n)^\chi := \{u \in S_\lambda(\mathbb{C}^n) \mid Xu = \prod_i x_i u, \forall X \in T\},$$

the weight space of $S_\lambda(\mathbb{C}^n)$ formed by the elements which are formally *multilinear*.

Since the character $\prod_i x_i$ is left invariant by conjugation by permutation matrices it follows that the symmetric group $S_n \subset GL(n, \mathbb{C})$ of permutation matrices acts on $S_\lambda(\mathbb{C}^n)^\chi$, we claim that:

Proposition. $S_\lambda(\mathbb{C}^n)^\times = 0$ unless $\lambda \vdash n$ and in this case $S_\lambda((\mathbb{C}^n)^*)^\times$ is identified with the irreducible representation M_λ of S_n .

Proof. In fact assume $Xu = \prod_i x_i u$. Clearly u is in a polynomial representation of degree n , on the other hand

$$S^n(\mathbb{C}^n \otimes V) = \oplus_{\lambda \vdash n} S_\lambda((\mathbb{C}^n)^*) \otimes S_\lambda(V)$$

hence

$$(6.4.3) \quad V^{\otimes n} := S^n(\mathbb{C}^n \otimes V)^\times = \oplus_{\lambda \vdash n} S_\lambda((\mathbb{C}^n)^*)^\times \otimes S_\lambda(V) = \oplus_{\lambda \vdash n} M_\lambda \otimes S_\lambda(V)$$

we get the required identification. \square

Therefore, given a polynomial representation P of $GL(n, \mathbb{C})$, if it is homogeneous of degree n , in order to determine its decomposition $P = \oplus_{\lambda \vdash n} m_\lambda S_\lambda(\mathbb{C}^n)$ we can equivalently restrict to the multilinear weight space $M := \{p \in P \mid X.p = \prod_i x_i p\}$ (for X diagonal with entries x_i) and see how it decomposes as representation of S_n since:

$$(6.4.4) \quad P = \oplus_{\lambda \vdash n} m_\lambda S_\lambda(\mathbb{C}^n) \iff M = \oplus_{\lambda \vdash n} m_\lambda M_\lambda.$$

7 Polynomial functors.

7.1 Schur functors Consider two vector spaces V, W , the space $\text{hom}(V, W) = W \otimes V^*$, and the ring of polynomial functions decomposed as:

$$(7.1.1) \quad \mathcal{P}[\text{hom}(V, W)] = S[W^* \otimes V] = \oplus_\lambda S_\lambda(W^*) \otimes S_\lambda(V).$$

A way to explicitly identify the spaces $S_\lambda(W^*) \otimes S_\lambda(V)$ as spaces of functions is obtained by a variation of the method of matrix coefficients.

We start by stressing the fact that the construction of the representation $S_\lambda(V)$ from V is in a sense *natural*, in the language of categories.

Recall that a map between vector spaces is called a *polynomial map* if in coordinates it is given by polynomials.

Definition. A functor F from the category of vector spaces to itself is called a **polynomial functor** if, given two vector spaces V, W the map $A \rightarrow F(A)$ from the vector space $\text{hom}(V, W)$ to the vector space $\text{hom}(F(V), F(W))$ is a polynomial map.

We say that F is homogeneous of degree k if $\text{hom}(V, W) \rightarrow \text{hom}(F(V), F(W))$ is a polynomial map homogeneous of degree k .

The functor $F : V \rightarrow V^{\otimes n}$ is clearly a polynomial functor, homogeneous of degree n . When $A : V \rightarrow W$ the map $F(A)$ is $A^{\otimes n}$.

We can now justify the word *Schur functor* in the definition 3.1.3, $S_\lambda(V) := e_T V^{\otimes n}$, e_T a Young symmetrizer, associated to a partition λ .

As V varies $V \mapsto S_\lambda(V)$ can be considered as a functor. In fact it is a subfunctor of the tensor power, since clearly, if $A : V \rightarrow W$ is a linear map, $A^{\otimes n}$ commutes with a_T . Thus $A^{\otimes n}(e_T V^{\otimes n}) \subset e_T W^{\otimes n}$ and we define:

$$(7.1.2) \quad S_\lambda(A) : S_\lambda(V) \longrightarrow V^{\otimes n} \xrightarrow{A^{\otimes n}} W^{\otimes n} \xrightarrow{e_T} S_\lambda(W).$$

Summarizing:

Proposition 1. *Given any partition $\mu \vdash n$, $V \rightarrow S_\mu(V)$ is a homogeneous polynomial functor on vector spaces of degree n , called a **Schur functor**.*

Remark The exterior and symmetric power $\wedge^k V, S^k(V)$ are examples of Schur functors.

Since the map $S_\mu : \text{hom}(V, W) \rightarrow \text{hom}(S_\mu(V), S_\mu(W))$ defined by $S_\mu : X \rightarrow S_\mu(X)$ is a homogeneous polynomial map of degree n , the dual map $S_\mu^* : \text{hom}(S_\mu(V), S_\mu(W))^* \rightarrow \mathcal{P}[\text{hom}(V, W)]$ defined by

$$S_\mu^*(\phi)(X) := \langle \phi | S_\mu(X) \rangle, \quad \phi \in \text{hom}(S_\mu(V), S_\mu(W))^*, \quad X \in \text{hom}(V, W)$$

is a $GL(V) \times GL(W)$ equivariant map into the homogeneous polynomials of degree n .

By the irreducibility of $\text{hom}(S_\mu(V), S_\mu(W))^* = S_\mu(V) \otimes S_\mu(W)^*$ it must be a linear isomorphism to an irreducible submodule of $\mathcal{P}[\text{hom}(V, W)]$ uniquely determined by Cauchy's formula. By comparing the isotypic component of type $S_\mu(V)$ we deduce:

Proposition 2. $\mathcal{P}[\text{hom}(V, W)] = \bigoplus_\mu \text{hom}(S_\mu(V), S_\mu(W))^*$ and we have the isomorphism $S_\mu(W^*) = S_\mu(W)^*$.

Let us apply the previous discussion to $\text{hom}(\wedge^i V, \wedge^i W)$.

Choose bases $e_i, i = 1, \dots, h, f_j, j = 1, \dots, k$ for V, W respectively and identify the space $\text{hom}(V, W)$ with the space of $k \times h$ matrices.

Thus the ring $\mathcal{P}[\text{hom}(V, W)]$ is the polynomial ring $\mathbb{C}[x_{ij}], i = 1, \dots, h, j = 1, \dots, k$ where x_{ij} are the matrix entries.

Given a matrix X the entries of $\wedge^i X$ are the determinants of all the minors of order i extracted from X , and:

Corollary. $\wedge^i V \otimes (\wedge^i W)^*$ can be identified to the space of polynomials spanned by the determinants of all the minors of order i extracted from X , which is thus irreducible as representation of $GL(V) \times GL(W)$.

7.2 Homogeneous functors We want to prove that any polynomial functor is equivalent to a direct sum of Schur functors. We start with:

Proposition 1. *A polynomial functor is a direct sum of homogeneous functors.*

Proof. The scalar multiplications by $\alpha \in \mathbb{C}^*$, on a space V induce, by functoriality, a polynomial representation of \mathbb{C}^* on $F(V)$ which then decomposes as $F(V) = \bigoplus_k F_k(V)$,

with $F_k(V)$ the subspace of weight α^k . Clearly $F_k(V)$ is a subfunctor and $F = \bigoplus_k F_k(V)$, moreover $F_k(V)$ is a homogeneous functor of degree k .

We can polarize a homogeneous functor of degree k as follows. Consider, for a k -tuple V_1, \dots, V_k of vector spaces their direct sum $\bigoplus_i V_i$ together with the action of a k -dimensional torus T with the scalar multiplication x_i on each summand V_i . T acts in a polynomial way on $F(\bigoplus_i V_i)$ and we can decompose by weights

$$F(\bigoplus_i V_i) = \bigoplus_\lambda F_\lambda(V_1, \dots, V_k), \quad \lambda = x_1^{h_1} x_2^{h_2} \dots x_k^{h_k}, \quad \sum h_i = k.$$

One easily verifies that the inclusion $V_i \rightarrow \bigoplus_j V_j$ induces isomorphism between $F(V_i)$ and $F_{x_i^k}(V_1, \dots, V_k)$. \square

Proposition 2. *Let $\eta : F \rightarrow G$ be a natural transformation of polynomial functors of degree k , then η is an isomorphism if and only if $\eta_{\mathbb{C}^k} : F(\mathbb{C}^k) \rightarrow G(\mathbb{C}^k)$ is an isomorphism.*

Proof. Since any vector space is isomorphic to \mathbb{C}^m for some m we have to prove isomorphism for these spaces.

On \mathbb{C}^m acts the diagonal torus which by functoriality acts also on $F(\mathbb{C}^m), G(\mathbb{C}^m)$. By naturality $\eta_{\mathbb{C}^m} : F(\mathbb{C}^m) \rightarrow G(\mathbb{C}^m)$ must preserve weight spaces with respect to the diagonal matrices. Now each weight involves at most k indices and so it can be deduced from the relative weight space for \mathbb{C}^k , for these weight spaces the isomorphism is guaranteed by the hypotheses. \square

For a homogeneous polynomial functor of degree k one can factor (Chap. 5, 2.3):

$$F(A) = L(A^k), \quad \text{hom}(V, W) \xrightarrow{A^k} S^k(\text{hom}(V, W)) \xrightarrow{L} \text{hom}(F(V), F(W))$$

with L a linear map, which by functoriality is also $GL(V) \times GL(W)$ linear. Now

$$S^k(\text{hom}(V, W)) = \bigoplus_\lambda \text{hom}(S_\lambda(V), S_\lambda(W)).$$

Let H be a finite dimensional polynomial representation of $GL(k, \mathbb{C})$ of degree k . This means that the homomorphism $\rho : GL(k, \mathbb{C}) \rightarrow GL(H)$ extends to a polynomial map, homogeneous of degree k , and which we still denote by $\rho : \text{hom}(\mathbb{C}^k, \mathbb{C}^k) \rightarrow \text{End}(H)$.

ρ induces a linear map $T : S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k)) \rightarrow \text{End}(H)$ and the map

$$\pi : S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k)) \otimes H \rightarrow H, \quad \pi(f^k \otimes u) := T(f^k)(u).$$

The homomorphism property of ρ implies that, if $A, B \in \text{hom}(\mathbb{C}^k, \mathbb{C}^k)$, we have $T((AB)^k) = T(A^k)T(B^k)$. If $h, g \in GL(n, \mathbb{C})$, the left and right action of (h, g) on $S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k))$ is defined so that $(h, g).B^k = (hBg^{-1})^k$, thus the map π has the following equivariance properties:

$$(7.2.1) \quad \pi((h, g).a \otimes gu) = h\pi(a \otimes u).$$

Identify

$$S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k)) \otimes H = \bigoplus_{\lambda \vdash k} S_\lambda(\mathbb{C}^k)^* \otimes S_\lambda(\mathbb{C}^k) \otimes H.$$

The right action on $S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k))$ equals on each summand the action on $S_\lambda(\mathbb{C}^k)^*$, while the left action is the one on $S_\lambda(\mathbb{C}^k)$.

Lemma 1. π induces an isomorphism between $\oplus_{\lambda \vdash k} (S_\lambda(\mathbb{C}^k)^* \otimes H)^{GL(n, \mathbb{C})} \otimes S_\lambda(\mathbb{C}^k)$ and H .

Proof. The map π is clearly surjective. From the equivariance properties it is clear that π factors through the projection to the invariants $\oplus_{\lambda \vdash k} (S_\lambda(\mathbb{C}^k)^* \otimes H)^{GL(n, \mathbb{C})} \otimes S_\lambda(\mathbb{C}^k)$, of $GL(n, \mathbb{C})$ acting by the tensor product of the left action on $S^k(\text{hom}(\mathbb{C}^k, \mathbb{C}^k))$ and the given action on H . By additivity we can reduce to the case that $H = S_\mu(\mathbb{C}^k)$ for some $\mu \vdash k$.

Recall that $(S_\lambda(\mathbb{C}^k)^* \otimes S_\mu(\mathbb{C}^k))^{GL(n, \mathbb{C})} = \text{hom}_{GL(n, \mathbb{C})}(S_\lambda(\mathbb{C}^k), S_\mu(\mathbb{C}^k))$ is 0, if $\lambda \neq \mu$ and 1-dimensional, generated by the identity, if $\lambda = \mu$. Thus

$$\oplus_{\lambda \vdash k} (S_\lambda(\mathbb{C}^k)^* \otimes S_\mu(\mathbb{C}^k))^{GL(n, \mathbb{C})} \otimes S_\lambda(\mathbb{C}^k) = (S_\mu(\mathbb{C}^k)^* \otimes S_\mu(\mathbb{C}^k))^{GL(n, \mathbb{C})} \otimes S_\mu(\mathbb{C}^k) = S_\mu(\mathbb{C}^k).$$

Since now π has become a surjective map between two irreducible modules, π is an isomorphism. \square

We apply this to the polynomial representation $F(\mathbb{C}^k)$ induced by the functor F .

Lemma 2. For any vector space V the linear map

$$\pi_V : S^k(\text{hom}(\mathbb{C}^k, V)) \otimes F(\mathbb{C}^k) \rightarrow F(V), \quad \pi_V(f^k \otimes u) := F(f)(u), \quad f \in \text{hom}(\mathbb{C}^k, V), \quad u \in F(\mathbb{C}^k)$$

is $GL(V)$ equivariant and $GL(k, \mathbb{C})$ invariant.

Proof. Given an element $g \in GL(k, \mathbb{C})$ it acts on $f^k \otimes u \in S^k(\text{hom}(\mathbb{C}^k, V)) \otimes F(\mathbb{C}^k)$ by $(f \circ g^{-1})^k \otimes F(g)u$. We have $F(f \circ g^{-1}) = F(f) \circ F(g)^{-1}$ and $\pi_V((f \circ g^{-1})^k \otimes F(g)u) = F(f \circ g^{-1})(F(g)u) = F(f)u$. \square

Theorem. The linear map

$$(7.2.2) \quad \pi_V : [S^k(\text{hom}(\mathbb{C}^k, V)) \otimes F(\mathbb{C}^k)]^{GL(k, \mathbb{C})} \rightarrow F(V), \quad \pi_V(f \otimes u) := F(f)(u),$$

is a functorial isomorphism.

Proof. From Proposition 2, it is enough to prove this statement for $V = \mathbb{C}^k$. This is the content of Lemma 2. \square

By the classification of polynomial representations, we have that $F(\mathbb{C}^k) = \oplus_{\lambda \vdash k} m_\lambda S_\lambda(\mathbb{C}^k)$ for some non negative integers m_i ; in we deduce:

Corollary. A polynomial functor F of degree k is of the form $F(V) = \oplus_{\lambda \vdash k} m_\lambda S_\lambda(V)$.

Polynomial functors can be summed (direct sum) multiplied (tensor product) and composed. All these operations can be extended to a ring whose elements are purely formal differences of functors (a Grothendieck type of ring). In analogy to the theory of characters an element of this ring is called a *virtual functor*.

Proposition 3. The ring of virtual functors is canonically isomorphic to the ring of infinite symmetric functions.

Proof. We identify the functor S_λ with the symmetric function $S_\lambda(x)$. \square

7.3 Plethysm The composition of functors becomes the *Plethysm operation* on symmetric functions. In general it is quite difficult to compute such compositions, even simple ones as $\wedge^i(\wedge^h V)$. There are formulas for $S^k(S^2(V))$, $S^k(\wedge^2(V))$ and some dual ones.

In general the computation $F \circ G$ should be done according to the following:

Algorithm. Apply a polynomial functor G to the space \mathbb{C}^m with its standard basis.

For the corresponding linear group and diagonal torus T , $G(\mathbb{C}^m)$ is a polynomial representation of some dimension N . It then has a basis of T -weight vectors with characters a list of monomials M_i , the character of T on $G(\mathbb{C}^m)$ is $\sum_i^N M_i$ a symmetric function $S_G(x_1, \dots, x_m)$.

If G is homogeneous of degree k this symmetric function is determined as soon as $m \geq k$.

When we apply F to $G(\mathbb{C}^m)$ we use the basis of weight vectors to see that the symmetric function

$$S_{F \circ G}(x_1, \dots, x_m) = S_F(M_1, \dots, M_N).$$

□

Some simple remarks are in order. First of all, given a fixed functor G the map $F \rightarrow F \circ G$ is clearly a ring homomorphism, therefore it is determined by the value on a set of generators. One can choose as generators the exterior powers, in this case the operation $\wedge^i \circ F$ as transformations in F are called λ -operations and written λ^i .

These operations satisfy the basic law: $\lambda^i(a+b) = \sum_{h+k=i} \lambda^h(a)\lambda^k(b)$.

It is also convenient to use as generators the Newton functions $\psi_k = \sum_i x_i^k$ since then:

$$\psi_k(S(x_1, \dots, x_m)) = S(x_1^k, \dots, x_m^k), \quad \psi_k(\psi_h) = \psi_{kh}.$$

All of this can be formalized, giving rise to the theory of λ -rings (cf. [Kn]).

8 Representations of the linear and special linear groups.

8.1 Representations of $SL(V)$, $GL(V)$ Given an n -dimensional vector space V we want to give the complete list of irreducible representations for the general and special linear groups $GL(n) = GL(V)$, $SL(n) = SL(V)$.

From Chap 7, Theorem 1.4 we know that all the irreducible representations of $SL(V)$ appear in the tensor powers $V^{\otimes m}$ and all the irreducible representations of $GL(V)$ appear in the tensor powers $V^{\otimes m}$ tensored with integer powers of the determinant $\wedge^n(V)$. For simplicity we will denote by $D := \wedge^n(V)$ and by convention $D^{-1} := \wedge^n(V)^*$. From what we have already seen the irreducible representations of $GL(V)$ which appear in the tensor powers are the modules $S_\lambda(V)$, $ht(\lambda) \leq n$. They are all distinct since they have distinct characters. Given $S_\lambda(V) \subset V^{\otimes m}$, $\lambda \vdash m$ consider $S_\lambda(V) \otimes \wedge^n(V) \subset V^{\otimes m+n}$. Since $\wedge^n(V)$ is 1-dimensional clearly $S_\lambda(V) \otimes \wedge^n(V)$ is also irreducible. Its character is $S_{\lambda+1^n}(x) = (x_1 x_2 \dots x_n) S_\lambda(x)$ hence (cf. Chap. 2, 6.2.1):

$$(8.1.1) \quad S_\lambda(V) \otimes \wedge^n(V) = S_{\lambda+1^n}(V).$$

We need now a simple Lemma. Let $\mathcal{P}_{n-1} := \{k_1 \geq k_2 \geq \dots \geq k_{n-1} \geq 0\}$ be the set of all partitions (of any integer) of height $\leq n-1$. Consider the polynomial ring $\mathbb{Z}[x_1, \dots, x_n, (x_1 x_2 \dots x_n)^{-1}]$ obtained by inverting $e_n = x_1 x_2 \dots x_n$

Lemma. *i) The ring of symmetric elements in $\mathbb{Z}[x_1, \dots, x_n, (x_1 x_2 \dots x_n)^{-1}]$ is generated by $e_1, e_2, \dots, e_{n-1}, e_n^{\pm 1}$ and it has as basis the elements:*

$$S_\lambda e_n^m, \lambda \in \mathcal{P}_{n-1}, m \in \mathbb{Z}.$$

ii) The ring $\mathbb{Z}[e_1, e_2, \dots, e_{n-1}, e_n]/(e_n - 1)$ has as basis the classes of the elements $S_\lambda, \lambda \in \mathcal{P}_{n-1}$.

Proof. i) Since e_n is symmetric it is clear that a fraction $\frac{f}{e_n^k}$ is symmetric if and only if f is symmetric, hence the first statement. Any element of $\mathbb{Z}[e_1, e_2, \dots, e_{n-1}, e_n^{\pm 1}]$ can be written in a unique way in the form $\sum_{k \in \mathbb{Z}} a_k e_n^k$ with $a_k \in \mathbb{Z}[e_1, e_2, \dots, e_{n-1}]$. We know that the Schur functions $S_\lambda, \lambda \in \mathcal{P}_{n-1}$ are a basis of $\mathbb{Z}[e_1, e_2, \dots, e_{n-1}, e_n]/(e_n)$ and the claim follows.

ii) follows from i). \square

Theorem. *i) The list of irreducible representations of $SL(V)$ is:*

$$(8.1.2) \quad S_\lambda(V), \text{ ht}(\lambda) \leq n-1.$$

ii) The list of irreducible representations of $GL(V)$ is:

$$(8.1.3) \quad S_\lambda(V) \otimes D^k, \text{ ht}(\lambda) \leq n-1, k \in \mathbb{Z}.$$

Proof. i) The group $GL(V)$ is generated by $SL(V)$ and the scalar matrices which commute with every element. Therefore in any irreducible representation of $GL(V)$ the scalars in $GL(V)$ also act as scalars in the representation. It follows immediately that the representation remains irreducible when restricted to $SL(V)$.

Thus we have to understand when two irreducible representations $S_\lambda(V), S_\mu(V)$, with $\text{ht}(\lambda) \leq n, \text{ht}(\mu) \leq n$ are isomorphic once restricted to $SL(V)$.

Any λ can be uniquely written in the form $(m, m, m, \dots, m) + (k_1, k_2, \dots, k_{n-1}, 0)$ or $\lambda = \mu + m 1^n, \text{ht}(\mu) \leq n-1$ and so $S_\lambda(V) = S_\mu(V) \otimes D^m$. Clearly $S_\lambda(V) = S_\mu(V)$ as representations of $SL(V)$. Thus to finish we have to show that, if $\lambda \neq \mu$, are two partitions of height $\leq n-1$ the two $SL(V)$ representations $S_\lambda(V), S_\mu(V)$ are not isomorphic. This follows from the fact that the characters of the representations $S_\lambda(V), \lambda \in \mathcal{P}_{n-1}$ are a basis of the invariant functions on $SL(V)$, by the previous lemma.

ii) We have seen at the beginning of this paragraph, that all irreducible representations of $GL(V)$ appear in the list 8.1.3. Now if two different elements of this list were isomorphic, by multiplying by a high enough power of D we would obtain two isomorphic polynomial representations belonging to two different partitions a contradiction. \square

8.2 The coordinate ring of the linear group

We can interpret the previous theory in terms of the coordinate ring $\mathbb{C}[GL(V)]$ of the general linear group.

Since $GL(V)$ is the open set of $End(V) = V \otimes V^*$ where the determinant $d \neq 0$ its coordinate ring is the localization at d of the ring $S(V^* \otimes V)$ which, under the two actions of $GL(V)$ decomposes as $\bigoplus_{ht(\lambda) \leq n} S_\lambda(V^*) \otimes S_\lambda(V) = \bigoplus_{ht(\lambda) \leq n-1, k \geq 0} d^k S_\lambda(V^*) \otimes S_\lambda(V)$. It follows immediately then that:

$$(8.2.1) \quad \mathbb{C}[GL(V)] = \bigoplus_{ht(\lambda) \leq n-1, k \in \mathbb{Z}} d^k S_\lambda(V^*) \otimes S_\lambda(V).$$

This of course is, for the linear group, the explicit form of formula 3.1.1 of Chap. 7. From it we deduce that:

$$(8.2.2) \quad S_\lambda(V^*) = S_\lambda(V)^*, \quad \forall \lambda, ht(\lambda) \leq n-1, \quad (d^* = d^{-1}).$$

Similarly:

$$(8.2.3) \quad \mathbb{C}[SL(V)] = \bigoplus_{ht(\lambda) \leq n-1} S_\lambda(V^*) \otimes S_\lambda(V).$$

8.3 Determinantal expressions for Schur functions

In this section we want

to remark a determinant development for Schur functions which is often used.

Recall that $V_\lambda = e_T(V^{\otimes n})$ is a quotient of $a_T(V^{\otimes n}) = \bigwedge^{k_1} V \otimes \bigwedge^{k_2} V \dots \otimes \bigwedge^{k_n} V$ where the k_i are the columns of a tableau T and is contained in $s_T(V^{\otimes n}) = S^{h_1} V \otimes S^{h_2} V \dots \otimes S^{h_n} V$ where the h_i are the rows of T . Here one has to interpret both antisymmetrization and symmetrization as occurring respectively in the columns and row indices.¹⁴ The composition $e_T = \frac{1}{p(\lambda)} s_T a_T$ can be viewed as the result of a map

$$\bigwedge^{k_1} V \otimes \bigwedge^{k_2} V \dots \otimes \bigwedge^{k_n} V \rightarrow s_T(V^{\otimes n}) = S^{h_1} V \otimes S^{h_2} V \dots \otimes S^{h_n} V.$$

As representations $\bigwedge^{k_1} V \otimes \bigwedge^{k_2} V \dots \otimes \bigwedge^{k_n} V$ and $S^{h_1} V \otimes S^{h_2} V \dots \otimes S^{h_n} V$ decompose in the direct sum of a copy of V_λ and other irreducible representations.

The character of the exterior power $\bigwedge^i(V)$ is the elementary symmetric function $e_i(x)$. The one of $S^i(V)$ is the function $h_i(x)$ sum of all monomials of degree i .

In the formal ring of symmetric functions there is a formal duality between the elements e_i and the h_j . From the definition of the e_i and from Molien's formula:

$$\sum_{i=0}^{\infty} h_i(x) q^i = \frac{1}{\prod (1 - x_i q)}, \quad \sum_{i=0}^{\infty} (-1)^i e_i(x) q^i = \prod (1 - x_i q), \quad 1 = \left(\sum_{i=0}^{\infty} (-1)^i e_i(x) q^i \right) \left(\sum_{i=0}^{\infty} h_i(x) q^i \right)$$

hence for $m > 0$ we have $\sum_{i+j=m} (-1)^i e_i(x) h_j(x) = 0$. These identity tell us that:

$$\mathbb{Z}[e_1, e_2, \dots, e_i, \dots] = \mathbb{Z}[h_1, h_2, \dots, h_i, \dots]$$

¹⁴it is awkward to denote symmetrization on non consecutive indices as we did, more correctly one should compose with the appropriate permutation which places the indices in the correct positions.

and also that we can present the ring of formal functions with generators e_i, h_j and the previous relations:

$$(8.3.1) \quad \mathbb{Z}[e_1, e_2, \dots, e_i, \dots; h_1, h_2, \dots, h_i, \dots] / \left(\sum_{i+j=m} (-1)^i e_i h_j \right).$$

The mapping $\tau : e_i(x) \rightarrow h_i(x), h_i(x) \rightarrow e_i(x)$ preserves these relations and gives an involutory automorphism in the ring of symmetric functions. Take the Cauchy identity

$$(8.3.2) \quad \sum S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_j \sum_{k=0}^{\infty} h_k(x) y_j^k$$

and multiply it by the Vandermonde determinant $V(y)$, getting

$$\sum S_\lambda(x) A_{\lambda+\varrho}(y) = \prod_j \sum_{k=0}^{\infty} h_k(x) y_j^k V(y).$$

For a given $\lambda = a_1, \dots, a_n$ we see that $S_\lambda(x)$ is the coefficient of the monomial $y_1^{a_1+n-1} y_2^{a_2+n-2} \dots y_n^{a_n}$ and we easily see that this is

$$(8.3.3) \quad \sum_{\sigma \in S_n} \epsilon_\sigma \prod_{i=1}^n h_{\sigma(i)-i+a_i}$$

thus

Proposition. *The Schur function S_λ is the determinant of the $n \times n$ matrix which in the position i, j has the element h_{j-i+a_i} with the convention that $h_k = 0, \forall k < 0$.*

8.4 Skew Cauchy formula We want to complete this discussion with an interesting variation of the Cauchy formula (which is used in the computation of the cohomology of the linear group cf. [AD]).

Given two vector spaces V, W we want to describe $\Lambda(V \otimes W)$ as representation of $GL(V) \times GL(W)$.

Theorem.

$$(8.4.1) \quad \Lambda(V \otimes W) = \sum_{\lambda} S_\lambda(V) \otimes S_{\tilde{\lambda}}(W)$$

Proof. $\tilde{\lambda}$ denotes as in Chap. 1, 1.1, the dual partition.

We argue in the following way, for very k , $\Lambda^k(V \otimes W)$ is a polynomial representation of degree k of both groups hence by the general theory $\Lambda^k(V \otimes W) = \bigoplus_{\lambda \vdash k} S_\lambda(V) \otimes P_\mu(W)$ for some representations $P_\mu(W)$ to be determined. To do it in the stable case where $\dim W = k$ we use the Proposition 6.4, through formula 6.4.4 and compute the multilinear elements of $P_\mu(W)$ as representation of S_n .

For this, as in 6.4, identify $W = \mathbb{C}^k$ with basis e_i and $\bigwedge^k(V \otimes W) = \bigwedge(\bigoplus_{i=1}^k V \otimes e_i) = \bigotimes_{i=1}^k \bigwedge(V \otimes e_i)$. When we restrict to the multilinear elements we have $V \otimes e_1 \wedge \cdots \wedge V \otimes e_k$ which, as representation of $GL(V)$ can be identified to $V^{\otimes n}$ except that the natural representation of the symmetric group $S_n \subset GL(n, \mathbb{C})$ is the canonical action on $V^{\otimes n}$ tensored by the sign representation.

Thus we deduce that, if χ is the multilinear weight,

$$\oplus_{\lambda \vdash k} S_\lambda(V) \otimes M_{\tilde{\lambda}} = \left(\bigwedge^k V \otimes \mathbb{C}^k \right)^\chi.$$

This implies $P_\lambda(W) = M_{\tilde{\lambda}}$ hence $P_\lambda(W) = S_{\tilde{\lambda}}(W)$ from which the claim follows. \square

Remark. In terms of characters formula 8.4.1 is equivalent to the Cauchy formula (C4) of Chap. 2, §7.1:

$$(8.4.2) \quad \prod_{i=1}^n \prod_{j=1}^m (1 + x_i y_j) = \sum_{\lambda} S_\lambda(x) S_{\tilde{\lambda}}(y).$$

There is a simple determinantal formula corollary of this identity as in 8.3.

Here we remark that $\prod_{i=1}^n (1 + x_i y) = \sum_{j=0}^n e_j(x) y^j$ where the e_j are the elementary symmetric functions.

The same reasoning as in 8.3 then gives the formula

$$(8.4.3) \quad S_\lambda(x) = \sum_{\sigma \in S_n} \epsilon_\sigma \prod_{i=1}^n e_{\sigma(i) - i + k_i}$$

where k_i are the columns of $\tilde{\lambda}$ i.e. the rows of λ .

Proposition. *The Schur function S_λ is the determinant of the $n \times n$ matrix which in the position i, j has the element $e_{j - i + k_i}$, the k_i are the rows of λ , with the convention that $e_k = 0$, $\forall k < 0$.*

From the two determinantal formulas found we deduce:

Corollary. *Under the involutive map $\tau : e_i \mapsto h_i$ we have $\tau : S_\lambda \mapsto S_{\tilde{\lambda}}$.*

Proof. In fact when we apply τ to the first determinantal formula for S_λ we find the second determinantal formula for $S_{\tilde{\lambda}}$. \square

9 Branching rules for S_n , standard diagrams

9.1 Murnhagam's rule We wish to describe now a fairly simple recursive algorithm, due to Murnhagam, to compute the numbers $c_\lambda(\mu)$. It is based on the knowledge of the multiplication of $\psi_k S_\lambda$ in the ring of symmetric functions.

We assume the number n of variables to be more than $k + |\lambda|$, i.e. to be in a stable range for the formula.

Let h_i denote the rows of λ , we may as well compute $\psi_k(x)S_\lambda(x)V(x) = \psi_k(x)A_{\lambda+\varrho}(x)$:

$$(9.1.1) \quad \psi_k(x)A_{\lambda+\varrho}(x) = \left(\sum_{i=1}^n x_i^k \right) \left(\sum_{s \in S_n} \epsilon_s x_{s1}^{h_1+n-1} x_{s2}^{h_1+n-2} \dots x_{sn}^{h_n} \right).$$

Indicate by $k_i = h_i + n - i$. We inspect the monomials appearing in the alternating function which is at the right of 9.1.1.

Each term is a monomial with exponents obtained from the sequence k_i by adding to one of them say k_j the number k .

If the resulting sequence has two numbers equal it cannot contribute a term to an alternating sum and so it must be dropped, otherwise reorder it getting a sequence:

$$k_1 > k_2 > \dots k_i > k_j + k > k_{i+1} > \dots k_{j-1} > k_{j+1} > \dots > k_n.$$

Then we see that the partition $\lambda' : h'_1, \dots, h'_i, \dots, h'_n$ associated to this sequence is:

$$h'_t = h_t, \text{ if } t \leq i \text{ or } t > j, \quad h'_t = h_{t-1} + 1 \text{ if } i + 2 \leq t \leq j, \quad h'_{i+1} = h_j + k - j + i + 1.$$

The coefficient of $S_{\lambda'}$ in $\psi_k(x)S_\lambda(x)$ is $(-1)^{j-1-i}$ by reordering the rows.

To understand the λ' which appear let us define the *rim* or boundary of a diagram λ as the set of points $(i, j) \in \lambda$ for which there is no point $(h, k) \in \lambda$ with $i < h, j < k$.

There is a simple way of visualizing the various partitions λ' which arise in this way.

Notice that we have modified $j - i$ consecutive rows, adding a total of k new boxes. Each row of this set, except the bottom row, has been replaced by the row immediately below it plus one extra box, to the bottom row we add the remaining boxes.

This property appears saying that the new diagram λ' is thus any diagram which contains the diagram λ and such that their difference is connected, made of k boxes of the rim of λ' . Intuitively it is like a *slinky*.¹⁵ So one has to think of a slinky made of k boxes, sliding in all possible ways down the diagram.

The sign to attribute to such a configuration is $+1$ if the number of rows occupied is odd, -1 otherwise. More formally we have:

Murnhagam's rule. $\psi_k(x)S_\lambda(x) = \sum \pm S_{\lambda'}$ where λ' runs over all diagrams, such that removing a connected set of k boxes of the rim of λ' we have λ .

The sign to attribute to λ' is $+1$ if the number of rows modified from λ is odd, -1 otherwise.

¹⁵this was explained to me by A. Garsia and refers to a toy sold in novelty shops.

Example Standard tableaux of shape $3, 2$ (compute $d(\lambda) = 5$):

$$\begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array} \quad \begin{array}{cc} 2 & 5 \\ 1 & 3 \end{array} \quad \begin{array}{cc} 4 & 5 \\ 1 & 2 \end{array} \quad \begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array} \quad \begin{array}{cc} 3 & 5 \\ 1 & 2 \end{array} .$$

9.2 Branching rule for S_n We want to draw another important consequence of the previous multiplication formula between Newton functions and Schur functions.

Consider a module M_λ for S_n and consider $S_{n-1} \subset S_n$, we want to analyze M_λ as a representation of the subgroup S_{n-1} . For this we perform a character computation.

We introduce first a simple notation, given two partitions $\lambda \vdash m$, $\mu \vdash n$ we say that $\lambda \subset \mu$ if we have an inclusion of the corresponding Ferrer's diagrams or equivalently if each row of λ is less or equal of the corresponding row of μ .

If $\lambda \subset \mu$ and $n = m + 1$ we will also say that λ, μ are **adjacent**,¹⁶ in this case clearly μ is obtained from λ removing a box lying in a corner.

With these remarks we notice a special case of 9.1:

$$(9.2.1) \quad \psi_1 S_\lambda = \sum_{\mu \vdash |\lambda|+1, \lambda \subset \mu} S_\mu.$$

Consider now an element of S_{n-1} to which is associated a partition ν . The same element as permutation in S_n has associated the partition $\nu 1$. Computing characters we have:

$$(9.2.2) \quad \begin{aligned} \sum_{\lambda \vdash n} c_\lambda(\nu 1) S_\lambda &= \psi_{\nu 1} = \psi_1 \psi_\nu = \sum_{\tau \vdash (n-1)} c_\tau(\nu) \psi_1 S_\tau \\ &= \sum_{\tau \vdash (n-1)} c_\tau(\nu) \sum_{\mu \vdash n, \tau \subset \mu} S_\mu. \end{aligned}$$

In other words:

$$(9.2.3) \quad c_\lambda(\nu 1) = \sum_{\mu \vdash (n-1), \mu \subset \lambda} c_\mu(\nu),$$

This identity between characters becomes in module notations:

Theorem Branching rule for the symmetric group. *When restricting from S_n to S_{n-1} we have:*

$$(9.2.4) \quad M_\lambda = \bigoplus_{\mu \vdash (n-1), \mu \subset \lambda} M_\mu.$$

A remarkable feature of this decomposition is that each irreducible S_{n-1} module appearing in M_λ has multiplicity 1, which implies in particular that the decomposition 9.2.4 is unique.

¹⁶adjacency is a general notion in a poset, here the order is inclusion

The correspondence is established by associating to a standard skew tableau T the sequence of diagrams μ_i where μ_i is obtained from λ by removing the boxes occupied by the numbers $n, n-1, \dots, n-i+1$.

When we apply the branch rule several times, passing from S_n to S_{n-k} we obtain a decomposition of M_λ into a sum of modules indexed by all possible skew standard tableaux of shape λ/μ filled with the numbers $n-k+1, n-k+2, \dots, n-1, n$.

In particular, for a given shape $\mu \vdash n-k$ the multiplicity of M_μ in M_λ equals the number of such tableaux.

In particular we may go all the way down to S_1 and obtain a canonical decomposition of M_λ into 1-dimensional spaces indexed by all the standard tableaux of shape λ . We recover in a more precise way what we discussed in the previous paragraph.

Proposition. *The dimension of M_λ equals the number of standard tableaux of shape λ .*

It is of some interest to discuss the previous decomposition in the following way.

For every k let S_k be the symmetric group on k elements contained in S_n , so that $\mathbb{Q}[S_k] \subset \mathbb{Q}[S_n]$ as subalgebra.

Let Z_k be the center of $\mathbb{Q}[S_k]$. The algebras $Z_k \subset \mathbb{Q}[S_n]$ generate a commutative subalgebra C . In fact for every k we have that the center of $\mathbb{Q}[S_k]$ has a basis of idempotents u_λ indexed by the partitions of k . On any irreducible representation this subalgebra, by the analysis made has a basis of common eigenvectors given by the decomposition into 1 dimensional spaces previously described.

EXERCISE Prove that the common eigenvalues of the u_λ are distinct and so this decomposition is again unique.

Remark. The decomposition just obtained is almost equivalent to selecting a basis of M_λ indexed by standard diagrams. Fixing an invariant scalar product in M_λ we immediately see by induction that the decomposition is orthogonal (because non isomorphic representations are necessarily orthogonal). If we work over \mathbb{R} we can select thus a vector of norm 1 in each summand. This leaves still some sign ambiguity which can be resolved by suitable conventions. The selection of a standard basis is in fact a rather fascinating topic, it can be done in several quite inequivalent ways suggested by very different considerations, we will see some in the next chapters.

A possible goal is to exhibit not only an explicit basis but also explicit matrices for the permutations of S_n or at least for a set of generating permutations (usually one chooses the Coxeter generators $(i \ i+1)$, $i = 1, \dots, n-1$). We will discuss this question when we will deal in a more systematic way with standard tableaux in Chap. 13.

10 Branching rules for the linear group, semistandard diagrams

10.1 Branching rule When we deal with representations of the linear group we can use the character theory which identifies the Schur functions $S_\lambda(x_1, \dots, x_n)$ as the

If we repeat these branching, and markings, we see that a sequence of branchings produces a *semistandard tableau* (cf. Chap. 12 1.1 for a formal definition) like:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 5 & & \\ 5 & & \end{array}$$

as in the case of the symmetric group we can deduce a basis of the representation indexed by semistandard tableaux. Conversely we shall see that one can start from such a basis and deduce a stronger branching theorem which is valid over the integers (Chap. 13, 5.4).

10.2 Pieri's formula Although we shall discuss the general Littlewood Richardson rule in Chap 12, we start with an example, the study of $S_\lambda(V) \otimes \bigwedge^i(V)$.

By the previous analysis this can be computed by computing the product $S_\lambda(x)e_i(x)$, where $e_i(x) = S_{1^i}(x)$ is the character of $\bigwedge^i(V)$. For this set $\lambda = h_1, h_2, \dots, h_r$ and $\{\lambda\}_i := \{\mu \mid \mu \supset \lambda, |\mu| = |\lambda| + i \text{ and each column } k_i \text{ of } \mu \text{ satisfies } h_i \leq k_i \leq h_i + 1\}$.

Theorem Pieri's formula.

$$(10.2.1) \quad S_\lambda(x)e_i(x) = \sum_{\mu \in \{\lambda\}_i} S_\mu(x), \quad S_\lambda(V) \otimes \bigwedge^i(V) = \bigoplus_{\mu \in \{\lambda\}_i} S_\mu(V).$$

Proof. Let $\lambda = h_1, h_2, \dots, h_n$ where we take n sufficiently large and allow some h_i to be 0, and multiply $S_\lambda(x)e_i(x)V(x) = A_{\lambda+\rho}(x)e_i(x)$. We must decompose the alternating function $A_{\lambda+\rho}(x)e_i(x)$ in terms of functions $A_{\mu+\rho}(x)$. Let $l_i = h_i + n - i$, the only way to obtain in $A_{\lambda+\rho}(x)e_i(x)$ a monomial $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ with $m_1 > m_2 > \dots > m_n$ is possibly by multiplying $x_1^{l_1} x_2^{l_2} \dots x_n^{l_n} x_{j_1} x_{j_2} \dots x_{j_i}$. This monomial has strictly decreasing exponents for the variables x_1, \dots, x_n if and only if, the following condition is satisfied. Set $k_a = h_a$ if a does not appear in the indices j_1, j_2, \dots, j_i and $k_a = h_a + 1$ otherwise. We must have that $k_1 \geq k_2 \geq \dots \geq k_n$, in other words $\mu := k_1 \geq k_2 \geq \dots \geq k_n$ is a diagram in $\{\lambda\}_i$. The coefficient of such monomial is 1 hence we deduce the claim:

$$A_{\lambda+\rho}(x)e_i(x) = \sum_{\mu \in \{\lambda\}_i} A_{\mu+\rho}(x).$$

□

We may now deduce also by duality, using the involutory map $\tau : e_i \rightarrow h_i$ (cf. 8.3) and the fact that $h_i(x) = S_i(x)$ is the character of $S^i(V)$, the formula for:

$$(10.2.2) \quad S_\lambda(x)h_i(x) = \sum_{\tilde{\mu} \in \{\tilde{\lambda}\}_i} S_\mu(x), \quad S_\lambda(V) \otimes S^i(V) = \bigoplus_{\tilde{\mu} \in \{\tilde{\lambda}\}_i} S_\mu(V).$$

In other words when we perform $S_\lambda(V) \otimes \bigwedge^i(V)$ we get a sum of $S_\mu(V)$ where μ runs over all diagrams obtained from λ by adding i cases and at most one case in each column, while

when we perform $S_\lambda(V) \otimes S^i(V)$ we get a sum of $S_\mu(V)$ where μ runs over all diagrams obtained from λ by adding i cases and at most one case in each row.¹⁸

10.3 Proof of the rule We can now discuss the branch rule from $GL(n, \mathbb{C})$ to $GL(n-1, \mathbb{C})$. From the point of view of characters it is clear that, if $f(x_1, \dots, x_n)$ is the character of a representation of $GL(n, \mathbb{C})$, $f(x_1, \dots, x_{n-1}, 1)$ is the character of the restriction of the representation to $GL(n-1, \mathbb{C})$. We want thus to compute $S_\lambda(x_1, \dots, x_{n-1}, 1)$. For this we use Cauchy's formula getting:

$$\begin{aligned} \sum_{\lambda} S_\lambda(x_1, \dots, x_{n-1}, 1) S_\lambda(y_1, \dots, y_{n-1}, y_n) &= \prod_{i=1}^{n-1} \prod_{j=1}^n \frac{1}{1 - x_i y_j} \prod_{j=1}^n \frac{1}{1 - y_j} = \\ &= \sum_{\mu} S_\mu(x_1, \dots, x_{n-1}) S_\mu(y_1, \dots, y_{n-1}, y_n) \sum_{j=0}^{\infty} h_j(y_1, \dots, y_{n-1}, y_n). \end{aligned}$$

Use 10.2.2 to get:

$$\begin{aligned} \sum_{\lambda} S_\lambda(x_1, \dots, x_{n-1}, 1) S_\lambda(y_1, \dots, y_{n-1}, y_n) &= \\ &= \sum_{\mu} S_\mu(x_1, \dots, x_{n-1}) \sum_{j=0}^{\infty} \sum_{\tilde{\lambda} \in \{\tilde{\mu}\}_j} S_{\tilde{\lambda}}(y_1, \dots, y_{n-1}, y_n). \end{aligned}$$

Comparing the coefficients of $S_\lambda(y_1, \dots, y_{n-1}, y_n)$ we obtain:

$$S_\lambda(x_1, \dots, x_{n-1}, 1) = \sum_{\mu | \tilde{\lambda} \in \{\tilde{\mu}\}_j} S_\mu(x_1, \dots, x_{n-1}).$$

In other words let $\{\lambda\}^j$ be the set of diagrams which are obtained from λ by removing j cases and at most one case in each row, then the branching of $S_\lambda(\mathbb{C}^n)$ to $GL(n-1)$ is $\bigoplus_{\mu \in \{\lambda\}^j} S_\mu(\mathbb{C}^{n-1})$. In particular we have the property that the irreducible representations which appear come with multiplicity 1.

Since $S_\lambda(x_1, \dots, x_{n-1}, x_n)$ is homogeneous of degree $|\lambda|$ while $\mu \in \{\lambda\}^j$ is homogeneous of degree $|\lambda| - j$ we must have:

$$S_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_{j=0}^{|\lambda|} x_n^j \sum_{\mu \in \{\lambda\}^j} S_\mu(x_1, \dots, x_{n-1}).$$

□

We may iterate the branching and get at each step the branching to $GL(n-i)$ is a direct sum of representations $S_\mu(\mathbb{C}^{n-1})$ with indexing a sequence of diagrams $\mu = \mu_i \subset \mu_{i-1} \subset$

¹⁸these two rules are sometimes referred to as Pieri's rule

$\cdots \subset \mu_0 = \lambda$ where each μ_j is obtained from μ_{j-1} removing u_j cases and at most one case in each row, furthermore we must have $ht(\mu_j) \leq n - j$. In correspondence we develop:

$$S_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_{\mu = \mu_i \subset \mu_{i-1} \subset \cdots \subset \mu_0 = \lambda} x_n^{u_1} x_{n-1}^{u_2} \cdots x_{n-i+1}^{u_i} S_\mu(x_1, \dots, x_{n-j}).$$

If we continue the branching all the way to 1, we decompose the space $S_\lambda(V)$ into 1-dimensional subspaces which are weight vectors for the diagonal matrices, each such weight vector is indexed by a complete *flag* of subdiagrams $\emptyset = \mu_n \subset \mu_1 \subset \cdots \mu_i \subset \cdots \subset \mu_0 = \lambda$ and weight $\prod_{i=1}^n x_i^{u_{n-i+1}}$.

A convenient way to encode such flags of subdiagram is by filling the diagram λ as semistandard tableau, placing $n - i$ in all the cases of μ_i not in μ_{i-1} . The restriction we have placed implies that all the rows are strictly increasing, since we remove at most one box from each row, while the columns are weakly increasing, since we may remove more than one box at each step but we fill with a strictly decreasing sequence of numbers. Thus we get a semistandard tableau T of (column-)shape λ filled with the numbers $1, 2, \dots, n$, conversely such a semistandard tableau corresponds to an allowed sequence of subdiagrams $\emptyset = \mu_n \subset \mu_1 \subset \cdots \mu_i \subset \cdots \subset \mu_0 = \lambda$. Then the monomial $\prod_{i=1}^n x_i^{u_{n-i+1}}$ is deduced directly from T , since u_{n-i+1} is the number of appearances of i in the tableau.

We set $x^T := \prod_{i=1}^n x_i^{u_{n-i+1}}$ and call it the *weight of the tableau* T . Finally we have:

Theorem.

$$(10.3.1) \quad S_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_{T \text{ semistandard of shape } \lambda} x^T.$$

Of course the set of semistandard tableaux depends on the set of numbers $1, \dots, n$. Since the rows have to be filled by strictly increasing numbers we must have a restriction on height. The rows have at most n -elements. Example, $S_{3,2,2}(x_1, x_2, x_3)$ is obtained from the tableaux:

$$\begin{array}{cccccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & & 1 & 3 & & 1 & 2 & & 1 & 3 & & 1 & 2 & & 1 & 2 & \\ 2 & 3 & & 2 & 3 & & 2 & 3 & & 1 & 3 & & 1 & 3 & & 1 & 2 & \end{array}$$

$$S_{3,2,2}(x_1, x_2, x_3) = x_1 x_2^3 x_3^3 + x_1^2 x_2^2 x_3^3 + x_1^2 x_2^3 x_3^2 + x_1^3 x_2 x_3^3 + x_1^3 x_2^2 x_3^2 + x_1^3 x_2^3 x_3.$$

Of course if we increase the number of variables also the number and types of monomials will increase.

We may apply at the same time the branching rule for the symmetric and the linear group. We take an n -dimensional vector space V and consider $V^{\otimes m} = \oplus_{\lambda \vdash m} M_\lambda \otimes S_\lambda(V)$. When we branch on both sides we decompose $V^{\otimes m}$ into a direct sum of 1-dimensional weight spaces indexed by pairs $T_1 | T_2$ where T_1 is a standard diagram of shape $\lambda \vdash m$ and T_2 is a semistandard diagram of shape λ filled with $1, 2, \dots, n$, we will see how this construction of a basis reflects into a combinatorial construction, the Robinson–Schensted correspondence, in Chap. 12, 1.

Notice that it is not evident from 10.3.1 that the function $S_\lambda(x_1, \dots, x_{n-1}, x_n)$ is even symmetric, nevertheless there is a purely combinatorial approach to Schur functions which takes 10.3.1 as definition. In this approach the proof of the symmetry of the formula is done by a simple *marriage* argument.