IV STANDARD MONOMIALS

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Contents

36 The Robinson Schensted correspondence

36.1 We start explaining some combinatorial aspects of representation theory by giving a beautiful combinatorial analogue of the decomposition of tensor space $V^{\otimes n} = \oplus_{\lambda \vdash n} M_\lambda \otimes V_\lambda$.

We thus start from a totally ordered set A which in combinatorics is called an *alphabet*, it plays the role of an ordered basis of V.

Consider next the set of *words* of length n in this alphabet, i.e. sequences $a_1 a_2 \dots a_n$, $a_i \in A$. If $|A| = m$ this is a set with m^n elements in correspondence with a basis of $V^{\otimes n}$.

Next we shall construct from A certain combinatorial objects called *columns, tableaux.* Let us use a pictorial language and illustrate with examples, we shall use as alphabet either the usual alphabet or the integers.

A standard column of length k consists in placing k distinct elements of A in a column (i.e. one in top of the other) so that they decrease from top to bottom:

Example

$$\begin{array}{ccc} & s & \\ t & & 10 \\ & p & \\ g & & 9 \\ & g & \\ e & & 6 \\ & e & \\ b & & 5 \\ & d & \\ a & & 1 \\ & c & \end{array}$$

Next a semistandard tableau will be given by a sequence of columns of non increasing length, which we will place one next to the other, so that we shall identify the *rows* of the tableau. For the rows we assume that the elements going from left to right are weakly increasing (i.e. they can be also equal) **Example**

$$
\begin{array}{llll}
 & & & \\
t & & & \\
g & g & j & \\
e & f & f & \\
b & c & d & u \\
a & b & b & f
\end{array}
\qquad
\begin{array}{lll}
s & & \\
p & & \\
g & h & p \\
e & f & g \\
d & e & f \\
c & d & e
\end{array}
\qquad
\begin{array}{lllll}
 & & & & \\
10 & & & & \\
9 & & & & \\
6 & & & & \\
5 & 5 & 5 & & \\
1 & 1 & 1 & 1 & 1
\end{array}
$$

The main algorithm which we need is that of

inserting a letter in a standard column

Thus we assume to have a letter x and a column c; we begin by placing x on top of the column, if the resulting column is standard this is the result of inserting x in c, otherwise we start going down the column attemting to replace the entry that we encounter with x and we stop at the first step in which this produces a standard column. We thus replace the corresponding letter y with x, obtaining a new column c' and an expelled letter y.

Example

$$
\text{Insert } h \text{ in} \quad
\begin{array}{l} t \\ g \\ e \\ b \\ a \end{array}
\quad \text{we obtain} \quad
\begin{array}{l} t \\ h \\ e \\ b \\ a \end{array}
\quad \text{expelling } g.
$$

It is not excluded that the entering and exiting letter be the same, for instance in the previous case if we wanted to insert g we would also extract g. A special case is when c is empty and then inserting x just creates a column consisting of only x.

Now the first remark is that, from the new column c' and, if present, the expelled letter y one can reconstruct c and x. In fact we try backwards to insert y in c' from bottom upwards, stopping at the first position that makes the new column standard and expelling the relative entry, this is the reconstruction of c, x.

The second point is that we can insert now a letter x in a semistandard tableau T as follows. T is a sequence of columns $c_1, c_2, \dots, c_i$ we first insert x in c_1; if we get an expelled element x_1 we insert it in c_2; if we get an expelled element x_2 we insert it in c_3 etc..

Example Insert d in

$$
\begin{array}{llll}
t & & & \\
g & g & j & \\
e & f & f & \\
b & c & d & u \\
a & b & b & f
\end{array}
$$

get

$$
\begin{array}{lllll}
t & & & & \\
g & g & j & & \\
d & e & f & & \\
b & c & d & u & \\
a & b & b & f & f
\end{array}
$$

insert d in

$$\begin{array}{lll} s & & \\ p & & \\ g & h & p \\ e & f & g \\ d & e & f \\ c & d & e \end{array}$$

and get

$$\begin{array}{llll} s & & & \\ p & & & \\ g & h & p & \\ e & f & g & \\ d & e & f & \\ c & d & d & e \end{array}$$

In any case the new tableau has one extra case occupied by some letter. By the previous remark the knowledge of this case allows recursively to reconstruct the original tableau and the inserted letter.

All this can be made into a recursive construction. Starting from a word $w = a_1 a_2 \dots a_k = a_1 w_1$ of length k, we construct two tableau $I(w)$, $D(w)$ of the same shape, with k entries. The first, called *the inserted tableu* is obtained recursively from the empty tableau inserting a_1 in the tableau $T(w_1)$ (constructed by recursion). This tableau is semistandard and contains as entries exactly the letters appearing in w.

The tableau $D(w)$ is a record of the way in which the tableau $T(w)$ has been recursively constructed. It is filled with all the numbers from 1 to k. Each number appears only once and we will refer to this property as standard[1].

$D(w)$ is constructed from $D(w_1)$, which by inductive hypothesis has the same shape as $T(w_1)$, by inserting in the position of the new case (occupied by the procedure of inserting a_1), the number k.

An example should illustrate the construction. We take the word *standard* and construct the sequence of tableaux $T(w_i)$ associated inserting its letters staring from the rigth, we get:

$$d \qquad \begin{array}[b]{l} r \\ d \end{array} \qquad \begin{array}[b]{ll} r & \\ a & d \end{array} \qquad \begin{array}[b]{ll} d & r \\ a & d \end{array} \qquad \begin{array}[b]{ll} n & \\ d & r \\ a & d \end{array} \qquad \begin{array}[b]{lll} n & & \\ d & r & \\ a & a & d \end{array} \qquad \begin{array}[b]{lll} t & & \\ n & & \\ d & r & \\ a & a & d \end{array} \qquad \begin{array}[b]{lll} s & & \\ n & t & \\ d & r & \\ a & a & d \end{array}$$

the sequence of record tableaux is

$$1 \qquad \begin{array}[b]{l} 2 \\ 1 \end{array} \qquad \begin{array}[b]{ll} 2 & \\ 1 & 3 \end{array} \qquad \begin{array}[b]{ll} 2 & 4 \\ 1 & 3 \end{array} \qquad \begin{array}[b]{ll} 5 & \\ 2 & 4 \\ 1 & 3 \end{array} \qquad \begin{array}[b]{lll} 5 & & \\ 2 & 4 & \\ 1 & 3 & 6 \end{array} \qquad \begin{array}[b]{lll} 7 & & \\ 5 & & \\ 2 & 4 & \\ 1 & 3 & 6 \end{array} \qquad \begin{array}[b]{lll} 7 & & \\ 5 & 8 & \\ 2 & 4 & \\ 1 & 3 & 6 \end{array}$$

[1]some authors prefer doubly standard for this restricted type and standard in place of semistandard.

Theorem Robinson Schensted correspondence. *The map $w \to (D(w), T(w))$ is a bijection between the set of words of lenth k and pairs of tableaux of the same shape of which $D(w)$ is standard and $T(w)$ semistandard.*

Proof. The proof follows from the sequence of previous remarks about the reversibility of the operation of inserting a case. The diagram $D(w)$ allows stepwise to decide which case has been filled at each step and so to reconstruct the insertion procedure and thus the original word.

Remark that, given a semistandard tableau T we may call its content the set of elements appearing in it with the respective multiplicity, similarly we may speak of the content of a given word. The Robinson Schensted correspondence preserves contents.

There is a special case to be observed, assume that the record tableau $D(w)$ is such that, if we read it starting from left to rigth and then from the bottom to the top we find the numbers 1,2, ... , k in increasing order e.g.:

$$\begin{array}{llll} 9 & & & \\ 7 & 8 & & \\ 5 & 6 & & \\ 1 & 2 & 3 & 4 \end{array}$$

Then the word w can be very quickly read off from $T(w)$ it is obtained by reading $T(w)$ from top to bottom and from left to rigth (like in usual language), e.g.[2]:

$$spuntatu \implies \begin{array}{lll} s & & \\ p & u & \\ n & t & \\ a & t & u \end{array} \qquad \begin{array}{lll} 8 & & \\ 6 & 7 & \\ 4 & 5 & \\ 1 & 2 & 3 \end{array}$$

36.2 Knuth equivalence A natural construction from the R-S correspondence is the *Knuth equivalence*

Definition. Two words w_1, w_2 are said to be Knuth equivalent if $T(w_1) = T(w_2)$.

For reasons that will appear in a moment it is important to describe first words of length 3 which are Knuth equivalent.

First consider the 6 words in a, b, c with the 3 letters appearing, we have the simple table of corresponding diagrams:

$$abc \to \begin{array}{lll} a & b & c \end{array} \quad acb \to \begin{array}{ll} c & \\ a & b \end{array} \quad bac \to \begin{array}{ll} b & \\ a & c \end{array} \quad bca \to \begin{array}{ll} b & \\ a & c \end{array} \quad cab \to \begin{array}{ll} c & \\ a & b \end{array} \quad cba \to \begin{array}{l} c \\ b \\ a \end{array}$$

we deduce $acb \equiv cab$, $bac \equiv cab$. For words of length 3 with repetition of letters we further have $aba \equiv baa$ and $bab \equiv bba$.

The main result is

[2] I tried to find a real word, this is maybe dialect

Theorem. *Knuth equivalence is compatible with multiplication of words and it is the minimal compatible equivalence generated by the previous equivalence on words of length 3.*

Dim. We shall prove both statements at the same time. First of all, by the construction of the equivalence, it is clear that if w_1, w_2 are equivalent words and z is a word zw_1 is equivalent to zw_2.

We need to prove conversely that $w_1 z$ is equivalent to $w_2 z$ and we prove it first when w_1, w_2 are of length 3.

For instance let us do the case $wuv \equiv uwv$ for 3 arbitrary letters $u < v < w$. We have to show that inserting these letters in the 2 given orderings in a semistandard tableau T produces the same result.

Let c be the first column of T and T' the tableau obtained from T removing the first column.

Suppose first that inserting in succession uwv in c we place these letters in 3 distinct cases expelling successively some letters f, g, e. From the analysis of the positions in which these letters were it is easily seen that $e < f < g$ and that, inserting wuv we expell f, e, g. Thus in both cases the first column is obtained from c replacing e, f, g with u, v, w. The tableau T' now is modified by inserting the word egf or gef thus we are in a case similar to the one we starded but for a smaller tableau and induction applies.

Some other cases are possible and are similarly analyzed.

If the top element of c is $< u$ the result of insertion is, in both cases, to place u, w on top of c and insert v in T'.

Similar analysis if w or u expels v.

37 Standard monomials

37.1 We start with a somewhat axiomatic approach. Suppose that we have a commutative[3] algebra R over a ring A and let us give a set $S := \{s_1, \dots, s_N\}$ of elements of R together with a partial ordering of S.

Definition.

(1) An ordered product $s_{i_1} s_{i_2} \dots s_{i_k}$ of elements of S is said to be standard if the elements appear in increasing order (with respect to the given partial ordering).

(2) We say that R has a standard monomial theory for S if the standard monomials form a basis of R over A.

Suppose that R has a standard monomial theory for S; given $s, t \in S$ which are not comparable, we have a unique expression, called straigthening law:

(37.1.1) $$st = \sum_i \alpha_i M_i, \ \ \alpha_i \in A, \ \ M_i \text{ standard.}$$

(3) We say that R has a straigthening algorithm if, given any monomial $s_{i_1} s_{i_2} \dots s_{i_k}$ of elements of S by applying in any way sequentially straightening laws, i.e. substitutions of a product of two non comparable elements with the corresponding

[3] this is not essential

combination of standard monomials, we have that this process stops after finitely many steps giving the expression of the given product in terms of standard monomials.

Our prime example will be the following:

$A = \mathbb{Z},\ R := \mathbb{Z}[x_{ij}],\ i = 1, \ldots, n;\ j = 1 \ldots m$ the polynomial ring in nm variables, S will be the set of determinants of minors of the $m \times n$ matrix with entries the x_{ij}.

Combinatorially it is useful to describe a determinant of a $k \times k$ minor as two sequences

$$(i_k\, i_{k-1}\, \ldots i_1 | j_1\, j_2 \ldots j_k)$$

where the i_t are the indeces of the rows while the j_s the indeces of the columns. It is custumary to write the i^s in decreasing and the j^s in increasing order.

The partial ordering will be defined as follows

$$(i_h\, i_{h-1}\, \ldots i_1 | j_1\, j_2 \ldots j_h) \leq (u_k\, u_{k-1}\, \ldots u_1 | v_1\, v_2 \ldots v_k) \text{ iff } h \leq k,\ i_s \geq u_s;\ j_t \geq v_t,\ \forall s, t \leq h$$

In other words if we display the two determinants as rows of a bitableau this is standard.

$$\begin{matrix} u_k\, \ldots u_h\ u_{h-1}\, \ldots u_1 | v_1\, v_2 \ldots v_h \ldots v_k \\ i_h\ i_{h-1}\, \ldots\ i_1\ | j_1\, j_2 \ldots\ j_h \end{matrix}$$

Let us give the full partially ordered set of the 19 minors of a 3×3 matrix:

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In the next sections we will show that $\mathbb{Z}[x_{ij}]$ has a standard monomial theory with respect to this partially ordered set of minors and will explicit the straightening algorithm.

38 Plücker coordinates

38.1 Combinatorial approach We start with a very simple combinatorial approach to which we will soon give a deeper geometrical meaning.

We consider an $n \times m$ matrix $X := (x_{ij})$ of indeterminates (the coordinates on the space $M_{n,m}$ of matrices), assume $n \leq m$. We denote by $x_1, x_2, \dots, x_m$ the columns of a matrix in $M_{n,m}$. We work inside the ring $A := \mathbb{Z}[x_{ij}]$ of polynomials in these variables, we may wish to consider an element in A as a function of the columns and then we will write it as $f(x_1, x_2, \dots, x_m)$. We use the following notation. Given n integers $i_1, i_2, \dots, i_n$ chosen between the numbers $1, 2, \dots, m$ by the symbol:

$$[i_1, i_2, \dots, i_n] \tag{38.1.1}$$

we denote the determinant of the matrix which has as columns the columns of indeces $i_1, i_2, \dots, i_n$ of the matrix X, we call such a polynomial a Plücker coordinate.

The first properties of these symbols are:

S1) $[i_1, i_2, \dots, i_n] = 0$ if and only if 2 indeces coincide.

S2) $[i_1, i_2, \dots, i_n]$ is *antisymmetric* (under permutation of the indeces).

S3) $[i_1, i_2, \dots, i_n]$ is multilinear as a function of the vector variables.

The next property is crucial for the theory. Assume $m \geq 2n$ and consider the product:

$$f(x_1, x_2, \dots, x_{2n}) := [1, 2, \dots, n][n+1, n+2, \dots, 2n]. \tag{38.1.2}$$

Select now an index $k \leq n$ and the $n+1$ variables $x_k, x_{k+1}, \dots, x_n, x_{n+1}, x_{n+2}, \dots, x_{n+k}$.

Next alternate the function f in these variables:

$$\sum_{\sigma \in S_{n+1}} \epsilon_\sigma f(x_1, \dots, x_{k-1} x_{\sigma(k)}, x_{\sigma(k+1)}, \dots, x_{\sigma(n)}, x_{\sigma(n+1)}, \dots, x_{\sigma(n+k)}, x_{n+k+1}, \dots, x_{2n})$$

The result is a multilinear and alternating expression in the $n+1$ vector variables

$$x_k, x_{k+1}, \dots, x_n, x_{n+1}, x_{n+2}, \dots, x_{n+k}.$$

This is necessarily 0 since the vector variables are n dimensional.

The symmetric group S_{2n} acts on the space of $2n$-tuples of vectors x_i by permuting the indeces. Then we have an induced action on functions by

$$(\sigma f)(x_1, x_2, \dots, x_{2n}) := f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(2n)})$$

The function $[1, 2, \dots, n][n+1, n+2, \dots, 2n]$ is alternating with respect to the subgroup $S_n \times S_n$ acting separately on the first n and last n indeces.

Given $k \leq n$ consider the symmetric group S_{n+1} (subgroup of S_{2n}), permuting only the indeces $k, k+1, \ldots, n+k$.

With respect to the action of this subgroup the function $[1, 2, \ldots, n][n+1, n+2, \ldots, 2n]$ is alternating with respect to the subgroup $S_{n-k+1} \times S_k$ of the permutations which permute separately the variables $k, k+1, \ldots, n$ and $n+1, n+2, \ldots, n+k$.

Thus if $g \in S_{n+1}, h \in S_{n-k+1} \times S_k$ we have $ghf(x_1, x_2, \ldots, x_{2n}) = \epsilon_h gf(x_1, x_2, \ldots, x_{2n})$. We deduce that, if $g_1, g_2, \ldots, g_N$ are representatives of the left cosets $g(S_{n-k+1} \times S_k)$:

$$0 = \sum_{i=1}^{N} \epsilon_{g_i} g_i f(x_1, x_2, \ldots, x_{2n}) \tag{38.1.3}$$

The function $gf(x_1, x_2, \ldots, x_{2n})$ is obtained from $f(x_1, x_2, \ldots, x_{2n})$ by replacing the variables x_i with $x_{g^{-1}i}$. As representatives we may choose some canonical elements. Remark that two elements g, k are in the same right coset with respect to $S_{n-k+1} \times S_k$ if and only if the numbers $k, k+1, \ldots, n$ and $n+1, n+2, \ldots, n+k$ correspond to the same sets of elements. Therefore we can choose as representatives for right cosets the following permutations:

choose a number h and h elements out of $k, k+1, \ldots, n$ and another h out of $n+1, n+2, \ldots, n+k$ then exchange in order the first set of h elements with the second, call this perputation an exchange, its sign is $(-1)^h$.

A better choice can be the one obtained by composing such an exchange with a reordering of the indeces in each Plücker coordinate. This is a *shuffle* since it is exactly the operation performed on a deck of cards by a single shuffle.

A shuffle in our case is a permutation σ such that:

$$\sigma(k) < \sigma(k+1) < \ldots < \sigma(n); \quad \text{and} \quad \sigma(n+1) < \sigma(n+2) < \ldots < \sigma(n+k).$$

Thus the basic relation is:

the sum (with signs) of all exchanges, in the polynomial $f(x_1, x_2, \ldots, x_{2n})$, of the variables $x_k, x_{k+1}, \ldots, x_n$, with the variables $x_{n+1}, x_{n+2}, \ldots, x_{n+k}$ equals to 0.

We can now choose any indeces $i_1, i_2, \ldots, i_n; j_1, j_2, \ldots, j_n$ and substitute in the basic relation 33.1.3 to the vector variables $x_h, h = 1, \ldots, n$ the variable x_{i_h} and to $x_{n+h}, h = 1, \ldots, n$ the variable x_{j_h}, the resulting relation will be denoted symbolycally by:

$$\sum \epsilon \begin{vmatrix} i_1, i_2, \ldots, \underline{i_k, \ldots, i_n} \\ \underline{j_1, j_2, \ldots, j_k}, \ldots, j_n \end{vmatrix} \cong 0 \tag{38.1.4}$$

where the symbol should remind us that we should sum over all exchanges of the underlined indeces with the sign of the exchange, and the 2 lines tableau represents the product of the two corresponding Plücker coordinates.

We want to work in a formal way and consider the polynomial ring in the symbols $|i_1, i_2, \ldots, i_n|$ as independent variables only subject to the symmetry conditions S1 and S2.

The expressions 33.1.4 are to be thought as quadratic polynomials in this polynomial ring.

When we substitute to the symbol $|i_1, i_2, \dots, i_n|$ the corresponding Plücker coordinate $[i_1, i_2, \dots, i_n]$ the quadratic polynomials 33.1.4 vanish, i.e. they are *quadratic equations.*

Remark It is possible that several terms of the quadratic relation vanish or cancel each other.

Let us define thus a ring A as the polynomial ring $\mathbb{Z}[\,|i_1, i_2, \dots, i_n|\,]$ modulo the ideal J generated by the quadratic polynomials 33.1.4. The previous discussion shows that we have a homomorphism:

$$j : A = \mathbb{Z}[\,|i_1, i_2, \dots, i_n|\,]/J \to \mathbb{Z}[\,[i_1, i_2, \dots, i_n]\,]$$

One of our goals is:

Theorem. *The map j is an isomorphism.*

38.2 Before we can prove Theorem 33.1 we need to draw a first consequence of the quadratic relations. For the moment when we speak of a Plücker coordinate $[i_1, i_2, \dots, i_n]$ we will mean only the class of $|i_1, i_2, \dots, i_n|$ in A. Of course with Theorem 33.1 this use will be consistent with our previous one.

Consider a product of m Plücker coordinates

$$[i_{11}, i_{12}, \dots, i_{1k}, \dots, i_{1n}][i_{33}, i_{34}, \dots, i_{2k}, \dots, i_{1n}] \dots [i_{m1}, i_{m2}, \dots, i_{mk}, \dots, i_{mn}]$$

and display it as an m lines tableau:

$$(38.2.1) \qquad \begin{vmatrix} i_{11}\ i_{12} & \dots\ i_{1k} & \dots\ i_{1n} \\ i_{33}\ i_{34} & \dots\ i_{2k} & \dots\ i_{1n} \\ & \dots & \\ & \dots & \\ i_{m1}\ i_{m2} & \dots\ i_{mk} & \dots\ i_{mn} \end{vmatrix}$$

Due to the antisymmetry properties of the coordinates let us assume that the indeces in each row are strictly increasing otherwise the product is either 0 or up to sign equals the one in which each row has been reordered.

Definition. *We say that a rectangular tableau is* **standard** *if its rows are strictly increasing and its columns are non decreasing (i.e. $i_{hk} < i_{h\ k+1}$ and $i_{hk} \le i_{h+1\ k}$). The corresponding monomial is then called a* **standard monomial**.

It is convenient, for what follows, to associate to a tableau the word obtained by reading sequentially the numbers on each row:

$$(38.2.2) \qquad i_{11}\ i_{12} \dots i_{1k} \dots i_{1n}, i_{33}\ i_{34} \dots i_{2k} \dots\ i_{1n} \dots \dots i_{m1}\ i_{m2} \dots i_{mk}\ \dots\ i_{mn}$$

and order these words lexicographically. It is then clear that, if the rows of a tableaux T are not strictly increasing, the tableaux T' obtained from T by reordering the rows in an increasing way is strictly smaller than T in the lexicographic order.

The main algorithm is given by:

Lemma. *A product T of two Plücker coordinates*

$$T := \begin{vmatrix} i_1, i_2, \ldots, i_k, \ldots, i_n \\ j_1, j_2, \ldots, j_k, \ldots, j_n \end{vmatrix}$$

can be expressed, through the quadratic relations 33.1.4 as a linear combination with integer coefficients of standard tableaux with 2 rows, preceding T in the lexicographic order and filled with the same indeces $i_1, i_2, \ldots, i_k, \ldots, i_n, j_1, j_2, \ldots, j_k, \ldots, j_n$.

Proof. We may assume first that the 2 rows are strictly increasing. Next, if the tableau is not standard, there is a position k for which $i_k > j_k$ and hence:

$$j_1 < j_2 < \ldots < j_k < i_k < \ldots < i_n.$$

We call such a position a **violation** of the standard form. We then apply the corresponding quadratic equation. In this equation every shuffle, different from the identity, replaces some of the indeces $i_k < \ldots < i_n$ with indeces from $j_1 < j_2 < \ldots < j_k$. It produces thus a tableau which is strictly lower lexicographically than T. Thus, if T is not standard it can be expressed, via the relations 33.1.4 as a linear combination of lexicographically smaller tableaux, we say that we have applied a step of a **straightening algorithm**.

Take the resulting expression, if it is a linear combination of standard tableaux we stop otherwise we can repeat the algorithm to all the non standard tableaux appearing, again each non standard tableau is replaced with a linear combination of strictly smaller tableaux; since the 2 lines tableaux filled with the indeces $i_1, i_2, \ldots, i_k, \ldots, i_n, j_1, j_2, \ldots, j_k, \ldots, j_n$ are a finite set, totally ordered lexicographically, the straightening algorithm must terminate after a finite number of steps, i.e. we obtain an expression with only standard 2 lines tableaux.

We can now pass to the general case:

Theorem. *Any rectangular tableau with m rows is a linear combination with integer coefficients of standard tableaux. The standard form can be obtained by a repeated application of the straightening algorithm to pairs of consecutive rows.*

Proof. The proof is essentially obvious. We first reorder each row then inspect the tableau for a possible violation in two consecutive rows. If there is no violation the tableau is standard otherwise we replace the two given rows with strictly lower two lines tableaux, then we repeat the algorithm. The same reasoning of the lemma shows that the algorithm stops after a finite number of steps.

38.3 Remarks Some remarks on the previous algorithm are in order. First of all we can express the same ideas in a slightly different way. Consider the set S of $\binom{m}{n}$ symbols $|i_1\ i_2\ \ldots i_n|$ where $1 \leq i_1 < i_2 < \ldots < i_n \leq m$. We give to S a partial ordering (the Bruhat order) by declaring:

$$|i_1\ i_2\ \ldots i_n| \leq |j_1\ j_2\ \ldots j_n|, \text{ if and only if, } i_k \leq j_k,\ \forall k = 1, \ldots, n.$$

Remark that $|i_1\ i_2\ \ldots i_n| \leq |j_1\ j_2\ \ldots j_n|$ if and only if the tableau:

$$\begin{vmatrix} i_1\ i_2\ \ldots i_n \\ j_1\ j_2\ \ldots j_n \end{vmatrix}$$

is standard.

In this language a standard monomial is a product

$$[i_{11}, i_{12}, \dots, i_{1k}, \dots, i_{1n}][i_{23}, i_{24}, \dots, i_{2k}, \dots, i_{1n}] \dots [i_{m1}, i_{m2}, \dots, i_{mk}, \dots, i_{mn}]$$

in which the coordinates appearing are increasing from left to right in the given order.

If $a = |i_1\ i_2\ \dots i_n|,\ b = |j_1\ j_2\ \dots j_n|$ and the product

$$ab = \begin{vmatrix} i_1\ i_2\ \dots i_n \\ j_1\ j_2\ \dots j_n \end{vmatrix}$$

is not standard then we can apply a quadratic equation and obtain $ab = \sum_i \lambda_i a_i b_i$ with λ_i coefficients and a_i, b_i obtained from a, b by the shuffle procedure of Lemma 33.2. The proof of that lemma shows in fact that $a < a_i,\ b > b_i$. It is useful to axiomatize the setting.

Definition. *1) Given a commutative algebra R over a commutative ring A a finite set $S \subset R$ and a partial ordering in S we say that a product $s_1 s_2 \dots s_m$ is a* **standard monomial** *if $s_i \in S$ and $s_1 \leq s_2 \leq \dots \leq s_m$ in the given partial ordering.*

2) We say that R is a **quadratic Hodge algebra** *over S if:*

i) The standard monomials are a basis of R over A.

ii) If $a, b \in S$ are not comparable then:

$$ab = \sum_i \lambda_i a_i b_i \tag{38.3.1}$$

with $\lambda_i \in A$ and $a < a_i,\ b > b_i$.

The quadratic relations 33.3.1 give the straightening law for R.

Our main goal is a Theorem which includes Theorem 33.1:

Theorem. *The standard tableaux form a $\mathbb{Z}$ basis of A and A is isomorphic to the ring $\mathbb{Z}[[i_1, i_2, \dots, i_n]] \subset \mathbb{Z}[x_{ij}]$.*

Proof. Since the standard monomials span linearly A and since by construction j is clearly surjective, it suffices to show that the standard monomials are linearly independent in the ring $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$. This point can be achieved in several different ways, we will follow first a combinatorial and then a geometric approach through Schubert cells.

The algebraic combinatorial proof starts as follows:

Remark first that, in a standard tableau, each index i can appear only in the first i columns.

Let us define a tableau to be $k-$ canonical if, for each $i \leq k$, the indeces i which appear are all on the i^{th} column. Of course a tableau (with n columns) is n canonical if and only if the i^{th} column is filled with i for each i, i.e. it is of type $|1\ 2\ 3\ \dots\ n-1\ n|^h$.

Suppose we are given a standard tableau T which is k canonical. Let $p = p(T)$ be the minimum index (greater than k) which appears in T in a column $j < p$. Set $m_p(T)$ be the minimum of such column indeces.

The entries before it in the corresponding row are then the indeces $1\ 2\ 3\ \dots\ j-1$. Given an index j let us consider the set $\mathcal{T}_{p,j,h}$ of k canonical standard tableaux for which p is the minimum index (greater than k) which appears in T in a column $j < p$ and $m_p(T) = j$

and in this column p occurs exactly h times. The main combinatorial remark we make is that, if we substitute p with j in all these positions we see that we have a map which to distinct tableaux associates distinct $k-$canonical tableaux T' with, either $p(T') > p(T)$ or $p(T') = p(T)$ and $m_p(T') > m_p(T)$.

To prove the injectivity it is enough to observe that, if a tableau T is transformed in a tableau T', the tableau T is obtained from T' by substituting with p the last h occurrences of j (which are in the j^{th} column).

The next remark is that if we substitute the variable x_i with $x_i + \lambda x_j$, $(i \neq j)$ in a Plücker coordinate $[i_1, i_2, \ldots, i_n]$.

The result of the substitution is $[i_1, i_2, \ldots, i_n]$, if i does not appear among the indeces $i_1, i_2, \ldots, i_n$ or if both indeces i, j appear.

If instead $i = i_k$ we get

$$[i_1, i_2, \ldots, i_n] + \lambda[i_1, i_2, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_n].$$

Suppose we make the same substitution in a tableau, i.e. in a product of Plücker coordinates; then by expanding the product of the transformed coordinates we obtain a polynomial in λ of degree equal to the number of entries i which appear in rows of T where j does not appear. The leading coefficient of this polynomial is the tableau obtained from T substituting with j all the entries i which appear in rows of T where j does not appear.

After these preliminary remarks we can give a proof of the linear independence of the standard monomials in the Plücker coordinates.

Let us assume by contradiction that:

$$0 = f(x_1, \ldots, x_m) = \sum_i c_i T_i \tag{38.3.2}$$

is a dependence relation among (distinct) standard tableaux.

At least one of the T_i must be different from a power $|1\ 2\ 3\ \ldots\ n-1\ n|^h$ since such a relation is not valid.

Let then p be the minimum index which appears in one of the T_i in a column $j < p$ and let j be the minimum of these column indeces. Let also h be the maximum number of such occurrences of p and assume that the tableaux T_i, $i \leq k$ are the ones for which this happens. This implies that, if in the relation 33.3.1 we substitute x_p with $x_p + \lambda x_j$, where λ is a parameter, we get a new relation which can be develeped as a polynomial in λ of degree h. Since this is identically 0, each coefficient must be zero. Its leading coefficient is:

$$\sum_{i=1}^{k} c_i T_i' \tag{38.3.3}$$

where T_i' is obtained from T_i replacing the h indeces p appearing on the j column with j.

According to our previous combinatorial remark the tableaux T_i' are distinct and thus 33.3.3 is a new relation. We are thus in an inductive procedure which terminates with a relation of type:

$$\sum_k a_k |1\ 2\ 3\ \ldots\ n-1\ n|^k$$

which is a contradiction.

39 The Grassmann variety and its Schubert cells

39.1 Schubert cells The theory of Schubert cells has several interesting features, we start now with an elementary treatment. Let us start with an m dimensional vector space V over a field F and consider $\wedge^n V$ for some $n \leq m$.

Proposition. *1) Given n vectors $v_1, v_2, \ldots, v_n \in V$, the decomposable vector*

$$v_1 \wedge v_2 \wedge \ldots \wedge v_n \neq 0$$

if and only if the vectors are linearly independent.

2) Given n linearly independent vectors $v_1, v_2, \ldots, v_n \in V$ and a vector v:

$$v \wedge v_1 \wedge v_2 \wedge \ldots \wedge v_n = 0$$

if and only if v lies in the subspace spanned by the vectors v_i.

3) If $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ are both linearly independent sets of vectors then:

$$w_1 \wedge w_2 \wedge \ldots \wedge w_n = \alpha v_1 \wedge v_2 \wedge \ldots \wedge v_n$$

with α a non zero scalar if and only if the two sets span the same n dimensional subspace W of V.

Proof. Clearly 2) is a consequence of 1). As for this statement, if the $v_i's$ are linearly independent they may be completed to a basis and then the statement follows from the fact that $v_1 \wedge v_2 \wedge \ldots \wedge v_n$ is one of the basis elements of $\wedge^n V$.

If conversely one of the v_i is a linear combination of the others we replace this expression in the product and have a sum of products with a repeated vector which is then 0.

3) Assume first that they span the same subspace. By hypothesis $w_i = \sum_j c_{ij} v_j$ with $C = (c_{ij})$ an invertible matrix hence:

$$w_1 \wedge w_2 \wedge \ldots \wedge w_n = det(C) v_1 \wedge v_2 \wedge \ldots \wedge v_n.$$

Conversely by 2) we see that

$$W := \{v \in V | v \wedge w_1 \wedge w_2 \wedge \ldots \wedge w_n = 0\}.$$

We have an immediate geometric corollary.

Given an n dimensional subspace $W \subset V$ with basis $v_1, v_2, \ldots, v_n$, the vector $w := v_1 \wedge v_2 \wedge \ldots \wedge v_n$ is non zero ans so it determines a point in the projective space $\mathbb{P}(\wedge^n(V))$ (whose points are the lines in $\wedge^n(V)$).

Part 3) shows that this point is independent of the basis chosen but depends only on the subspace W, thus we can indicate it by the symbol $[W]$.

Part 2) shows that the subspace W is recovered by the point $[W]$ we get:

Corollary. *The map $W \to [W]$ is a 1-1 correspondence between the set of all $n-$dimensional subspaces of V and the points in $\mathbb{P}(\wedge^n V)$ corresponding to decomposable elements.*

Let us denote by $Gr_n(V)$ the set of $n-$dimensional subspaces of V or its image in $\mathbb{P}(\wedge^n(V))$. In order to understand the construction we will be more explicit.

Consider the set $S_{n,m}$ of $n - tuples$ $v_1, v_2, \dots, v_n$ of linearly independent vectors in V.

$$S_{n,m} := \{(v_1, v_2, \dots, v_n) | v_1 \wedge v_2 \wedge \dots \wedge v_n \neq 0\}. \tag{39.1.1}$$

Chosen a basis $e_1, \dots, e_m$ of V, associate to it the basis $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$ of $\wedge^n V$ where $(i_1 < i_2 \dots < i_n)$.

Represent in coordinates an $n - tuple$ $v_1, v_2, \dots, v_n$ of vectors in V as the rows of an $n \times m$ matrix X (of rank n).

In the basis $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$ of $\wedge^n V$ the coordinates of $v_1 \wedge v_2 \wedge \dots \wedge v_n$ are then the determinants of the maximal minors of X.

Explicitely let us denote by $[i_1 i_2 \dots i_n](X)$ the determinant of the maximal minor of X extracted from the columns $i_1\ i_2 \dots\ i_n$ then:

$$v_1 \wedge v_2 \wedge \dots \wedge v_n = \sum_{1 \leq i_1 < i_2 \dots < i_n \leq m} [i_1 i_2 \dots i_n](X) e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}. \tag{39.1.2}$$

Thus $S_{n,m}$ can be identified to the open set of $n \times m$ matrices of maximal rank.

$S_{n,m}$ is called the (algebraic) Stiefel manifold.[4]

Let us indicate by $W(X)$ the subspace of V spanned by the rows of X. The group $Gl(n, F)$ acts by left multiplication on $S_{n,m}$ and if $A \in Gl(n, F),\ X \in S_{n,m}$ we have:

$$W(X) = W(Y), \text{ if and only if, } Y = AX,\ A \in Gl(n, F)$$
$$[i_1 i_2 \dots i_n](X) = det(A)[i_1 i_2 \dots i_n](X).$$

In particular:

$Gr_n(V)$ can be identified to the set of orbits of $Gl(n, F)$ acting by left multiplication on $S_{n,m}$. We want to understand the nature of $Gr_n(V)$ as variety. We need:

Lemma. *Given a map between two affine spaces $\pi : F^k \to F^{k+h}$, of the form:*

$$\pi(x_1, x_2, \dots, x_k) = (x_1, x_2, \dots, x_k, p_1, \dots, p_h)$$

with $p_i = p_i(x_1, x_2, \dots, x_k)$ polynomials, its image is a closed subvariety of F^{k+h} and π is an isomorphism of F^k onto its image.

Proof. The image is the closed subvariety given by the equations:

$$x_{n+i} - p_i(x_1, x_2, \dots, x_k) = 0.$$

The inverse of the map π is the projection

$$(x_1, x_2, \dots, x_k, \dots, x_{k+h}) \to (x_1, x_2, \dots, x_k).$$

We can now state and prove the main result of this section:

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[4]The usual Stiefel manifold is, over , the set of $n - tuple$ $v_1, v_2, \dots, v_n$ of orthonormal vectors in ^{m}, it is homotopic to $S_{n,m}$.

Theorem. *1) $Gr_n(V)$ is a smooth projective subvariety of $\mathbb{P}(\wedge^n(V))$.*

2) The map $X \to W[X]$ from $S_{n,m}$ to $Gr_n(V)$ is a principal $Gl(n,F)$ bundle (locally trivial in the Zariski topology).

Proof. The proof will in fact show something more. Consider the affine open set U of $\mathbb{P}(\wedge^n(V))$ where one of the projective coordinates is not 0 and intersect it with $Gr_n(V)$. We claim that $U \cap Gr_n(V)$ is closed in U and isomorphic to an $n(m-n)$ dimensional affine space and that on this open set the bundle of point 2) is trivial.

To prove this let us assume for simplicity of notations that U is the open set where the coordinate of $e_1 \wedge e_2 \wedge \ldots \wedge e_n$ is not 0. We use in this set the affine coordinates obtained by setting the corresponding projective coordinate equal to 1.

The condition that $W(X) \in U$ is clearly, $[1\ 2 \ldots\ n](X) \neq 0$ i.e. that the submatrix A of X formed from the first n columns is invertible.

Since we have selected this particular coordinate it is useful to display the elements of $S_{n,m}$ in block form as $X = (A\ T)$, (A, T respectively $n \times n$, $n \times m - n$ matrices).

Consider the matrix $Y = A^{-1}X = (1_n\ Z)$ with Z an $n \times m - n$ matrix and $T = AZ$.

It follows that the map $i : Gl(n,F) \times M_{n,m}(F) \to S_{n,m}$:

$$i(A, Z) = (A\ AZ)$$

is an isomorphism of varieties to the open set $S^0_{n,m}$ of $n \times m$ matrices X such that $W(X) \in U$, its inverse is $j : S^0_{n,m} \to Gl(n,F) \times M_{n,m}(F)$ given by:

$$j(A\ T) = (A, A^{-1}T)$$

Thus we have that the set of matrices of type $(1_n\ Z)$ is a set of representatives for the $Gl(n,F)$ orbits of matrices X with $W(X) \in U$. In other words in a vector space W such that $[W] \in U$ there is a unique basis which in matrix form is of type $(1_n\ Z)$. This will also give the required trivialization of the bundle.

Let us now understand in affine coordinates the map from the space of $n \times m - n$ matrices to $U \cap Gr_n(V)$. It is given by computing the determinants of the maximal minors of $X = (1_n\ Z)$. A simple computation shows that:

$$[1\ 2 \ldots i-1\ n+k\ i+1 \ldots\ n](X) = \begin{vmatrix} 1 & 0 & \ldots & 0 & z_{1\ k} & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & z_{2\ k} & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & 1 & z_{i-1\ k} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & z_{i\ k} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & z_{i+1\ k} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & 0 & z_{n\ k} & 0 & 0 & \ldots & 1 \end{vmatrix}$$

This determinant is z_{ik}. Thus Z maps to a point in U in which $n \times (m-n)$ of the coordinates are, up to sign, the coordinates z_{ik}, the remaining coordinates are instead polynomials in these variables. Now we can invoke the previous lemma and conclude.

39.2 Let us display a matrix $X \in S_{nm}$ as a sequence $(v_1, v_2, \dots, v_m)$ of column vectors. So that $AX = (Av_1, Av_2, \dots, Av_m)$. Therefore if $i_1 < i_2 < \dots < i_k$ are indeces the property that the corresponding columns in X are linearly independent is invariant in the $Gl(n, F)$ orbit and depends only on the space $W(X)$ spanned by the rows. In particular we will consider the sequence $i_1 < i_2 < \dots < i_n$ where v_{i_1} is the first non zero column and inductively $v_{i_{k+1}}$ is the first column vector which is linearly independent from $v_{i_1}, v_{i_2}, \dots v_{i_k}$. For an $n-$dimensional subspace W we will set $s(W)$ to be the sequence thus constructed from a matrix X for which $W = W(X)$. We finally set:

$$C_{i_1,i_2,\dots,i_n} = \{W \in Gr_n(V) |\ s(W) = i_1, i_2, \dots, i_n\}. \tag{39.2.1}$$

$C_{i_1,i_2,\dots,i_n}$ is contained in the open set $U_{i_1,i_2,\dots,i_n}$ of $Gr_n(V)$ where the Plücker coordinate $[i_1, i_2, \dots, i_n]$ is not zero. In 34.1 we have seen that this open set can be identified to the set of $n \times m - n$ matrices X for which the submatrix, extracted from the columns $i_1 < i_2 < \dots < i_n$, is the identity matrix. We wish thus to represent our set $C_{i_1,i_2,\dots,i_n}$ by these matrices. We have that the columns $i_1, i_2, \dots, i_n$ are the columns of the identity matrix, the columns before i_1 are 0 and between i_k, i_{k+1} are vectors in which all coordinates greater that k are 0, we will refer to such a matrix as a canonical representative; e.g. $n = 4, m = 11,\ i_1 = 2, i_2 = 6, i_3 = 9, i_4 = 11$:

$$\begin{vmatrix} 0 & 1 & a_1 & a_2 & a_3 & 0 & b_{11} & b_{12} & 0 & c_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{33} & b_{34} & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \tag{39.2.2}$$

Thus $C_{i_1,i_2,\dots,i_n}$ is an affine subspace of $U_{i_1,i_2,\dots,i_n}$ given by the vanishing of certain coordinates. Precisely the free parameters appearing in the columns between i_k, i_{k+1} are displayed in a $k \times (i_{k+1} - i_k - 1)$ matrices, and the ones in the columns after i_n in an $n \times (m - i_n)$ matrix. Thus:

Proposition. *$C_{i_1,i_2,\dots,i_n}$ is a closed subspace of the open set $U_{i_1,i_2,\dots,i_n}$ of the Grassmann variety called a Shubert cell. Its dimension is:*

$$dim(C_{i_1,i_2,\dots,i_n}) = \sum_{k=1}^{n-1} k(i_{k+1} - i_k - 1 - 1) + n(m - i_n) = nm - \frac{n(n-1)}{2} - \sum_{j=1}^{n} i_j. \tag{39.2.3}$$

39.3 Let us make an important remark. By definition of the indeces $i_1, i_2, \dots, i_n$ associated to a matrix X we have that, given a nuber $j < i_k$, the submatrix formed by the first j columns has rank at most $k - 1$. This implies immediately that, if we give indeces $j_1, j_2, \dots, j_n$ for which the corresponding Plücker coordinate is non zero then $i_1, i_2, \dots, i_n \leq j_1, j_2, \dots, j_n$, in other words:

Proposition. *$C_{i_1,i_2,\dots,i_n}$ is the subset of the Grassmann variety where vanish all Plücker coordinates $j_1, j_2, \dots, j_n$ which are not greater or equal than $i_1, i_2, \dots, i_n$ and $i_1, i_2, \dots, i_n$ is non zero.*

We have thus decomposed the Grassmann variety into cells indexed by the elements $i_1, i_2, \dots, i_n$. We have already seen that this set of indeces has a natural total ordering and we wish to understand this order in a geometric fashion, let us indicate by $P_{n,m}$ this partially ordered set. First let us make a simple remark based on the following:

Definition. *In a partially ordered set P we will say that 2 elements a, b are* **adjacent** *if:*

$$a < b, \text{ and if } a \leq c \leq b, \text{ then } a = c, \text{ or } c = b.$$

Remark. *The elements adjacent to $i_1, i_2, \ldots, i_n$ are obtained by selecting any index i_k such that $i_k + 1 < i_{k+1}$ and replacing it by $i_k + 1$ (if $k = n$ the condition is $i_k < m$).*

Proof. The proof is a simple exercise left to the reader.

39.4 There is a geometric meaning of the Schubert cells related to the relative position with respect to a *standard flag.*

Definition. *A flag in a vector space V is an increasing sequence of subspaces:*

$$V_1 \subset V_2 \subset \ldots \subset V_k.$$

A complete flag in an $n-$dimensional space V is a flag:

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V \tag{39.4.1}$$

With $dim(V_i) = i, \ i = 1, \ldots, n$.

Sometimes it is better to use a projective language, so that V_i gives rise to an $i - 1$ dimensional linear subspace in the projective space $\mathbb{P}(V)$.

A complete flag in an n dimensional projective space is a sequence: $\pi_0 \subset \pi_1 \subset \pi_2 \ldots \subset \pi_n$ with π_i linear subspace of dimension i.

We fix the standard flag, with F_i the set of vectors with the first $m-i$ coordinates equal to 0 spanned by the last i vectors of the basis $e_1, \ldots, e_m$.

Given a space $W \in C_{i_1, i_2, \ldots, i_n}$ let $v_1, \ldots, v_n$ be the corresponding normalized basis as rows of an $n \times m$ matrix X for which the submatrix, extracted from the columns $i_1, i_2, \ldots, i_n$, is the identity matrix. A linear combination $\sum_{k=1}^n c_k v_k$ has thus the number c_k as i_k coordinate $1 \leq k \leq n$. Thus for any i we see that

$$W \cap F_i = \{\sum_{k=1}^n c_k v_k | c_k = 0, \text{ if } i_k < m - i\} \tag{39.4.2}$$

thus:

$$dim(F_i \cap W) = n - k, \text{ if and only if } i_k < m - i \leq i_{k+1}.$$

In other words, setting $d_i := dim(F_i \cap W)$ this sequence of numbers is completely determined and determines the numbers $\underline{i} := i_1 < i_2 < \ldots < i_n$, let us denote by $\underline{d}[\underline{i}]$ the sequence thus defined, it has the properties:

$$d_m = n, \ d_1 \leq 1, d_i \leq d_{i+1} \leq d_i + 1.$$

The numbers $m - i_k + 1$ are the ones in which the sequence jumps by 1. E.g. for the example given in (34.2.2) we have the sequence:

$$1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4$$

We observe that, given two sequences

$$\underline{i} := i_1 < i_2 < \ldots < i_n,\ \underline{j} := j_1 < j_2 < \ldots < j_n$$

we have:

$$\underline{i} \leq \underline{j},\quad \text{iff}\quad \underline{d}[\underline{i}] \leq \underline{d}[\underline{j}].$$

39.5 We pass now to a second fact:

Definition.

$$S_{i_1,i_2,\ldots,i_n} := \{W | dim(F_i \cap W) \leq d_i\}.$$

From the previous remarks:

$$C_{i_1,i_2,\ldots,i_n} := \{W | dim(F_i \cap W) = d_i\},\quad S_{\underline{i}} = \cup_{\underline{j} \geq \underline{i}} C_{\underline{j}}.$$

We need now to interpret these notions in a group theoretic way.

We define T to be the subgroup of $GL(m, F)$ of diagonal matrices. Let $I_{i_1,i_2,\ldots,i_n}$ be the $n \times m$ matrix with the identity matrix in the columns $i_1, i_2, \ldots, i_n$ and 0 otherwise. We call this the *center* of the Schubert cell.

Lemma. *The* $\binom{m}{n}$ *decomposable vectors associated to the matrices* $I_{i_1,i_2,\ldots,i_n}$ *are the vectors* $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_n}$. *These are a basis of weight vectors for the group of* T *acting on* $\wedge^n F^m$. *The corresponding points in projective space* $P(\wedge^n F^m)$ *are the fixed points of the action of* T, *the corresponding subspaces are the only* $T-$*stable subspaces of* F^m.

Proof. Given an action of a group G on a vector space the fixed points in the corresponding projective space are the stable 1-dimensional subspaces. If the space has a basis of weight vectors of distinct weights any G stable subspace is spanned by a subset of these vectors, the lemma follows.

We define B to be the subgroup of $GL(m, F)$ which stabilizes the flag F_i. A matrix $X \in B$ if and only if Xe_i is a linear combination of the elements e_j with $j \geq i$. This means that B is the group of lower triangular matrices. From the definitions we have clearly that the sets $C_{i_1,i_2,\ldots,i_n}, S_{i_1,i_2,\ldots,i_n}$ are stable under the action of B in fact we have:

Theorem. $C_{i_1,i_2,\ldots,i_n}$ *is a* B *orbit.*

Proof. Represent the elements of $C_{i_1,i_2,\ldots,i_n}$ by their matrices whose rows are the canonical basis. Consider for any such matrix X an associated matrix $\tilde{X}$ which has the i_k row equal to the k^{th} row of X and otherwise the rows of the identity matrix, for instance for X the matrix of 34.2.2 we have:

(39.5.1)
$$\tilde{X} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & 0 & b_{11} & b_{12} & 0 & c_{11} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{33} & b_{34} & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

We have:

$$X = I_{i_1,i_2,\dots,i_n}\tilde{X}$$

and $\tilde{X}^t \in B$. This implies the theorem.

Finally:

Proposition. *$S_{i_1,i_2,\dots,i_n}$ is the Zariski closure of $C_{i_1,i_2,\dots,i_n}$.*

Proof. $S_{i_1,i_2,\dots,i_n}$ is defined by the vanishing of all Plücker coordinates not greater or equal to $i_1, i_2, \dots, i_n$, hence it is closed and contains $C_{i_1,i_2,\dots,i_n}$.

Since $C_{i_1,i_2,\dots,i_n}$ is a B orbit its closure is a union of B orbits hence a union of Schubert cells. To prove the theorem it is enough by 34.3 to show that, if for some k we have $i_k + 1 < i_{k+1}$, then $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$ is in the closure of $C_{i_1,i_2,\dots,i_n}$. For this consider the matrix $I_{i_1,i_2,\dots,i_n}(b)$ which differs from $I_{i_1,i_2,\dots,i_n}$ only in the $i_k + 1$ column which has 0 in all entries except b in the $i_k + 1$ row. The space defined by this matrix equals the one defined by the matrix which equals $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$ except in the i_k column which has 0 in all entries except b^{-1} in the i_k row. The limit as $b \to \infty$ of this matrix tends to $I_{i_1,i_2,\dots,i_{k-1},i_k+1,i_{k+1},i_n}$.

e.g.

$$W(\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}) = W(\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix})$$

$$\lim_{b\to\infty} W(\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}) = W(\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix})$$

39.6 Standard monomials We want to apply now the theory developed to standard monomials.

We have seen that the Schubert variety $S_{i_1,i_2,\dots,i_n} = S_{\underline{i}}$ is the intersection of the Grassmann variety with the subspace where the coordinates $\underline{j}$ which are not greater or equal than $\underline{i}$ vanish.

We say that a standard monomial is standard on $S_{i_1,i_2,\dots,i_n}$ if it is a product of Plücker coordinates greater or equal than $\underline{i}$.

Theorem. *The monomials standard on $S_{\underline{i}}$ are a basis of the projective coordinate ring of $S_{\underline{i}}$.*

Proof. The monomials which are not standard on $S_{\underline{i}}$ vanish on this variety hence it is enough to show that the monomials standard on $S_{\underline{i}}$ restricted on this variety are linearly independent. Assume by contradiction that $\sum_{k=1}^{n} c_k T_k$ vanishes on $S_{\underline{i}}$, by induction assume that the degree of this relation is minimal.

Let us consider for each monomial T_k its minimal coordinate p_k and write $T_k = p_k T'_k$; then select, among the Plücker coordinates p_k, a maximal coordinate $p_{\underline{j}}$ and decompose the sum as:

$$\sum_{k=1}^{m} c_k p_k T'_k + p_{\underline{j}}(\sum_{k=m+1}^{n} c_k T'_k).$$

By hypothesis $\underline{i} \leq \underline{j}$. Restrict the relation to $S_{\underline{j}}$, all the standard monomials which contain coordinates not greater than $\underline{j}$ vanish so, by choice of $\underline{j}$, we have that $p_{\underline{j}}(\sum_{k=m+1}^{n} c_k T'_k)$ vanishes on $S_{\underline{j}}$. Since $S_{\underline{j}}$ is irreducible and $p_{\underline{j}}$ is non zero on $S_{\underline{j}}$, we must have that $(\sum_{k=m+1}^{n} c_k T'_k)$ vanishes on $S_{\underline{j}}$. This relation has a lower degree and we reach a contradiction.

Of course this theorem is more precise than the standard monomial theorem for the Grassmann variety.

39.7 The theory just developed has deep generalizations, we want to give here a further refinement in the language of roots which is the one suitable for the general theory.

We work over the group $GL(n, \mathbb{C})$ and consider its adjoint action on the space of matrices $M_n(\mathbb{C})$. Let T denote the subgroup of diagonal matrices, if $t \in T$ the diagonal entries of t will be written $t_1, t_2, \dots, t_n$ in order. Consider the elementary matrices $e_{i,j}$, under conjugation by diagonal matrices we have $te_{i,j}t^{-1} = t_i t_j^{-1} e_{i,j}$. If $i \neq j$ the character $t \to t_i t_j^{-1}$ is not 1 and $e_{i,j}$ is (up to scalars) the unique weight vector for this weight, these weights are called the **roots** of the group $GL(n, \mathbb{C})$. It is conveniant to use the additive notation and write:

$$t_i t_j^{-1} = t^{\alpha_{i,j}}.$$

In this notation we consider the Lie adjoint action of the subalgebra $\underline{t}$ of diagonal matrices (the Cartan subalgebra). If h is diagonal with entries h_i then $[h, e_{i,j}] = (h_i - h_j)e_{i,j}$ and we consider $\alpha_{i,j}$ as the linear function on $\underline{t}$ given by $\alpha_{i,j} : h \to h_i - h_j$. If $t = exp(h)$ then $t_i t_j^{-1} = exp(\alpha_{i,j}(h))$. It is important to separate the roots into positive and negative according wether $i < j$ or $i > j$ we indicate these two sets by $\Phi^+, \ \Phi^-$.

Consider the group B^- of lower triangular matrices and in it the subgroup U^- of strictly lower triangular matrices, i.e. triangular matrices with 1 on the diagonal. Similarly we can consider the upper triangular matrices B^+, U^+.

In U^- there are $\binom{n}{2}$ remarkable subgroups, the ones defined for any pair $1 \leq j < i \leq n$, by $U_{i,j} := \{exp(\lambda e_{i,j}) = 1 + \lambda e_{i,j}\}$.

Let $u^- := Lie(U^-)$ denote the Lie algebra of lower triangular matrices with 0 on the diagonal. It is the linear span of the matrices $e_{i,j},\ i > j$. Every matrix $X \in u^-$ satisfies $X^n = 0$ thus the exponential map, restricted to u^- ia a polynomial map:

$$exp : u^- \to U^-,\ X \to \sum_{i=0}^{n-1} \frac{1}{i!} X^i.$$

Similarly any matrix in U^- is of the form $1 - X$ with $X^n = 0$ and thus

$$log : U^- \to u^-,\ 1 - X \to -\sum_{i=1}^{n-1} \frac{X^i}{n}.$$

The two algebraic maps, exp, log are inverse isomorphism. In this case we have:

Proposition. *The exponential of a Lie subalgebra in u^- is an algebraic subgroup. Conversely the logarithm of an algebraic subgroup of U^- is a Lie subalgebra.*

Proof. It is a simple application of the correspondence between Lie algebras and Lie groups. Let M be a Lie algebra. $H := exp(M)$ is a subvariety of U^-, a neighborhood A of 1 in H has the property that $AA \subset H,\ A^{-1} \subset H$. Clearly the Zariski closure of A is H so by continuity of the product and the inverse in the Zariski topology we get $HH \subset H, H^{-1} \subset H$ so H is an algebraic subgroup. The converse is even more trivial.

Consider the conjugation action under T, which normalizes u^- and U^-. We have $exp(tAt^{-1}) = t\,exp(A)t^{-1}$ and so under the correspondence between subalgebras of u^- and subgroups of U^- the subgroups of U^- normalized by T correspond to the subalgebras of u^- with a basis a set of $e_\alpha, \alpha \in S$, if α, β are negative roots $[e_\alpha, e_\beta] = e_{\alpha+\beta}$ if $\alpha + \beta$ is a root, otherwise $[e_\alpha, e_\beta] = 0$. The condition that a set S of roots is such that the span of the e_α is a subalgebra is:

*) If $\alpha, \beta \in S$ and $\alpha + \beta$ is a root then $\alpha + \beta \in S$.

Definition. *A* **root subgroup** *of U^- is an algebraic subgroup normalized by T.*

The previous analysis shows that root subgroups are classified by the special sets of roots S which satisfy the property *).

There is a remarkable class of root subgroups associated to permutations. Let σ be a permutationin S_n, identified with the permutation matrix in $GL(n, \mathbb{C})$. We have $\sigma e_{i,j} \sigma^{-1} = e_{\sigma(i),\sigma(j)}$.

Consider $U^- \cap \sigma^{-1} U^+ \sigma$. This is clearly a root subgroup with Lie algebra spanned by the $e_{i,j} | i > j,$ and $\sigma(i) < \sigma(j)$, it corresponds to the set of roots $\alpha \in \Phi^-$ such that $\sigma(\alpha) \in \Phi^+$.

Consider now the center $I_{i_1,i_2,\dots,i_n}$ of a Schubert cell $C_{i_1,i_2,\dots,i_n}$. Since this point is fixed under T its stabilizer in U^- is a root subgroup. To see which roots belong to the stabilizer apply the element $1 + \lambda e_{i,j}$ to it and see that $I_{i_1,i_2,\dots,i_n}$ is fixed if and only if either $j \notin i_1, i_2, \dots, i_n$ or if $i, j \in i_1, i_2, \dots, i_n$.

We see that the set of roots which do not belong to the stabilizer are

$$\{\alpha_{i,j} | j \in i_1, i_2, \ldots, i_n,\ i \notin i_1, i_2, \ldots, i_n\},$$

it satisfies property *) and so it defines a subgroup $U_{i_1,i_2,\ldots,i_n}$, it is easily verified that this is exactly the set of matrices used in theorem 34.5, of which we gave an example in 34.5.1. We summarize:

Theorem. *Given the center $I_{\underline{i}}$ of a Schubert cell $\underline{i}$, the set of negative roots is partitioned in two sets which correspond to two root subgroups $U^0_{\underline{i}}, U^1_{\underline{i}}$ of U^-. $U^0_{\underline{i}}$ is the stabilizer of $I_{\underline{i}}$ while the orbit map $g \to gI_{\underline{i}}$ establishes an algebraic isomorphism between $U^1_{\underline{i}}$ and the cell $C_{\underline{i}}$.*

Finally we want to study the tangent space to the Grassmann variety in $I_{\underline{i}}$, in particular since this point is T stable we want to study the linear action of T on this tangent space. We consider the open set $U_{\underline{i}}$ represented by $n \times m$ matrices with the identity matrix in the $\underline{i}$ position. It can can be identified to the space of $n \times m - n$ matrices and the point $I_{\underline{i}}$ corresponds to 0. A matrix $X \in T$ with diagonal entries x_i acts on these matrices as follows. Pick a $n \times m$ matrix Y with the identity matrix in the $\underline{i}$ position, multiply it YX^t obtaining a matrix which has in the in the $\underline{i}$ positions the diagonal matrix D with entries $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Thus $D^{-1}YX$ id the transformed element in $U_{\underline{i}}$. Having identified this open set with a space of matrices we see that the given action is linear and so it can be thought as the actin on the tangent space. Finally we see that this tangent space can be thought as spanned by the elementary matrices $e_{k,j},\ k = 1, \ldots, n;\ 1 \leq j \leq m,\ j \notin \underline{i}$. On this matrix the diagonal matrix X acts as $x_{i_k}^{-1} x_j$. These are weights corresponding to negative roots when $i_k < j$ to positive otherwise.

The subspace where the coordinates relative to the positive roots are zero is the orbit under U^- of the center $I_{\underline{i}}$.

We could as well have taken the U^+ orbits and see that:

The subspace where the coordinates relative to the negative roots are zero is the orbit under U^+ of the center $I_{\underline{i}}$.

We have thus two decompositions of the Grassmann variety in cells, Schubert cells and dual cells, the two opposite cells having as center a given point $I_{\underline{i}}$ have complementary dimension and intersect transversally. They decompose the affine space $U_{\underline{i}}$ as a product of two affine spaces. Finally there is a final general point. Fix a 1-parameter group $x_i = t^{h_i}$ with $h_1 > h_2 > h_3 > \ldots > h_n$. The character $x_i x_j^{-1}$ restricted to this subgroup is $t^{h_i - h_j}$. The exponent is thus positive if and only is the weight is a positive root.

39.8 The reader can now verify by the local description of the Grassmann variety around a point $I_{\underline{i}}$ that:

Theorem.
$$C_{\underline{i}} = \{p \in Gr_n(V) | \lim_{t \to \infty} tp = I_{\underline{i}}\}.$$

A similar statement holds for the opposite cells taking the limit as $t \to 0$.

There is a general theory, the theory of Bialinicki-Birula, which gives a general decomposition of projective varieties under torus actions which generalizes the theory we have described here.

Remark also that the symmetric group permutes the centers of the Shubert cells, and $\sigma(I_{i_1,i_2,\dots,i_n}) = I_{\sigma(i_1),\sigma(i_2),\dots,\sigma(i_n)}$ so that their stabilizers in $GL(n,\mathbb{C})$ are conjugate under the symmetric group.

40 Double tableaux

40.1 We want to consider the polynomial ring of functions $\mathbb{Z}[x_{i,j}], i = 1,n;\ j = 1,m$ in the entries x_{ij} of a matrix X of variables, which we think as polynomial functions on the space of $n \times m$ matrices (we can work on $\mathbb{Z}$ since the methods will be combinatorial).

In this ring we will study the relations among the special polynomials obtained as determinants of minors of the matrix X. In order to establish notations we indicate by:

$$(i_1, i_2, \dots, i_k | j_1, j_2, \dots, j_k)$$

the determinat of the minor extracted from the rows $i_1, i_2, \dots, i_k$ and the columns $j_1, j_2, \dots, j_k$. In these notations a variable $x_{i,j}$ is indicated as $(i|j)$.

$$\text{e.g.} \quad (1\,2|1\,3) := x_{11}x_{23} - x_{21}x_{13}.$$

Consider the Grassmann variety $Gr_n(m+n)$ and in it the open set A where the Plücker coordinate extracted from the last n columns is non zero. In §33 we have seen that this open set can be identified with the space $M_{n,m}$ of $n \times m$ matrices.

To a matrix X being associated the space spanned by the rows of $X\ 1_n$.

Remark In more intrinsic terms, given two vector spaces V, W we identify $\hom(V,W)$ to an open set of the Grassmannian in $V \oplus W$ by associating to a map $f : V \to W$ its graph $\Gamma(f) \subset V \oplus W$.

The point 0 corresponds thus to the unique 0 dimensional Schubert cell, which is also the only closed Schubert cell. Thus every Schubert cell has a non empty intersection with this open set.

We use as coordinates in X the variables x_{ij} but we display them as

$$\overline{X} := \begin{vmatrix} x_{n1} & x_{n2} & \dots & x_{n,m-1} & x_{nm} \\ x_{n-1,1} & x_{n-1,2} & \dots & x_{n-1,m-1} & x_{n-1,m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{11} & x_{12} & \dots & x_{1,m-1} & x_{1m} \end{vmatrix}$$

Let us compute a Plücker coordinate $[i_1, i_2, \dots, i_n]$ for $\overline{X}\ 1_n$.

We must distinguish among the indeces i_k appearing, the ones $\leq m$ say $i_1, i_2, \dots, i_h$ and the ones bigger than m, so $i_{h+t} = m + j_t$ where $t = 1, \dots, n-h;\ 1 \leq j_t \leq n$.

The last $n-t$ columns of the submatrix of $X\ 1_n$ extracted from the columns $i_1, i_2, \dots, i_h$ are thus the solumns of indeces $j_1, j_2, \dots, j_{n-h}$ of the identity matrix.

Let Y be an $n \times (n-1)$ matrix, and e_i the i^{th} column of the identity matrix.

The determinant $det(Ye_i)$ of the $n \times n$ matrix obtained from Y adding e_i as last column, equals $(-1)^{n+i} det(Y_i)$, where Y_i is the $n-1 \times n-1$ matrix extracted from Y by deleting the i^{th} row.

In our case therefore we obtain that $[i_1, i_2, \dots, i_h, m+j_1, \dots, m+j_{n-h}]$ equals, up to sign, the determinant $(u_1, u_2, \dots, u_h | i_1, i_2, \dots, i_h)$ where the indeces $u_1, u_2, \dots, u_h$ are complementary, in $1, 2, \dots, n$, to the indeces $n-j_1, n-j_2, \dots, n-j_{n-h}$ (because of our choice of X).

We have defined a bijictive map between the set of Plücker coordinates $[i_1, i_2, \dots, i_n]$ in $1, 2, \dots, n+m$ distinct from the last coordinate and the minors of the $n \times m$ matrix.

40.2 Since the Plücker coordinates are naturally partially ordered we want to understand the same ordering transported on the minors.

Suppose thus that we are given two coordinates:

$$[i_1, i_2, \dots, i_h, m+j_1, \dots, m+j_{n-h}] \le [u_1, u_2, \dots, u_k, m+s_1, \dots, m+s_{n-k}] \tag{40.2.1}$$

corresponding to the minors

$$(v_1, v_2, \dots, v_h | i_1, i_2, \dots, i_h), (w_1, w_2, \dots, w_k | u_1, u_2, \dots, u_k). \tag{40.2.2}$$

From the inequality 35.2.1 we deduce $k \le h$ and $j_{n-a} \le s_{n-k+h-a}$.

For this one should remark that, if from the list $n, n_1, \dots, 2\,1$ we remove the numbers in the positions $j_1, j_2, \dots, j_{n-h}$ obtaining a list $u_1, u_2, \dots, u_h$ and then we delete the numbers in the positions $s_1, s_2, \dots, s_{n-k}$, $k \le h$, with $j_{n-a} \le s_{n-k+h-a}$, obtaining a list $v_1, v_2, \dots, v_k$ displaying the indeces as:

$$\begin{array}{r} v_1, v_2, v_3, v_4, \dots\dots, v_{h-1}, v_h \\ w_1, w_2, \dots, w_{k-1}, w_k \end{array}$$

the resulting tableau has indeces strictly increasing in the rows from right to left while they are non increasing in the columns from top to bottom.

The formal implication is that a standard product of Plücker coordinates, interpreted (up to sign) as a product of determinants of minors, appears as a double tableau, in which the shape of the left side is the reflection of the shape on the right. The rows are strictly increasing and the columns are non decreasing in the right tableau and non increasing in the left, as example. Let $n = 3, m = 5$, consider a tableau:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 7 \\ 2 & 4 & 8 \\ 2 & 6 & 8 \\ 3 & 7 & 8 \end{vmatrix}$$

to this corresponds the double tableau:

$$
\begin{array}{r|l}
3\ 2\ 1 & 1\ 2\ 3 \\
3\ 2\ 1 & 1\ 2\ 4 \\
3\ 1 & 1\ 4 \\
3\ 2 & 2\ 4 \\
2 & 2 \\
3 & 3
\end{array}
$$

We will call such a tableau a double standard tableau.

Of course together with the notion of double standard tableau we also have that of double tableau or *bitableau*, which can be either thought as a product of determinants of minors of decreasing sizes or as a pair of tableaux, called left (or row) and right (or column) tableau of the same size.

If one takes the second point of view, which is useful when analyzing formally the straigthening laws, one may think that the space of 1 line tableaux of size k is a vector space M_k with basis the symbols $(v_1, v_2, \dots, v_h | i_1, i_2, \dots, i_h)$.

For a partition $\lambda := m_1 \geq m_2 \geq \cdots \geq m_t$ the tableaux of shape λ can be thought as the tensor product $M_{m_1} \otimes M_{m_2} \otimes \cdots \otimes M_{m_t}$, when we evaluate a formal tableau into a product of determinants we have a map with non trivial kernel (the space spanned by the straigthening laws).

We want to interpret now the theory of Tableaux in terms of representation theory. For this we want to think of the space of $n \times m$ matrices as $hom(V, W) = W \otimes V^*$ where V is $n-$dimensional and W is $m-$dimensional (as $\mathbb{Z}$ free modules if we work over $\mathbb{Z}$). The algebra R of polynomial functions on $hom(V, W)$ is the symmetric algebra on $W^* \otimes V$.

$$R = S[V^* \otimes W] \tag{40.2.3}$$

The two linear groups $GL(V),\ GL(W)$ act on the space of matrices and on R.

In matrix notations the action of an element $(A, B) \in GL(n) \times GL(m)$ on an $n \times m$ matrix Y is BYA^{-1}. If $e_i,\ i = 1, \dots, n$ is a basis of W and $f_j,\ j = 1, \dots, m$ one of V under the identification $R = S[W^* \otimes V] = \mathbb{Z}[x_{ij}]$, the element $e^i \otimes f_j$ corresponds to x_{ij}:

$$< e^i \otimes f_j | X >:=< e^i | X f_j >=< e^i | \sum_h x_{hj} e_h >= x_{ij}.$$

Geometrically we can think as follows. On the Grassmannian $G_{m,m+n}$ acts the linear group $GL(m+n)$ the action is induced by the action on $n \times m+n$ matrices Y by YC^{-1}, $C \in GL(m+n)$.

The space of $n \times m$ matrices is identifyed to the cell $X1_n$ and is stable under the diagonal subgroup $GL(m) \times GL(n)$. Thus if $C = \begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}$ we have

$$(Y\ 1_n)C^{-1} = (XA^{-1}\ B^{-1}) \equiv (BYA^{-1}\ 1_n) \tag{40.2.4}$$

If now we want to understand the dual action on polynomials we can use the standard dual form $(gf)(u) = f(g^{-1}u)$ for the action on a vector space as follows:

Remark. *The trasforms of the coordinate functions* x_{ij} *under* A, B *are the entries of* $B^{-1}XA$ *where* $X = (x_{ij})$ *is the matrix having as entries the variables* x_{ij}*.*

The action of the two linear groups on rows and columns induces in particular an action of the two groups of diagonal matrices and a double tableau is clearly a weight vector under both groups.

Its weight is read off from the row and column indeces appearing.

We may encode the number of appearences of each index on the row and column tableau as two sequences

$$1^{h_1}\, 2^{h_2} \, \dots \, n^{h_n};\; 1^{k_1}\, 2^{k_2} \, \dots \, m^{k_m}$$

when one wants to stress the combinatorial point of view one calls these two sequences the *content* of the double tableau.

According to the definition of the action of a group on functions we see that the weight of a diagonal matrix in $GL(n)$ acting on rows and with entries b_i is $\prod_{i=1}^n b^{-h_i}$ while the weight of a diagonal matrix in $GL(m)$ acting on columns and with entries a_i is $\prod_{i=1}^m a^{k_i}$.

We come now to the main theorem:

Theorem. *The double standard tableaux are a* $\mathbb{Z}$ *basis of* $\mathbb{Z}[x_{i,j}]$*.*

Proof. Since the standard monomials in the Plücker coordinates are a basis of $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$ we have that the double standard tableaux span the polynomial algebra $\mathbb{Z}[x_{i,j}]$ over $\mathbb{Z}$.

We need to show that they are linearly independent.

One could give a proof in the same spirit as for the ordinary Plücker coordinates or one can argue as follows.

We have identified the space of $n \times m$ matrices with the open set of the Grassmann variety where the Plücker coordinate $p = [m+1, m+2, \dots, m+n]$ is non zero.

There are several remarks to be made:

1. The coordinate p is the maximal element of the ordered set of coordinates, so that, if T is a standard monomial so is Tp.

2. Since a $\mathbb{Z}$ basis of $\mathbb{Z}[[i_1, i_2, \dots, i_n]]$ is given by the tableaux Tp^k where T is a standard tableau not containing p, we have that these tableaux not containing p are a basis over the polynomial ring $\mathbb{Z}[p]$.

3. The algebra $\mathbb{Z}[x_{i,j}]$ equals the quotient algebra $\mathbb{Z}[[i_1, i_2, \dots, i_n]]/(p-1)$.

From 2) and 3) it follows that the image in $\mathbb{Z}[x_{i,j}]$ of the standard monomials which do not end with p are a $\mathbb{Z}$ basis.

But the images of these monomials are the double standard tableaux and the theorem follows.

Point 1 and 2 are clear.

Point 3 is a general fact on projective varieties, if $W \subset P^n$ is a projective variety and A is its homogeneous coordinate ring, the coordinate ring of the affine part of W where a coordinate x is not zero is $A/(x-1)$.

40.3

We now want to see how representation theory runs in a characteristic free fashion using tableaux.

First of all we need to analyze a basic quadratic relation for a two lines tableau. We have to understand the quadratic relation 33.1.4 in terms of double tableaux. Assume thus we have a product of two Plücker coordinates giving a double tableau with two rows of length $a \geq b$ (if not we exchange these two rows). From the previous hypothesis follows that there are two possibilities for the point where is the violation, either the two indeces i_k, j_k are both column indeces or both row indeces. Let us treat the first case, the other is similar. In this case all indeces $i_1, \dots, i_k$ are row indeces while among the $j_k, \dots, j_n$ there can be also row indeces.

In each summand of 33.1.4 some top indeces are exchanged with bottom indeces so we can separate the sum into two contributions, the first in which no row indeces are exchanged and the second with the remaining terms. Thus in the first we have a sum of tableaux always of type a, b while in the second the possible types are $a+t, b-t,\ t > 0$.

Summarizing

Proposition. *A straightening law on the column indeces for a product*

$$T = (i_k \dots i_1 | j_1 \dots j_k)(u_h \dots u_1 | v_1 \dots v_h)$$

of 2 determinants of sizes $k \geq h$ is the sum of two terms $T_1 + T_2$, where T_2 is a sum of tableaux of types $a+t, b-t,\ t > 0$ and T_1 is the sum of the tableaux obtained from T by selecting an index i such that $j_i > v_i$ and performing all possible shuffles among $j_i \dots j_k$ and $v_1 \dots v_i$ while leaving fixed the row indeces

$$\sum \epsilon \begin{array}{r|l} i_n, i_{n-1}, \dots, i_m,\ \dots, i_2, i_1 & j_1, j_2, \dots, \underline{j_k, \dots, j_m, \dots j_n} \\ u_m, \dots, u_2, u_1 & \underline{v_1, v_2, \dots, v_k}, \dots, v_m \end{array} + T_2$$

(similar statement for row straightening).

This, in characteristic 0, is closely connected with a special case of Pieri's formula.

$$a \geq b, \qquad \wedge^a V \otimes \wedge^b V = \oplus_{t=0}^{b} S_{a+t,b-t}(V).$$

In fact when $a = b$ the basic quadratic equation between Plücker coordinates is just the fact that multiplying such coordinnates one obtains only the representation $S_{a,a}(V)$ (??? spiegare meglio).

If we write the tableau as $A|B$ where A, B represent the two tableaux of row and column indeces, we see that the contribution from the first part of the sum is of type:

$$\sum_C c_{B,C} A|C \tag{40.3.1}$$

where the coefficients $c_{B|C}$ are independent of A. Then we give the following:

Definition. *The dominant order for sequences of real numbers is:*

$$(a_1, \dots, a_m) \geq (b_1, \dots, b_n), \qquad \textit{iff}\ , \sum_{i=1}^{h} a_i \geq \sum_{i=1}^{h} b_i,\ \forall h = 1, \dots, n.$$

In particular we obtain a (partial) ordering on partitions.

Remark If we take a vector $(b_1, \dots, b_n)$ and construct $(a_1, \dots, a_m)$ by reordering the entries in decreasing order then $(a_1, \dots, a_m) \geq (b_1, \dots, b_n)$.

Corollary. *1) Given a double tableau of shape λ by the straightening algorithm it is expressed as a linear combination of standard tableaux of shapes $\geq \lambda$ and of the same double weight.*

2) Let S_λ resp. A_λ denote the linear span of all tableaux of shape $\geq \lambda$ resp. of standard tableaux of shape λ. We have

$$S_\mu := \oplus_{\lambda \geq \mu,\ |\lambda|=|\mu|} A_\lambda.$$

Denote by $S'_\mu := \oplus_{\lambda > \mu,\ |\lambda|=|\mu|} A_\lambda$ (which has as basis the double standard tableaux of shape $> \lambda$ in the dominant ordering).

3) The space S_μ / S'_μ is a representation of $GL(V) \times GL(W)$ equipped with a natural basis indexed by double standard tableax $A|B$ of shape μ. When we take an operator $X \in GL(V)$ we have $X(A|B) = \sum_C c_{B,C} A|C$ where C runs over the standard tableaux and the coefficients are independent of A, similarly for $GL(W)$.

Proof (sketch). The first fact follows from the analysis made for two rows and from the previous remark.

By definition, if $\lambda := k_1, k_2, \dots, k_i$ is a partition we have that $T_\lambda := M_{k_1} M_{k_2} \dots M_{k_i}$ is the span of all double tableaux of shape λ. Thus $S_\mu = \sum_{\lambda \geq \mu,\ |\lambda|=|\mu|} T_\lambda$ by the first fact proved.

Part 3 follows from the previous lemma.

Before computing explicitely we relate our work Cauchy's formula.

Let us study the subspace M_k, of the ring of polynomials spanned by the determinants of the $k \times k$ minors.

Given an element $A \in hom(V, W)$ it induces a map $\wedge^k A : \wedge^k V \to \wedge^k W$ thus we have a map:

$$i_k : hom(\wedge^k V, \wedge^k W)^* = \wedge^k W^* \otimes \wedge^k V \to R = S[V^* \otimes W],\ i_k(\phi \otimes u)(A) :=< \phi | Au > .$$

It is clear that M_k is the image of i_k.

Revert for a moment to characteristic 0. Take a Schur functor associated to a partition λ and define:

$$i_\lambda : hom(V_\lambda, W_\lambda)^* = W_\lambda^* \otimes V_\lambda \to R = S[V^* \otimes W],\ i_\lambda(\phi \otimes u)(A) :=< \phi | Au > .$$

Set $M_\lambda = i_\lambda(W_\lambda^* \otimes V_\lambda)$, since the map i_λ is $GL(V) \times GL(W)$ equivariant we can identify simply $M_\lambda = W_\lambda^* \otimes V_\lambda$ since the last one is irreducible.

To connect with our present theory we shall compute the invariants

$$(W_\lambda^* \otimes V_\lambda)^{U^- \times U^+} = (W_\lambda^*)^{U^-} \otimes (V_\lambda)^{U^+}$$

from the highest weigth theory of Chap. 3 we know that V_λ has a unique U^+ fixed vector of weight λ (or ω_λ with the notation 28.3.1) while W_λ^* has a unique U^- fixed vector of weight $-\lambda$, it follows that the space $(W_\lambda^*)^{U^-} \otimes (V_\lambda)^{U^+}$ is formed by the multiples of the bicanonical tableau K_λ.

Theorem. *In characteristic 0, if $\lambda \vdash p$:*

$$S_\lambda = \oplus_{|\lambda| \leq min(m,n),\ \mu \geq \lambda,\ \mu \vdash p} W^*_\lambda \otimes V_\lambda$$

S_μ / S'_μ *is isomorphic to* $W^*_\mu \otimes V_\mu$.

Proof. This comes from Cauchy's formula and the characters of the representations $W^*_\lambda \otimes V_\lambda$ and the remarks on the highest weights since the $U^- \times U^+$ fixed vectors in S_λ are the linear combinations of the bicanonical tableaux K_μ for $|\lambda| \leq min(m,n),\ \mu \geq \lambda,\ \mu \vdash p$.

40.4 U invariants. We work now on the space of $n \times m$ matrices and the polynomial ring $\mathbb{Z}[x_{ij}],\ i = 1, \dots, n;\ j = 1 \dots m$.

Consider the root subroups, which we denoted by $a + \lambda b$, acting on matrices by adding to the a^{th} column the b^{th} column multiplied by λ.

This is the result of the multiplication

$$X(1 + \lambda e_{ba}).$$

A single determinant of a minor $D := (i_1, \dots, i_k | j_1, \dots, j_k)$ is transformed according to the following rule (cf. 33.3):

If a does not appear among the elements j_s or if both a, b appear among these elements D is left invariant.

If $a = j_s$ and b does not appear D is transformed into $D + \lambda D'$ where D' is obtained from D by substitutiong a in the column indeces with b.

Of course a similar analysis is valid for row action.

This implies a combinatorial description of the group action of $G = GL(m) \times GL(n)$ on the space of tableaux.

Let us do it for $\mathbb{Z}$ or a field F, so that the special linear group over F or $\mathbb{Z}$ is generated by the elements $a + \lambda b$. We have described the action of such an element on a single determinant.

The space M_k generated by determinants of size k we have seen is isomorphic to $W^* \otimes V$, the combinatorial action of G on M_k extends to a combinatorial action on $M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_r}$.

Then if $\lambda := k_1, k_2, \dots, k_r$ we can apply multiplication of determinants to obtain an equivariant map

$$M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_r} \to S_\lambda$$

we can view the straigthening laws as a combinatorial description of a set of generators for the kernel of this map, thus we have a combinatorial description by generators and relations of the group action on S_λ.

An argument similar to the one performed in 33.3 shows that.

Given a linear combination $C := \sum_i c_i T_i$ of double standard tableaux, apply to it the transformation $2 + \lambda 1$ we see that we obtain a polynomial in λ. The degree k of this polynomial is the maximum of the number of occurrences of 2 in a tableau T_i as column index not preceded by 1, i.e. 2 occurs on the first column.

Its leading term is of the form $\sum c_i T_i'$ where the sum extends to all the indeces of tableaux T_i where 2 appears in the first column k times and T_i' is obtained from T_i by replacing 2 with 1 in these positions. It is clear that to distinct tableaux T_i correspond distinct tableaux T_i' and thus this leading coefficient is non 0. It follows that:

The element C is invariant under $2+\lambda 1$ if and only if in the column tableau 2 appears only on the second column.

Let us indicate by $A^{1,2}$ this ring of invariant elements under $2+\lambda 1$.

We can now repeat the argument using $3+\lambda 1$ on the elements of $A^{1,2}$ and see that

An element $C \in A^{1,2}$ is invariant under $3+\lambda 1$ if and only if in the column tableau each occurrence of 3 is preceded by 1.

By induction we can define $A^{1,k}$ the ring of invariants under all the root subgroups $i+\lambda 1,\ i \leq k$.

$A^{1,k}$ is spanned by the elements such that in the column tableau no element $i \leq k$ appears on the first column.

We can go up to $k = m$ and obtain tableaux with 1 on the first column of the rigth tableau.

Next we can repeat the argument, on $A^{1,m}$, using the root subgroups $i+\lambda 2,\ i \leq k$. We define thus $A^{2,k}$ to be the ring of invariants under all the root subgroups $i+\lambda 1$ and all the root subgroups $i+\lambda 2,\ i \leq k$.

$A^{2,k}$ is spanned by the elements with 1 on the first column of the rigth tableau and no element $2 < i \leq k$ appears on the second column.

In general, given $i < j \leq m$ consider the subgroup $U_{i,j}$ of upper triangular matrices generated by the root subgroups

$$b+\lambda a,\ a \leq i-1,\ b \leq m;\ b+\lambda i, \quad b \leq j$$

and denote by $A^{i,j}$ the corresponding ring of invariants then:

Theorem. *$A^{i,j}$ is spanned by the elements in which the first $i-1$ columns of the rigth tableau are filled respectively with the numbers $1, 2, \ldots, i-1$ while no number $i < k \leq j$ is on the i column.*

Corollary. *The ring of polynomial invariants under the full group U^+ of upper triangular matrices, acting on the columns, is spanned by the double standard tableaux whose column side has the i^{th} column filled with i for all i. We call such a tableau canonical.*

The main remark is that, given a shape λ there is a *unique* canonical tableau of that given shape characterized by having 1 on the first column, 2 on the second etc. we denote it by C_λ. e.g, m=5:

$$C_{33211} := \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & \\ 1 & & \end{matrix} \quad , \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & \\ 1 & 2 & & & \\ 1 & & & & \\ 1 & & & & \end{matrix}$$

One could have done a similar procedure starting from the subgroups $m+\lambda i$ and getting:

Corollary. *The ring of polynomial invariants under the full group U^- of lower triangular matrices, acting on the columns, is spanned by the double standard tableaux whose column side has the property property that each index $i < m$ appearing is followed by $i+1$. We call such a tableau anticanonical.*

Again given a shape λ there is a *unique* anticanonical tableau of that given shape e.g, m=5:

$$\begin{matrix} 3 & 4 & 5 \\ 3 & 4 & 5 \\ 4 & 5 & \\ 5 & & \\ 5 & & \end{matrix} \quad , \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & \\ 4 & 5 & & & \\ 5 & & & & \\ 5 & & & & \end{matrix}$$

Remark that a tableau can be at the same time canonical and anticanonical if and only if all its rows have length m:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$$

Of course we have a similar statement for the action on rows (the left action) except that the invariants under left action by U^- are left canonical and instead by U^+ action are left anticanonical.

Now we will obtain several interesting corollaries.

40.5 $SL(n)$ invariants

Theorem. *The ring generated by the Plücker coordinates $[i_1, \ldots, i_n]$ extracted from an $n \times m$ matrix, is the ring of invariants under the action of the special linear group on the columns.*

Proof. This is the consequence of the previous corollaries and remarks.

Classically this is used to prove the projective normality of the Grassmann variety and the factoriality of the ring of Plücker coordinates, necessary for the definition of the Chow variety.

The invariants under right U^+ action decompose as

$$\oplus_\lambda V_\lambda$$

where V_λ is the span of all double standard tableaux of shape λ with rigth canonical tableau.

If we act with a diagonal matrix t with entry a_i in the ii position by rigth multiplication this multiplies the i^{th} column by a_i and thus transforms a double tableau T which is rigth canonical and of shape λ into $T \prod a_i^{k_i}$ where k_i is the length of the i^{th} column.

Thus the decomposition $\oplus_\lambda V_\lambda$ is a decomposition into weigth spaces under the Borel subgroup of upper triangular matrices.

If we consider now the left action by $GL(n)$ it commutes with the rigth action and thus each V_λ is a $GL(n)$ submodule.

Assume for instance $n \leq m$. The $U^- \times U^+$ invariants are spanned by those tableaux which are canonical on the left and the rigth and will be called *bicanonical*. These tableax are the polynomials in the determinants $d_k := (k, k-1, \dots, 1|1, 2, \dots, k)$.

A monomial $d_1^{h_1} d_2^{h_2} \dots d_n^{h_n}$ is a bicanonical tableau whose shape λ is determined by the sequence h_i and will be denoted by K_λ.

An argument similar to the previous analysis of U invariants shows that:

Proposition. *1) Any U^- fixed vector in V_λ is multiple of the bicanonical tableau K_λ of shape λ and every U^- stable subspace of V_λ contains K_λ.*

2) V_λ is an indecomposable U^- or $GL(n)$ module.

3) $V_\lambda V_\mu = V_{\lambda+\mu}$ (Cartan multiplication).

Proof. 1) and 2) follow from the previous analysis as for 3) we have to specify the meaning of $\lambda + \mu$. Its correct meaning is by interpreting the partitions as weights for the torus then it is clear that a product of two weight vectors as as weight the sum of the weights. Thus $V_\lambda V_\mu \subset V_{\lambda+\mu}$, to show equality we observe that a standard tableau of shape $\lambda + \mu$ can be written as the product of two standard tableaux of shapes λ and μ.

Remark This is basically the theory of the highest weigth vector in this case. The reader is invited to complete the representation theory of the general linear group in characteristic 0 by this combinatorial approach (as alternative to the one developed in Chapter 3).

An important remark.

Suppose we take a bitableau (not necessarily standard) $T = A|C_\lambda$ (with column tableau canonical) clearly T is invariant by right U^+ action and it has weight λ thus $T \in V_\lambda$.

We should remark that, when we apply to it the row straigthening relations the terms appearing are all of shape λ and no higher.

We may express this fact in the following way. Let V_i denote the space of row tableaux representing determinants of $i \times i$ minors left canonical.

If $\lambda = k_1 \geq k_2 \geq \dots \geq k_r$ the tableaux of shape λ can be viewed as the natural tensor product basis of $V_{k_1} \otimes V_{k_2} \dots \otimes V_{k_r}$.

The straigthening laws for V_λ can be viewed as elements of this tensor product, and we will call the subspace spanned by these elements R_λ. Then

$$V_\lambda := V_{k_1} \otimes V_{k_2} \dots V_{k_r} / R_\lambda$$

Similarly on the rows

$$W^\lambda := W^{k_1} \otimes W^{k_2} \dots W^{k_r} / R^\lambda$$

41Characteristic free Invariant Theory

41.1 Now the characteristic free proof of the first fundamental Theorem.

Let F be an infinite field[5] we want to show the FFT of the linear group for vectors and forms with coefficients in F.

We want now to show that:

[5]one could relax this by working on formal invariants

FFT Theorem. *The ring of polynomial functions on $M_{p,m}(F) \times M_{m,q}(F)$ which are $Gl(m, \mathbb{F})$ invariant is given by the polynomial functions on $M_{p,q}(F)$ composed with the product map, which has as image the determinantal variety of matrices of rank at most m.*

Let us first establish the correct notations. We display a matrix A in $M_{p,m}(F)$ as p rows ϕ_i:

$$A := \begin{vmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_p \end{vmatrix}$$

while a a matrix B in $M_{m,q}(F)$ as q columns x_i:

$$B := \begin{vmatrix} x_1 & x_2 & \dots & x_p \end{vmatrix}$$

The entries of the product are the *scalar products*

$$\overline{x}_{ij} :=< \phi_i | x_j >$$

The theory developed for the determinantal variety implies that the double standard tableaux in these elements $\overline{x}_{ij}$ with at most m columns are a basis of the ring A_m generated by these elements.

Lemma. *Assume that an element $p := \sum c_i T_i \in A_m$, with T_i distinct double standard tableaux, vanishes when we compute it on the variety C_m formed by those pairs A, B of matrices for wich the first m columns x_i of B are linearly dependent; then the column tableau of each T_i starts with the row $1, 2, \dots, m$.*

Similarly if it vanishes when we compute it on the variety R_m formed by those pairs A, B of matrices for wich the first m rows ϕ_i of A are linearly dependent; then the row tableau of each T_i starts with the row $m, m-1, \dots, 1$.

Proof. First of all it is clear that every double standard tableau with column tableau starting with the row $1, 2, \dots, m$ vanishes on C_m and if we split $p = p_0 + p_1$ with p_0 of the previous type also p_1 vanishes on C_m and we must show that $p_1 = 0$ and can assume $p = p_1$.

We observe that, if 1 does not appear in some T_i then evaluating in the subvariety of $M_{p,m}(F) \times M_{m,q}(F)$ where $x_1 = 0$ we get that p vanishes as well as all the elements that contain 1.

We deduce that a non trivial relation on the double standard tableaux in the indeces $1, \dots, p; 2, \dots, q$ which is a contradiction.

Next by substituting $x_1 \to x_1 + \lambda x_2$ in p we have a polynomial vanishing identically on C_m hence its leading term vanishes on C_m, this leading term is a linear combination of double standard tableaux obtained by some of the T_i by substituting all 1 not followed by 2 with 2.

Next we perform $x_1 + \lambda x_3, \dots, x_1 + \lambda x_m$ and in a similar fashion we deduce a new leading term in which the 1 not followed by $2, 3, \dots, m$ are been replaced with larger indeces.

Formally this step does not produce immediately a standard tableau, for instance if we have a row $1\,2\,3\,7 \dots$ and replace 1 by 4 we get $4\,2\,3\,7 \dots$, but this can be mmediately rearranged up to sign to $2\,3\,4\,7 \dots$.

Since by hypothesis p does not contain any tableau with first row in the rigth side equal to $1, 2, 3, \ldots, m$ at the end of this procedure we must get a non trivial linear combination of double standard tableaux in which 1 does not appear in the column indeces and vanishing on C_m. This, we have seen, is a contradiction.

The proof for the rows is identical.

We may assume $p \geq m,\ q \geq m$ and consider $d := (m, m-1, \ldots, 1|1, 2, \ldots, m)$.

Let $\mathcal{A}$ be the open set in the variety of matrices of rank $\leq m$ in $M_{p,q}(F)$ where $d \neq 0$.

Similarly let $\mathcal{B}$ be the open set of elements in $M_{p,m}(F) \times M_{m,q}(F)$ which, under multipication, map to $\mathcal{A}$.

The space $\mathcal{B}$ can be described as pairs of matrices in block form

$$\begin{vmatrix} A \\ B \end{vmatrix}, \ \begin{vmatrix} C & D \end{vmatrix}$$

with multiplication

$$\begin{vmatrix} AC & AD \\ BC & BD \end{vmatrix}$$

and AC invertible.

The complement of $\mathcal{B}$ is formed by those pair of matrices (A, B) in which, either the first m columns x_i of B or the first m rows ϕ_j of A are linearly dependent, i.e. in the notations of the Lemma it is $C_m \cup R_m$.

Thus, setting $\mathcal{B}' := \{(\begin{vmatrix} 1_m \\ B \end{vmatrix}, \ \begin{vmatrix} C & D \end{vmatrix})\}$ with C invertible, we get that $\mathcal{B}$ is isomorphic to the product $GL(m, F) \times \mathcal{B}'$.

By multiplication we get

$$\begin{vmatrix} C & D \\ BC & BD \end{vmatrix}$$

this clearly implies that the matrices $\mathcal{B}'$ are isomorphic to $\mathcal{A}$ under multiplication and that they form a section of the quotient.

It follows that the invariant functions on $\mathcal{B}$ are just the coordinates of $\mathcal{A}$ in other words: After inverting d the ring of invariants is the ring of polynomial functions on $M_{p,q}(F)$ composed with the product map.

We want to use the theory of standard tableaux to show that this denominator can be eliminated.

Let then f be a polynomial invariant that by hypothesis can be multiplied by some power of d to get a polynomial on $M_{p,q}(F)$.

Now we take a minimal such power of d and will show that it is 1.

For this we remark that fd^h for $h \geq 1$ vanishes on the complement of $\mathcal{B}$ and so on the complement of $\mathcal{A}$. Now we only have to show that a polynomial on the determinantal variety that vanishes on the complement of $\mathcal{A}$ is a multiple of d.

By the previous lemma applied to columns and rows we see that each first row of each double standard tableau T_i in the developement of fd^h is $(m, m-1, \ldots, 1|1, 2, \ldots, m)$ i.e. d divides this polynomial as desired.

42Representation theory

42.1 Now revert to representation theory. as for U^+ acting on the right we can analyze U^- acting on the left and decompose the invariants ${}^{U^-}A := \oplus_\mu W^\mu$ where W^μ is the span of the double standard tableaux of shape μ and such that the left tableau is canonical.

Since in V_λ resp. W^μ the right or left tableau is fixed we can describe the elements of these spaces by just one toublau. We have remarked already that the standard basis plus the straigthening algorithm determine combinatorially these modules as representations of the corresponding linear groups.

Now fix a shape $\lambda \vdash k$ and consider as before the spaces $S_\lambda,\ S'_\lambda$ spanned respectively by all double standard tableaux of shapes $\mu \vdash k,\ \mu \geq \lambda$ and $\mu \vdash k,\ \mu > \lambda$.

Both these subspaces are $GL(m) \times GL(n)$ submodules and in a natural way S_λ/S'_λ has a basis indexed by all double standard tableaux of shape λ.

We establish a combinatorial linear isomorphism j_λ between $W^\lambda \otimes V_\lambda$ and S_λ/S'_λ by setting

$$j_\lambda(A \otimes B) := A|B$$

where A is a standard row tableau, B a standard column tableau and $A|B$ the corresponding double tableau.

Theorem. *j_λ is an isomorphism of $G := GL(m) \times GL(n)$ modules.*

Proof. Let $\lambda = k_1 \ldots k_r$. The space of row tableaux of size i is isomorphic to $W^i \otimes V_i$ and the space S_λ/S'_λ is naturally a quotient, as G module, of

$$W^{k_1} \otimes V_{k_1} \otimes W^{k_2} \otimes V_{k_2} \cdots \otimes W^{k_r} \otimes V_{k_r} := W^\lambda \otimes V_\lambda$$

modulo the straigthening relations.

Thus the module structures are deduced from the straightenoing algorithms, thus it is enough to remark that, in the straightening algorithm for a double tableau for instance for a column vialation, the part which does not involve strictly larger tableaux does not change the row tableau and it is independent of the row tableau, this implies that the relations defining S_λ/S'_λ in $(W^{k_1} \otimes W^{k_2} \otimes \cdots \otimes W^{k_r}) \otimes (V_{k_r} \otimes V_{k_1} \otimes V_{k_2} \cdots \otimes V_{k_r})$ are exactly

$$R^\lambda \otimes (V_{k_r} \otimes V_{k_1} \otimes V_{k_2} \cdots \otimes V_{k_r}) + (W^{k_1} \otimes W^{k_2} \otimes \cdots \otimes W^{k_r}) \otimes R_\lambda$$

and is exactly the statement requested.

Now we want to apply this theory to the special linear group.

So we take double tableaux for an $n \times n$ matrix $X = (x_{ij})$, call $A := F[x_{ij}]$ and remark that $d = \det(X) = (n, \ldots, 1|1, \ldots, n)$ is the first coordiante so the double standard tableaux with at most $n-1$ columns are a basis of A over the polynomial ring $F[d]$ hence, setting $d = 1$ in the quotient ring $A/(d-1)$ the double standard tableaux with at most $n-1$ columns are a basis over F.

Moreover d is invariand under the action of $SL(n) \times SL(n)$ and thus $A/(d-1)$ is an $SL(n) \times SL(n)$ module.

We leave to the reader to verify that:

$A/(d-1)$ is the coordinate ring of $SL(n)$ and its $SL(n) \times SL(n)$ module action corresponds to the let and rigth group actions.

The image of the V_λ for λ with at most $n-1$ columns give a decomposition of $A/(d-1)^{U^+}$ (similarly for W^μ).

We want now to analyze the map $\overline{f}(g) := f(g^{-1})$ which exchanges left and rigth actions on standard tableaux.

For this remark that the inverse of a matrix X of determinant 1 is the adjugate $\wedge^{n-1}X$. More generally consider the pairing $\wedge^k F^n \times \wedge^{n-k}F^n \to \wedge^n F^n = F$ under which

$$< \wedge^k X u_1 \wedge \dots u_k | \wedge^{n-k} X v_1 \wedge \dots n_{n-k} >= \wedge^n X u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_{n-k} = u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_{n-k}$$

if we write everything in matrix notations the pairing between basis elements of the two exterior powers is a diagonal $\binom{n}{k}$ matrix of signs ± 1 that we denote by J_k. We thus have:

Lemma. *There is an identification between $(\wedge^k X^{-1})^t$ and $J_k \wedge^{n-k} X$.*

Proof. From the previous pairing and compatibility of the product with the operators $\wedge X$ we have:

$$(\wedge^k X)^t J_k \wedge^{n-k} X = 1_{\binom{n}{k}}$$

thus

$$(\wedge^k X^{-1})^t = J_k \wedge^{n-k} X$$

this implies that under the map $f \to \overline{f}$ a determinant of a k minor of indeces $i_1 \dots i_k | j_1 \dots j_k$ is transformed up to sign, into the $n-k$ minor with complementary row and column indeces.

Corollary. *$f \to \overline{f}$ maps isomorphically V_λ into W^μ where if λ has rows $k_1, k_2, \dots, k_r$ then μ has rows $n-k_r, n-k_{r-1} < \dots, n-k_1$.*

42.2 Symmetric group We want to recover now, in a characteristic free way, the theory developed in Chap. 3.

There are several points to that theory.

Theorem. *If V is a finite dimensional vector space over a field F with at least $m+1$ elements the centralizer of $G := GL(V)$ acting on $V^{\otimes m}$ is spanned by the symmetric group.*

Proof. We have as usual the identification $End_G V^{\otimes m}$ with the invariants $(V^{*\otimes m} \otimes V^{\otimes m})^G$.

Now we claim that the elements of $(V^{*\otimes m} \otimes V^{\otimes m})^G$ are invariants for any extension of the field F and so are multilinear invariants in the sense of Theorem 36.1. Then we have that the multilinear invariants as described by that theorem are spanned by the products $\prod_{i=1}^m < \alpha_{\sigma(i)} | x_i >$ which corresponds to σ and the theorem is proved.

To see that the invariants $u \in (V^{*\otimes m} \otimes V^{\otimes m})^G$ are invariants over any field G remark that it is enough to show that u is invariant under the elementary transformations $1 + \lambda e_{ij},\ \lambda \in G$.

If we write the condition of invariance $u(1+\lambda e_{ij}) = (1+\lambda e_{ij})u$ we see that it is a polynomial in λ of degree $\leq m$ and by hypothesis vanishes on F. By the assomption that F has at least $m+1$ elements it follows that this polynomial is identically 0.

Next we have seen in corollary 34.4 that the space of double tableaux of given weigth has as basis the standard bitableaux of the same weight, we want to apply this idea to *multilinear tableax.*

Let us start with a remark on tensor calculus.

Let V be an $n-$dimensional vector space and consider $V^{*\otimes m}$ the space of multilinear functions on V. If $e_i,\ i=1,\dots,n$ is a basis of V and e^i the dual basis, then $e^{i_1}\otimes e^{i_2}\otimes \cdots \otimes e^{i_m}$ is the induced basis of $V^{*\otimes m}$.

In functional notation $V^{*\otimes m}$ is the space of multilinear functions $f(x_1,\dots,x_m)$ in the arguments $x_i \in V$.

Writing $x_i := \sum x_{ji}e_j$ we have

$$< e^{i_1}\otimes e^{i_2}\otimes\cdots\otimes e^{i_m}|x_1\otimes\cdots\otimes x_m >= \prod_{h=1}^{m} x_{i_h h} \tag{42.2.1}$$

thus the space $V^{*\otimes m}$ is identified to the subspace of the polynomials in the variables $x_{ij},\ i=1,\dots n;\ j=1,\dots,m$ which are *multilinear* in the right indeces $1,2,\dots,m$. From the theory of double standard tableaux it follows immediately that:

Theorem. *$V^{*\otimes m}$ has as basis the double standard tableaux T of size m which are filled with all the indeces $1,2,\dots,m$ and without repetitions, in the column tableau and with the indeces from $1,2,\dots,n$ (with possible repetitions) in the row tableau.*

To these tableau we can apply the theory of 35.4. One should remark that on $V^{*\otimes m}$ we do not obviously have the full action of $GL(n)\times GL(m)$ but only of $GL(n)\times S_m$, where $S_m \subset GL(m)$ as permutation matrices.

Corollary. *1) Given a multilinear double tableau of shape λ by the straightening algorithm it is expressed as a linear combination of multilinear standard tableaux of shapes $\geq \lambda$.*

2) Let S^0_λ resp. A^0_λ denote the linear span of all multilinear tableaux of shape $\geq \lambda$ resp. of multilinear standard tableaux of shape λ. We have

$$S^0_\mu := \oplus_{\lambda\geq\mu,\ |\lambda|=|\mu|}A^0_\lambda.$$

Denote by $S^1_\mu := \oplus_{\lambda>\mu,\ |\lambda|=|\mu|}A^0_\lambda$ (which has as basis the multilinear double standard tableaux of shape $>\lambda$ in the dominant ordering).

3) The space S^0_μ/S^1_μ is a representation of $GL(n)\times S_m$ equipped with a natural basis indexed by double standard tableax $A|B$ of shape μ and with B doubly standard (or multilinear).

It is isomorphic to the tensor product $V^\lambda\otimes M_\lambda$ with V^λ representation of $GL(n)$ with basis the standard tableaux of shape λ and M_λ representation of S_m with basis the multilinear standard tableaux of shape λ.

Proof. It is similar to 35.4 and so we omit it.

In both cases the straightening laws give combinatorial rules to determine the actions of the corresponding groups on the basis of standard diagrams.

42.3 Finally let us consider in $\mathbb{Z}[x_{ij}],\ i,j=1,\dots,n$ the space Σ_n spanned by the monomials of degree n which are multilinear both in the right and left indeces.

These monomials have as basis the $n!$ monomials $\prod_{i=1}^n x_{\sigma(i)i} = \prod_{j=1}^n x_{j\sigma^{-1}(j)},\ \sigma\in S_n$ and also the double standard tableaux which are multilinear or doubly standard both on left and right.

Proposition. *The map $\phi:\sigma\to\prod_{i=1}^n x_{\sigma(i)i},\ \phi:\mathbb{Z}[S_n]\to\Sigma_n$ is an $S_n\times S_n$ linear isomorphism. Where on the group algebra $\mathbb{Z}[S_n]\to\Sigma_n$ we have the usual left and right actions while on Σ_n we have the two actions on left and right indeces.*

Proof. By construction it is an isomorphism of abelian groups and $\phi(abc^{-1})=\prod_{i=1}^n x_{(abc^{-1})(i)i}=\prod_{i=1}^n x_{a(b(i))\,c(i)}$.

As in the previous theory we have a filtration by the shape of double standard tableaux (this time multilinear on both sides or *bimultilinear*) which is stable under the $S_n\times S_n$ action, the factors are tensor products $M^\lambda\otimes M_\lambda$. It corresponds, in a characteristic free form, to the decomposition of the group algebra in its simple ideals.

Corollary. *1) Given a bimultilinear double tableau of shape λ by the straightening algorithm it is expressed as a linear combination of bimultilinear standard tableaux of shapes $\geq\lambda$.*

2) Let S_λ^{00} resp. A_λ^{00} denote the linear span of all bimultilinear tableaux of shape $\geq\lambda$ resp. of bimultilinear standard tableaux of shape λ. We have

$$S_\mu^{00} := \oplus_{\lambda\geq\mu,\ |\lambda|=|\mu|}A_\lambda^{00}.$$

Denote by $S_\mu^{11} := \oplus_{\lambda>\mu,\ |\lambda|=|\mu|}A_\lambda^{00}$ (which has as basis the multilinear double standard tableaux of shape $>\lambda$ in the dominant ordering).

3) The space S_μ^{00}/S_μ^{11} is a representation of $S_n\times S_n$ equipped with a natural basis indexed by double doubly standard (or bimultilinear) tableax $A|B$ of shape μ.

It is isomorphic to the tensor product $M^\lambda\otimes M_\lambda$ with M^λ a representation of S_n with basis the left multilinear standard tableaux of shape λ and M_λ representation of S_n with basis the right multilinear standard tableaux of shape λ.

Proof. It is similar to 35.4 and so we omit it.

Again one could completely reconstruct the characteristic 0 theory from this approach.

43Second fundamental theorem for GL and S_m

43.1 Consider now the more general theory of standard tableaux on a Schubert variety. We have remarked at the beginning of 35.1 that every Schubert cell intersects the affine set A which we have identified to the space $M_{n,m}$ of $n\times m$ matrices. The intersection of a Schubert variety with A will be called an affine Schubert variety. It is indesed by a minor a of the matrix X and indicated by S_a. The proof given in 35.5 and the remarks on the connection between projective and affine coordinate rings give:

Theorem. *Given a minor a of X the ideal of the variety S_a is generated by the determinants of the minors b which are not greater than equal than the minor a. Its affine coordinate ring has a basis formed by the standard monomials in the determinants of the remaining minors.*

There is a very remarkable special case of this theorem. Choose the $k \times k$ minor whose row and column indeces are the first indeces $1, 2, \dots, k$. One easily verifies:

A minor b is not greater or equal than a if and only if it is a minor or rank $> k$. Thus S_a is the determinantal variety of matrices of rank at most k. We deduce:

Theorem. *The ideal I_k generated by the determinants of the $k+1 \times k+1$ minors is prime (in the polynomial ring $A[x_{i,j}]$ over any integral domain A).*

The standard tableaux which contain at least a minors of rank $\geq k+1$ are a basis of the ideal I_k.

The standard tableaux formed with minors of rank at most k are a basis of the coordinate ring $A[x_{i,j}]/I_k$.

Proof. The only thing to be remarked is that a determinant of a minor of rank $s > k+1$ can be expanded, by Laplace rule as a linear combination of determinants of $k+1$ minors. So these elements generate the ideal defined by the Plücker coordinates which are not greater than a.

Over a field the variety defined is the determinantal variety of matrices of rank at most k.

The first fundamental theorem for the general linear group over a field F has been formulated in §15.4 and in the previous paragraph as follows.

We are given an $m-$dimensional vector space V.

The ring of polynomial functions on $(V^*)^p \times V^q$ which are $GL(V)$ invariant is generated by the functions $< \alpha_i | v_j >$.

Equivalently the ring of polynomial functions on $M_{p,m} \times M_{m,q}$ which are $Gl(m, F)$ invariant is given by the polynomial functions on $M_{p,q}$ composed with the product map, which has as image the determinantal variety of matrices of rank at most m. Thus the theorem 35.3 can be interpreted as:

Theorem. *(Second fundamental theorem for the linear group).*

Every relation among the invariants $< \alpha_i | v_j >$ is in the ideal of the determinants of the $m+1$ minors of the matrix formed by the $< \alpha_i | v_j >$.

43.2 The second fundamental theorem for the symmetric group We have seen that the space of $GL(V)$ endomorphisms of $V^{\otimes n}$ is spanned by the symmetric group S_n, we have a linear isomorphism between the space of operators on $V^{\otimes n}$ spanned by the permutations and the space of multilinear invariant functions.

To a permutation σ corresponds f_σ.

$$f_\sigma(\alpha_1, \alpha_2, \dots, \alpha_n, v_1, v_2, \dots, v_n) = \prod_{i=1}^{n} < \alpha_{\sigma i} | v_i > .$$

In more formal words f_σ is obtained by evaluating the variables x_{hk} in the invariants $< \alpha_h | v_k >$ the monomial $\prod_{i=1}^{n} x_{\sigma i, i}$. We want to analyze the relations among these invariants. We know that such relations are the intersection of the linear span of the given monomials with the determinantal ideal.

Now the span of the multilinear monomials $\prod_{i=1}^{n} x_{\sigma i,i}$ is the span of the double tableaux with n boxes in which both the right and left tableau are filled with the n distinct integers $1, \dots, n$.

Theorem. *The intersection of the ideal I_k with the span of multilinear monomials corresponds to the two sided ideal, of the algebra of the symmetric group S_n, generated by the antisymmetrizer $\sum_{\sigma \in S_{k+1}} \epsilon_\sigma \sigma$ in $k+1$ elements.*

Proof. By the previous paragraph it is enough to remark that this antisymmetrizer corresponds to the polynomial

$$(k+1, k, \dots, 2, 1 | 1, 2, \dots, k, k+1) \prod_{j=k+2}^{m} (j|j)$$

43.3 More standard monomial theory We have seen in §the two plethysm formulas for $S[S^2(V)]$ and $S[\wedge^2[V]$, we want to give now a combinatorial interpretation of these formulas.

We think of the first algebra over $\mathbb{Z}$ as the polynomial ring $\mathbb{Z}[x_{ij}]$ is a set of variables x_{ij} subject to the symmetry condition $x_{ij} = x_{ji}$ while the second algebra is the polynomial ring $\mathbb{Z}[y_{ij}]$ is a set of variables $y_{ij},\ i \neq j$ subject to the skew symmetry condition $y_{ij} = -y_{ji}$.

In the first case we wil display the determinant of a $k \times k$ minor extracted from the rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ as a two rows tableau

$$\begin{vmatrix} i_1, i_2, \dots, i_k \\ j_1, j_2, \dots, j_k \end{vmatrix}$$

the main combinatorial identity is that:

Lemma. *If we fix any index a and consider the $k+1$ indeces $i_a, i_{a+1}, \dots, i_k, j_1, j_2, \dots, j_a$ then alternating the minor in these indeces produces 0.*

Proof. We prove it by decreasing induction on a. Since this is a formal identity in $\mathbb{Z}[x_{ij}]$ we can work in $\mathbb{Q}[x_{ij}]$.

Start from

$$\begin{vmatrix} i_1, i_2, \dots, i_{k-1}, s \\ j_1, j_2, \dots, j_k \end{vmatrix} = \sum_{p=1}^{k} \begin{vmatrix} i_1, i_2, \dots, \dots, i_{k-1}, j_p \\ j_1, j_2, \dots, j_{p-1}, s, j_{p+1}, \dots, j_k \end{vmatrix}$$

to prove this develop the determinants appearing with respect to the last row:

$$\sum_{p=1}^{k}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1},j_p \\ j_1,j_2,\dots,j_{p-1},s,j_{p+1},\dots,j_k\end{vmatrix} =$$

$$\sum_{p=1}^{k}(\sum_{u=1}^{p-1}(-1)^{n+u}\begin{vmatrix} j_p \\ j_u\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,\check{j}_u,\dots,j_{p-1},s,j_{p+1},\dots,j_k\end{vmatrix}$$

$$+(-1)^{n+p}\begin{vmatrix} j_p \\ s\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,j_{p-1},j_{p+1},\dots,j_k\end{vmatrix}$$

$$+\sum_{u=p+1}^{k}(-1)^{n+u}\begin{vmatrix} j_p \\ j_u\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,j_{p-1},s,j_{p+1},\dots,\check{j}_u,\dots,j_k\end{vmatrix})$$

or in other words

$$\sum_{p=1}^{k}(\sum_{u=1}^{p-1}(-1)^{n+u}\begin{vmatrix} j_p \\ j_u\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,\check{j}_u,\dots,j_{p-1},s,j_{p+1},\dots,j_k\end{vmatrix})$$

$$+\sum_{u=1}^{k}(\sum_{p=u+1}^{k}(-1)^{n+p}\begin{vmatrix} j_u \\ j_p\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,j_{u-1},s,j_{u+1},\dots,\check{j}_p,\dots,j_k\end{vmatrix})$$

$$+\sum_{p=1}^{k}((-1)^{n+p}\begin{vmatrix} j_p \\ s\end{vmatrix}\begin{vmatrix} i_1,i_2,\dots,\dots,i_{k-1} \\ j_1,j_2,\dots,j_{p-1},j_{p+1},\dots,j_k\end{vmatrix})$$

the first terms cancel and the last is the development of $\begin{vmatrix} i_1,i_2,\dots,i_{k-1},s \\ j_1,j_2,\dots,j_k\end{vmatrix}$.

As a consequence let us take any product of minors displayed now as a tableau with each type of row appearing appears an even number of times, in other words the columns of the tableau are all even, we deduce:

Theorem. *The standard tableaux with even columns form a $\mathbb{Z}$ basis of $\mathbb{Z}[x_{ij}]$.*

Proof. A product of variables x_{ij} is a tableau (with just one column), we show first that every tableau is a linear combination of standard ones.

So we look at a violation of standardness in the tableau.

This can occur in two different ways since a tableau is a product $d_1 d_2 \dots d_s$ of determinants of minors.

The first case is when the violation appears in two indeces $i_a > j_a$ of a minor $d_k = \begin{vmatrix} i_1,i_2,\dots,i_k \\ j_1,j_2,\dots,j_k\end{vmatrix}$. The previous identity implies immediately that this violation can be removed replacing the tableau with lexicographically smaller ones. The second case is when the violation occurs between a column index of some d_k and the corresponding row index of d_{k+1}. Here we can use the fact that by symmetry in a minor we can exchange the rows with the column indeces and then we can apply the identity on double tableax discussed in 35.3. The final result is to express the given tableau as a linear combination of tableaux which are either of strictly higher shape or lexicographycally inferior to the given one. Thus this straightening algorithm terminates.

In order to prove that the standard tableaux so obtained are linearly independent one could procede as in the previous paragraphs but also we can remark that, since standard tableaux of a given shape are, in characteristic 0, in correspondence with a basis of the corresponding linear representation of the linear group, the proposed basis is in each degree k (by the plethysm formula) of cardinality equal to the dimension of $S^k[S^2(V)]$ and so being a set of linear generators it must be a basis.

For the symplectic case $\mathbb{Z}[y_{ij}], i, j = 1, \dots, n$ subject to the skew symmetry, we define, for every sequence $1 \leq i_1 < i_2 < \dots i_{2k} \leq n$ formed by an even number of indeces, the symbol $|i_1, i_2, \dots, i_{2k}|$ to denote the Pfaffian of the principal minor of the skew matrix $Y = (y_{ij})$.

A product of such Pfaffians can be displaied as a tableau with even rows.

Here the main combinatorial identity is:

Lemma. *Take the product of two Pfaffians*

Theorem. *The standard tableaux with even rows form a $\mathbb{Z}$ basis of $\mathbb{Z}[y_{ij}]$.*

Proof. A variable $y_{ij},\ i < j$ equals the Pfaffian that we have indicated by $|ij|$ thus a product of variables y_{ij} is a tableau with two columns, we show again first that every tableau is a linear combination of standard ones.

So we look at a violation of standardness in the tableau.

We need an identity between Pfaffians, next we use the straigthening algorithm and finally the same argument with the Plethysm formula.