THREE QUICK RECIPES WITH FULLY EXTENDED ORIENTED 2D TQFTS

DOMENICO FIORENZA

Abstract. It always happens: you have a talk for dinner and nothing prepared. Your signature dish never fails, but you have served it too many times already and you'd like to surprise your guests with something new. Try these quick, light and colourful reinterpretations of haute cuisine classics, like (nonabelian) Fourier transforms and the Plancherel theorem for finite groups.

Contents

1. The ingredients

First thing you need to become familiar with are the ingredients you are going to cook. All of them can be easily obtained in pre-cooked form from the literature in a wide variety of examples. Yet, as the best bread is the one from one's own oven, we'll see how to prepare a few of these out of raw ingredients such as finite groups.

- a symmetric monoidal $(\infty, 2)$-category $\mathcal{C}$;[1]
- a pair of fully dualizable objects in $\mathcal{C}$: the blue object B and the red object R, together with an *invertible* oriented defect line between them;
- the *unit* object $\mathbf{1}$ of $\mathcal{C}$, corresponding to the invisible colour;
- boundary conditions $L, M, N, \ldots,$ for the blue and the red object (defect lines between the unit object and the blue/red object or vice versa);
- defect points $\phi : L \to M$, $\psi : N \to N$, $\ldots$, on the boundaries.

[1] There was a time when the newspapers said that only twelve men[2] understood symmetric monoidal $(\infty, 1)$-categories. I do not believe there ever was such a time. There might have been a time when only one man did, because he was the only guy who caught on, before he wrote his paper. But after people read the paper a lot of people kind of understood the theory of symmetric monoidal $(\infty, 1)$-categories in some way or other, but more than twelve. On the other hand, I think I can safely say that nobody understands symmetric monoidal $(\infty, 2)$-categories.

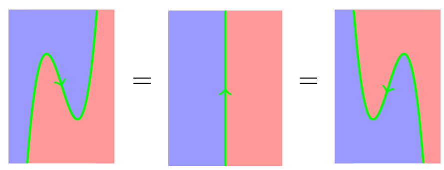
[2] The use of the masculine "men" for "people" witnesses the acute gender bias at the time of the original quote (Richard Feynman, Cornell University Lectures, 1964).

All these data are required to be homotopy invariant for the $SO(2)$-action on $\mathcal{C}$.

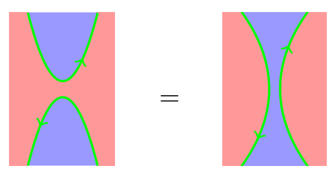
2. Basic preparations

In order to be able to prepare a recipe with the above ingredients, you need become familiar with a few basic preparations with fancy names.

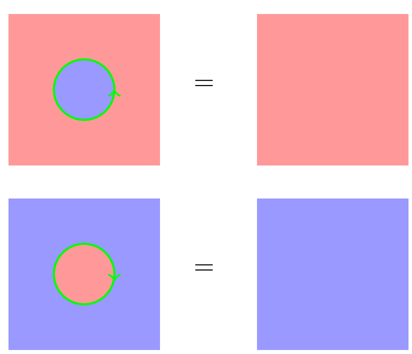
Zorro moves:



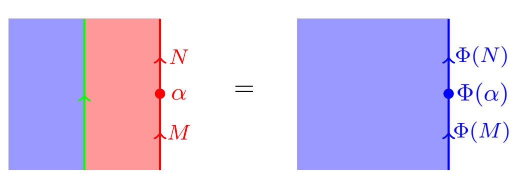
Pinch until split:

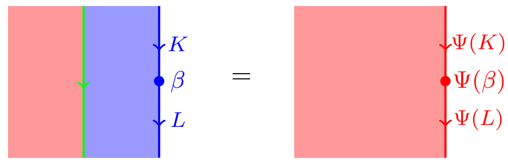


Bubble (dis)appearances:

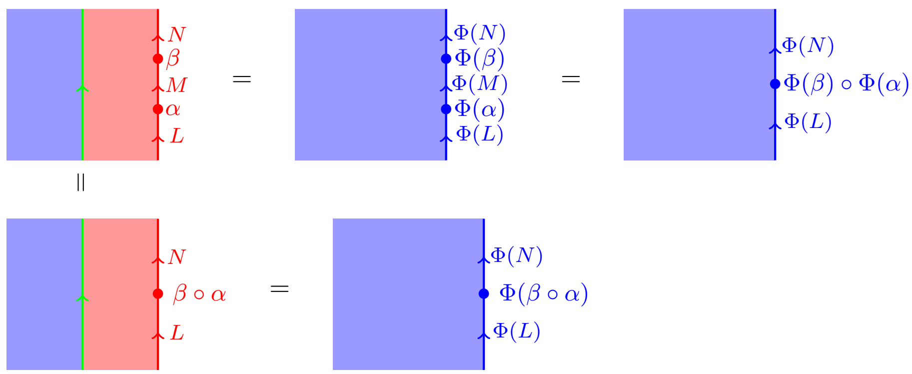


Pushing the defect line to the boundary:





Pushing and glueing:



and (of course) the same for blue and red exchanged.

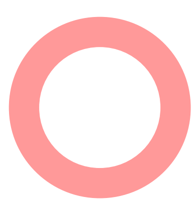
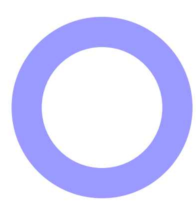
3. Amuse-bouche: the isomorphism between the centres

Ingredients: as this is a very simple recipe, you will only need the general ingredients listed at the beginning of the cookbook (→ *Ingredients*).

Difficulty: easy.

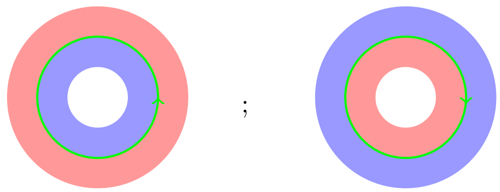
Cooking time: 3 minutes.

Preparation: First, you need to cook the centres:[3]

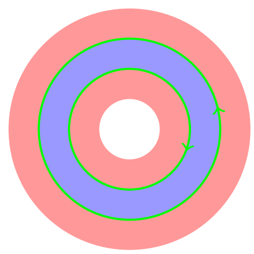
$Z(R) =$  ; $Z(B) =$ 

[3]These are called centres as, when A is a semisimple symmetric Frobenius algebra, seen as a fully dualizable and $SO(2)$-homotopy fixed point object in the the symmetric monoidal $(\infty, 2)$-category Alg_2 of finite dimensional $\mathbb{K}$-algebras, bimodules and morphisms of bimodules, then $Z(A)$ is precisely the centre of the algebra A.

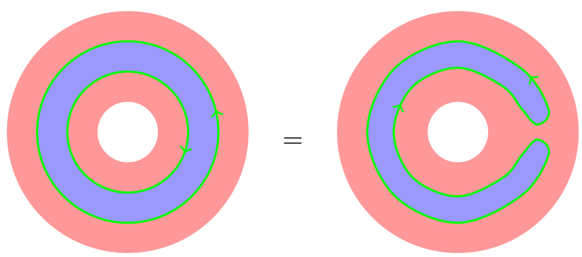
Now, you have to prepare isomorphisms between these. To do so, first prepare natural morphisms between them using the invertible defect line:



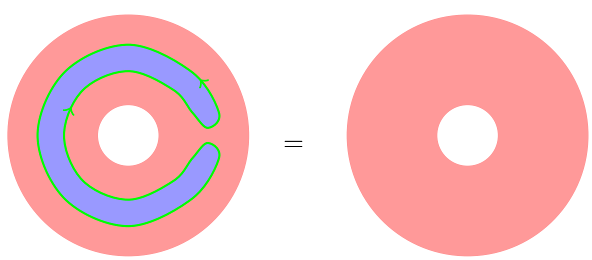
Now show that these are indeed isomorphisms (after all, your defect line is invertible, isn't it?). I'll show you only one composition, the other one is identical. First cook the composition



Next pinch the region between the defect lines until it doesn't split (→ *Basic preparations*).



Now make the blue bubble disappear[4] (→ *Basic preparations*):



[4]An US voter may read this as a Republican slogan. I can assure you this was not the intention: I've unsuccessfully tried other options: "make the red bubble disappear" could have been more appreciated in traditionally liberal places like US universities but could have been read as anti-communist in Europe; "make the black bubble disappear" would be a great democratic anti-fascist statement in Europe, but it would take a nasty white suprematism flavour in US; "make the pink bubble disappear" could at the same time be accused of misogyny and cause a debate on why a given colour should be associated with a gender or a sexual identity. In the end, the best thing is probably not to make any bubble disappear here: are you really so much interested in showing that the centre of B is isomorphic to the centre of R?

Notice that in preparing the morphisms between the centres, the invertibility of the defect line has not been used: invertibility has only been needed in making the bubble disappear, i.e., in showing that those morphisms were actually isomorphisms. So for noninvertible defect lines one still has distinguished morphisms between the centres, which however will generally not be invertible. This does not mean they will not be interesting: a classical example is the character of a finite dimensional representation V of a finite group G obtained this way by looking at the pair (V, V^*) as a noninvertible defect line between $\mathbb{K}$-modules (i.e., $\mathbb{K}$-vector spaces) and $\mathbb{K}[G]$-modules.

4. Hors d'oeuvre: nonabelian Fourier transforms

This is actually a particular case of one of the basic preparations you have learned to cook in full generality. Here we'll see how to serve it so to infuse it with one of its most classic flavours.

Ingredients:

- A characteristic zero algebraically closed field $\mathbb{K}$;
- A finite group G;
- the symmetric monoidal $(\infty, 2)$-category $\mathcal{C} = \mathrm{Alg}_2$ of finite dimensional $\mathbb{K}$-algebras, bimodules and morphisms of bimodules;
- $R = \mathbb{K}[G]$, the semisimple symmetric Frobenius algebra given by the group algebra of G with trace the coefficient of the unit element 1_G of G;
- $B = \mathcal{F}un_{\mathbb{K}}(\hat{G})$, the semisimple symmetric Frobenius algebra of $\mathbb{K}$-valued functions on the set $\hat{G}$ of isomorphism classes of irreducible representations of G with trace given by the integral with respect to the Plancherel measure $\mu(i) = (\dim_{\mathbb{K}} V_i)^2/|G|$;
- the invertible defect line given by $(\bigoplus_{i\in\hat{G}} V_i, \bigoplus_{i\in\hat{G}} V_i^*)$, where each representation of G is naturally seen as a left $\mathbb{K}[G]$-module, and the linear dual of a representation is naturally seen as a right $\mathbb{K}[G]$-module; the right and left $\mathcal{F}un_{\mathbb{K}}(\hat{G})$-module structures on $\bigoplus_{i\in\hat{G}} V_i$ and $\bigoplus_{i\in\hat{G}} V_i^*$, respectively, are the obvious ones;

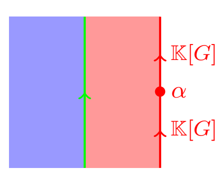
Difficulty: easy.

Cooking time: 2 minutes.

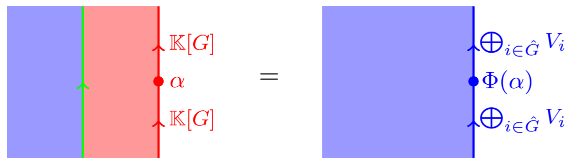
Preparation: Look at $\mathbb{K}[G]$ as boundary condition for the red object, by thinking of it as a left $\mathbb{K}[G]$ and a right $\mathbb{K}$-module. Pick any

$$\alpha \in \mathbb{K}[G] = \mathrm{Hom}_{\mathbb{K}[G],\mathbb{K}}(\mathbb{K}[G], \mathbb{K}[G])$$

and use it as a defect point on the boundary. Draw



Push the defect line to the boundary ($\to$ *Basic preparations*):



Pick

$$\Phi(\alpha) \in \mathrm{Hom}_{\mathcal{F}un_{\mathbb{K}}(\hat{G}),\mathbb{K}}(\bigoplus_{i\in\hat{G}} V_i, \bigoplus_{i\in\hat{G}} V_i) = \bigoplus_{i\in\hat{G}} \mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i).$$

You have obtained the isomorphism of algebras (the compatibility with compositions is given by pushing and glueing $\to$ *Basic preparations*)

$$\Phi\colon \mathbb{K}[G] \xrightarrow{\sim} \bigoplus_{i\in\hat{G}} \mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i).$$

known as the nonabelian Fourier transform. Note that the trace on the right is the integral with respect to the Plancherel measure of the traces of the various endomorphisms normalized by the dimension of the representation:

$$(\varphi_i)_{i\in\hat{G}} \mapsto \sum_{i\in\hat{G}} \frac{(\dim_{\mathbb{K}} V_i)^2}{|G|}\frac{\mathrm{tr}(\varphi_i)}{\dim_{\mathbb{K}} V_i} = \frac{1}{|G|}\sum_{i\in\hat{G}} \dim_{\mathbb{K}} V_i \cdot \mathrm{tr}(\varphi_i).$$

An explicit formula for $\Phi(\alpha)$ is as follows: if $\alpha = \sum_{g\in G}\alpha_g g$, then

$$(\Phi(\alpha)_i)_{i\in\hat{G}} = \left(\sum_{g\in G}\alpha_g\rho_i(g)\right)_{i\in\hat{G}},$$

where $\rho_i\colon G \to \mathrm{Aut}_{\mathbb{K},\mathbb{K}}(V_i)$ are the (chosen representatives for the) irreducible representations of G.

Presentation suggestion: Recall that $\mathbb{K}[G]$ is naturally isomorphic as a $\mathbb{K}$-algebra to the vector space $\mathcal{F}un_{\mathbb{K}}(G)$ of $\mathbb{K}$-valued functions on the set underlying the group G, endowed with the convolution product. The trace of $\mathbb{K}$ is translated by this isomorphism into the evaluation on the unit element 1_G of G. This way you can present the nonabelian Fourier transform as an algebra isomorphism

$$\Phi\colon (\mathcal{F}un_{\mathbb{K}}(G), *) \xrightarrow{\sim} \bigoplus_{i\in\hat{G}} \mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i)$$

4.1. **Raw ingredients.** The above preparation was so simple since we started with pre-cooked ingredients: we took from the literature shelves the fact that $(\bigoplus_{i\in\hat{G}} V_i, \bigoplus_{i\in\hat{G}} V_i^*)$ is an $SO(2)$-homotopy invariant invertible defect line. If we want to start with the raw ingredients, i.e., without knowing this, then what we have is only that $(\bigoplus_{i\in\hat{G}} V_i, \bigoplus_{i\in\hat{G}} V_i^*)$ is a defect line between $\mathbb{K}[G]$ and $\mathcal{F}un_{\mathbb{K}}(\hat{G})$, and we have to prove its $SO(2)$-homotopy invariance and its invertibility. The $SO(2)$-homotopy invariance corresponds to the Zorro moves and so, ultimately, to Schur's lemma. As far as concerns the invertibility, verbatim repeating the steps above, we get the algebra homomorphism $\Phi\colon \mathbb{K}[G] \xrightarrow{\sim} \bigoplus_{i\in\hat{G}} \mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i)$ and proving that our defect line is invertible is *equivalent* to showing this is an isomorphism.

This can be done as usual by means of classical characters theory. So, from the point of view of this cookbook, one could say that the main result of character theory is the invertibility of the $(\bigoplus_{i\in\hat{G}} V_i, \bigoplus_{i\in\hat{G}} V_i^*)$ defect.

4.2. **Nonabelian is too spicy? here's the abelian recipe.** When G is abelian, the set $\hat{G}$ is the underlying set of the *dual group* of G, i.e., of the group of characters

$$\chi\colon G \to \mathbb{K}^*$$

endowed with poinwise multiplication (this corresponds to tensor product of 1-dimensional representations). As all of the irreducible representations of G are 1-dimensional in this case, we have canonical isomorphisms $\mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i) \cong \mathbb{K}$ for any $i \in \hat{G}$ and the Fourier transform becomes an isomorphism

$$\Phi\colon (\mathcal{F}un_{\mathbb{K}}(G), *) \xrightarrow{\sim} (\mathcal{F}un_{\mathbb{K}}(\hat{G}), \cdot).$$

Written out explicitly, if $\alpha \in \mathcal{F}un_{\mathbb{K}}(G)$, then

$$\Phi(\alpha)\colon \hat{G} \to \mathbb{K}$$

is the function defined by

$$\Phi(\alpha)\colon \chi \mapsto \sum_{g\in G} \alpha(g)\chi(g).$$

4.3. **Getting high with higher Fourier transforms.** If you are willing to play with "foams" and other techniques from modern haute cuisine, then you need to upgrade from symmetric monoidal $(\infty, 2)$-categories to symmetric monoidal $(\infty, 3)$-categories. Here you can pick the same kind of ingredients as in the $(\infty, 2)$-categorical case, and prepare the same recipes: they will be beautifully enhanced by the richness of flavour of the $(\infty, 3)$-categorical setting. We are not going to dwell into any detail here, but let me at least mention that the one-level-higher version of the pair $(\mathbb{K}[G], \mathcal{F}un_{\mathbb{K}}(\hat{G}))$ is the pair of monoidal categories $(\mathrm{Vect}_{\mathbb{K}}[G], \mathrm{Rep}_{\mathbb{K}}(G))$. The invertible defect surface[5] in this case is the pair $(\mathrm{Vect}_{\mathbb{K}}(G\backslash\!\backslash G), \mathrm{Vect}_{\mathbb{K}}(G/\!/G))$ of finite-dimensional G-graded $\mathbb{K}$-vector spaces G-equivariant for the left (resp., right) multiplication action of G on itself.

5. The main course: the Plancherel theorem

Like with Fourier transforms, here we will start with a very general and verstile recipe, and then we'll cook it with ingredients derived from finite groups to get its most classic flavour.

Ingredients: the general ingredients listed at the beginning of the cookbook ($\rightarrow$ *Ingredients*).

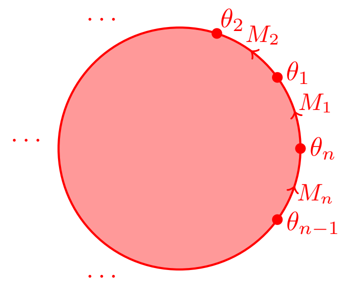
Difficulty: easy.

Cooking time: 5 minutes.

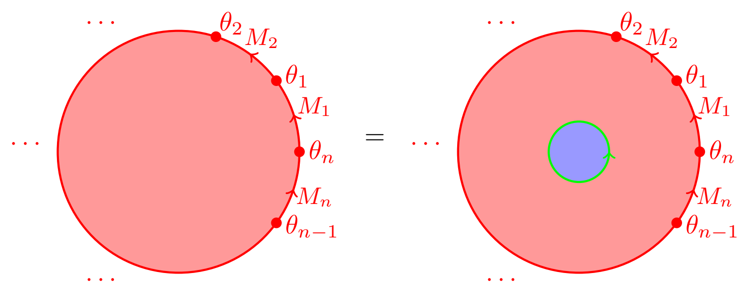
Preparation: Pick n boundary conditions $M_1, \ldots, M_n$ for R and n defect points

[5]Having moved to $(\infty, 3)$-categories, the relevant TQTFs are now 3-dimensional.

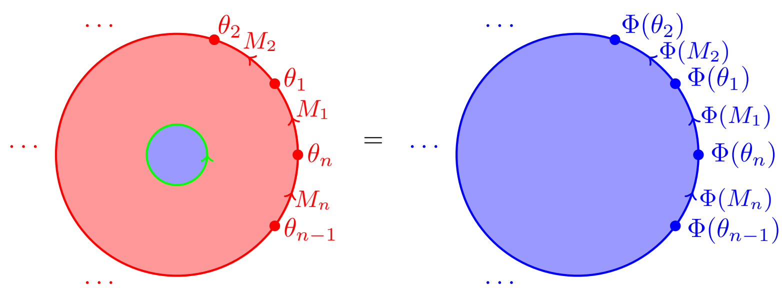
$\theta_i : M_i \to M_{i+1}$ on the boundaries, where the index i is taken modulo n. Prepare a red disk decorated with these data:



Add a blue bubble in the middle ($\to$ *Basic preparations*):



Push the defect line to the boundary ($\to$ *Basic preparations*)



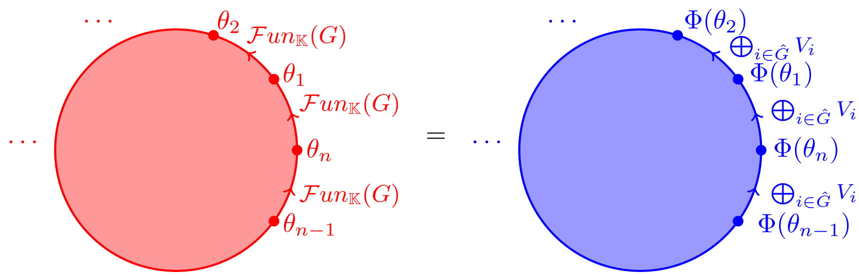
That's it. As this may leave too a strong abstract aftertaste, here is the classical recipe for a finite group G.

Ingredients: same as for the nonabelian Fourier transform ($\to$ *Hors d'oeuvre*)

Difficulty: easy.

Cooking time: 3 minutes.

Preparation: Look at $\mathbb{K}[G]$ as boundary condition for the red object, by thinking of it as a left $\mathbb{K}[G]$ and a right $\mathbb{K}$-module. Use the the canonical isomorphism $\mathbb{K}[G] \cong (\mathcal{F}un_{\mathbb{K}}(G), *)$ to think of $\mathcal{F}un_{\mathbb{K}}(G)$ as a boundary condition for the red object. Chose $M_i = \mathcal{F}un_{\mathbb{K}}(G)$ for any $i = 1, \ldots, n$. Pick arbitrary functions $\theta_1, \ldots, \theta_n \in \mathcal{F}un_{\mathbb{K}}(G)$ and use them as defect points. The general recipe then specialises to



i.e., to the identity

$$(\theta_1 * \theta_2 * \cdots * \theta_n)(1_G) = \frac{1}{|G|}\sum_{i\in\hat{G}}(\dim V_i)\mathrm{tr}(\Phi(\theta_1)_i \circ \Phi(\theta_2)_i \circ \cdots \circ \Phi(\theta_n)_i).$$

The convolution product is explicitly given by

$$(\theta_1 * \theta_2)(h) = \sum_{g\in G}\theta_1(g)\theta_2(g^{-1}h)$$

and it is associative. So, inductively one has

$$(\theta_1 * \theta_2 * \cdots * \theta_n)(h) = \sum_{g_1,\ldots,g_{n-1}\in G}\theta_1(g_1)\theta_2(g_2)\cdots\theta_{n-1}(g_{n-1})\theta_n(g_{n-1}^{-1}\cdots g_2^{-1}g_1^{-1}h).$$

Hence one gets the identity

$$\sum_{g_1,\ldots,g_{n-1}\in G}\theta_1(g_1)\cdots\theta_{n-1}(g_{n-1})\theta_n(g_{n-1}^{-1}\cdots g_1^{-1}) = \frac{1}{|G|}\sum_{i\in\hat{G}}(\dim V_i)\mathrm{tr}(\Phi(\theta_1)_i\circ\cdots\circ\Phi(\theta_n)_i)$$

giving the n-point Plancherel theorem. The traditional recipe for this has $n = 2$, and therefore reads

$$\sum_{g\in G}\theta_1(g)\theta_2(g^{-1}) = \frac{1}{|G|}\sum_{i\in\hat{G}}(\dim V_i)\mathrm{tr}(\Phi(\theta_1)_i \circ \Phi(\theta_2)_i)$$

Presentation suggestion: For any fixed k_0, by setting $k_i = k_{i-1}g_i$ for $i = 1,\ldots,n$, you have the identity

$$\sum_{g_1,\ldots,g_{n-1}\in G}\theta_1(g_1)\cdots\theta_{n-1}(g_{n-1})\theta_n(g_{n-1}^{-1}\cdots g_1^{-1})$$
$$= \sum_{k_1,\ldots,k_{n-1}\in G}\theta_1(k_0^{-1}k_1)\theta_2(k_1^{-1}k_2)\cdots\theta_{n-1}(k_{n-2}^{-1}k_{n-1})\theta_n(k_{n-1}^{-1}k_0).$$

Summing over k_0 you therefore get

$$\sum_{g_1,\ldots,g_{n-1}\in G}\theta_1(g_1)\cdots\theta_{n-1}(g_{n-1})\theta_n(g_{n-1}^{-1}\cdots g_1^{-1})$$
$$= \frac{1}{|G|}\sum_{k_0,k_1,\ldots,k_{n-1}\in G}\theta_1(k_0^{-1}k_1)\theta_2(k_1^{-1}k_2)\cdots\theta_{n-1}(k_{n-2}^{-1}k_{n-1})\theta_n(k_{n-1}^{-1}k_0)$$

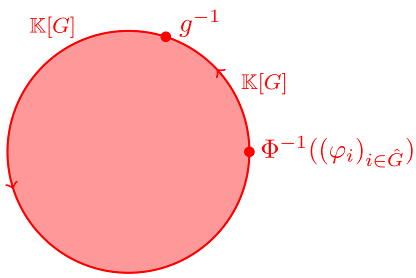
and, renaming $k_i = g_{i+1}$, the n-point Plancherel theorem takes the elegant cyclic form

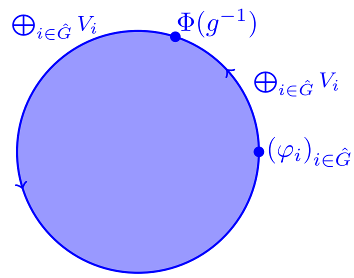
$$\sum_{g_1,\ldots,g_n\in G} \theta_1(g_1^{-1}g_2)\cdots\theta_{n-1}(g_{n-1}^{-1}g_n)\theta_n(g_n^{-1}g_1) = \sum_{i\in\hat{G}}(\dim V_i)\mathrm{tr}(\Phi(\theta_1)_i\circ\cdots\circ\Phi(\theta_n)_i)$$

5.1. **The inverse Fourier transform.** Writing an explicit expression for

$$\Phi^{-1}\colon \bigoplus_{i\in\hat{G}}\mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i) \xrightarrow{\sim} \mathbb{K}[G]$$

is slightly less immediate than one may expect. Namely, repeating the steps to prepare the Fourier transform, with R and B exchanged and starting with $\bigoplus_{i\in\hat{G}} V_i$ on the boundary, one does not end up with an explicit $\mathbb{K}[G]$ on the boundary as one would have hoped for, but rather with $\bigoplus_{i\in\hat{G}}\mathrm{End}_{\mathbb{K},\mathbb{K}}(V_i)$ on the boundary. And the identification of this with $\mathbb{K}[G]$ is precisely the seeked for Φ^{-1}, which therefore remains unexplicited if we do so. So here's the chef's trick. Knowing an element $\alpha\in\mathbb{K}[G]$ means knowing its coefficients α_g, and the coefficient of g in α is nothing but the coefficient of 1_G in $g^{-1}\cdot\alpha$. Therefore we find

$(\Phi^{-1}((\varphi_i)_{i\in\hat{G}})_g =$ 

$=$ 

From this, recalling that

$$(\Phi(g^{-1})_i)_{i\in\hat{G}} = \left(\rho_i(g^{-1})\right)_{i\in\hat{G}},$$

we get the explicit formula

$$(\Phi^{-1}((\varphi_i)_{i\in\hat{G}})_g = \frac{1}{|G|}\sum_{i\in\hat{G}}(\dim_{\mathbb{K}} V_i)\mathrm{tr}\left(\rho_i(g^{-1})\circ\varphi_i\right).$$

In the abelian case the inverse transform is immediately seen to reduce to the following. If $f\in\mathcal{F}un_{\mathbb{K}}(\hat{G})$ then

$$\Phi^{-1}(f)\colon G\to\mathbb{K}$$

is the function defined by

$$\Phi^{-1}(f)\colon g\mapsto \frac{1}{|G|}\sum_{\chi\in\hat{G}}\chi(g^{-1})f(\chi).$$

6. Sweet ending: the 1d Ising model

This is actually a very particular case of the n-point Plancherel theorem. But if you serve the classic $n=2$ Plancherel theorem as main, then this specialisation of the arbitrary n case can be a very nice ending (especially if you have statistical mechanics scholars over for dinner), as it realises the 1-dimensional Ising model as a boundary theory for a fully extended 2-dimensional TQFT. The analogous statement is true in any dimension, but already the 2-dimensional Ising model realisation as a boundary theory for a fully extended 3d TQFT goes far beyond the cooking abilities assumed here. Should you be interested in it, I'm providing references in the list of suggested readings at the end of this note.

Ingredients:

- The field $\mathbb{C}$ of complex numbers;
- A complex number β;
- The finite group $\boldsymbol{\mu}_2 = \{1, -1\} \subseteq \mathbb{C}^*$ (the multiplicative group of square roots of 1 in $\mathbb{C}$);

Difficulty: easy.

Cooking time: 2 minutes.

Preparation: To prepare the 1d Ising model as a boundary 2d TQFT, notice that $\boldsymbol{\mu}_2$ acts on $\mathbb{C}$ by multiplication and define $\theta_\beta \in \mathcal{F}un(\boldsymbol{\mu}_2, \mathbb{C})$ as

$$\theta_\beta(\sigma) = e^{\beta\sigma},$$

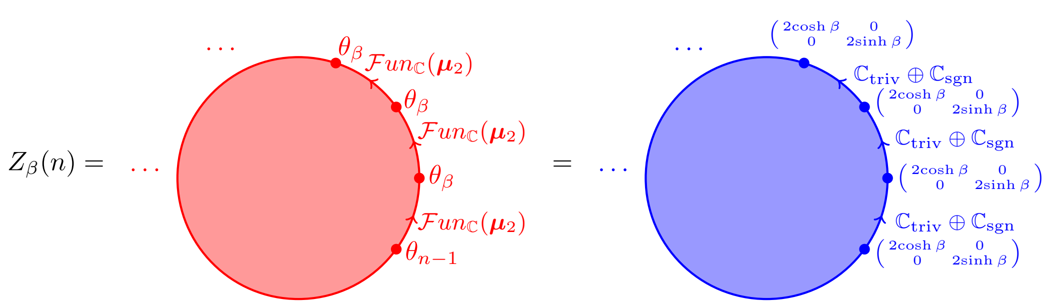
i.e., $\theta_\beta(\pm 1) = e^{\pm\beta}$. Using $\sigma^{-1} = \sigma$ for any $\sigma \in \boldsymbol{\mu}_2$, compute

$$\begin{aligned}\sum_{\sigma_1,\sigma_2,\ldots,\sigma_n \in \boldsymbol{\mu}_2} \theta_\beta(\sigma_1^{-1}\sigma_2)\cdots\theta_\beta(\sigma_n^{-1}\sigma_1) &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n \in \{1,-1\}} \theta_\beta(\sigma_1\sigma_2)\cdots\theta_\beta(\sigma_n\sigma_1)\\ &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n \in \{1,-1\}} e^{\beta\sigma_1\sigma_2}\cdots e^{\beta\sigma_n\sigma_1}\\ &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n \in \{1,-1\}} e^{\beta\sum_{j=1}^n \sigma_j\sigma_{j+1}},\end{aligned}$$

where in the sum over j the indices are taken modulo n. In the rightmost term you have obtained *partition function* $Z_\beta(n)$ of the periodic 1d Ising model with n nodes. Now recall that the two irreducible complex representations of $\boldsymbol{\mu}_2$ are the trivial representation and the defining (or sign) representation, and compute

$$\Phi(\theta_\beta) = \begin{pmatrix} e^\beta + e^{-\beta} & 0 \\ 0 & e^\beta - e^{-\beta}\end{pmatrix} = \begin{pmatrix} 2\cosh\beta & 0 \\ 0 & 2\sinh\beta\end{pmatrix}.$$

Complete the preparation by writing



i.e.,

$$Z_\beta(n) = (2\cosh\beta)^n + (2\sinh\beta)^n.$$

7. Acknowledgments and further readings

This little cookbook could not exist without several conversations on fully extended TQFTs I had over the years with Nils Carqueville and Alessandro Valentino. It is no exaggeration to say that everything I know about fully extended TQFTs is thanks to them, so it is a pleasure to end this note by thanking them for all the beautiful mathematics they taught to me (among many other things too personal to be mentioned here).

If you enjoyed the recipes presented here and are curios of preparing yourself more sophisticated ones, there are basically two ways. The first, and more effective, is to pester Alessandro and Nils as I did. If you feel to shy for this, then I can recommend a few pleasant text to read. The list here below is necessarily incomplete, I apologise for any omission.

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Sapienza Università di Roma; Dipartimento di Matematica “Guido Castelnuovo”, P.le Aldo Moro, 5 - 00185 - Roma, Italy
E-mail address: fiorenza@mat.uniroma1.it