An Introduction to the Language of Operads

Domenico Fiorenza

Solo dopo aver conosciuto la superficie delle cose
ci si può spingere a cercare quel che c'è sotto.
Ma la superficie delle cose è inesauribile.
— *Italo Calvino - Palomar*

1 Introduction

The aim of this note is to give an informal introduction to the language of operads, which has lately become ubiquitous in both modern abstract algebra and theoretical physics. The basic idea is that an operad $\mathcal{O}$ is a collection $\mathcal{O}(n)$ of "spaces" of n-ary operations, where by the generic word "space" we mean a set or a vector space or a topological space or a variety (topological, differential, analytic, algebraic), or, in the maximal generality, an object in a suitable tensor category. An n-ary operation has n inputs and just one output; this asymmetry makes the combinatorial definition of operads quite obscure at a first sight. To remedy such an asymmetry, we consider operations with n inputs and m outputs; this leads to the definition of MacLane's PROPs[1], which are collections $\mathcal{P}(n,m)$ of spaces of operations with n inputs and m outputs. In this more general context, an operad is simply the n-to-1 part of a PROP.

2 What is a PROP?

By definition, a PROP is simply a tensor category whose Hom-spaces have some additional structure, i.e. are objects of some other category. Before making the somehow colloquial definition above completely rigourous, let us make some example to show that objects like these are extremely common in mathematics:

A, B	Hom(A,B)
Finite sets	Finite set
$\mathbb{K}$-vector spaces	$\mathbb{K}$-vector space
Topological spaces	Topological space
Metric spaces	Metric space
$\mathbb{K}$-algebras	$\mathbb{K}$-vector space
Hilbert spaces	Banach space
Differential manifolds	Frechét manifold

[1]PROP is an acronym for PROducts and Permutations category

Now observe that for a generic category, Hom is a functor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Sets}$$

So we can formalize the concept of Hom-sets with structure by saying that a PROP is a pair of categories $\mathcal{C}_{\mathrm{ob}}$ (the category of the objects) and $\mathcal{C}_{\mathrm{Hom}}$ (the category of the morphisms) together with a functor

$$\mathcal{P} : (\mathcal{C}_{\mathrm{ob}})^{\mathrm{op}} \times \mathcal{C}_{\mathrm{ob}} \to \mathcal{C}_{\mathrm{Hom}}$$

To look at $\mathcal{P}(A, B)$ as a space of morphisms between the objects A and B of $\mathcal{C}_{\mathrm{ob}}$, we need some more axioms, generalizing what happens for Hom-spaces of generic categories.

Composition

The composition map

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

$$(f, g) \mapsto f \circ g$$

can be seen as a natural transformation

$$\circ_{A,B,C} : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

It satisfies the following associativity condition: the diagrams

$$\begin{tikzcd}
(\mathrm{Hom}(C,D) \times \mathrm{Hom}(B,C)) \times \mathrm{Hom}(A,B) \arrow[r, "\circ\times\mathrm{Id}"] \arrow[d, "\sim"'] & \mathrm{Hom}(B,D) \times \mathrm{Hom}(A,B) \arrow[d, "\circ"] \\
\mathrm{Hom}(C,D) \times (\mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B)) \arrow[d, "\mathrm{Id}\times\circ"'] & \mathrm{Hom}(A,D) \\
\mathrm{Hom}(C,D) \times \mathrm{Hom}(A,C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

commute for every choice of A, B, C, D. A blind generalization would be requiring the existence of a natural transformation

$$\circ_{A,B,C} : \mathcal{P}(B, C) \times \mathcal{P}(A, B) \to \mathcal{P}(A, C)$$

such the all diagrams as the one above would commute. But it would not be the right thing to do. For instance, if A, B, C are $\mathbb{K}$-vector spaces (one of the basic examples above), the composition

$$\circ_{A,B,C} : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

is *not* linear. So $\circ_{A,B,C}$ is not a morphism of vector spaces between $\mathrm{Hom}(B,C)\times \mathrm{Hom}(A,B)$ and $\mathrm{Hom}(A,C)$. But it is bilinear i.e. it is a morphism of vector spaces

$$\circ_{A,B,C} : \mathrm{Hom}(B,C)\otimes \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$$

So what we really have to ask is that $\mathcal{C}_{\mathrm{Hom}}$ be a tensor category and that $\circ$ be a natural transformation

$$\circ_{A,B,C} : \mathcal{P}(B,C)\otimes \mathcal{P}(A,B) \to \mathcal{P}(A,C)$$

such that all the diagrams

$$\begin{tikzcd}
(\mathcal{P}(C,D)\otimes\mathcal{P}(B,C))\otimes\mathcal{P}(A,B) \arrow[r, "\circ\otimes\mathrm{Id}"] \arrow[d, "a"'] & \mathcal{P}(B,D)\otimes\mathcal{P}(A,B) \arrow[d, "\circ"] \\
\mathcal{P}(C,D)\otimes(\mathcal{P}(B,C)\otimes\mathcal{P}(A,B)) \arrow[d, "\mathrm{Id}\otimes\circ"'] & \mathcal{P}(A,D) \\
\mathcal{P}(C,D)\otimes\mathcal{P}(A,C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

commute, where a denotes the associativity isomorphism of the tensor category $\mathcal{C}_{\mathrm{Hom}}$. Note that we are not requiring $\mathcal{C}_{\mathrm{Hom}}$ to have products.

Identity element

The set $\mathrm{Hom}(A,A)$ has a distinguished element Id_A that is the neutral element with respect to composition. Generalizing this concept requires a generalization of the concept of element so that we could speak of "elements" of $\mathcal{P}(A,A)$ and of the composition of them. A first attemp could consist in requiring $\mathcal{C}_{\mathrm{Hom}}$ to be a category of sets, i.e. a category endowed with a faithful functor

$$\pi : \mathcal{C}_{\mathrm{Hom}} \to \mathsf{Sets}$$

("forget the structure" functor), but then, to define the composition of $f \in \pi(\mathcal{P}(B,C))$ with $g \in \pi(\mathcal{P}(A,B))$ via $\circ$ we would need a natural map

$$\pi(\mathcal{P}(B,C)) \times \pi(\mathcal{P}(A,B)) \to \pi(\mathcal{P}(B,C)\otimes\mathcal{P}(A,B))\,.$$

This works very well in many situations (sets, groups, topological spaces, vector spaces...), but consider the category Top_X of topological spaces over a fixed space X (with fibre product as tensor product). What are the "elements" of $\varphi : Y \to X$? There is no natural answer and, though $\pi(\varphi : Y \to X) \mapsto Y$ as a set is a faithful functor from Top_X to Sets, there is no natural map

$$Y_1 \times Y_2 \to Y_1 \times_X Y_2\,.$$

We must seek for an alternative definition. Note that, if S is a set, giving an element of S is the same thing as giving a morphism $j : * \to S$, where $*$ denotes the set with only one element, i.e. the functor "elements" is representable. This happens in all the basic examples we are trying to generalize. In fact, we have:

Category of morphisms	**Elements**
Finite sets	$\mathrm{Hom}(*, -)$
$\mathbb{K}$-vector spaces	$\mathrm{Hom}(\mathbb{K}, -)$
Topological spaces	$\mathrm{Hom}(*, -)$
Metric spaces	$\mathrm{Hom}(*, -)$
Banach spaces	$\mathrm{Hom}(\mathbf{C}, -)$
Frechét manifolds	$\mathrm{Hom}(*, -)$

This happens also in other tensor categories we could decide to use as categories of morphisms. Note that in these cases there is no "natural" category having such categories of morphisms. Anyway these somehow "artificial" PROPs will be important in what follows.

Category of morphisms	**Elements**
Finite sets	$\mathrm{Hom}(*, -)$
$\mathbb{K}$-algebras	$\mathrm{Hom}(\mathbb{K}[x], -)$
Hilbert spaces	$\mathrm{Hom}(*, -)$
Differential manifolds	$\mathrm{Hom}(*, -)$
$\mathrm{Schemes}/_{\mathbb{K}}$	$\mathrm{Hom}(\mathrm{Spec}(\mathbb{K}), -)$

Denote by E the object representing the functor "elements". In all the examples above, E is a co-associative co-algebra in $\mathcal{C}_{\mathrm{Hom}}$, i. e., there is a map $\Delta : E \to E \otimes E$ such that the diagram

$$\begin{tikzcd}
 & E \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
E \otimes E \arrow[d, "\Delta \otimes \mathrm{Id}_E"] & & E \otimes E \arrow[d, "\mathrm{Id}_E \otimes \Delta"'] \\
(E \otimes E) \otimes E \arrow[rr, "a"] & & E \otimes (E \otimes E)
\end{tikzcd}$$

commutes (the only non completely trivial co-associative co-algebra structure is the one on $\mathbb{K}[x]$ in the category of $\mathbb{K}$-algebras; it is induced by the natural algebraic group structure on $\mathrm{Spec}(\mathbb{K}[x]) = \mathbf{A}^1_{\mathbb{K}}$). Having such an E at our disposal we can define the composition of two elements $f : E \to \mathcal{P}(B, C)$ and $g : E \to \mathcal{P}(A, B)$ via $\circ$ as the composition

$$E \xrightarrow{\Delta} E \otimes E \xrightarrow{f \otimes g} \mathcal{P}(B, C) \otimes \mathcal{P}(A, B) \xrightarrow{\circ} \mathcal{P}(A, C)$$

We write $f \circ g$ to denote the composition of f and g.

Finally we can give a meaning to the identity element of $\mathcal{P}(A, A)$. It will be a natural map

$$j_A : E \to \mathcal{P}(A, A)$$

which is the neutral element respect to the composition. Due to the co-asociativity of Δ, the composition defined above is associative.

This means that we can define a new category structure on $\mathcal{C}_{\mathrm{ob}}$ setting:

$$\mathrm{Hom}^{\mathrm{new}}(A,B) := \mathrm{Hom}(E, \mathcal{P}(A,B))$$

This new category is called the category "underlying" the PROP $\mathcal{P}$ (of course the underlying category depends on E and Δ, but these are data of the PROP). To ease notations we will write $f \in \mathcal{P}(A,B)$ to say that f is an element of $\mathcal{P}(A,B)$.
Another completely natural request is that the elements of $\mathcal{P}(B,C)$ act as left operators on $\mathcal{P}(A,B)$ and that the elements of $\mathcal{P}(A,B)$ acts as right operators on $\mathcal{P}(B,C)$, i.e., that there are maps

$$\mathrm{Hom}(E, \mathcal{P}(B,C)) \to \mathrm{Hom}(\mathcal{P}(A,B), \mathcal{P}(A,C))$$

and

$$\mathrm{Hom}(E, \mathcal{P}(A,B)) \to \mathrm{Hom}(\mathcal{P}(B,C), \mathcal{P}(A,C))$$

To have this, we just need two natural trasformations

$$\varphi^l : \mathrm{Id}_{\mathcal{C}_{\mathrm{Hom}}} \to E \otimes \mathrm{Id}_{\mathcal{C}_{\mathrm{Hom}}}$$

$$\varphi^r : \mathrm{Id}_{\mathcal{C}_{\mathrm{Hom}}} \to \mathrm{Id}_{\mathcal{C}_{\mathrm{Hom}}} \otimes E$$

such that φ^l_X and φ^r_X induce an E-co-bi-module structure on X. We further require

$$\varphi^l_E = \varphi^r_E = \Delta\,.$$

The maps φ^l's allow to assign to each element of $\mathcal{P}(B,C)$ a morphism between $\mathcal{P}(A,B)$ and $\mathcal{P}(A,C)$; since the objects of $\mathcal{C}_{\mathrm{Hom}}$ are left E-co-modules, this defines an action. The same considerations are valid for the φ^r's. A further request we make is that j_A acts trivially on both $\mathcal{P}(X,A)$ and $\mathcal{P}(A,X)$ for every X. Note that in any tensor category $\mathcal{C}$, one can take the unit object $\mathbf{1}_{\mathcal{C}}$ as E. Unless it is explicitly said, we will always assume $E = \mathbf{1}_{\mathcal{C}}$ in the following sections.

Tensor products

In a generic tensor category $\mathcal{C}$, the tensor product is a bi-functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

satisfying some associativity constrain. The functoriality of the tensor product means that there is a natural transform

$$\otimes_{A,B,C,D} : \mathrm{Hom}(A,B) \times \mathrm{Hom}(C,D) \to \mathrm{Hom}(A \otimes C, B \otimes D)$$

Translating this in the setting of PROPs, in which Hom-spaces have some structure, we are going to ask for a natural transform

$$\otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \to \mathcal{P}(A \otimes C, B \otimes D)\,,$$

satysfying associativity constrains that generalize those of ordinary tensor products. It is important to remark that, in order to generalize the diagram

$$\begin{tikzcd}
(\mathrm{Hom}(A,B)\otimes \mathrm{Hom}(F,A))\otimes(\mathrm{Hom}(C,D)\otimes \mathrm{Hom}(G,C)) \arrow[r, "\circ\otimes\circ"] \arrow[d, leftrightarrow, "\text{shuffle}"'] & \mathrm{Hom}(F,B)\otimes \mathrm{Hom}(G,D) \arrow[d, "\otimes"] \\
(\mathrm{Hom}(A,B)\otimes \mathrm{Hom}(C,D))\otimes(\mathrm{Hom}(F,A)\otimes \mathrm{Hom}(G,C)) \arrow[d, "(-\otimes-)\otimes(-\otimes-)"'] & \mathrm{Hom}(F\otimes G, B\otimes D) \\
\mathrm{Hom}(A\otimes C, B\otimes D)\otimes \mathrm{Hom}(F\otimes G, A\otimes C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

to

$$\begin{tikzcd}
(\mathcal{P}(A,B)\otimes \mathcal{P}(F,A))\otimes(\mathcal{P}(C,D)\otimes \mathcal{P}(G,C)) \arrow[r, "\circ\otimes\circ"] \arrow[d, leftrightarrow, "\text{shuffle}"'] & \mathcal{P}(F,B)\otimes \mathcal{P}(G,D) \arrow[d, "\otimes"] \\
(\mathcal{P}(A,B)\otimes \mathcal{P}(C,D))\otimes(\mathcal{P}(F,A)\otimes \mathcal{P}(G,C)) \arrow[d, "(-\otimes-)\otimes(-\otimes-)"'] & \mathcal{P}(F\otimes G, B\otimes D) \\
\mathcal{P}(A\otimes C, B\otimes D)\otimes \mathcal{P}(F\otimes G, A\otimes C) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

we must require the category $\mathcal{C}_{\mathrm{Hom}}$ to be *symmetric*. We are finally ready to give the formal definition of a PROP.

A formal definition

A PROP is $(\mathcal{C}_{\mathrm{ob}}, \mathcal{C}_{\mathrm{Hom}}, \mathcal{P}, \circ, E, \Delta, j, \varphi, \otimes)$ where

1. $\mathcal{C}_{\mathrm{ob}}$ is a tensor category, called the *category of objects*;
2. $\mathcal{C}_{\mathrm{Hom}}$ is a symmetric tensor category, called the *category of morphisms*;
3. $\mathcal{P}\colon (\mathcal{C}_{\mathrm{ob}})^{op} \times \mathcal{C}_{\mathrm{ob}} \to \mathcal{C}_{\mathrm{Hom}}$ is a functor, called the Hom-*space* functor;
4. $\circ_{A,B,C} : \mathcal{P}(B,C) \otimes \mathcal{P}(A,B) \to \mathcal{P}(A,C)$ is a natural transform, called the *composition map*;
5. (E, Δ) is a co-associative co-algebra in $\mathcal{C}_{\mathrm{Hom}}$;
6. j is a functorial map $j_A : E \to \mathcal{P}(A,A)$, called the *identity element*;
7. $\otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \to \mathcal{P}(A\otimes C, B\otimes D)$ is a natural transform inducing a tensor category structure on the underlying category of the PROP.

Braided and symmetric PROPs are defined adding the following axiom

8. $\sigma_{A,B} : E \to \mathcal{P}(A \otimes B, B \otimes A)$ is a natural map inducing a braided (symmetric) tensor category structure on the category underlying the PROP.

3 PROPs over $\mathbb{N}$ and Operads

This section is dedicated to PROPs having the natural numbers as category of objects. They were actually the first structures to be called "PROPs". For this reason we will completely restate the definition given above for such PROPs to find back the definition of operad one commonly finds in literature (see, for instance [Ad]). Let $\{\mathsf{N}, +\}$ be the discrete tensor category having the natural numbers (0 included) as objects and the sum as tensor product, and let $\mathcal{C}$ be a symmetric tensor category. Then, the data for a PROP with N as category of objects and $\mathcal{C}$ as category of morphisms[2] (a "$\mathcal{C}$-PROP over $\mathbb{N}$" for short) reduce to:

1. A map $\mathcal{P} : \mathbb{N} \times \mathbb{N} \to \mathcal{C}$;
2. For each $m, n, k \in \mathbb{N}$, composition maps
$$\circ_{mnk} : \mathcal{P}(n,k) \otimes \mathcal{P}(m,n) \to \mathcal{P}(m,k)$$
3. For each $m, n, k, l \in \mathbb{N}$, a tensor product
$$\otimes_{mnkl} : \mathcal{P}(m,n) \otimes \mathcal{P}(k,l) \to \mathcal{P}(m+k, n+l)$$
4. For each $n \in \mathbb{N}$ a distinguished element (identity) $j_n \in \mathcal{P}(n,n)$;

such that all the natural diagrams commute.

Braided and symmetric PROPs are defined by adding

5. For each $m, n \in \mathbb{N}$, a distinguished element $\sigma_{m,n} \in \mathcal{P}(m+n, n+m)$ inducing a braided (symmetric) tensor category structure on the category underlying the PROP.

Note that the commutativity of diagrams required above implies
$$j_n = j_1 \otimes \cdots \otimes j_1 = j_1^{\otimes n}$$
and that the braid relation tells us that the $\sigma_{m,n}$ are generated by the elementary braidings $\sigma_i^{(n+m)}$ defined by
$$\sigma_i^{(n)} := j_1 \otimes \cdots \otimes j_1 \otimes \sigma_{1,1} \otimes j_1 \otimes \cdots \otimes j_1 = j_1^{\otimes(i-1)} \otimes \sigma_{1,1} \otimes j_1^{\otimes(n-i-1)}$$
Moreover, the σ's induce natural actions of the braid (symmetric) group B_n (S_n) on the spaces $\mathcal{P}(n,m)$ and $\mathcal{P}(m,n)$ for each m.

[2] We are assuming $E = \mathbf{1}_{\mathcal{C}}$

Operads

An operad is simply the n-to-1 part of a symmetric PROP having N as category of the objects: assume that $\mathcal{P}$ be a symmetric $\mathcal{C}$-PROP, and set

$$\mathcal{O}(n) := \mathcal{P}(n,1), \qquad n \geq 1$$

Then we have a collection $\{\mathcal{O}(n), n \geq 1\}$ of objects of $\mathcal{C}$ equipped with the following set of data:

1. An action of the symmetric group S_n on $\mathcal{O}(n)$ for each $n \geq 1$.
2. Morphisms (called compositions)

   $$\gamma_{m_1,\ldots,m_l} : \mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \mathcal{O}(m_2) \otimes \cdots \otimes \mathcal{O}(m_l) \to \mathcal{O}(m_1 + m_2 + \cdots + m_l)$$

   defined as the composition

   $$\mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l) := \mathcal{P}(l,1) \otimes \mathcal{P}(m_1,1) \otimes \cdots \otimes \mathcal{P}(m_l,1) \xrightarrow{\mathrm{Id}_{\mathcal{P}(l,1)} \otimes \bigotimes}$$

   $$\xrightarrow{\mathrm{Id}_{\mathcal{P}(l,1)} \otimes \bigotimes} \mathcal{P}(l,1) \otimes \mathcal{P}(m_1 + m_2 + \cdots m_l, l) \xrightarrow{\circ}$$

   $$\xrightarrow{\circ} \mathcal{P}(m_1 + m_2 + \cdots m_l, 1)$$

   $$=: \mathcal{O}(m_1 + m_2 + \cdots m_l)$$

3. An element j_1 of $\mathcal{O}(1)$, called the unit, such that $\gamma_{1,1,\ldots,1}(\mu, j_1, j_1, \ldots, j_1) = \mu$ for any l and any $\mu \in \mathcal{O}(l)$.

These data satisfy conditions of associativity and equivariance with respect to the symmetric groups, so that the collection $\{\mathcal{O}(n), n \geq 1\}$ is an operad according to the original definition given by May in [Ma]. Vice versa, if $\{\mathcal{O}(n), n \geq 1\}$ is an operad, we can define a PROP by the following construction. Start setting

$$\mathcal{P}(n,1) := \mathcal{O}(n), \qquad n \geq 1$$

The spaces $\mathcal{P}(m,n)$ have to be constructed in such a way that we have natural maps

$$\otimes : \mathcal{P}(m_1,n_1) \otimes \mathcal{P}(m_2,n_2) \otimes \cdots \otimes \mathcal{P}(m_l,n_l) \to \mathcal{P}(\sum_i m_i, \sum_i n_i)$$

In particular, we must have

$$\otimes : \mathcal{P}(m_1,1) \otimes \mathcal{P}(m_2,1) \otimes \cdots \otimes \mathcal{P}(m_l,1) \to \mathcal{P}(\sum_i m_i, l)$$

i.e. for every $(m_1, m_2, \ldots, m_l)$ with $\sum_l m_l = m$, a map

$$\otimes : \mathcal{P}(m_1,1) \otimes \mathcal{P}(m_2,1) \otimes \cdots \otimes \mathcal{P}(m_l,1) \to \mathcal{P}(m,l)$$

Assume that $\mathcal{C}$ has direct sums, then set

$$\begin{aligned}\mathcal{P}(m,l) &:= \bigoplus_{(m_1,m_2,\ldots,m_l)|\sum_i m_i=m} \mathcal{P}(m_1,1)\otimes\cdots\otimes\mathcal{P}(m_l,1)\\ &= \bigoplus_{(m_1,m_2,\ldots,m_l)|\sum_i m_i=m} \mathcal{O}(m_1)\otimes\cdots\otimes\mathcal{O}(m_l)\,.\end{aligned}$$

Then the tensor products

$$\otimes_{mnkl} : \mathcal{P}(m,n)\otimes\mathcal{P}(k,l)\to\mathcal{P}(m+k,n+l)$$

are defined by the inclusion

$$\begin{aligned}&\left\{(m_1,m_2,\ldots,m_n):\sum_i m_i=m\right\}\times\left\{(k_1,k_2,\ldots,k_l):\sum_j k_j=k\right\}\hookrightarrow\\ &\qquad\hookrightarrow\left\{(s_1,s_2,\ldots,s_{n+l}):\sum_i s_i=m+k\right\}\\ &((m_1,m_2,\ldots,m_n),(k_1,k_2,\ldots,k_l))\mapsto(m_1,m_2,\ldots,m_n,k_1,k_2,\ldots,k_l)\end{aligned}$$

while the compositions

$$\circ_{mnk} : \mathcal{P}(n,k)\otimes\mathcal{P}(m,n)\to\mathcal{P}(m,k)$$

are defined component-wise as

$$\begin{aligned}&\big(\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)\big)\otimes\big(\mathcal{O}(m_1)\otimes\cdots\otimes\mathcal{O}(m_n)\big)\\ &\quad\simeq(\mathcal{O}(n_1)\otimes\mathcal{O}(m_1)\otimes\cdots\otimes\mathcal{O}(m_{n_1}))\otimes\cdots\otimes\\ &\qquad\otimes\big(\mathcal{O}(n_k)\otimes\mathcal{O}(m_{(n_1+\cdots+n_{k-1})+1})\otimes\cdots\otimes\mathcal{O}(m_n)\big)\xrightarrow{\gamma\otimes\cdots\otimes\gamma}\\ &\quad\xrightarrow{\gamma\otimes\cdots\otimes\gamma}\mathcal{O}(m_1+\cdots+m_{n_1})\otimes\cdots\otimes\mathcal{O}(m_{(n_1+\cdots+n_{k-1})+1}+\cdots+m_n)\end{aligned}$$

We refer to the PROP defined above as to the PROP generated by the operad $\mathcal{O}$, and denote it by the symbol $\mathcal{P}_{\mathcal{O}}$.

A basic example: the endomorphisms PROP and Operad

The basic example of a PROP is the following. Let V be a vector space over some fixed base field $\mathbb{K}$. Its *endomorphisms PROP* is by definition the symmetric $\mathbb{K}$-vector spaces PROP given by

$$\underline{\mathrm{End}}(V)(m,n) := \mathrm{Hom}(V^{\otimes m},V^{\otimes n})$$

According to the definitions above, the *endomorphisms operad* of V is the operad of vector spaces

$$\underline{\mathrm{End}}(V)(n) := \mathrm{Hom}(V^{\otimes n},V)\,.$$

4 Representations of PROPs (algebras over operads)

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two PROPs with the same category of morphisms $\mathcal{C}_{\mathrm{Hom}}$. Then a morphism

$$\rho : \mathcal{P}_1 \to \mathcal{P}_2$$

is simply a tensor functor

$$\rho_{\mathrm{ob}} : \mathcal{C}_{\mathrm{ob},1} \to \mathcal{C}_{\mathrm{ob},2}$$

together with natural trasformations

$$\rho_{\mathrm{Hom},A,B} : \mathcal{P}_1(A,B) \to \mathcal{P}_2(\rho_{\mathrm{ob}}A, \rho_{\mathrm{ob}}B)$$

satisfying natural conditions so to generalize the concept of tensor functor between tensor categories to this setting of "tensor categories with structure on the Hom-spaces". For instance, we require that the identity element (and the braiding, if the PROPs $\mathcal{P}_1$ and $\mathcal{P}_2$ are braided) go to the identity element and to the braiding respectively. Morphisms between operads are defined as morphisms between the PROPs they generate. Now let us stop with the abstract nonsense an go on to concrete examples. Take $\mathcal{P}_2$ as the vector space PROP given by the category of vector spaces over some fixed base field $\mathbb{K}$, and let $\mathcal{P}$ be any PROP of vector spaces over $\mathbb{N}$. Since $\mathbb{N}$ is generated as a tensor category by 1, the image of ρ_{ob} for a morphism

$$\rho : \mathcal{P} \to \mathsf{Vect}_{\mathbb{K}}$$

will be generated by the vector space $V = \rho_{\mathrm{ob}}(1)$. So the data of our morphism will be a family of morphisms

$$\rho_{m,n} : \mathcal{P}(m,n) \to \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}),$$

satisfying some comatibility axioms. Note that these data are equivalent to a morphism

$$\rho : \mathcal{P} \to \underline{\mathrm{End}}(V)$$

Elements of $\mathcal{P}(m,n)$ act as multilinear operators on V. A vector space V together with a morphism $\rho\colon \mathcal{P} \to \underline{\mathrm{End}}(V)$ is called a $\mathcal{P}$-algebra. If $\mathcal{O}$ is an operad of vector spaces, an $\mathcal{O}$-algebra is, by definition, a $\mathcal{P}_{\mathcal{O}}$-algebra. It can be seen as the datum of a collection of morphisms

$$\rho_n : \mathcal{O}(n) \to \mathrm{Hom}(V^{\otimes m}, V),$$

i.e. as an operad morphism $\mathcal{O} \to \underline{\mathrm{End}}(V)$. Elements of $\mathcal{O}(n)$ acts as n-ary products on V.

5 Free Operads

¿From now on, we will work with PROPs and operads of vector spaces, i.e. such that their category of morphisms is the category of $\mathbb{K}$-vector spaces, for some fixed base field $\mathbb{K}$. As usual, we will use the same symbol to denote a vector space and its underlying set. The problem we are going to solve in this section is the following: given a collection of sets $X = \{X(n)\}$, is there an operad of vector spaces $\mathcal{O}$ such that for any operad of vector spaces $\tilde{\mathcal{O}}$ and any family of maps of sets

$$\rho_n : X(n) \to \tilde{\mathcal{O}}$$

there is a unique lifting of ρ to an operad morphism $\mathcal{O} \to \tilde{\mathcal{O}}$? It will come out that this operad exists (and is clearly unique up to isomorphism); it will be called the *free operad* generated by the collection X and will be denoted by the symbol $\mathcal{F}(X)$. In particular, we obtain that for any vector space V, the choice of a set $X(n)$ of n-ary multiplications on V defines an $\mathcal{F}(X)$-algebra structure on V.

Trees and forests

A Reshetikhin-Turaev's tree is a directed tree embedded in $\mathbb{R} \times [0,1]$ in such a way that all the imputs lie on $\mathbb{R} \times \{0\}$, its only output lies on $\mathbb{R} \times \{1\}$, and the height function is strictly increasing on each edge in the direction given by the orientation on the edge. Define the tensor product of two trees T_1 and T_2 as the object obtained drawing T_2 on the right of T_1. Such an object is not a tree anymore (is not connected). Since it is a disjoint union of trees, it is called a "forest". Extending the tensor product from trees to forests we get a tensor category, called the category of forests (the unit object being the void forest). We denote it by the symbol $\mathcal{F}$. Let $\mathcal{F}(m,n)$ be the set of all forests with m inputs and n outputs. Defining the composition of a forest F_1 in $\mathcal{F}(n,k)$ with a forest F_2 in $\mathcal{F}(m,n)$ as the forest $F_1 \circ F_2$ in $\mathcal{F}(m,k)$ obtained putting F_1 over F_2 one gets a PROP of sets having N as category of the objects; such a PROP is called the Reshetikhin-Turaev's forest PROP. The identity element is represented by the stright line in $\mathcal{F}(1,1)$. It is a symmetric PROP, the braiding given by two crossing lines in $\mathcal{F}(2,2)$, with an identification of undercrossings and overcrossings. Call *vertex of type* $(n,1)$ a tree with only one $n+1$ valent vertex and containing no braiding.[3] Any forest can be built by composing and taking tensor products of vertices of type $(n,1)$ with $n \in \mathbb{N}$, the identity element and the braiding. A fundamental result by Reshetikhin and Turaev (see [RT]) is that $\mathcal{F}$ is freely generated as a symmetric PROP by these elementary pieces. Being $\mathcal{F}$ a PROP of sets, we can immediately get from it a PROP of vector spaces, simply taking the free vector spaces generated by the $\mathcal{F}(m,n)$ on some fixed base field $\mathbb{K}$. When it causes no confusion we will denote the PROP of vector spaces obtained in this way by the same symbol $\mathcal{F}$.

[3]Some authors call it an n-corolla.

Coloured forests

Now consider the elements of the set $X(n)$ as a set of colours for the $n+1$-valent vertices of the trees in the forests; this way we obtain a new PROP, called the PROP of X-coloured forests and denoted by $\mathcal{F}(X)$. Note that non-coloured forests can therefore be thought as being coloured with just one colour for each valency. If $X(n) = \emptyset$ we mean that there are no vertices of valence $n+1$ in the forests of $\mathcal{F}(X)$. The PROP $\mathcal{F}(X)$ is freely generated as a symmetric PROP by vertices of type $(n, 1)$ (coloured with all its possible colorations), the identity element and the braiding. Again we can consider the PROP of vector spaces generated by $\mathcal{F}(X)$ over $\mathbb{K}$ (and we will denote it by the same symbol). Since $\mathcal{F}(X)$ is freely generated by the vertices, the braidings and the identity element, the datum of a morphism of symmetric PROPs of vector spaces

$$\rho : \mathcal{F}(X) \to \tilde{\mathcal{P}}$$

reduces to the datum of a family of maps of sets

$$\rho_n : \left\{ \text{Vertices of type } (n, 1) \right\} \to \tilde{\mathcal{P}}(n, 1)$$

(for any morphism of symmetric PROPs, the images of the identity element and of the braiding of the source PROP are prescribed to be the identity element and the braiding of the target PROP). But, by construction, there is a bijection

$$X(n) \leftrightarrow \left\{ \text{Vertices of type } (n, 1) \right\} .$$

This shows that the n-to-1 part of the PROP $\mathcal{F}(X)$ is the free operad we were looking for; we will denote this operad by the same symbol $\mathcal{F}(X)$. More explicitly, an element of the free operad $\mathcal{F}(X)$ is a linear combination of Reshetikhin-Turaev's trees with the vertices decorated by the elements of X.

6 Ideals and quotients

Let $\mathcal{P}$ be a PROP of vector spaces. An ideal of $\mathcal{P}$ is a collection $\mathcal{I} = \{\mathcal{I}(n, m)\}$ of subspaces of $\mathcal{P}(n, m)$ which is closed with respect to composition and tensor products with $\mathcal{P}$. It is immediate to check that, if $\mathcal{I}$ is an ideal of $\mathcal{P}$, then the collection of vector spaces

$$(\mathcal{P}/\mathcal{I})((n, m) := \mathcal{P}(m, n)/\mathcal{I}(n, m)$$

defines a PROP $\mathcal{P}/\mathcal{I}$, that will be called the *qouotient* PROP of $\mathcal{P}$ by $\mathcal{I}$. If $\mathcal{O}$ is an operad, and $\mathcal{I}$ is an ideal of the PROP $\mathcal{P}_{\mathcal{O}}$, then the n-to-1 part of the quotient PROP $\mathcal{P}_{\mathcal{O}}/\mathcal{I}$ will be called the quotient operad of $\mathcal{O}$ by $\mathcal{I}$ and will be denoted by $\mathcal{O}/\mathcal{I}$. If $I = \{I(n, m)\}$ is a collection of subsets of $\mathcal{P}(n, m)$, the smallest ideal of $\mathcal{P}$ containing I is called the ideal generated by I.

As in standard algebra representation theory, if $\mathcal{I}$ is an ideal of the PROP $\mathcal{P}$ there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{PROP}}(\mathcal{P}/\mathcal{I}, \tilde{\mathcal{P}}) \simeq \left\{\rho \in \mathrm{Hom}_{\mathrm{PROP}}(\mathcal{P}, \tilde{\mathcal{P}}) \text{ such that } \rho(\mathcal{I}) = 0\right\}$$

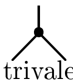
In particular, if $\mathcal{I}$ is an ideal of $\mathcal{F}(X)$, a representation

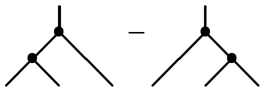
$$\mathcal{F}(X)/\mathcal{I} \to \underline{\mathrm{End}}(V)$$

is the same thing as defining on V a set of multiplications which satisfy the relations prescribed by $\mathcal{I}$.

7 Examples

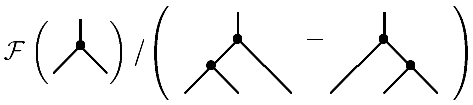
The operad Assoc

Let $X(2) = \{$  $\}$ and $X(n) = \emptyset$ for $n \neq 2$. The operad $\mathcal{F}(X)$ is simply the operad of trivalent trees; its representations are algebras with one binary operation. Let now $\mathcal{I}$ be the ideal of $\mathcal{F}(X)$ generated by



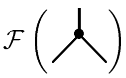
(associativity)

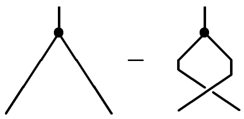
and let **Assoc** be the quotient operad



Then **Assoc**-algebras are exactly associative algebras.

The operad Comm

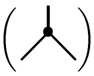
It is the quotient operad of  by the ideal generated by

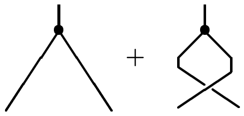


(commutativity)

Clearly, **Comm**-algebras are exactly commutative associative algebras.

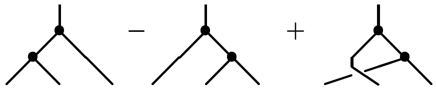
The operad Lie

It is the quotient operad of $\mathcal{F}$  by the ideal generated by



(anticommutativity)

and



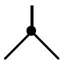
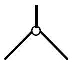
(Jacobi)

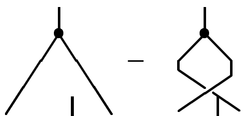
As the name says, **Lie**-algebras are exactly Lie algebras.

The operad Poisson

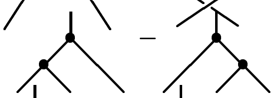
We end this short introduction to the language of operads with an example of an operad describing Poisson algebras. These are algebras with two binary operations: a commutative associative multiplication $\cdot$ and a Lie braket $\{\,,\}$ (Poisson bracket) which satysfy the compatibility realtion

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

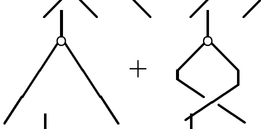
Since there are two binary operations, let $X(2) = \{$  ,  $\}$ and $X(n) = \emptyset$ for $n \neq 2$.
Let $\mathcal{I}$ be the ideal of $\mathcal{F}$ generated by



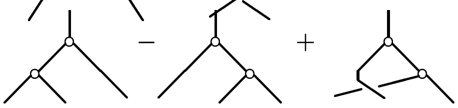
(commutativity)



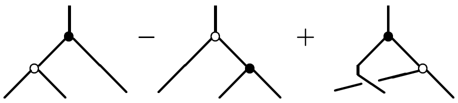
(associativity)



(anticommutativity)



(Jacobi)



(compatibility)

and let **Poisson** be the quotient operad of $\mathcal{F}(X)$ by $\mathcal{I}$. Then **Poisson**-algebras are exactly Poisson algebras.

References

[Ad] John Frank Adams. *Infinite loop spaces.* Princeton University Press, Princeton, N.J., 1978.

[Ma] J. P. May. *The geometry of iterated loop spaces.* Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[RT] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.