Deformations of algebras and cohomology

Domenico Fiorenza

forse di Irene ho già parlato sotto altri nomi;
forse non ho parlato che di Irene.
— *Italo Calvino - Le città invisibili*

1 Introduction

In this short note we show how each algebra deformation problem is described by a suitable cohomology theory. The language of *operads* allows us to describe such a cohomology theory in an universal way (i.e. independent of the particular algebra we are studying). As an example we will show how this general theory specializes, in the case of associative algebras, to the well known Hochshild cohomology. All ideas and tecniques used in this note are taken from the beautiful paper *"Cotangent cohomology of a category and deformations"* by Martin Markl.

Let us start in a *naive* way. Let V be a $\mathbb{K}$-vector space and let

$$\varphi : V^{\otimes n} \to V^{\otimes m}$$

A *formal deformation* of φ will be

$$\varphi_\hbar = \varphi + \hbar\varphi_1 + \hbar^2\varphi_2 + \cdots$$

where

$$\varphi_i : V^{\otimes n} \to V^{\otimes m}$$

and the *deformation parameter* $\hbar$ is to be thought so little as to ensure the convergence of the series. Actually, we are not interested at all in the convergence of the series defining $\varphi_\hbar$, but we treat it simply as a formal series (this is the reason why $\varphi_\hbar$ was called formal). If we now write

$$\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\hat{\otimes}\mathbb{K}[[\hbar]]$$

and by

$$\mathrm{mod}\ \hbar : \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \to \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$$

the canonical projection, we can restate what we have written above by sayng that the formal deformations of an element $\varphi \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ are

$$\left\{\varphi_\hbar \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \text{ such that } \varphi_\hbar = \varphi \mod \hbar\right\}$$

We can make one more step further: if $\langle *_{n,m}\rangle$ denotes the one-dimensional $\mathbb{K}$-vector space generated by the element $*_{n,m}$, then we have a canonical isomorphism

$$\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \tilde{=} \mathrm{Hom}(\langle *_{n,m}\rangle, \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}))$$

so that we can think of φ as

$$\varphi : \langle *_{n,m}\rangle \to \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$$

Thus, the deformations of φ can be seen as liftings:

$$\begin{tikzcd}
 & \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})[[\hbar]] \arrow[d, "\mathrm{mod}\ \hbar"] \\
\langle *_{n,m}\rangle \arrow[ur, "\varphi_\hbar"] \arrow[dr, "\varphi"'] & \\
 & \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})
\end{tikzcd}$$

2 $\mathcal{O}$-algebras and their deformations.

Recall that, if $\mathcal{O}$ is an operad of vector spaces, then an $\mathcal{O}$-algebra is simply an operad morphism

$$A : \mathcal{O} \to \underline{\mathrm{End}}(V)$$

where V is some vector space on a fixed base field $\mathbb{K}$ and $\underline{\mathrm{End}}(V)$ is its endomorphisms operad. Let now $\mathcal{C}$ be any PROP of $\mathbb{K}$-vector spaces. By $\mathcal{C}[[\hbar]]$ we denote the PROP defined as follows:

1. The objects of $\mathcal{C}[[\hbar]]$ are the same as the objects of $\mathcal{C}$
2. $\mathrm{Hom}_{\mathcal{C}[[\hbar]]}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)\hat{\otimes}\mathbb{K}[[\hbar]]$

Note that the essential here is that the *morphisms* are $\mathbb{K}$-vector spaces (whatever the objects are). In particular $\underline{\mathrm{End}}(V)$ is an operad of vector spaces, so we can consider the operad $\underline{\mathrm{End}}(V)[[\hbar]]$. The reduction modulus $\hbar$ is a functor

$$\mathrm{mod}\ \hbar : \mathcal{C}[[\hbar]] \to \mathcal{C}$$

Then, *formal deformations* of an algebra A are the liftings

$$\begin{tikzcd}
 & \underline{\mathrm{End}}(V)[[\hbar]] \arrow[d, "\mathrm{mod}\ \hbar"] \\
\mathcal{O} \arrow[ur, "A_\hbar"] \arrow[dr, "A"'] & \\
 & \underline{\mathrm{End}}(V)
\end{tikzcd}$$

Since

$$\mathbb{K}[[\hbar]] = \lim_{\leftarrow} \mathbb{K}[[\hbar]]/(\hbar^n) = \lim_{\leftarrow} \mathbb{K}[\hbar]/(\hbar^n)$$

A formal deformation of A can be thought as a projective limit

$$\begin{tikzcd}
 & \underline{\mathrm{End}}(V)[[\hbar]] \arrow[d] \\
 & \vdots \arrow[d] \\
 & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^3) \arrow[d, "\mathrm{mod}\ \hbar^2"] \\
\mathcal{O} \arrow[uuur, "A_\hbar"] \arrow[ur, "A_2"] \arrow[r, "A_1"] \arrow[dr, "A"'] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2) \arrow[d, "\mathrm{mod}\ \hbar"] \\
 & \underline{\mathrm{End}}(V)
\end{tikzcd}$$

The deformations A_n are called *infinitesimal deformations* of A. In particular, A_n is said to be a n^{th} order infinitesimal deformation of A. Then we can try to solve the problem of describing formal deformations of A by a *step-by-step* approach: starting from A, describe the 1^{st} order deformations of A; given a 1^{st} order deformation of A, try to lift it to a 2^{nd} order deformation, and so on (notice that the problem of the *existence* of formal deformations is trivial: the *trivial* deformation $A_\hbar = A + \hbar \cdot 0 + \hbar^2 \cdot 0 + \dots$ works. What we are interested in is describing the space of *all* formal deformations of A.). For any n, given an n^{th} order deformation A_n, there are two questions we have to answer:

1. Does A_n lift to a $(n+1)^{th}$ order deformation A_{n+1}?

2. If so, which are these liftings?

In the next section we will show how the *obstruction* to the answer *yes* in the first question is given by an element in the second cohomology group of a certain complex, and that, when this obstruction vanishes, the liftings are parametrized by elements in the first cohomology group of that complex. Such a cohomological theory is called the *cotangent cohomology* of A. Describing it is the aim of the next section.

3 Resolutions of operads and cotangent cohomology.

As we have seen in [Fi], any operad of vector spaces can be realized as a quotient of a free operad:

$$0 \longleftarrow \mathcal{O} \overset{\pi}{\longleftarrow} \mathcal{F}(X) \longleftarrow \mathcal{I} \longleftarrow 0$$

The kernel $\mathcal{I}$ of the natural projection $\pi : \mathcal{F}(X) \to \mathcal{O}$ is an ideal of $\mathcal{F}$. In particular it is an $\mathcal{F}(X)$-bi-module. Since any $\mathcal{F}(X)$-bi-module can be realized as the quotient of a free $\mathcal{F}(X)$-bi-module, the short exact sequence above can be extended to a resolution

$$\mathcal{R}: \ 0 \longleftarrow \mathcal{O} \xleftarrow{\pi} \mathcal{F}(X) \xleftarrow{\alpha} \mathcal{F}(X)\langle Y\rangle \xleftarrow{\beta} \mathcal{F}(X)\langle Z\rangle \longleftarrow \cdots$$

of $\mathcal{F}(X)$-bi-modules ($\mathcal{O}$ is a $\mathcal{F}(X)$-bi-module via π). Let now M be an $\mathcal{O}$-bi-module. Then M is an $\mathcal{F}(X)$-bi-module via π. Denote by

$$\mathrm{Der}(\mathcal{F}(X), M)$$

the space of *derivations* of $\mathcal{F}(X)$ with values in M, i.e. the set of all linear maps $\theta : \mathcal{F}(X) \to M$ such that

$$\begin{aligned} \theta(\varphi \circ \psi) &= \theta(\varphi) \cdot \psi + \varphi \cdot \theta(\psi) \\ \theta(\varphi \otimes \psi) &= \theta(\varphi) \otimes \psi + \varphi \otimes \theta(\psi) \end{aligned}$$

Since $\mathrm{Der}(\mathcal{F}(X), M)$ is a subspace of the space $\mathrm{Hom}(\mathcal{F}(X), M)$ of all linear maps from $\mathcal{F}(X)$ to M, the composition with α defines a morphism

$$\alpha^* : \mathrm{Der}(\mathcal{F}(X), M) \to \mathrm{Hom}_{\mathsf{Vect}}(\mathcal{F}(X)\langle Y\rangle, M)$$

What is crucial here is that the image of α^* is actually contained in the subspace $\mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle, M)$ of $\mathcal{F}(X)$-bi-module homomorfisms from $\mathcal{F}(X)\langle Y\rangle$ to M. In fact, if $\theta \in \mathrm{Der}(\mathcal{F}(X), M)$, $\varphi \in \mathcal{F}(X)$ and $\psi \in \mathcal{F}(X)\langle Y\rangle$, we have

$$(\alpha^*\theta)(\varphi \circ \psi) = \theta\left(\alpha(\varphi \circ \psi)\right) =$$

by $\mathcal{F}(X)$-linearity of α

$$= \theta\left(\varphi \circ \alpha(\psi)\right) = \theta(\varphi) \cdot \alpha(\psi) + \varphi \cdot \theta\left(\alpha(\psi)\right) =$$

since $\mathcal{F}(X)$ acts on M via π

$$\begin{aligned} &= \theta(\varphi) \cdot \pi\left(\alpha(\psi)\right) + \varphi \cdot \theta\left(\alpha(\psi)\right) = \\ &= \theta(\varphi) \cdot (\pi \circ \alpha)(\psi) + \varphi \cdot (\alpha^*\theta)(\psi) = \\ &= \varphi \cdot (\alpha^*\theta)(\psi) \end{aligned}$$

(the proof for the tensor product is completely analogous). Since β is an homomorphism of $\mathcal{F}(X)$-bi-modules, it induces (by composition) a map

$$\beta^* : \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Y\rangle, M) \to \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle Z\rangle, M)$$

Finally, being $\mathcal{R}$ a resolution, is $\beta^* \circ \alpha^* = 0$, so the sequence

$$0 \to \mathrm{Der}(\mathcal{F}(X), M) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Y\rangle, M) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Z\rangle, M) \tag{1}$$

is a cohomological complex. We set

$$\begin{aligned}
\mathcal{E}^1(\mathcal{R}, M) &:= \mathrm{Der}(\mathcal{F}(X), M) \\
\mathcal{E}^2(\mathcal{R}, M) &:= \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Y\rangle, M) \\
\mathcal{E}^3(\mathcal{R}, M) &:= \mathrm{Hom}_{\mathcal{F}(X)}(\mathcal{F}(X)\langle(Z\rangle, M) \\
\delta^1 := \alpha^*; &\quad \delta^2 := \beta^*
\end{aligned}$$

so that we can rewrite (1) as

$$0 \longrightarrow \mathcal{E}^1(\mathcal{R}, M) \xrightarrow{\delta^1} \mathcal{E}^2(\mathcal{R}, M) \xrightarrow{\delta^2} \mathcal{E}^3(\mathcal{R}, M) \tag{2}$$

The relosution of the operad $\mathcal{O}$ is not unique (it depends, among other things, on a choice of the set of generators X). But usual arguments show that the cohomology of $\{\mathcal{E}^*(\mathcal{R}, M), \delta^*\}$ does not depend on the particular resolution chosen. So we can define

$$\mathcal{T}^*(\mathcal{O}, M) := H^*(\mathcal{E}^*(\mathcal{R}, M), \delta^*)\,.$$

It is called the *cotangent cohomology* of $\mathcal{O}$ with coefficients in the $\mathcal{O}$-module M. Directly from the definition of the complex, it follows that $\mathcal{T}^1(\mathcal{O}, M) = \mathrm{Der}(\mathcal{O}, M)$. Notice that, if $A : \mathcal{O} \to \underline{\mathrm{End}}(V)$ is an algebra, then $\underline{\mathrm{End}}(V)$ is in a natural way an $\mathcal{O}$-module via A, so the cohomology groups

$$\mathcal{T}^*(\mathcal{O}, \underline{\mathrm{End}}(V))$$

are defined.

4 First order deformations.

The algebra A clearly lifts to a first order deformation, since the trivial deformation provides such a lifting. Since A is the unique 0^{th} order deformation of A, the answer to the first of our questions is trivially *yes*. To describe all the possible first order deformations, notice that, if A_1 is a first order deformation of A, we can write

$$A_1 = A + \hbar A_{(1)} \mod \hbar^2$$

The condition "$A_1 : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2)$ is a morphism" is equivalent to

$$\begin{aligned}
(A + \hbar A_{(1)})(\varphi \circ \psi) &= (A + \hbar A_{(1)})(\varphi) \circ (A + \hbar A_{(1)})(\psi) \mod \hbar^2, \\
(A + \hbar A_{(1)})(\varphi \otimes \psi) &= (A + \hbar A_{(1)})(\varphi) \otimes (A + \hbar A_{(1)})(\psi) \mod \hbar^2, \quad \forall \varphi, \psi \in \mathcal{O}
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
A(\varphi \circ \psi) + \hbar A_{(1)}(\varphi \circ \psi) &= \left(A(\varphi) + \hbar A_{(1)}(\varphi)\right) \circ \left(A(\psi) + \hbar A_{(1)}(\psi)\right) \mod \hbar^2 \\
&= A(\varphi \circ \psi) + \hbar \left(A_{(1)}(\varphi) \circ \psi + \varphi \circ A_{(1)}(\psi)\right) \mod \hbar^2
\end{aligned}$$

This gives

$$A_{(1)}(\varphi \circ \psi) = A_{(1)}(\varphi) \circ \psi + \varphi \circ A_{(1)}(\psi)$$

(exactly the same argument works for $\otimes$ in place of $\circ$), so $A_{(1)}$ is a *derivation* of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$. Vice versa, the argument above shows that if $A_{(1)}$ is a derivation of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$, the map $A_1 := A + \hbar A_{(1)}$ is an homomorphism $A_1 : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^2)$, and is clearly a lifting of A. Then we have proved

$$\{\text{First order deformations of } A\} \leftrightarrow \mathrm{Der}(\mathcal{O}, \underline{\mathrm{End}}(V)) = \mathcal{T}^1(\mathcal{O}, \underline{\mathrm{End}}(V))$$

5 Higher order deformations.

Assume now $A_n : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})$ is a n^{th} order deformation. We wonder if it lifts to a $(n+1)^{th}$ order deformation A_{n+1}. This time there is no section

$$\underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$$

so we have no trivial answer as in the case $n = 0$. Here we use the power of free objects. Being $\mathcal{F}(X)$ free, we can lift A_n to $\tilde{A}_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. This way we have

$$\begin{tikzcd}
\mathcal{F}(X)\langle Y\rangle \arrow[d, "\alpha"] & \\
\mathcal{F}(X) \arrow[r, "\tilde{A}_{n+1}"] \arrow[d, "\pi"] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \arrow[d, "\mathrm{mod}\ \hbar^{n+1}"] \\
\mathcal{O} \arrow[r, "A_n"] & \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})
\end{tikzcd}$$

In particular $\tilde{A}_{n+1} \circ \alpha$ maps all of $\mathcal{F}(X)\langle Y\rangle$ to $0 \mod \hbar^{n+1}$. This means

$$\begin{aligned}\alpha^*\tilde{A}_{n+1} : \mathcal{F}(X)\langle Y\rangle \to & \ker\{\mathrm{mod}\hbar^{n+1} : \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})\} \\ & \simeq \underline{\mathrm{End}}(V)\end{aligned}$$

We denote this composition as

$$\Omega(\tilde{A}_{n+1}) : \mathcal{F}(X)\langle Y\rangle \to \underline{\mathrm{End}}(V)$$

The map $\Omega(\tilde{A}_{n+1})$ is an homomorphism of $\mathcal{F}(X)$-modules ($\tilde{A}_{n+1}$ and α are), so $\Omega(\tilde{A}_{n+1})$ is an element of $\mathcal{E}^2(\mathcal{R}, \underline{\mathrm{End}}(V))$. We have $\delta^2\Omega(\tilde{A}_{n+1}) = \iota \circ \tilde{A}_{n+1} \circ \alpha \circ \beta = 0$ (where ι denotes the isomorphism between $\ker\{\mod \hbar^{n+1} : \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2}) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+1})\}$ and $\underline{\mathrm{End}}(V)$); so $\Omega(\tilde{A}_{n+1})$ is a 2-cocycle. The element

$\Omega(\tilde{A}_{n+1})$ depends on the lifting $\tilde{A}_{n+1}$. But, if we chose another lifting $\tilde{A}'_{n+1}$, we have

$$\tilde{A}'_{n+1} = \tilde{A}_{n+1} + \hbar^{n+1} A_{(n+1)} \mod \hbar^{n+2}$$

Since both $\tilde{A}'_{n+1}$ and $\tilde{A}_{n+1}$ are homomorphisms $\mod \hbar^{n+2}$, $A_{(n+1)}$ is a derivation of $\mathcal{F}(X)$ with values in $\underline{\mathrm{End}}(V)$. We have

$$\begin{aligned}\Omega(\tilde{A}'_{n+1}) &= \iota\alpha^* \tilde{A}_{n+1} + \iota\left(\hbar^{n+1}\alpha^* A_{(n+1)}\right)\\ &= \Omega(\tilde{A}_{n+1}) + \alpha^* A_{(n+1)}\\ &= \Omega(\tilde{A}_{n+1}) + \delta^1 A_{(n+1)}\end{aligned}$$

i.e. $\Omega(\tilde{A}'_{n+1})$ and $\Omega(\tilde{A}_{n+1})$ differ by a coboundary. This means that the cohomology class $[\Omega(\tilde{A}_{n+1})]$ is independent of the particular lifting chosen. We write

$$[\Omega(A_n)]$$

to denote it. Assume now that A_n has a lifting $A_{n+1} : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. Then $\pi^* A_{n+1}$ is a lifting of A_n from $\mathcal{F}(X)$ to $\underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$. We have $\alpha^*(\pi^* A_{n+1}) = 0$, so $[\Omega(A_n)] = [\Omega(\pi^* A_{n+1})] = 0$. This means that, if A_n lifts to a $(n+1)^{th}$ deformation, then the *obstruction class* $[\Omega(A_n)]$ vanishes. The converse is true: assume $[\Omega(A_n)] = 0$. Then there exists a lifting $\tilde{A}_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$ and a derivation $A_{(n+1)} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)$ such that

$$\Omega(\tilde{A}_{n+1}) = \delta^1 A_{(n+1)}$$

We can rewrite this equation as

$$\iota\alpha^* \tilde{A}_{n+1} = \iota\hbar^{n+1}\delta^1 A_{(n+1)} = \iota\hbar^{n+1}\alpha^* A_{(n+1)}$$

that is

$$\iota\left(\alpha^*(\tilde{A}_{n+1} - \hbar^{n+1} A_{(n+1)})\right) = 0$$

Since ι is an isomorphism, this gives

$$\alpha^*(\tilde{A}_{n+1} - \hbar^{n+1} A_{(n+1)}) = 0$$

Let now $A_{n+1} := \tilde{A}_{n+1} - \hbar^{n+1} A_{(n+1)} \mod \hbar^{n+2}$. Since $\tilde{A}_{n+1}$ is an homomorphism $\mod \hbar^{n+2}$ and $A_{(n+1)}$ is a derivation, A_{n+1} is an homomorphism

$$A_{n+1} : \mathcal{F}(X) \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$$

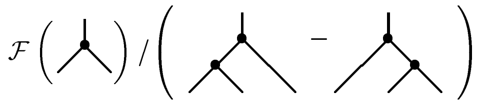
and lifts A_n. Moreover $\alpha^* A_{n+1} = 0$, so A_{n+1} defines an homomorphism $A_{n+1} : \mathcal{O} \to \underline{\mathrm{End}}(V)[[\hbar]]/(\hbar^{n+2})$ which lifts A_n. This way we have proved that the cohomology class $[\Omega(A_n)]$ is a complete obstruction to the lifting of A_n to an $(n+1)^{th}$ order deformation, i.e., the answer to the first of our two questions is given by an element in the second cohomology group

$$\mathcal{T}^2(\mathcal{O}, \underline{\mathrm{End}}(V))$$

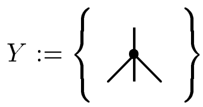
Assume now that a lift A_{n+1} exists. Then all the others are of the form $A_{n+1} + \hbar^{n+1}A_{(n+1)} \mod \hbar^{n+2}$, with $A_{(n+1)}$ derivation of $\mathcal{O}$ with values in $\underline{\mathrm{End}}(V)$ (the proof of this statement is completely analogue to the one given for the case $n = 0$). This means that, when the answer to the first question is yes, the answer to the second is: $\mathcal{T}^1(\mathcal{O}, \underline{\mathrm{End}}(V))$.

6 Deformations of associative algebras.

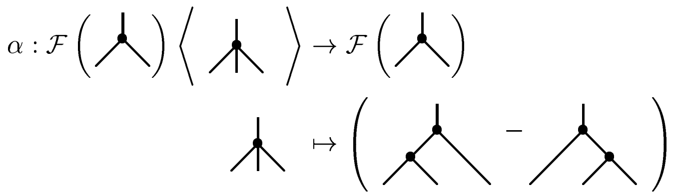
We must find a resolution (at least the first three terms) of the operad **Assoc** governing associative algebras. We already know that **Assoc** can be presented as the quotient



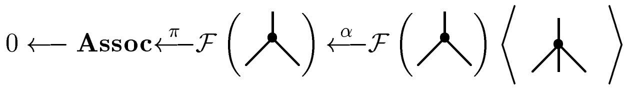
So the kernel of the canonical projection can be generated by a single element, which is an element with 3 inputs and one output. Thus, if we set



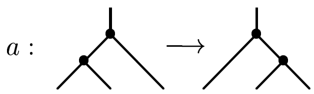
and define $\alpha : \mathcal{F}(X)\langle Y\rangle \to \mathcal{F}(X)$ as



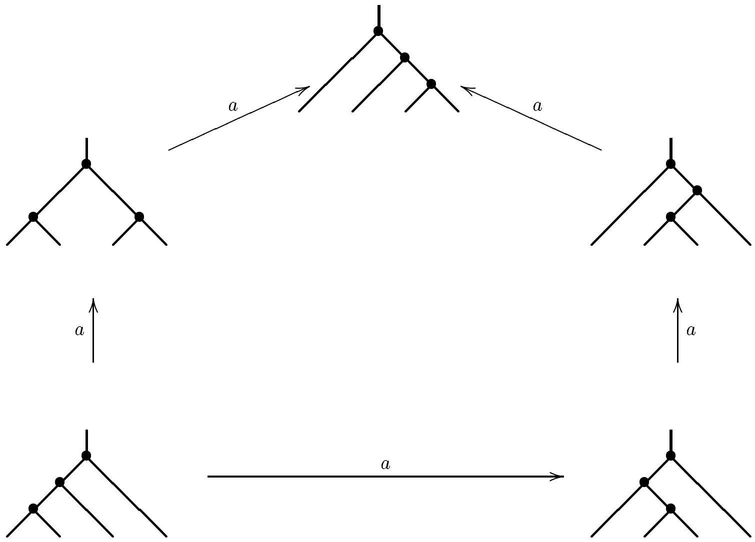
the sequence



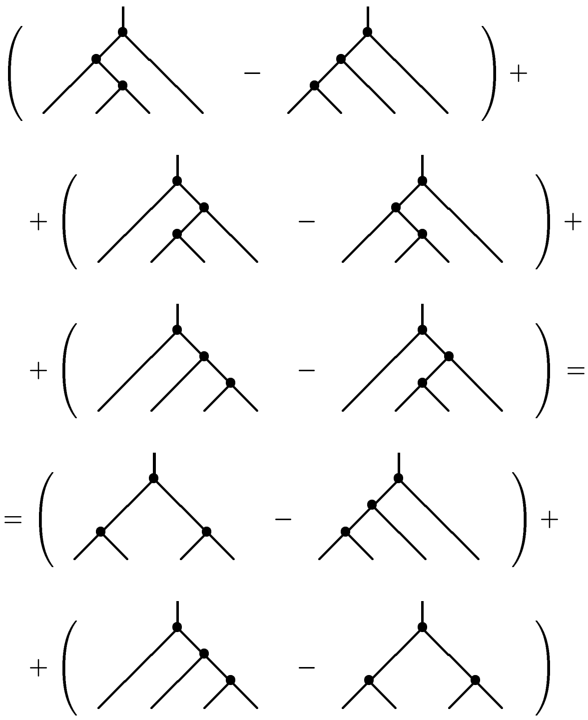
is a resolution of **Assoc**. We need one more term. To describe the kernel of α we use McLane coherence theorem. If we denote by a the *move*



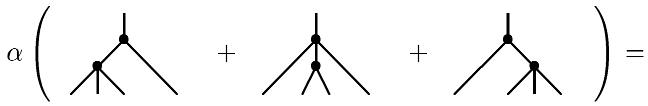
then we have the well known pentagon relation:

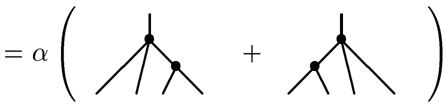


This gives:

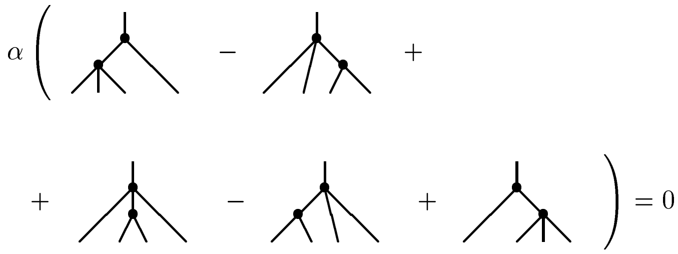


that can be rewritten as:

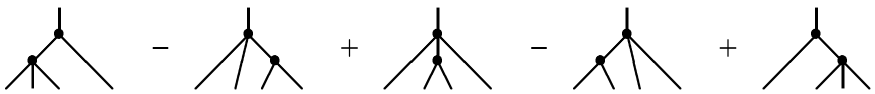




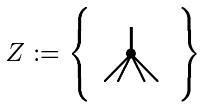
that is



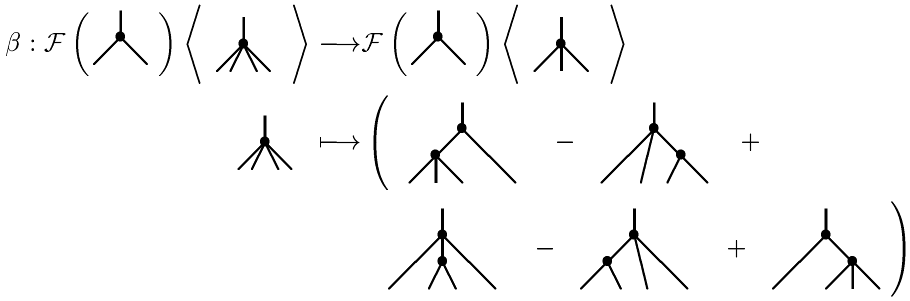
The McLane coherence theorem tells us that the pentagon generates all possible relations among the a moves. This implies that the element



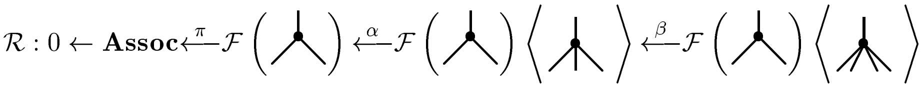
generates $\ker \alpha$. Thus, if we set

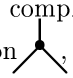


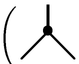
and define



then



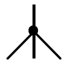
is a resolution of **Assoc**. We can now compute the complex $\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V))$ for an associative algebra $\{V, \mu\}$. Since $\mathcal{F}(X)$ is free on , to assign a derivation θ, we just have to assign the value

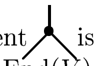
$$m := \theta \left(\right.$$  $$\left. \right)$$

Then θ is completely determined by the Leibnitz rule. Since m can be arbitrarly chosen in $\mathrm{Hom}(V^{\otimes 2}, V)$, we have a canonical isomorphism

$$\mathcal{E}^1(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 2}, V)$$

In the same way, $\mathcal{F}(X)$-module maps $\varphi : \mathcal{F}(X)\langle Y \rangle \to \underline{\mathrm{End}}(V)$ are completely determined by the image

$$f := \varphi \left(\right.$$  $$\left. \right)$$

which is an element of $\mathrm{Hom}(V^{\otimes 3}, V)$ (the element  is forced to act as μ since φ is an $\mathcal{F}(X)$-modules map, and $\mathcal{F}(X)$ acts on $\underline{\mathrm{End}}(V)$ via π). Thus we have a canonical isomorphism

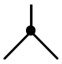
$$\mathcal{E}^2(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 3}, V)$$

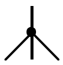
A completely analogous argument gives

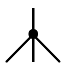
$$\mathcal{E}^3(\mathcal{R}, \underline{\mathrm{End}}(V)) = \mathrm{Hom}(V^{\otimes 4}, V)$$

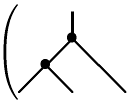
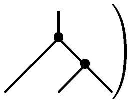
This means that, at the level of cochain spaces, we have

$$\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)) = C^*(V, V)[1]$$

where the latter is the cochain space of the well known Hochshild complex. Now we come to the differentials. Let $m \in \mathrm{Hom}(V^{\otimes 2}, V)$. Then there exists θ such that $m = \theta \left(\right.$  $\left. \right)$. We have

$$\delta^1 m = (\delta^1 \theta) \left(\right.$$  $$\left. \right)$$

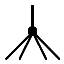
$$= \theta \left(\alpha \left(\right. \right.$$  $$\left. \left. \right) \right)$$

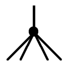
$$= \theta \left(\right.$$  $$-$$  $$\left. \right)$$

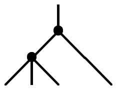
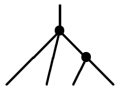
$$= \mu(m \otimes Id) - \mu(Id \otimes Id) + m(\mu \otimes Id) - m(Id \otimes \mu)$$

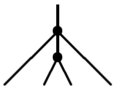
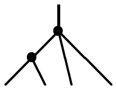
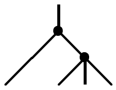
$$= d^1_{Hoch}\, m$$

For $f \in \mathrm{Hom}(V^{\otimes 3}, V)$, results

$\delta^2 f = (\delta^2\varphi)\Bigg($  $\Bigg)$

$= \varphi\Bigg(\beta\Bigg($  $\Bigg)\Bigg)$

$= \varphi\Bigg($  $-$  $+$

 $-$  $+$  $\Bigg)$

$$= \mu(f \otimes Id) - f(Id^{\otimes 2} \otimes \mu) + f(Id \otimes \mu \otimes Id) - f(\mu \otimes Id^{\otimes 2}) + \mu(Id \otimes f)$$
$$= d^2_{Hoch}\ f$$

Then also the differentials coincide with the Hochshild ones, and we have proved that the complex $\{\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)), \delta^*\}$ is canonically isomorphic to the Hochshild complex:

$$\{\mathcal{E}^*(\mathcal{R}, \underline{\mathrm{End}}(V)), \delta^*\} = \{C^*(V,V)[1], d^*_{Hoch}\}$$

In particular we have found again the well known result that the obstruction to the lifting for an infinitesimal deformation of an associative algebra (V, μ) lies in $H^2_{Hoch}(V,V)$ and that, when this obstruction vanishes, the liftings are parametrized by $H^1_{Hoch}(V,V)$.

References

[Fi] D. Fiorenza. An Introduction to the Language of Operads, *available at* `http:www.mat.uniroma2.it/~fiorenza/publications.html`

[Ma1] M. Markl. Cotangent cohomology of a category and deformation. *J. Pure and Applied Algebra*, 113:195–218, 1996.

[Ma2] M. Markl. Homotopy Algebras via Resolution of Operads. `math.AT/9808101`.