WEAK KAM THEORY TOPICS
IN THE STATIONARY ERGODIC SETTING

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ABSTRACT. We perform a qualitative analysis of the critical equation associated with a stationary ergodic Hamiltonian through a stochastic version of the metric method, where the notion of closed random stationary set, issued from stochastic geometry, plays a major role. Our purpose is to give an appropriate notion of random Aubry set, to single out characterizing conditions for the existence of exact or approximate correctors, and write down representation formulae for them. For the last task, we make use of a Lax–type formula, adapted to the stochastic environment. This material can be regarded as a first step of a long–term project to develop a random analog of Weak KAM Theory, generalizing what done in the periodic case or, more generally, when the underlying space is a compact manifold.

1. INTRODUCTION

For a given a probability space Ω, on which $\mathbb{R}^N$ acts ergodically, we consider the family of Hamilton–Jacobi equations

$$H(x, Dv, \omega) = a \qquad \text{in } \mathbb{R}^N,$$

where a varies in $\mathbb{R}$, and H is a continuous Hamiltonian, convex and superlinear in the momentum variable, and stationary with respect to the action of $\mathbb{R}^N$. As it is well known, this framework includes the periodic [16], quasi–periodic [2] and almost–periodic cases [12] as particular instances.

A *stationary critical value*, denoted by c, can be defined in this setting as the minimal value a for which the above equation possesses *admissible* subsolutions, that is Lipschitz random functions that have stationary gradient with mean 0 and that are almost sure subsolutions either in the viscosity sense or, equivalently, almost everywhere in $\mathbb{R}^N$. The condition on the gradient implies almost sure sublinear growth at infinity, see [9, 10]. The stationary critical value is in general distinct from the *free critical value* c_f, i.e. the minimal value a for which the above equation admits subsolutions, without any further qualification. More precisely $c_f(\omega)$ is a random variable, almost surely constant because of the ergodicity assumption. Clearly $c \geq c_f$.

The relevance of the stationary critical value c relies on the fact that it is the only level of H for which the corresponding critical equation can have admissible exact or approximate solutions, also named *exact* and *approximate correctors* for the role they play in associated homogenization problems, see Section 3 for precise definitions.

The aim of the paper is to perform a qualitative study of the critical equation, in any space dimensions, through the metric approach, by developing the ideas of [9, 10]. The adaptation of this pattern to the stationary ergodic setting requires the use of some tools from random set theory, the leading idea being that the stationary ergodic structure of the Hamiltonian induces a stochastic geometry in the space of

the state variable, where the fundamental entities are the closed random stationary sets which, somehow, play the same role as the points in the deterministic case.

More specifically, our purpose is to give an appropriate notion of random Aubry set, to single out characterizing conditions for the existence of exact or approximate correctors, and write down representation formulae for them. This material can be regarded as a first step of a long–term project to develop a random analog of Weak KAM Theory, generalizing what done in the periodic case or, more generally, when the underlying space is a compact manifold, see [11].

We recall that the random version of the metric method has allowed to completely clarify the setup in the one–dimensional case [9], where it has been proved the existence of approximate or exact correctors via Lax representation formulae, depending on whether 0 belongs or not to the interior of the flat part of the effective Hamiltonian obtained via homogenization [19, 22]. This permits, among other things, to carry out the homogenization procedures through Evans' perturbed test function method.

Even if in the multidimensional analysis [10] many analogies with the one–dimensional setting appear, the topic is definitely more involved, due to the increased degrees of freedom, so that the picture is far from being complete. In particular the issue of the existence of approximate correctors is a relevant open problem, see Section 6.

Our investigation can be briefly described as follows. We associate to the critical equation a Finsler–type random semidistance S on $\mathbb{R}^N$, and we consider the family of fundamental (critical) admissible subsolutions obtained via the Lax formula

$$\inf\{g(y,\omega) + S(y,x,\omega) \,:\, y \in C(\omega)\}, \tag{1}$$

where $C(\omega)$ is a closed random stationary set and g is an admissible critical subsolution.

We first address our attention to detect characterizing conditions on g and $C(\omega)$ under which the above formula defines an exact corrector. In case $c = c_f$, this holds true if $C(\omega) \subset \mathcal{A}_f(\omega)$ almost surely, where $\mathcal{A}_f(\omega)$ is the classical Aubry set, made up, as in the deterministic case, by points around which some degeneracy of S takes place. It can be defined through conditions on cycles, see Section 3. If instead $c > c_f$ or $c = c_f$ and $C(\omega) \cap \mathcal{A}_f(\omega) = \emptyset$ a.s., we find that formula (1) gives a solution if and only if any point y_0 in $C(\omega)$ is connected with the "infinity" through a curve along which $g(\cdot,\omega)$ is equal to $g(\cdot,\omega) + S(\cdot,y_0,\omega)$. This, in turn, implies that the asymptotic norm associated to S is degenerate.

The subsequent step is to use this information to propose a suitable notion of random Aubry set and to explore its properties. Our choice, in analogy with the periodic setting, is to define the *random Aubry set* $\mathcal{A}$ as the maximal stationary closed random set that plugged into (1) in place of C defines a corrector for any choice of the admissible subsolution g. We find that if $c = c_f$ then $\mathcal{A}_f(\omega) \subset \mathcal{A}(\omega)$ a.s. and if, in addition, no metric degeneracy occurs at infinity or, in other term, the stable norm associated with S is strictly positive in any direction, then $\mathcal{A}(\omega)$ and $\mathcal{A}_f(\omega)$ almost surely coincide.

Further we prove, generalizing a property holding in the deterministic case, the existence of an admissible critical subsolution $\bar{v}$ which is *weakly strict* in $\mathbb{R}^N \setminus \mathcal{A}(\omega)$, i.e. almost surely satisfying

$$\bar{v}(x,\omega) - \bar{v}(y,\omega) < S(y,x,\omega) \qquad \text{for every } x, y \in \mathbb{R}^N \setminus \mathcal{A}(\omega) \quad \text{with } x \neq y.$$

We are also able to extend to the stationary ergodic case some dynamical properties of the Aubry set. More precisely, we show that the random Aubry set is almost surely foliated by curves defined in $\mathbb{R}$ along which any critical subsolution agrees with the semidistance S, up to additive constants. These curves turn out to be global minimizers for the action of the Lagrangian in duality with H, and, when H is regular enough, they are in addition integral curves of the Hamiltonian flow.

The results on the random Aubry set are obtained under the crucial hypothesis that Ω is *separable* from the measure theoretic viewpoint, meaning that $L^2(\Omega)$ is separable. This assumption, while standard in the probabilistic literature, would exclude here the almost–periodic case. Following the usual approach, in fact, an almost–periodic function can be seen as the restriction on $\mathbb{R}^N$ of a continuous map defined on $\mathbb{G}^N$, the *Bohr compactification* of $\mathbb{R}^N$. The associated normalized Haar measure is a probability measure which is ergodic with respect to the action of $\mathbb{R}^N$. This allows to include the almost–periodic case within the stationary ergodic framework, but the problem is that $\mathbb{G}^N$ is non–separable, see [1] for similar issues.

Thus we have to resort to a different construction, exposed in the Appendix, that we believe of independent interest. We basically exploit that any almost–periodic function on $\mathbb{R}^N$ is the uniform limit of a sequence of quasi–periodic functions, which, in turn, can be seen as specific realizations of stationary ergodic maps defined on k–dimensional tori, with k suitably chosen.

By properly defining the objects we work with, we obtain that a given almost–periodic Hamiltonian can be seen as a specific realization of a stationary ergodic one, with Ω equal to a *countable* product of finite dimensional tori. The latter, endowed with the product distance, is a compact metric space, thus separable both from a topological and a measure–theoretic viewpoint. Some attention must be paid in the previous construction in order to preserve the ergodicity of the action of $\mathbb{R}^N$ on Ω.

The paper is organized as follows: in Section 2 we fix notations and expose some preliminary material, in particular we present definitions and properties of stationary closed random sets and random functions that are relevant for our analysis. Section 3 is focused on stochastic Hamilton–Jacobi equations, we introduce the metric tools we will need, and recall some basic facts about Aubry–Mather theory in the deterministic setting. Section 4 is devoted to Lax formulae in the stationary ergodic setting, in particular to derive characterizing conditions on the source set and on the trace under which the corresponding Lax formula defines an exact corrector. In Section 5 we define the random Aubry set and study its properties. In Section 6 we discuss some questions left open by our study. The Appendix contains the construction outlined above, addressed to include the almost–periodic case in our framework.

2. Preliminaries

We write below a list of symbols used throughout this paper.

N	an integer number
$B_R(x_0)$	the closed ball in $\mathbb{R}^N$ centered at x_0 of radius R
B_R	the closed ball in $\mathbb{R}^k$ centered at 0 of radius R
$\langle \,\cdot\, , \cdot\, \rangle$	the scalar product in $\mathbb{R}^N$
$|\cdot|$	the Euclidean norm in $\mathbb{R}^N$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathcal{B}(\mathbb{R}^k)$	the σ–algebra of Borel subsets of $\mathbb{R}^k$
χ_E	the characteristic function of the set E

Given a subset U of $\mathbb{R}^N$, we denote by $\overline{U}$ its closure. We furthermore say that U is *compactly contained* in a subset V of $\mathbb{R}^N$ if $\overline{U}$ is compact and contained in V. If E is a Lebesgue measurable subset of $\mathbb{R}^N$, we denote by $|E|$ its N–dimensional Lebesgue measure, and qualify E as *negligible* whenever $|E| = 0$. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^N$ if it holds up to a negligible set. We will write $\varphi_n \rightrightarrows \varphi$ on $\mathbb{R}^N$ to mean that the sequence of functions $(\varphi_n)_n$ uniformly converges to φ on compact subsets of $\mathbb{R}^N$.

With the term *curve*, without any further specification, we refer to a Lipschitz–continuous function from some given interval $[a, b]$ to $\mathbb{R}^N$. The space of all such curves is denoted by $\mathrm{Lip}([a, b], \mathbb{R}^N)$, while $\mathrm{Lip}_{x,y}([a, b], \mathbb{R}^N)$ stands for the family of curves γ joining x to y, i.e. such that $\gamma(a) = x$ and $\gamma(b) = y$, for any fixed x, y in $\mathbb{R}^N$. The Euclidean length of a curve γ is denoted by $\mathcal{H}^1(\gamma)$.

Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a *separable probability space*, where $\mathbb{P}$ is the probability measure and $\mathcal{F}$ the σ–algebra of $\mathbb{P}$–measurable sets. Here separable is understood in the measure theoretic sense, meaning that the Hilbert space $L^2(\Omega)$ is separable, cf. [23] also for other equivalent definitions. A property will be said to hold *almost surely* (*a.s.* for short) on Ω if it holds up to a subset of probability 0. We will indicate by $L^p(\Omega)$, $p \geq 1$, the usual Lebesgue space on Ω with respect to $\mathbb{P}$. If $f \in L^1(\Omega)$, we write $\mathbb{E}(f)$ for the mean of f on Ω, i.e. the quantity $\int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}(\omega)$.

We qualify as *measurable* a map from Ω to itself, or to a topological space $\mathcal{M}$ with Borel σ–algebra $\mathcal{B}(\mathcal{M})$, if the inverse image of any set in $\mathcal{F}$ or in $\mathcal{B}(\mathcal{M})$ belongs to $\mathcal{F}$. The latter will be also called *random variable* with values in $\mathcal{M}$.

We will be particulary interested in the case where the range of a random variable is a *Polish space*, namely a complete and separable metric space. By $\mathrm{C}(\mathbb{R}^N)$ and $\mathrm{Lip}_\kappa(\mathbb{R}^n)$, we will denote the Polish space of continuous and Lipschitz–continuous real functions (with Lipschitz constant less than or equal to $\kappa > 0$), defined in $\mathbb{R}^N$, both endowed with the metric d inducing the topology of uniform convergence on compact subsets of $\mathbb{R}^N$. We will use the expressions *continuous random function*, κ–*Lipschitz random function*, respectively, for the previously introduced random variables. We will more simply say *Lipschitz random function* to mean a κ–Lipschitz random function for some $\kappa > 0$. See [9] for more detail on this point.

We proceed by recalling some basic facts on convergence in probability. Given a Polish space $(\mathbb{F}, d)$ and a sequence $(f_n)_n$ of random variables taking values in $\mathbb{F}$, we will say that f_n converge to f *in probability* if, for every $\varepsilon > 0$,

$$\mathbb{P}\left(\{\omega \in \Omega \,:\, d(f_n(\omega), f(\omega)) > \varepsilon\}\right) \to 0 \quad \text{as } n \to +\infty.$$

The limit f is still a random variable. Since $\mathbb{F}$ is a separable metric space, almost sure convergence, i.e. $d\left(f_n(\omega), f(\omega)\right) \to 0$ a.s. in ω, implies convergence in probability, while the converse is not true in general. However, the following characterization holds:

Theorem 2.1. *Let f_n, f be random variables with values in $\mathbb{F}$. Then $f_n \to f$ in probability if and only if every subsequence $(f_{n_k})_k$ has a subsequence converging to f a.s..*

We denote by $L^0(\Omega, \mathbb{F})$ the space made up by the equivalence classes of random variables with value in $\mathbb{F}$ for the relation of almost sure equality. For every $f, g \in L^0(\Omega, \mathbb{F})$, we set

$$\alpha(f, g) := \inf\{\varepsilon \geq 0 \,:\, \mathbb{P}\big(\{\omega \in \Omega \,:\, d(f(\omega), g(\omega)) > \varepsilon\}\big) \leq \varepsilon\}.$$

Theorem 2.2. *α is a metric, named after Ky Fan, which metrizes convergence in probability, i.e. $\alpha(f_n, f) \to 0$ if and only if $f_n \to f$ in probability, and turns $L^0(\Omega, \mathbb{F})$ into a Polish space.*

An *N–dimensional dynamical system* $(\tau_x)_{x \in \mathbb{R}^N}$ is defined as a family of mappings $\tau_x : \Omega \to \Omega$ which satisfy the following properties:

(1) the *group property:* $\tau_0 = id, \quad \tau_{x+y} = \tau_x \circ \tau_y$;
(2) the mappings $\tau_x : \Omega \to \Omega$ are measurable and measure preserving, i.e. $\mathbb{P}(\tau_x E) = \mathbb{P}(E)$ for every $E \in \mathcal{F}$;
(3) the map $(x, \omega) \mapsto \tau_x \omega$ from $\mathbb{R}^N \times \Omega$ to Ω is jointly measurable, i.e. measurable with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F}$.

We will moreover assume that $(\tau_x)_{x \in \mathbb{R}^N}$ is *ergodic,* i.e. that one of the following equivalent conditions hold:

(i) every measurable function f defined on Ω such that, for every $x \in \mathbb{R}^N$, $f(\tau_x \omega) = f(\omega)$ a.s. in Ω, is almost surely constant;
(ii) every set $A \in \mathcal{F}$ such that $\mathbb{P}(\tau_x A \,\Delta\, A) = 0$ for every $x \in \mathbb{R}^N$ has probability either 0 or 1, where Δ stands for the symmetric difference.

Given a random variable $f : \Omega \to \mathbb{R}$, for any fixed $\omega \in \Omega$ the function $x \mapsto f(\tau_x \omega)$ is said to be a *realization of f*. The following properties follow from Fubini's Theorem, see [14]: if $f \in L^p(\Omega)$, then $\mathbb{P}$–almost all its realizations belong to $L^p_{loc}(\mathbb{R}^N)$; if $f_n \to f$ in $L^p(\Omega)$, then $\mathbb{P}$–almost all realizations of f_n converge to the corresponding realization of f in $L^p_{loc}(\mathbb{R}^N)$. The Lebesgue spaces on $\mathbb{R}^N$ are understood with respect to the Lebesgue measure.

The next lemma guarantees that a modification of a random variable on a set of zero probability does not affect its realizations on sets of positive Lebesgue measure on $\mathbb{R}^N$, almost surely in ω. The proof is based on Fubini's Theorem again, see Lemma 7.1 in [14].

Lemma 2.3. *Let $\widehat{\Omega}$ be a set of full measure in Ω. Then there exists a set of full measure $\Omega' \subseteq \widehat{\Omega}$ such that for any $\omega \in \Omega'$ we have $\tau_x \omega \in \widehat{\Omega}$ for almost every $x \in \mathbb{R}^N$.*

A jointly measurable function v defined in $\mathbb{R}^N \times \Omega$ is said *stationary* if, for every $z \in \mathbb{R}^N$, there exists a set Ω_z with probability 1 such that for every $\omega \in \Omega_z$

$$v(\cdot + z, \omega) = v(\cdot, \tau_z \omega) \quad \text{on } \mathbb{R}^N$$

It is clear that a real random variable ϕ gives rise to a stationary function v by setting $v(x, \omega) = \phi(\tau_x \omega)$. Conversely, according to Proposition 3.1 in [9], a stationary

function v is, a.s. in ω, the realization of the measurable function $\omega \mapsto v(0, \omega)$. More precisely, there exists a set Ω' of probability 1 such that for every $\omega \in \Omega'$

$$v(x, \omega) = v(0, \tau_x \omega) \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2}$$

With the term *(graph–measurable) random set* we indicate a set–valued function $X : \Omega \to \mathcal{B}(\mathbb{R}^N)$ with

$$\Gamma(X) := \left\{ (x, \omega) \in \mathbb{R}^N \times \Omega \; : \; x \in X(\omega) \right\}$$

jointly measurable in $\mathbb{R}^N \times \Omega$. A random set X will be qualified as *stationary* if for every for every $z \in \mathbb{R}^N$, there exists a set Ω_z of probability 1 such that

$$X(\tau_z \omega) = X(\omega) - z \qquad \text{for every } \omega \in \Omega_z. \tag{3}$$

We use a stronger notion of measurability, which is usually named in the literature after Effros, to define a *closed random set*, say $X(\omega)$. Namely we require $X(\omega)$ to be a closed subset of $\mathbb{R}^N$ for any ω and

$$\{\omega \; : \; X(\omega) \cap K \neq \emptyset\} \in \mathcal{F}$$

with K varying among the compact (equivalently, open) subsets of $\mathbb{R}^N$. This condition can be analogously expressed by saying that X is measurable with respect to the Borel σ–algebra related to the Fell topology on the family of closed subsets of $\mathbb{R}^N$. This, in turn, coincides with the Effros σ–algebra. If $X(\omega)$ is measurable in this sense then it is also graph–measurable, see [18] for more details.

A closed random set X is called stationary if it, in addition, satisfies (3). Note that in this event the set $\{\omega \; : \; X(\omega) \neq \emptyset\}$, which is measurable by the Effros measurability of X, is invariant with respect to the group of translations $(\tau_x)_{x \in \mathbb{R}^N}$ by stationarity, so it has probability either 0 or 1 by the ergodicity assumption.

Proposition 2.4. *Let f be a continuous random function and C a closed subset of $\mathbb{R}$. Then*

$$X(\omega) := \{x \; : \; f(x, \omega) \in C\}$$

is a closed random set in $\mathbb{R}^N$. If in addition f is stationary, then X is stationary.

See [9] for a proof.

For a random stationary set X it is immediate, by exploiting that the maps $\{\tau_x\}_{x \in \mathbb{R}^N}$ are measure preserving, that $\mathbb{P}(X^{-1}(x))$ does not depend on x, where

$$X^{-1}(x) = \{\omega \; : \; x \in X(\omega)\}.$$

Such quantity will be called *volume fraction* of X and denoted by q_X. Note that to any measurable subset Ω' of Ω it can be associated a stationary set Y through the formula

$$Y(\omega) := \{x \; : \; \tau_x \omega \in \Omega'\}.$$

In this case $Y^{-1}(x) = \tau_{-x} \Omega'$, and so $q_Y = \mathbb{P}(\Omega')$. By exploiting the ergodicity assumption and Birkhoff Ergodic Theorem it is possible to derive an interesting information on the asymptotic structure of closed stationary sets.It says, in particular, they are spread with some uniformity in the space. We refer the reader to [9] for a proof.

Proposition 2.5. *Let X be an almost surely nonempty stationary closed random set in $\mathbb{R}^N$. Then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\lim_{r \to +\infty} \frac{|(X(\omega) + B_R) \cap B_r|}{|B_r|} \geq 1 - \varepsilon \qquad \text{a.s. in } \Omega,$$

whenever $R \geq R_\varepsilon$.

Given a Lipschitz random function v, we set

$$\Delta_v(\omega) := \left\{ x \in \mathbb{R}^N \; : \; v(\cdot, \omega) \text{ is differentiable at } x \right\}.$$

Definition 2.6. A random Lipschitz function v is said to have *stationary increments* if, for every $z \in \mathbb{R}^N$, there exists a set Ω_z of probability 1 such that

$$v(x + z, \omega) - v(y + z, \omega) = v(x, \tau_z\omega) - v(y, \tau_z\omega) \quad \text{for all } x, y \in \mathbb{R}^N \qquad (4)$$

for every $\omega \in \Omega_z$.

The following holds:

Proposition 2.7. *Let v be a Lipschitz random function, then Δ_v is a random set. In addition, it is stationary with volume fraction 1 whenever v has stationary increments.*

Let v be a Lipschitz random function with stationary gradient. For every fixed $x \in \mathbb{R}^N$, the random variable $Dv(x, \cdot)$ is well defined on $\Delta_v^{-1}(x)$, which has probability 1 since Δ_v is a stationary set with volume fraction 1. Accordingly, we can define the mean $\mathbb{E}(Dv(x, \cdot))$, which is furthermore independent of x by the stationary character of Dv. In the sequel, we will be especially interested in the case when this mean is zero.

Definition 2.8. A Lipschitz random function will be called *admissible* if it has stationary increments and gradient with mean 0.

We state two characterizations of admissible random functions, and a result that guarantees that stationary Lipschitz random functions are admissible.

Theorem 2.9. *A Lipschitz random function v with stationary increments has gradient with vanishing mean if and only if it is almost surely sublinear at infinity, namely*

$$\lim_{|x| \to +\infty} \frac{v(x, \omega)}{|x|} = 0 \qquad a.s. \ in \ \omega. \qquad (5)$$

Theorem 2.10. *A Lipschitz random function v with stationary increments has gradient with vanishing mean if and only if*

$$x \mapsto \mathbb{E}(v(y, \cdot) - v(x, \cdot)) = 0 \quad for \ any \ x, y \in \mathbb{R}^N. \qquad (6)$$

Theorem 2.11. *Any stationary Lipschitz random function v is admissible.*

Notice that the mean $\mathbb{E}(v(x, \cdot))$ of a Lipschitz random function is independent of x, so when such a quantity is finite Theorem 2.11 is just a consequence of Theorem 2.10.

3. Stochastic Hamilton–Jacobi equations

We consider an Hamiltonian

$$H : \mathbb{R}^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$$

satisfying the following conditions:

(H1) the map $\omega \mapsto H(\cdot, \cdot, \omega)$ from Ω to the Polish space $C(\mathbb{R}^N \times \mathbb{R}^N)$ is measurable;

(H2) for every $(x, \omega) \in \mathbb{R}^N \times \Omega$, $H(x, \cdot, \omega)$ is convex on $\mathbb{R}^N$;

(H3) there exist two superlinear functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\alpha(|p|) \leq H(x, p, \omega) \leq \beta(|p|) \qquad \text{for all } (x, p, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega;$$

(H4) for every $(x, \omega) \in \mathbb{R}^N \times \Omega$, the set of minimizers of $H(x, \cdot, \omega)$ has empty interior;

(H5) $H(\cdot + z, \cdot, \omega) = H(\cdot, \cdot, \tau_z \omega)$ for every $(z, \omega) \in \mathbb{R}^N \times \Omega$.

Remark 3.1. Condition (H3) is equivalent to saying that H is superlinear and locally bounded in p, uniformly with respect to (x, ω). We deduce from (H2)

$$|H(x, p, \omega) - H(x, q, \omega)| \leq L_R |p - q| \quad \text{for all } x, \omega, \text{ and } p, q \text{ in } B_R, \qquad (7)$$

where $L_R = \sup\{ |H(x, p, \omega)| : (x, \omega) \in \mathbb{R}^N \times \Omega, |p| \leq R + 2 \}$, which is finite thanks to (H3). For a comment on hypothesis (H4), see Remark 3.6.

Remark 3.2. Any given periodic, quasi–periodic or almost–periodic Hamiltonian $H_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ can be seen as a specific realization of a suitably defined stationary ergodic Hamiltonian, cf. Remark 4.2 in [9]. In the periodic and quasi–periodic cases we take as Ω a k–dimensional torus, with k suitably chosen, which is separable both from the topological and the measure theoretic viewpoint. In the almost–periodic case, the usual construction is to take as Ω the Bohr compactification of $\mathbb{R}^N$, which however is not separable, cf. [1]. In order to include this interesting case in our treatment, we will show in the Appendix that, for a given almost–periodic Hamiltonian H_0, it is possible to construct a separable probability space Ω, equipped with an ergodic group of translations, such that H_0 can be seen as a specific realization of a stationary ergodic Hamiltonian.

For every $a \in \mathbb{R}$, we are interested in the stochastic Hamilton–Jacobi equation

$$H(x, Dv(x, \omega), \omega) = a \qquad \text{in } \mathbb{R}^N. \qquad (8)$$

The material we are about to expose has been already presented in [9, 10], to which we refer for the details. Here we just recall the main items.

We say that a Lipschitz random function is a *solution* (resp. *subsolution*) of (8) if it is a viscosity solution (resp. a.e. subsolution) a.s. in ω (see [3, 4] for the definition of viscosity (sub)solution in the deterministic case). Notice that any such subsolution is almost surely in $\mathrm{Lip}_{\kappa_a}(\mathbb{R}^n)$, where

$$\kappa_a := \sup\{ |p| : H(x, p, \omega) \leq a \text{ for some } (x, \omega) \in \mathbb{R}^N \times \Omega \}, \qquad (9)$$

which is finite thanks to (H3). We are interested in the class of *admissible subsolutions*, hereafter denoted by $\mathcal{S}_a$, i.e. random functions taking values in $\mathrm{Lip}_{\kappa_a}(\mathbb{R})$ with stationary increments and zero mean gradient that are subsolutions of (8). An admissible solution will be also named *exact corrector*, remembering its role in homogenization. Further, for any $\delta > 0$, a random function v_δ will be called a δ–*approximate corrector* for the equation (8) if it belongs to $\mathcal{S}_{a+\delta}$ and satisfies the inequalities

$$a - \delta \leq H(x, Dv_\delta(x, \omega), \omega) \leq a + \delta$$

in the viscosity sense a.s. in ω. We say that (8) has *approximate correctors* if it admits δ–approximate correctors for any $\delta > 0$.

We proceed by defining the *free* and the *stationary critical value*, denoted by $c_f(\omega)$ and c respectively, as follows:

$$c_f(\omega) \;=\; \inf\left\{a \in \mathbb{R} \;:\; \text{(8) has a subsolution } v \in \mathrm{Lip}(\mathbb{R}^N)\right\}, \qquad (10)$$

$$c \;=\; \inf\{a \in \mathbb{R} \;:\; \mathcal{S}_a \neq \emptyset\}. \qquad (11)$$

We emphasize that in definition (10) we are considering *deterministic* a.e. subsolutions v of the equation (8), where ω is treated as a fixed parameter. Furthermore, we note that $c_f(\tau_z\omega) = c_f(\omega)$ for every $(z, \omega) \in \mathbb{R}^N \times \Omega$, so that, by ergodicity, the random variable $c_f(\omega)$ is almost surely equal to a constant, still denoted by c_f. Hereafter we will write Ω_f for the set of probability 1 where $c_f(\omega)$ equals c_f.

Concerning the definition of the critical value c, we notice that the set appearing at the right–hand side of (11) is non void, since it contains the value $\sup_{(x,\omega)} H(x, 0, \omega)$, which is finite thanks to (H3). Moreover, the infimum is attained. In fact, see [10, 17]

Theorem 3.3. $\mathcal{S}_c \neq \emptyset$.

It is apparent by the definitions that $c \geq c_f$. A more precise result, establishing the relation with the effective Hamiltonian obtained via the homogenization [19, 22], will be discussed in the next section.

In the sequel, we mostly focus our attention on the *critical equation*

$$H(x, Dv(x, \omega), \omega) = c \qquad \text{in } \mathbb{R}^N. \qquad (12)$$

The relevance of the critical value is given by the following result, see Theorem 4.5 in [9] for the proof.

Theorem 3.4. *The critical equation (12) is the unique among the equations (8) for which either an exact corrector or approximate correctors may exist.*

Following the so called metric method for the analysis of (8), see [11], we introduce an intrinsic path distance. In next formulae we assume that $a \geq c_f$ and $\omega \in \Omega_f$. We start by defining the sublevels

$$Z_a(x, \omega) := \{p \;:\; H(x, p, \omega) \leq a\},$$

and the related support functions

$$\sigma_a(x, q, \omega) := \sup\left\{\langle q, p \rangle \;:\; p \in Z_a(x, \omega)\right\}.$$

It comes from (7) (cf. Lemma 4.6 in [9]) that, given $b > a$, we can find $\delta = \delta(b, a) > 0$ with

$$Z_a(x, \omega) + B_\delta \subseteq Z_b(x, \omega) \qquad \text{for every } (x, \omega) \in \mathbb{R}^N \times \Omega_f. \qquad (13)$$

This property is needed in the proof of Theorem 3.4. It is straightforward to check that σ_a is convex in q, upper semicontinuous in x and, in addition, continuous whenever $Z_a(x, \omega)$ has nonempty interior or reduces to a point. We extend the definition of σ_a to $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ by setting $\sigma_a(\cdot, \cdot, \omega) \equiv 0$ for every $\omega \in \Omega \setminus \Omega_f$. With this choice, the function σ_a is jointly measurable in $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ and enjoys the stationarity property

$$\sigma_a(\cdot + z, \cdot, \omega) = \sigma_a(\cdot, \cdot, \tau_z\omega) \quad \text{for every } z \in \mathbb{R}^N \text{ and } \omega \in \Omega.$$

We define the semidistance S_a as

$$S_a(x, y, \omega) = \inf\left\{\int_0^1 \sigma_a(\gamma(s), \dot{\gamma}(s), \omega)\, ds \;:\; \gamma \in \mathrm{Lip}_{x,y}([0, 1], \mathbb{R}^N)\right\}, \qquad (14)$$

The function S_a is measurable on $\mathbb{R}^N \times \mathbb{R}^N \times \Omega$ with respect to the product σ–algebra $\mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{F}$, and satisfies the following properties:

$$
\begin{aligned}
S_a(x, y, \tau_z \omega) &= S(x + z, y + z, \omega) \\
S_a(x, y, \omega) &\leq S_a(x, z, \omega) + S_a(z, y, \omega) \\
S_a(x, y, \omega) &\leq \kappa_a |x - y|
\end{aligned}
$$

for all $x, y, z \in \mathbb{R}^N$ and $\omega \in \Omega$.

In the study of equation (8), a special role is played by the classical *(projected) Aubry set* (cf. [11]), defined for every $\omega \in \Omega_f$ as the collection of points $y \in \mathbb{R}^N$ such that

$$
\inf \left\{ \int_0^1 \sigma_a(\gamma, \dot{\gamma}, \omega) \, \mathrm{d}s \; : \; \gamma \in \mathrm{Lip}_{y,y}([0, 1], \mathbb{R}^N), \, \mathcal{H}^1(\gamma) \geq \delta \right\} = 0
$$

for some $\delta > 0$, or, equivalently (cf. [11, Lemma 5.1]), for any $\delta > 0$. From the Aubry–Mather theory for deterministic Hamiltonians we know that, when $a > c_f$, this set is empty for all $\omega \in \Omega_f$, i.e. almost surely. Hence, the only interesting case is the one corresponding to $a = c_f$. Hereafter we will denote by $\mathcal{A}_f(\omega)$ the collection of points y of $\mathbb{R}^N$ enjoying the above condition with $a = c_f$. The set $\mathcal{A}_f(\omega)$ is closed for every $\omega \in \Omega$.

We will also use later an equivalent definition of $\mathcal{A}_f(\omega)$, see [6]. For every $\omega \in \Omega$, let

$$
L(x, q, \omega) := \max_{p \in \mathbb{R}^N} \left\{ \langle p, q \rangle - H(x, p, \omega) \right\}, \qquad (x, q) \in \mathbb{R}^N \times \mathbb{R}^N
$$

and, for every $t > 0$,

$$
h_t(x, y, \omega) := \inf \left\{ \int_0^t (L(\gamma, \dot{\gamma}, \omega) + c) \, \mathrm{d}s \; : \; \gamma(0) = x, \, \gamma(t) = y \right\}, \qquad x, y \in \mathbb{R}^N.
$$

Then

$$
\mathcal{A}_f(\omega) = \{ y \in \mathbb{R}^N \; : \; \liminf_{t \to +\infty} h_t(y, y, \omega) = 0 \}. \tag{15}
$$

In the next theorem we outline the main deterministic properties linking $\mathcal{A}_f(\omega)$ to equation (8), see [11].

Theorem 3.5. *Let $\omega \in \Omega_f$. The following holds:*

(i) *Assume that $\mathcal{A}_f(\omega) \neq \emptyset$. If w_0 is a function defined on $C \subset \mathcal{A}_f(\omega)$ such that*

$$
w_0(x) - w_0(y) \leq S_{c_f}(y, x, \omega) \quad \text{for every } x, y \in C,
$$

then the function

$$
w(x) := \min_{y \in C} \left(w_0(y) + S_{c_f}(y, x, \omega) \right) \qquad x \in \mathbb{R}^N
$$

is the maximal subsolution of (8) with $a = c_f$ equaling w_0 on C, and a solution as well.

(ii) *Let U be a bounded open subset of $\mathbb{R}^N$, and assume that either $a > c_f$, or $a = c_f$ and $U \cap \mathcal{A}_f(\omega) = \emptyset$. Let w_0 be a function defined on ∂U such that*

$$
w_0(x) - w_0(y) \leq S_a(y, x, \omega) \quad \text{for every } x, y \in \partial U.
$$

Then the function

$$
w(x) := \inf_{y \in \partial U} \left(w_0(y) + S_a(y, x, \omega) \right) \qquad x \in U
$$

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is the unique viscosity solution of the Dirichlet Problem:

$$\begin{cases} H(x, D\phi(x), \omega) = a & \text{in } U \\ \phi(x) = w_0(x) & \text{on } \partial U. \end{cases}$$

(iii) *Assume that $a = c_f$ and let U be a bounded open subset of $\mathbb{R}^N$ with $U \cap \mathcal{A}_f(\omega) \neq \emptyset$. Let w_0 be a function defined in $\partial U \cup \mathcal{A}_f$ 1–Lipschitz continuous with respect to S_a. Then the function*

$$w(x) := \inf \left\{ w_0(y) + S_a(y, x, \omega) \,:\, y \in \partial U \cup (U \cap \mathcal{A}_f) \right\} \qquad x \in U \setminus \mathcal{A}_f$$

is the unique viscosity solution of the Dirichlet Problem:

$$\begin{cases} H(x, D\phi(x), \omega) = a & \text{in } U \setminus \mathcal{A}_f \\ \phi(x) = w_0(x) & \text{on } \partial U \cup (U \cap \mathcal{A}_f). \end{cases}$$

We define, for every $\omega \in \Omega$, the *set of equilibria*, as follows:

$$\mathcal{E}(\omega) := \{ y \in \mathbb{R} \,:\, \min_p H(y, p, \omega) = c_f \}.$$

The set $\mathcal{E}(\omega)$ is a (possibly empty) closed subset of $\mathcal{A}_f(\omega)$ (cf. [11, Lemma 5.2]). It is apparent that $c_f \geq \sup_{x \in \mathbb{R}^N} \min_{p \in \mathbb{R}^N} H(x, p, \omega)$ a.s. in ω; we point out that $\mathcal{E}(\omega)$ is nonempty if and only if the previous formula holds with an equality. In this case, $\mathcal{E}(\omega)$ is made up by the points y where the maximum is attained.

Remark 3.6. The inclusion $\mathcal{E}(\omega) \subseteq \mathcal{A}_f(\omega)$ depends on the fact that the c_f–sublevel $\{p \,:\, H(y, p, \omega) \leq c_f\}$ is non–void and has empty interior when $y \in \mathcal{E}(\omega)$. The latter is a consequence of (H4), and this is actually the unique point where such condition is used.

We recall for later use a result from [10].

Proposition 3.7. $\mathcal{E}(\omega)$ *and* $\mathcal{A}_f(\omega)$ *are stationary closed random sets.*

4. Lax formula and closed random sets

In this section we give a stochastic version of Lax formula and investigate when it provides an exact corrector.

Let $C(\omega)$ be an almost surely nonempty stationary closed random set in $\mathbb{R}^N$. Take a Lipschitz random function g and set, for $a \geq c_f$,

$$u(x, \omega) := \inf\{g(y, \omega) + S_a(y, x, \omega) \,:\, y \in C(\omega)\} \quad x \in \mathbb{R}^N, \tag{16}$$

where we agree that $u(\cdot, \omega) \equiv 0$ when either $C(\omega) = \emptyset$ or the above infimum is equal to $-\infty$. The following holds, see [9, 10]:

Proposition 4.1. *Let $a \geq c_f$ and $C(\omega)$, u as above.*

(i) *Let g be a stationary random function and assume that the infimum in (16) is a.s. finite. Then u is a stationary random variable belonging to $\mathcal{S}_a$ and satisfies $u(\cdot, \omega) \leq g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω. Moreover, u is a viscosity solution of (8) in $\mathbb{R}^N \setminus C(\omega)$ a.s. in ω.*

(ii) *Assume $g \in \mathcal{S}_a$. Then the random Lipschitz function u belongs to $\mathcal{S}_a$ and satisfies $u(\cdot, \omega) = g(\cdot, \omega)$ on $C(\omega)$ a.s. in ω. Moreover, u is a viscosity solution of (8) in $\mathbb{R}^N \setminus C(\omega)$ a.s. in ω.*

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We recall that the effective Hamiltonian $\overline{H}$ is the function associating to any $P \in \mathbb{R}^N$ the critical value of the Hamiltonian $H(x, p + P, \omega)$, equivalently it can be defined by homogenization, see [19, 22]. It can be proved, see [9, 10], that it is convex and superlinear, and $\min_{\mathbb{R}^N} \overline{H} = c_f$. For any $a \geq c_f$ we denote by $\overline{Z}_a$ the a–sublevel of $\overline{H}$. By making use of Propositions 4.1 and 3.7, the following result has been proved in [10]

Theorem 4.2.

  (i) *If $c = c_f$ and the classical Aubry set $\mathcal{A}_f(\omega)$ is almost surely nonempty, then the extension of any $g \in \mathcal{S}_c$ from $\mathcal{A}_f$ through Lax formula with distance S_c provides an exact corrector for (12);*

  (ii) *If $0 \in \operatorname{Int}(\overline{Z}_c)$, then $c = c_f$ and there exists an exact corrector for (12) if and only if the classical Aubry set $\mathcal{A}_f(\omega)$ is almost surely nonempty.*

To ease notations, from now on we will always write S, σ and $\mathcal{S}$ in place of S_c, σ_c and $\mathcal{S}_c$, respectively.

The next result shows that the property that the Lax formula with source a random set $C(\omega)$ and trace $g \in \mathcal{S}$ gives an exact corrector can be solely detected looking at the behavior of g on C. This will be used in the next section for studying the random Aubry set.

Theorem 4.3. *Let $C(\omega)$ and g be a stationary closed random set and a critical subsolution, respectively. Assume that either $c > c_f$, or $c = c_f$ and $\mathcal{A}_f(\omega) \cap C(\omega) = \emptyset$ a.s. in ω. Then the Lax extension of g from $C(\omega)$ with distance S, denoted by u, is an exact corrector if and only if*

$$\text{for any } y_0 \in C(\omega) \text{ there exists a diverging sequence } (y_n)_n \text{ in } C(\omega)$$
$$\text{such that} \tag{17}$$
$$g(y_0, \omega) = \lim_n g(y_n, \omega) + S(y_n, y_0, \omega),$$

a.s. in ω.

Proof. (17) *holds $\Rightarrow u$ is an exact corrector*

In view of Proposition 4.1, we can select a subset Ω' of Ω with $\mathbb{P}(\Omega') = 1$ such that $C(\omega) \neq \emptyset$, (17) holds and $u(\cdot, \omega)$ is a viscosity solution to (12) in $\mathbb{R}^N \setminus C(\omega)$, whenever $\omega \in \Omega'$. Let us fix ω in Ω'. If $u(\cdot, \omega)$ is not a critical solution, there exist $x_0 \in C(\omega)$ and a strict C^1 subtangent φ to $u(\cdot, \omega)$ at x_0 with

$$H(x_0, D\varphi(x_0), \omega) < c.$$

By the usual technique of pushing up such test function, we can construct a deterministic subsolution v to $H(x, Du, x, \omega) = c$ such that

$$v(x_0) > u(x_0, \omega) \quad \text{and} \quad v(y_n) = u(y_n, \omega) \quad \text{definitively in } n.$$

For n sufficiently large we then get

$$v(y_n) + S(y_n, x_0, \omega) < v(x_0),$$

which is impossible by the subsolution property of v.

  u *is an admissible solution $\Rightarrow$ (17) holds*

Let us fix $\omega \in \Omega$ such that $C(\omega) \neq \emptyset$, $C(\omega) \cap \mathcal{A}_f(\omega) \neq \emptyset$, and $u(\cdot, \omega)$ and $g(\cdot, \omega)$ are an admissible critical solution and subsolution, respectively. These properties hold

in a subset of Ω with probability 1. We introduce a partial order relation in $C(\omega)$ by setting
$$y_1 \succ y_2 \iff g(y_2, \omega) = g(y_1, \omega) + S(y_1, y_2, \omega).$$
We exploit the triangle inequality on S and the fact that $g(\cdot, \omega)$ is a subsolution, to see that this relation enjoys the transitivity property. To prove that it is also antisymmetric, we consider y_1, y_2 with $y_1 \succ y_2$ and $y_2 \succ y_1$, accordingly
$$g(y_2, \omega) = g(y_1, \omega) + S(y_1, y_2, \omega) \quad \text{and} \quad g(y_1, \omega) = g(y_2, \omega) + S(y_2, y_1, \omega).$$
By summing up, we get $S(y_1, y_2, \omega) + S(y_2, y_1, \omega) = 0$, which gives $y_1 = y_2$, as desired, since $\mathcal{A}_f(\omega) \cap C(\omega) = \emptyset$.

For a fixed $y_0 \in C(\omega)$, let
$$C_{y_0}(\omega) = \{y \in C(\omega) \,:\, y \succ y_0 \}.$$
Using the continuity of $g(\cdot, \omega)$, $S(\cdot, y_0, \omega)$ and the closed character of $C(\omega)$, it is easy to check that this set is closed. If we show that $C_{y_0}(\omega)$ is unbounded, the assertion is obtained.

Let us then assume, for purposes of contradiction, that $C_{y_0}(\omega)$ is compact. We show that in this case $C_{y_0}(\omega)$ admits a maximal element with respect to $\succ$. Thanks to Zorn lemma, it suffices to prove:

Claim: any totally ordered subset E of $C_{y_0}(\omega)$ admits an upper bound in $C_{y_0}(\omega)$.

We first show that $\overline{E}$ is totally ordered, i.e. $y \succ y'$, $y' \succ y$ or $y = y'$ for any pair y, y' of elements of $\overline{E}$. Let
$$y = \lim_n y_n, \quad y' = \lim_n y'_n, \quad \text{with } y_n, y'_n \in E \text{ for every } n \in \mathbb{N}.$$
If definitively $y_n \succ y'_n$, then passing to the limit in the equality
$$g(y'_n, \omega) = g(y_n, \omega) + S(y_n, y'_n, \omega)$$
we get $y \succ y'$. Similarly $y' \succ y$ if $y'_n \succ y_n$ definitively. Finally, if there exist two subsequences with
$$y'_{n_j} \succ y_{n_j} \text{ and } y_{n_k} \succ y'_{n_k},$$
we get $y = y'$, for both $y \succ y'$ and $y' \succ y$ hold, and $\succ$ enjoys the antisymmetric property.

Since $\overline{E}$ is compact, for every $\varepsilon > 0$ we find a finite number $m = m(\varepsilon)$ of points $y_1^\varepsilon, \ldots, y_m^\varepsilon$ in $\overline{E}$ such that $\overline{E} \subset \cup_i B_\varepsilon(y_i^\varepsilon)$. Up to a reordering, we can as well assume $y_1^\varepsilon \succ y_j^\varepsilon$ for all $j \neq 1$. For every $y \in \overline{E}$ and a suitable $i \in \{1, \ldots, m\}$ we have
$$g(y, \omega) \geq g(y_i^\varepsilon, \omega) + S(y_i^\varepsilon, y, \omega) - 2\kappa_c \varepsilon$$
$$\geq g(y_1^\varepsilon, \omega) + S(y_1^\varepsilon, y_i^\varepsilon, \omega) + S(y_i^\varepsilon, y, \omega) - 2\kappa_c\varepsilon \geq g(y_1^\varepsilon, \omega) + S(y_1^\varepsilon, y, \omega) - 2\kappa_c\varepsilon.$$
Taking the limit as $\varepsilon \to 0$ of y_1^ε and using the compactness of $\overline{E}$, we get an upper bound for E, as it was claimed.

We denote by $\widetilde{y}$ a maximal element in $C_{y_0}(\omega)$ with respect to $\succ$. Since $u(\cdot, \omega)$ agrees with $g(\cdot, \omega)$ on $C(\omega)$ and is a viscosity solution of (12), and $\mathcal{A}_f(\omega) \cap C(\omega) = \emptyset$, Theorem 3.5 yields that there exists $y' \neq \widetilde{y}$ with
$$g(\widetilde{y}, \omega) = u(y', \omega) + S(y', \widetilde{y}, \omega).$$
If
$$u(y', \omega) = g(z, \omega) + S(z, y', \omega) \qquad \text{for some } z \in C(\omega),$$
then
$$g(\widetilde{y}, \omega) = g(z, \omega) + S(z, y', \omega) + S(y', \widetilde{y}, \omega) \geq g(z, \omega) + S(z, \widetilde{y}, \omega),$$

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which, in turn, implies $z = \widetilde{y}$ since $z \in C_{y_0}(\omega)$ by the transitivity property of $\succ$ and $\widetilde{y}$ is maximal; consequently $S(\widetilde{y}, y', \omega) + S(y', \widetilde{y}, \omega) = 0$ which is impossible since $\widetilde{y} \neq y'$ and $\mathcal{A}_f(\omega) \cap C(\omega) = \emptyset$. Therefore

$$u(y', \omega) = \lim_n \big(g(y_n, \omega) + S(y_n, y', \omega) \big),$$

for some diverging sequence $(y_n)_n$ in $C(\omega)$. We derive

$$
\begin{aligned}
g(\widetilde{y}, \omega) &= \lim_n \big(g(y_n, \omega) + S(y_n, y', \omega) + S(y', \widetilde{y}, \omega) \big) \\
&\geq \lim_n \big(g(y_n, \omega) + S(y_n, \widetilde{y}, \omega) \big) \geq g(\widetilde{y}, \omega),
\end{aligned}
$$

and, since $\widetilde{y} \succ y_0$,

$$
\begin{aligned}
g(y_0, \omega) &= g(\widetilde{y}, \omega) + S(\widetilde{y}, y_0, \omega) = \lim_n g(y_n, \omega) + S(y_n, \widetilde{y}, \omega) + S(\widetilde{y}, y_0, \omega) \\
&\geq \lim_n g(y_n, \omega) + S(y_n, y_0, \omega).
\end{aligned}
$$

Since the converse inequality also holds as $g(\cdot, \omega)$ is a critical subsolution, we finally obtain that $y_n \in C_{y_0}(\omega)$ for any n, which is impossible since y_n is a diverging sequence and $C_{y_0}(\omega)$ is a compact set, by assumption. $\qquad\square$

We point out that, in the previous theorem, the argument for deriving from (17) that u is an exact corrector, can be used to get a slight more general assertion, that we write down below for later use.

Corollary 4.4. *Let $C(\omega)$ and u be a stationary closed random set and the Lax extension of some critical subsolution from $C(\omega)$ with distance S, respectively. If for any $y_0 \in C(\omega)$ there exists $y_1 \neq y_0$ with*

$$u(y_0, \omega) = u(y_1, \omega) + S(y_1, y_0, \omega),$$

a.s. in ω, then u is an exact corrector.

We derive a further corollary of Theorem 4.3:

Corollary 4.5. *Let $C(\omega)$ and g be a stationary closed random and an admissible critical subsolution, respectively. Assume that either $c > c_f$, or $c = c_f$ and $\mathcal{A}_f(\omega) \cap C(\omega) = \emptyset$ a.s. in ω. If the Lax extension of g from $C(\omega)$ with distance S is an exact corrector then*

for any $y_0 \in C(\omega)$ there exists a diverging sequence $(z_n)_n$ in $\mathbb{R}^n$

$$\text{such that} \tag{18}$$

$$g(y_0, \omega) = g(z_n, \omega) + S(z_n, y_0, \omega) \quad \text{for any } n,$$

a.s. in ω.

Proof. Given ω in a subset of Ω with probability 1 and $y_0 \in C(\omega)$, there is, by Theorem 4.3, a diverging sequence $(y_n)_n$ in $C(\omega)$ satisfying (17). Given $k \in \mathbb{N}$, we can assume, without loss of generality, that $|y_n| > k$, for any n. Let $(\xi_n)_n$ a sequence of curves, defined in $[0,1]$, joining y_n to y_0 with

$$\int_0^1 \sigma(\xi_n, \dot{\xi}_n, \omega)\, ds + g(y_n, \omega) \leq g(y_0, \omega) + 1/n \quad \text{for any } n \in \mathbb{N}. \tag{19}$$

Since $|y_n| > k$, there is, for any n, $t_n \in [0,1]$ with

$$|\xi(t_n)| = k.$$

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From (19) we derive

$$S(y_n, \xi_n(t_n), \omega) \quad + \quad S(\xi_n(t_n), y_0, \omega)$$
$$\leq \quad \big(g(\xi_n(t_n), \omega) - g(y_n, \omega)\big) + \big(g(y_0, \omega) - g(\xi_n(t_n), \omega)\big) + 1/n$$

and taking into account that g is a critical subsolution, we get

$$\lim_n S(\xi_n(t_n), y_0, \omega) = \lim_n \; g(y_0, \omega) - g(\xi_n(t_n), \omega)$$

For any limit point z_k of $(\xi_n(t_n))$, we find

$$g(y_0, \omega) - g(z_k, \omega) = S(z_k, y_0, \omega) \quad \text{where} \quad |z_k| = k,$$

and since $k \in \mathbb{N}$ was arbitrarily chosen, the assertion follows. $\qquad\square$

5. RANDOM AUBRY SET

We start by introducing a notion of Aubry set adapted to the stationary ergodic setting, see also Remark 6.9 in [9]. To motivate it, we recall that in the deterministic case the Aubry set can be characterized by the property that the critical intrinsic distance from any of its points is a critical solution. Roughly speaking, the idea underlying the next definition is to replace points by random stationary closed subsets and make use of the Lax formula taking as trace any critical admissible subsolutions.

Definition 5.1. *A stationary closed random set $\mathcal{A}(\omega)$ is called* random Aubry set *if*

(i) *the extension of any admissible critical subsolution from $\mathcal{A}(\omega)$ via the Lax formula (16) yields an exact corrector;*
(ii) *any closed random stationary set $C(\omega)$ enjoying the previous property is almost surely contained in $\mathcal{A}(\omega)$.*

We also need the following

Definition 5.2. *An admissible critical subsolution is called* weakly strict *on some random set $X(\omega)$ if a.s. in ω*

$$v(x, \omega) - v(y, \omega) < S(y, x, \omega) \qquad \text{for every } x, y \in X(\omega) \text{ with } x \neq y.$$

The main result of the first part of the section is

Theorem 5.3. *Assume that $c > c_f$ or $c = c_f$ and $\mathcal{A}_f(\omega) = \emptyset$ a.s. in ω. Then there exists a critical admissible subsolution which is weakly strict in $\mathbb{R}^N \setminus \mathcal{A}(\omega)$ a.s. in ω.*

This, in particular, implies the existence of a critical admissible subsolution, weakly strict on the whole $\mathbb{R}^N$, if the random Aubry set is almost surely empty.

We postpone the proof after some preliminary analysis. When $c = c_f$ it is clear by Theorem 4.2 that $\mathcal{A}_f(\omega) \subseteq \mathcal{A}(\omega)$, and this inclusion can be strict a.s. in ω. This occurs even in the periodic setting. Albert Fathi provided us with an example of a periodic Hamiltonian for which $\mathcal{A}_f$ is empty, while, of course, $\mathcal{A}$ is not. In this example, however, $0 \in \partial \overline{Z}_{c_f}$. Actually we have:

Proposition 5.4. *Assume that $0 \in \mathrm{Int}\,\big(\overline{Z}_{c_f}\big)$ and, consequently, that $c = c_f$. Then $\mathcal{A}(\omega) = \mathcal{A}_f(\omega)$ a.s. in ω. Moreover $\mathcal{A}_f(\omega)$ is a uniqueness set for (12).*

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Proof. If $\mathcal{A}_f = \emptyset$ a.s. in ω, then $\mathcal{A}$ is also almost surely empty since no correctors can exist by Theorem 4.2 (ii). Let us assume that $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω, and, in addition, for purposes of contradiction, that $\mathcal{A}_f(\omega) \subsetneq \mathcal{A}(\omega)$ a.s in ω. We claim that, in this case, there exists a closed random stationary a.s. nonempty set $C(\omega)$ with

$$C(\omega) \subset \mathcal{A}(\omega) \quad \text{and} \quad C(\omega) \cap \mathcal{A}_f(\omega) = \emptyset \quad \text{a.s. in } \omega.$$

For this, we denote by $f(x, \omega)$ the Euclidean distance of x from $\mathcal{A}_f(\omega)$, for any x, ω (with the convention that it is equal to $-\infty$ whenever $\mathcal{A}_f(\omega)$ is empty) and, for any $n \in \mathbb{N}$, and consider the random set

$$C_n(\omega) := \mathcal{A}(\omega) \cap \{x \,:\, f(x, \omega) \geq 1/n\}.$$

We see that it is closed stationary taking into account that f is a stationary continuous random function, Proposition 2.4, and the fact that the intersection of two closed random stationary sets inherits the same property. If $C_n(\omega) = \emptyset$ a.s. in ω, for any n, then

$$\mathcal{A}(\omega) \subset \bigcap_n \{x \,:\, f(x, \omega) < 1/n\} = \mathcal{A}_f(\omega) \quad \text{a.s. in } \omega,$$

which is in contrast with our assumption. Accordingly, there exists n_0 with $C_{n_0} \neq \emptyset$ a.s in ω. The claim is proved by taking $C = C_{n_0}$.

Let now u be any critical admissible subsolution. By the very definition of random Aubry set, the Lax extension of u from C via S yields an exact corrector, then, according to Theorem 4.3 and (17), we find a.s. in ω

$$u(y_0, \omega) = \lim_n u(y_n, \omega) + S(y_n, y_0, \omega)$$

for any $y_0 \in C(\omega)$ and some diverging sequence $(y_n)_n$ of elements of $C(\omega)$. On the other side, since $0 \in \text{int} \overline{Z}_c$, we have a.s. in ω

$$\lim_{|y| \to +\infty} u(y, \omega) + S(y, y_0, \omega) = +\infty \quad \text{for any } y_0 \in \mathbb{R}^N, \tag{20}$$

which yields a contradiction.

Let us finally prove the asserted uniqueness property of $\mathcal{A}_f$. Let v be an exact corrector, we fix ω such that $\mathcal{A}_f(\omega) \neq \emptyset$, $v(\cdot, \omega)$ is a solution to $H(x, Du, \omega) = c$ and (20) holds true. We consider the sequence of Dirichlet problems.

$$\begin{cases} H(x, Du, \omega) = c & \text{in } B_n \setminus \mathcal{A}_f(\omega) \\ u(x) = v(x, \omega) & \text{in } \partial B_n \cup (B_n \cap \mathcal{A}_f(\omega)). \end{cases}$$

According to Theorem 3.5, we find, for any n, the relation

$$v(0, \omega) := \inf \{v(y, \omega) + S(y, 0, \omega) \,:\, y \in \partial B_n \cup (B_n \cap \mathcal{A}_f)\}.$$

Letting n go to infinity and taking into account (20), we deduce the existence of $y_0 \in \mathcal{A}_f(\omega)$ satisfying

$$v(0, \omega) = v(y_0, \omega) + S(y_0, 0, \omega).$$

By applying the previous argument to any $x \in \mathbb{R}^N$ in place of 0, we finally get

$$v(x, \omega) = \inf\{v(y, \omega) + S(y, x, \omega) \,:\, x \in \mathcal{A}_f(\omega), \}$$

which says that any exact corrector is the Lax extension of its trace on $\mathcal{A}_f(\omega)$ a.s. in ω. This ends the proof. $\qquad \square$

We assume from now on that $c > c_f$ or $c = c_f$ and $\mathcal{A}_f(\omega) = \emptyset$ a.s. in ω. For any $v \in \mathcal{S}$, we define

$$r_v(x, \omega) = \max\{r \geq 0 : \inf_{y \in \partial B_r(x)} \big(v(y, \omega) + S(y, x, \omega)\big) = v(x, \omega)\}, \quad \omega \in \Omega. \quad (21)$$

Proposition 5.5. *Let $v \in \mathcal{S}$. The following properties hold:*

(i) *the map $r_v : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is jointly measurable in $\Omega \times \mathbb{R}^N$;*

(ii) *r_v is stationary;*

(iii) *$r_v(\cdot, \omega)$ is upper semicontinuous on $\mathbb{R}^N$ for every $\omega \in \Omega$;*

(iv) *for every $z \in \mathbb{R}^N$, $r_v(\cdot, \tau_z \omega) = r_v(\cdot + z, \omega)$ a.s. in ω;*

(v) *if $v(\cdot, \omega)$ is the local uniform limit in $\mathbb{R}^N$ of a sequence $v_n(\cdot, \omega)$, with $v_n \in \mathcal{S}$ for every n, then*

$$\limsup_{n \to +\infty} r_{v_n}(x, \omega) \leq r_v(x, \omega) \quad \text{for every } x \in \mathbb{R}^N.$$

(vi) *if $\hat{v} := v - v(0, \omega)$ for every ω, then $r_{\hat{v}} = r_v$ in $\mathbb{R}^N \times \Omega$.*

Proof. Let us denote by $\psi_r(x, \omega)$ the infimum appearing in formula (21). Fix $r > 0$ and let $(z_n)_n$ be a dense subset of ∂B_r. It is clear that

$$\psi_r(x, \omega) = \inf_{n \in \mathbb{N}} (v(x + z_n, \omega) + S(x + z_n, x, \omega)),$$

which implies that ψ_r is measurable on $\mathbb{R}^N \times \Omega$. Let now $(r_n)_n$ be a dense subset of $\mathbb{R}_+$ and, for each $n \in \mathbb{N}$, set

$$E_n := \{(x, \omega) \in \mathbb{R}^N \times \Omega : \psi_{r_n}(x, \omega) = v(x, \omega)\}.$$

Then $r_v(x, \omega) = \sup_n r_n \chi_{E_n}(x, \omega)$ on $\mathbb{R}^N \times \Omega$, and this proves *(i)*. Assertions *(ii)–(vi)* follow from the very definition of r_v and the fact that v has stationary increments. $\qquad\square$

We will also need the following:

Lemma 5.6. *Let $v \in \mathcal{S}$ and $\alpha > 0$. Then the sets*

$$C_\alpha(\omega) := \{x \in \mathbb{R}^N : r_v(x, \omega) \geq \alpha\}, \quad C_\infty(\omega) := \{x \in \mathbb{R}^N : r_v(x, \omega) = +\infty\}$$

are stationary closed random sets.

Proof. It is clear by Proposition 5.5 *(ii)* that C_α is stationary. In order to prove that C_α is a closed random set, we note that $C_\alpha(\omega) = \{x \in \mathbb{R}^N : G_\alpha(x, \omega) = v(x, \omega)\}$, where

$$G_\alpha(x, \omega) := \min_{y \in \partial B_\alpha} \big(v(x + y, \omega) + S(x + y, x, \omega)\big), \quad (x, \omega) \in \mathbb{R}^N \times \Omega.$$

It is easily seen that G_α is jointly measurable and continuous in x for any fixed ω, thus proving the asserted property for $C_\alpha(\omega)$ in view of Proposition 2.4. The remainder of the statement follows since $C_\infty(\omega) = \bigcap_n C_n(\omega)$ and the intersection of a countable family of stationary closed random sets is still a stationary closed random set. $\qquad\square$

Theorem 4.3 suggests that the following identity should hold

$$\mathcal{A}(\omega) = \bigcap_{v \in \mathcal{S}} \{x \in \mathbb{R}^N : r_v(x, \omega) = +\infty\},$$

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a.s in ω. However this need not be true, the main difficulty being that the above intersection is not countable in general. To avoid this problem, we essentially exploit the separability assumption on Ω. According to Theorems 2.1 and 2.2, the family of renormalized critical subsolution

$$\widehat{\mathcal{S}} := \{\hat{v} \in \mathcal{S} \ : \ \hat{v}(0, \omega) = 0 \text{ for every } \omega\}$$

is a subspace of $L^0(\Omega, \mathrm{C}(\mathbb{R}^N))$, in particular it is separable with respect to the Ky Fan metric. Therefore there exists a sequence of Lipschitz random functions $(v_n)_n$ which is dense in $\widehat{S}$ with respect to the convergence in probability. That implies, in view of Theorem 2.1, that $(v_n)_n$ is also dense for the almost sure convergence in $\mathrm{C}(\mathbb{R}^N)$. We have

Theorem 5.7. *Let $(v_n)_n$ as above. Then*

$$\mathcal{A}(\omega) = \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^N \ : \ r_{v_n}(x, \omega) = +\infty\} \qquad a.s. \ in \ \omega. \tag{22}$$

Proof. Let us denote by $C(\omega)$ the set appearing at the right–hand side of (22), for every $\omega \in \Omega$. The fact that $C(\omega)$ is a stationary closed random set follows from Lemma 5.6. Let us show that $C(\omega) \subseteq \mathcal{A}(\omega)$ a.s. in ω. By definition of Aubry set, we need to show that $C(\omega)$ enjoys item *(i)* in Definition 5.1. According to Corollary 4.4, this amounts to requiring the following identity to hold almost surely:

$$r_v(\cdot, \omega) \ > 0 \quad \text{on } C(\omega), \tag{23}$$

whenever v is the Lax extension of some admissible trace from $C(\omega)$. We set $\hat{v} = v - v(0, \omega)$ for every ω. Clearly $\hat{v} \in \widehat{S}$, so there exists a sequence $(v_{n_k})_k$ and a set Ω_0 of probability 1 such that $v_{n_k}(\cdot, \omega) \rightrightarrows \hat{v}(\cdot, \omega)$ in $\mathbb{R}^N$ for every $\omega \in \Omega_0$. By Proposition 5.5 for any such ω we get

$$\limsup_{k \to +\infty} r_{v_{n_k}}(x, \omega) \le r_{\hat{v}}(x, \omega) = r_v(x, \omega) \qquad \text{for every } x \in \mathbb{R}^N,$$

thus proving (23) by the definition of $C(\omega)$. Conversely, since $\mathcal{A}(\omega)$ enjoys item *(i)* in Definition 5.1, we have in particular that $r_{v_n}(\cdot, \omega) \equiv +\infty$ on $\mathcal{A}(\omega)$ a.s. in ω for every $n \in \mathbb{N}$ by Corollary 4.5. That implies $\mathcal{A}(\omega) \subseteq C(\omega)$ and concludes the proof. $\qquad\square$

We proceed by showing the existence of a random function $\overline{v}$ in $\mathcal{S}$ enjoying a minimality property.

Proposition 5.8. *There exist $\overline{v} \in \mathcal{S}$ such that, for every $v \in \mathcal{S}$, the following inequality holds almost surely:*

$$r_{\overline{v}}(x, \omega) \le r_v(x, \omega) \quad in \ \mathbb{R}^N.$$

In particular, $\mathcal{A}(\omega) = \{x \in \mathbb{R}^N \ : \ r_{\overline{v}}(x, \omega) = +\infty\}$ a.s. in ω.

Proof. Let us take a sequence of positive real numbers $(\lambda_n)_n$ with $\sum_n \lambda_n = 1$ and set

$$\overline{v}(x, \omega) = \sum_{n=1}^{+\infty} \lambda_n v_n(x, \omega), \qquad \text{for every } (x, \omega) \in \mathbb{R}^N \times \Omega, \tag{24}$$

where v_n are the renormalized critical subsolutions appearing in (22). It is easy to check that $\overline{v} \in \mathcal{S}$. Let Ω_0 be a set of probability 1 such that for $\omega \in \Omega_0$ all the

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functions $v_n(\cdot, \omega)$ are subsolutions of the critical equation (12). Let us fix $\omega \in \Omega_0$ and $x \in \mathbb{R}^N$. If $|y - x| > r_{v_n}(x, \omega)$ for some $n \in \mathbb{N}$, then

$$
\begin{aligned}
\overline{v}(x, \omega) - \overline{v}(y, \omega) &= \sum_{k \neq n} \lambda_k \left(v_k(x, \omega) - v_k(y, \omega) \right) + \lambda_n \left(v_n(x, \omega) - v_n(y, \omega) \right) \\
&< \sum_{k \neq n} \lambda_k S(y, x, \omega) + \lambda_n S(y, x, \omega) = S(y, x, \omega).
\end{aligned}
$$

We derive

$$
r_{\overline{v}}(x, \omega) \leq r_{v_n}(x, \omega) \quad \text{for every } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}. \tag{25}
$$

To show that (25) holds true almost surely when v_n is replaced by any $v \in \mathcal{S}$, set $\hat{v} = v - v(0, \omega)$. Clearly $\hat{v} \in \widehat{S}$, and being $(v_n)_n$ dense in $\widehat{S}$ with respect to the almost sure convergence, we derive by Proposition 5.5 *(iii)*, *(vi)* that

$$
\liminf_{n \to +\infty} r_{v_n}(x, \omega) \leq r_{\hat{v}}(x, \omega) = r_v(x, \omega) \qquad \text{for every } x \in \mathbb{R}^N
$$

a.s. in ω. This shows the minimality of $r_{\overline{v}}$ and, consequently, that $\{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) = +\infty\} \subseteq \mathcal{A}(\omega)$ a.s. in ω. The opposite inclusion holds as well since $r_{\overline{v}}(\cdot, \omega) \equiv +\infty$ on $\mathcal{A}(\omega)$ a.s. in ω by definition of Aubry set, as already remarked in the proof of Theorem 5.7. $\qquad \square$

Proof of Theorem 5.3. More precisely, we will prove that there exists $\overline{v} \in \mathcal{S}$ such that

$$
r_{\overline{v}}(\cdot, \omega) \equiv +\infty \quad \text{on } \mathcal{A}(\omega), \qquad r_{\overline{v}}(\cdot, \omega) \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{A}(\omega),
$$

a.s. in ω.

Let $\overline{v}$ the admissible critical subsolution given by Proposition 5.8, see (24). If $\overline{v}$ is weakly strict on the whole $\mathbb{R}^N$, we derive from Proposition 5.8 that $\mathcal{A}(\omega)$ is almost surely empty and the assertion follows. If, on the other hand, $\overline{v}$ is not weakly strict on $\mathbb{R}^N$, for a suitable $\alpha > 0$ the set

$$
C_\alpha(\omega) := \{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) \geq \alpha\}
$$

is a.s. nonempty, and is in addition a closed stationary random set by Lemma 5.6. In view of Proposition 5.8

$$
C_\alpha(\omega) \supseteq \{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) = +\infty\} = \mathcal{A}(\omega) \qquad \text{a.s in } \omega. \tag{26}
$$

To show that the opposite inclusion holds as well, let u be the random function obtained via the Lax–formula (16) with $C_\alpha(\omega)$ in place of $C(\omega)$ and $\overline{v}$ in place of g. By the minimality property of $r_{\overline{v}}$, we derive that $r_u(\cdot, \omega)$ is strictly positive on $C_\alpha(\omega)$ a.s in ω, so combining Corollary 4.4 with Proposition 4.1 we get that u is an exact corrector with trace $\overline{v}(\cdot, \omega)$ on $C_\alpha(\omega)$. Then we invoke Theorem 4.3 to see that $r_{\overline{v}}(\cdot, \omega) \equiv +\infty$ on $C_\alpha(\omega)$ a.s. in ω. This proves that $C_\alpha(\omega)$ agrees with $\mathcal{A}(\omega)$ a.s. in ω, and that $\mathcal{A}(\omega)$ is almost surely nonempty. As a consequence we deduce that $C_\alpha(\omega)$ is almost surely nonempty *for every* $\alpha > 0$, see (26). We can therefore iterate the above argument to prove that, for every $\alpha > 0$, $C_\alpha(\omega) = \mathcal{A}(\omega)$ a.s. in ω, i.e.

$$
\{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) \geq \alpha\} = \{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) = +\infty\} \qquad \text{a.s. in } \omega.
$$

This readily gives $\mathbb{R}^N \setminus \mathcal{A}(\omega) = \{x \in \mathbb{R}^N : r_{\overline{v}}(x, \omega) = 0\}$, as it was to be shown.
$\qquad \square$

In the second part of the section we prove that the random Aubry set is almost surely foliated by curves defined in $\mathbb{R}$ enjoying some minimality conditions, where the critical admissible subsolutions coincide up to an additive constant. When H is regular enough, such curves turn out to be integral curves of the Hamiltonian flow. This generalizes properties holding in the deterministic setting.

Theorem 5.9. *Assume $\mathcal{A}(\omega) \neq \emptyset$ a.s. in ω. Then there exists a set Ω_0 of probability 1 such that for any $\omega \in \Omega_0$ and any $x \in \mathcal{A}(\omega)$ we can find a curve $\eta_x : \mathbb{R} \to \mathcal{A}(\omega)$ (depending on ω) with $\eta_x(0) = x$ satisfying the following properties:*

(i)   for every $a < b$ in $\mathbb{R}$

$$S(\eta_x(a), \eta_x(b), \omega) = \int_a^b (L(\eta_x, \dot{\eta}_x, \omega) + c)\,\mathrm{d}s;$$

(ii) $\displaystyle\lim_{t \to \pm\infty} |\eta_x(t)| = +\infty;$

(iii)   for every $v \in \mathcal{S}$ the following equality holds a.s. in ω:

$$\int_a^b (L(\eta_x, \dot{\eta}_x, \omega) + c)\,\mathrm{d}s = v(\eta_x(b), \omega) - v(\eta_x(a), \omega) \qquad \textit{for every } a < b \textit{ in } \mathbb{R}.$$

We start by some preliminary remarks.

Let $\check{H}(x, p, \omega) = H(x, -p, \omega)$, and denote by $\check{c}$, $\check{\mathcal{S}}$ and $\check{\mathcal{A}}(\omega)$ the associated critical value, the family of admissible subsolutions of $\check{H}(x, Dv, \omega) = \check{c}$ and the Aubry set, respectively. It is easy to see that $\check{c} = c$. We also have:

Proposition 5.10. $\check{\mathcal{A}}(\omega) = \mathcal{A}(\omega)$ *a.s. in ω.*

Proof. Let $\check{S}$ the semi–distance associated to $\check{H}$. It is easy to check that

$$\check{S}(x, y, \omega) = S(y, x, \omega) \qquad \text{for every } x, y \in \mathbb{R}^N \text{ and } \omega \in \Omega.$$

Let $\overline{v}$ be a random function of $\mathcal{S}$ weakly strict outside the Aubry set, see Theorem 5.3. Clearly $-\overline{v} \in \check{\mathcal{S}}$. Let Ω_0 be a set of probability 1 such that for every $\omega \in \Omega_0$ the function $\overline{v}(\cdot, \omega)$ is a critical subsolution and

$$\mathbb{R}^N \setminus \mathcal{A}(\omega) = \{x \in \mathbb{R}^N \, : \, r_{\overline{v}}(x, \omega) = 0\}.$$

We claim that the stationary random function $\check{r}_{-\overline{v}}(\cdot, \omega)$, defined through (21) with $-\overline{v}$ in place of v and $\check{S}$ in place of S, vanishes in $\mathbb{R}^N \setminus \mathcal{A}(\omega)$ for every $\omega \in \Omega_0$. This would imply $\check{\mathcal{A}}(\omega) \subset \mathcal{A}(\omega)$ a.s. in ω, and arguing analogously the opposite inclusion can be obtained as well.

To prove the claim, we argue by contradiction by assuming that there exist an $\omega \in \Omega_0$ and a point $x \in \mathbb{R}^N \setminus \mathcal{A}(\omega)$ such that $\check{r}_{-\overline{v}}(x, \omega) > 0$. Then there exist an $r > 0$ and a point $y \in \partial B_r(x)$ such that

$$-\overline{v}(x, \omega) = -\overline{v}(y, \omega) + \check{S}(y, x, \omega). \tag{27}$$

Since $\mathcal{A}(\omega)$ is closed and $-\overline{v}(\cdot, \omega)$ is a critical subsolution for $\check{H}$, we can choose $r > 0$ small enough such that $\partial B_r(x) \subset \mathbb{R}^N \setminus \mathcal{A}(\omega)$. From (27) we obtain

$$\overline{v}(y, \omega) = \overline{v}(x, \omega) + S(x, y, \omega),$$

yielding $r_{\overline{v}}(y, \omega) > 0$ with $y \in \mathbb{R}^{\mathbb{N}} \setminus \mathcal{A}(\omega)$, a contradiction to the choice of Ω_0. $\square$

Let

$$L(x, q, \omega) := \max_{p \in \mathbb{R}^N} \{\langle p, q \rangle - H(x, p, \omega)\}.$$

The inequality

$$L(x, q, \omega) \geq \max_{H(x,p,\omega) \leq c} \{\langle p, q \rangle - H(x, p, \omega)\} = \sigma(x, q, \omega) - c$$

yields

$$\int_a^b (L(\gamma, \dot{\gamma}, \omega) + c) \, dt \geq \int_a^b \sigma(\gamma, \dot{\gamma}, \omega) \, dt$$

for every curve $\gamma : [a, b] \to \mathbb{R}^N$. We also recall the relation

$$S(y, x, \omega) = \inf \left\{ \int_0^t (L(\gamma, \dot{\gamma}, \omega) + c) \, ds \, : \, \gamma(0) = x, \, \gamma(t) = y, \, t > 0 \right\}$$

for every $x, y \in \mathbb{R}^N$ and $\omega \in \Omega$.

We approach the proof of Theorem 5.9 by proving a weaker version of it.

Proposition 5.11. *Assume $\mathcal{A}(\omega) \neq \emptyset$ a.s. in ω and let $v \in \mathcal{S}$ be weakly strict outside the Aubry set. Then there exists a set Ω_v of probability 1 such that for any $\omega \in \Omega_v$ and any $x \in \mathcal{A}(\omega)$ we can find a curve $\eta_x : \mathbb{R} \to \mathcal{A}(\omega)$ (depending on ω) with $\eta_x(0) = x$ satisfying*

$$S(\eta_x(a), \eta_x(b), \omega) = \int_a^b (L(\eta_x, \dot{\eta}_x, \omega) + c) \, ds = v(\eta_x(b), \omega) - v(\eta_x(a), \omega),$$

whenever $a < b$ in $\mathbb{R}$. In addition $\lim_{t \to \pm\infty} |\eta_x(t)| = +\infty$.

Proof. We take Ω_v such that for every $\omega \in \Omega_v$ the function $v(\cdot, \omega)$ is a critical subsolution, $\mathcal{A}(\omega) \neq \emptyset$ and

$$\mathcal{A}(\omega) = \{x \in \mathbb{R}^N \, : \, r_v(x, \omega) = +\infty\}, \quad \mathbb{R}^N \setminus \mathcal{A}(\omega) = \{x \in \mathbb{R}^N \, : \, r_v(x, \omega) = 0\}.$$

Fix $\omega \in \Omega_v$. The function

$$u(x) := \inf\{v(x, \omega) + S(y, x, \omega) \, : \, y \in \mathcal{A}(\omega)\}, \qquad x \in \mathbb{R}^N,$$

is a viscosity solution of

$$H(x, Du, \omega) = c \quad \text{in } \mathbb{R}^N,$$

and consequently $u(x) - ct$ is a solution of the time–dependent Hamilton–Jacobi Cauchy problem

$$\begin{cases} \partial_t w + H(x, Dw, \omega) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ w(0, x, \omega) = u(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Hence the following Lax–Oleinik representation formula holds for every $x \in \mathbb{R}^N$ and $t > 0$:

$$u(x) = \inf \left\{ u(\gamma(-t)) + \int_{-t}^0 (L(\gamma(s), \dot{\gamma}(s), \omega) + c) \, ds \, : \, \gamma(0) = x \right\}, \qquad (28)$$

where γ varies in the family of absolutely continuous curves from $[-t, 0]$ to $\mathbb{R}^N$, see [8]. By standard arguments of the Calculus of Variations [5], a minimizing absolutely continuous curve does exist for any fixed $t > 0$ thanks to the coercivity and lower semicontinuity properties of L. Moreover such curves turn out to be equi–Lipschitz continuous, see [7]. Given an increasing sequence t_n with $\lim_n t_n = +\infty$, we denote by γ_n the corresponding minimizers and extend them on the whole interval $(-\infty, 0]$ by setting $\gamma_n(t) = \gamma_n(-t_n)$ in $(-\infty, -t_n)$, for any n. Thanks to Ascoli Theorem, the sequence γ_n so defined has a local uniform limit, denoted by γ_x, in $(-\infty, 0]$, up to

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subsequences. Taking into account the optimality of the γ_n and the fact that u is a critical (sub)solution, we get for any $t > 0$

$$u(\gamma_x(0)) - u(\gamma_x(-t)) = \int_{-t}^{0} (L(\gamma_x, \dot\gamma_x, \omega) + c)\,\mathrm{d}s = S(\gamma_x(-t), \gamma_x(0), \omega), \qquad (29)$$

and for any $a < b \le 0$,

$$u(\gamma_x(b)) - u(\gamma_x(a)) = \int_{a}^{b} (L(\gamma_x, \dot\gamma_x, \omega) + c)\,\mathrm{d}s = S(\gamma_x(a), \gamma_x(b), \omega).$$

If, in particular, $x = \gamma_x(0) \in \mathcal{A}(\omega)$, we have

$$u(\gamma_x(0)) = v(\gamma_x(0), \omega), \qquad u(\gamma_x(-t)) \ge v(\gamma_x(-t), \omega) \quad \text{for every } t > 0.$$

From (29) we then derive

$$S(\gamma_x(-t), \gamma_x(0), \omega) \le v(\gamma_x(0), \omega) - v(\gamma_x(-t), \omega),$$

which in turn implies that $v(\cdot, \omega)$ and $u(\cdot)$ coincide on γ_x. Since $r_v(\cdot, \omega)$ vanishes outside $\mathcal{A}(\omega)$, we conclude the support of γ_x is contained in $\mathcal{A}(\omega)$, as claimed.

The same argument can be applied to the function $-v(\cdot, \omega)$ and to the Hamiltonian $\check{H}(x, p, \omega) := H(x, -p, \omega)$. In view of Proposition 5.10, we can assume, without any loss of generality, that $\check{\mathcal{A}}(\omega) = \mathcal{A}(\omega)$. Taking into account the relations

$$\check{L}(x, q, \omega) = L(x, -q, \omega), \quad \check{\sigma}(x, q, \omega) = \sigma(x, -q, \omega), \quad \check{S}(x, y, \omega) = S(y, x, \omega)$$

for every x, y, $q \in \mathbb{R}^N$, we deduce as above that for every $x \in \mathcal{A}(\omega)$ there exists a curve $\xi_x : (-\infty, 0] \to \mathcal{A}(\omega)$ with $\xi_x(0) = x$ satisfying

$$-v(\xi_x(b), \omega) + v(\xi_x(a), \omega) = \int_{a}^{b} \check{L}(\xi_x, \dot\xi_x, \omega)\,\mathrm{d}s = \check{S}(\xi_x(a), \xi_x(b), \omega)$$

for every $a < b \le 0$. The curve η_x with the claimed properties is obtained by setting

$$\eta_x(t) := \begin{cases} \xi_x(-t) & \text{if } t \ge 0 \\ \gamma_x(t) & \text{if } t \le 0. \end{cases}$$

Finally, if there is a limit point $\widetilde{x}$ of η_x for $t \to \pm\infty$, we can find a sequence of compact intervals $[a_n, b_n]$ with $b_n - a_n \ge n$ such that $\eta_x(a_n)$ and $\eta_x(b_n)$ both converge to $\widetilde{x}$ and

$$\int_{a_n}^{b_n} (L(\eta_x, \dot\eta_x, \omega) + c)\,\mathrm{d}s = S(\eta_x(a_n), \eta_x(b_n), \omega) \to 0.$$

By joining $\widetilde{x}$ to $\eta_x(a_n)$ and to $\eta_x(b_n)$ with two segments, we can define a sequence of loops $\xi_n : [0, t_n] \to \mathbb{R}^N$ with $\widetilde{x}$ as base point such that $t_n \to +\infty$ and

$$\int_{0}^{t_n} (L(\xi_n, \dot\xi_n, \omega) + c)\,\mathrm{d}s \to 0.$$

This would imply that $\widetilde{x} \in \mathcal{A}_f(\omega)$ in view of (15). Since $\mathcal{A}_f(\omega)$ is almost surely empty by hypothesis, we see that no such points can exist and the limit relation at infinity asserted in the statement follows. $\qquad \square$

We proceed to show that the minimal curves η_x can be chosen independently of $v \in \mathcal{S}$.

Proof of Theorem 5.9. The critical subsolution $\overline{v}$ appearing in the statement of Proposition 5.8 is weakly strict outside the Aubry set and has the form

$$\overline{v}(x,\omega) = \sum_n \lambda_n v_n(x,\omega) \qquad \text{for every } (x,\omega) \in \mathbb{R}^N \times \Omega,$$

where the $(\lambda_n)_n$ are positive constants satisfying $\sum_n \lambda_n = 1$, and $(v_n)_n$ is a sequence dense in $\widehat{\mathcal{S}}$ with respect to the the almost sure convergence in $C(\mathbb{R}^N)$. By Proposition 5.11 there exists a set Ω_0 of probability 1 such that, for every $\omega \in \Omega_0$ and every $x \in \mathcal{A}(\omega)$, we can find a curve $\eta_x : \mathbb{R} \to \mathbb{R}$ satisfying $\eta_x(0) = x$, $\lim_{t \to \pm\infty} |\eta_x(t)| = +\infty$ and

$$\overline{v}(\eta_x(b),\omega) - \overline{v}(\eta_x(a),\omega) = \int_a^b (L(\eta_x, \dot{\eta}_x, \omega) + c)\, \mathrm{d}s = S(\eta_x(a), \eta_x(b), \omega) \qquad (30)$$

whenever $a < b$ in $\mathbb{R}$. Since $v_n \in \mathcal{S}$, up to removing from Ω_0 a set of probability 0, we can furthermore assume that, for any $\omega \in \Omega_0$, each function $v_n(\cdot,\omega)$ is a subsolution of (12). This readily implies that, for any such ω, equality (30) holds with v_n in place of $\overline{v}$, for every $n \in \mathbb{N}$.

To prove *(iii)*, fix $v \in \mathcal{S}$ and set $\hat{v} = v - v(0,\omega)$. Clearly, it suffices to show the assertion for $\hat{v}$. Since $\hat{v} \in \widehat{\mathcal{S}}$, there exists a subsequence $(v_{n_k})_k$ and a set $\widehat{\Omega} \subseteq \Omega_0$ of probability 1 such that $v_{n_k}(\cdot,\omega) \rightrightarrows v(\cdot,\omega)$ for any $\omega \in \widehat{\Omega}$. By passing to the limit, we derive that equality (30) holds with $\hat{v}$ in place of $\overline{v}$ for any such $\omega \in \widehat{\Omega}$, as it was to be shown. $\qquad\square$

6. Open questions

This is the third of a series of papers we have devoted to the analysis of critical equations for stationary ergodic Hamiltonians, see [9, 10], by using the metric approach combined with some tools from Random Set Theory. This method has allowed to get a complete picture of the setup when the state variable space is 1–dimensional, as specified in the introduction, and, we think, has revealed to be effective also in the multidimensional setting, highlighting some interesting analogies with the compact case. However many crucial problems are still to be clarified. The more striking is:

(1) *In case of existence of an exact corrector, is the random Aubry set almost surely nonempty ?*

In view of Theorem 5.3, we can put it more dramatically:

(1$'$) *Is it impossible the simultaneous existence of an exact corrector and a global weakly strict admissible critical subsolution ?*

In this respect, it should be helpful to strengthen Theorem 5.3, as in periodic case. So we would also like to know:

(2) *If the Aubry set is a.s. empty, there exist* strict *global critical subsolutions? Can we find one of such subsolution which is, in addition, smooth?*

If the answer to (1), (1$'$) is positive, another question urges itself upon us:

(3) *Is any exact corrector the Lax extension from the Aubry set of an admissible trace ?*

Or, in other terms, is the Aubry set an uniqueness set for the critical equation, as in the deterministic compact case? Notice that both questions (1) and (3) have

positive answer when $N = 1$, see [9], and in any space dimension when $c = c_f = \sup_x \min_p H(x, p, \omega)$ and the critical stable norm in nondegenerate, see [10].

On the contrary, if $c = c_f$, $\mathcal{A}_f(\omega)$ is a.s. empty and the critical stable norm is nondegenerate, then no exact solutions can exist. We stress that, as far as we know, all counterexamples published in the literature to the existence of exact correctors are in this frame. It should be interesting to find, if possible, counterexamples in cases where the previous conditions are not satisfied.

The above nonexistence result morally says that we can hope to have exact correctors only if some metric degeneracy of S_c takes place either at finite points (i.e. when $\mathcal{A}_f(\omega) \neq \emptyset$ a.s. in ω) or at infinity (i.e. when the stable norm vanishes in some directions).

The converse is partially true, in the sense that when $c = c_f$ and the classical Aubry set $\mathcal{A}_f(\omega)$ is almost surely nonempty, we know that exact correctors do exist. One should wonder if a corrector does exist in case of sole metric degeneracy at infinity. Indeed, this is unclear even if $c = c_f$ and, evidently, $\mathcal{A}_f(\omega)$ is a.s. empty. We have exhibited an example in [10], see Example 6.8, of Eikonal equation of the type

$$|Du(x, \omega)|^2 = V(x, \omega) \qquad \text{in } \mathbb{R}^N, \tag{31}$$

where the potential V is a random continuous stationary bounded positive function with infimum a.s. equal to 0, with the peculiarity that the corresponding critical stable norm is equal to 0, i.e. vanishes in any direction. Note that here $c = c_f = 0$ since the null function is a strict admissible subsolution, and no subsolutions of (8) exist for $a < 0$. Here we face a dilemma: either an exact corrector does exist, and then the question (1), (1') has a negative answer, since the Aubry set $\mathcal{A}(\omega)$ must be a.s. empty (for the null function is a strict admissible subsolution); or we have to recognize that metric degeneracy at infinity is not sufficient for yielding critical solutions.

We remark that a negative answer to questions (1), (1') would come from the finiteness of

$$\liminf_{|y| \to +\infty} S(y, 0, \omega) \quad \text{a.s. in } \omega,$$

where S, as usual, is the critical distance associate to (31). In fact, if such limit is less than $+\infty$, then, by the triangle inequality and other properties enjoyed by S, it is easily seen that

$$\Omega_0 := \{\omega : \liminf_{|y| \to +\infty} S(y, x, \omega) < +\infty \quad \text{for some } x \in \mathbb{R}^N\},$$

has probability 1, and so a finite–valued random function u can be defined by setting

$$u(x, \omega) = \liminf_{|y| \to +\infty} S(y, x, \omega) \qquad \text{for } \omega \in \Omega_0$$

and $u(x, \omega) = 0$ otherwise, for every $x \in \mathbb{R}^N$. Via standard arguments, it can be then proved that $u(\cdot, \omega)$ is a solution of (31).

Another subject of interest is about approximate correctors. So far we don't have any counterexamples to their existence when exact correctors do not exist. Hence the main question is:

(4) *Do approximate correctors always exist?*

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This issue is also strongly related to homogenization problems and a positive answer would be an important step towards generalizations of the results proved in [19, 22] to more general Hamiltonians.

As usual, the answer is positive if $N = 1$, or in any space dimension if $c = c_f = \sup_x \min_p H(x, p, \omega)$. In this setting, we have in addition proved that approximate correctors can be represented by Lax formula (16), taking as random source set the δ–maximizers over $\mathbb{R}^N$ of the function $x \mapsto \min_p H(x, p, \omega)$. This result essentially exploits the assumption that c_f is the supremum of such function, which is always the case in the 1–dimensional setting.

To extend it in more general setups, at least when $c = c_f$, the idea could be to replace the δ–minimizers by some sort of *approximate* Aubry set. But such a set seems not easy to define even assuming the existence of a smooth strict critical subsolution, and so this attempt has not given, till now, any output.

Note that the existence results of [12] for approximate correctors in the almost–periodic case are based on an ergodic approximation of the Hamilton–Jacobi equation, and so are not constructive. A final question, which stems from the previous discussion, then is

(5) *At least in the almost–periodic case, are the approximate correctors representable through Lax formulae?*

Appendix A

We begin recalling that a function f defined on $\mathbb{R}^N$ is said to be *almost–periodic* if it is bounded, continuous and if it can be approximated, uniformly on $\mathbb{R}^N$, by finite linear combinations of functions in the set $\{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \mathbb{R}^N\}$, see [1, 21] for instance.

This appendix is devoted to show that any almost–periodic Hamiltonian is a specific realization of a stationary ergodic Hamiltonian, with underlying probability space Ω *separable* in a measure theoretic sense. Generalizing the construction of the quasi–periodic case, we more precisely prove that Ω can be taken as the infinite–dimensional torus with $\mathbb{R}^N$ appropriately acting on it. Therefore Ω is in addition a *compact metric space* with the product topology, and is as well separable from a topological viewpoint.

The statement of the main result is the following:

Theorem A.1. *Let $H_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous Hamiltonian satisfying the following assumptions:*

(B1)   *$H_0(\cdot, p)$ is almost–periodic in $\mathbb{R}^N$ for every fixed $p \in \mathbb{R}^N$;*

(B2)   *$H_0(x, \cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \mathbb{R}^N$;*

(B3)   *there exist two superlinear continuous functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\alpha(|p|) \le H_0(x, p) \le \beta(|p|) \qquad \text{for all } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N;$$

(B4) *the set of minimizers of $H_0(x, \cdot)$ has empty interior for every $x \in \mathbb{R}^N$.*

Then there exist a separable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an ergodic group of translations $(\tau_x)_{x \in \mathbb{R}^N}$ and an Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ satisfying assumptions (H1)–(H5) of Section 3 such that

$$H(x, p, \omega_0) = H_0(x, p) \qquad \text{for every } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N,$$

for some $\omega_0 \in \Omega$.

The proof of Theorem A.1 will require some preliminary work. We start by some classical definitions and results from the theory of dynamical systems, see [13].

A continuous map $\tau : \Omega \to \Omega$ defined on a *Hausdorff topological space* Ω will be said to be *minimal* if the orbit

$$\mathrm{orb}(\omega) := \{\tau^n(\omega) \,:\, n \in \mathbb{Z}\}$$

of every point $\omega \in \Omega$ is dense in Ω. A Borel probability measure μ on Ω is called τ–*invariant* if $\mu(\tau^{-1}(E)) = \mu(E)$ for every μ–measurable set E. A measurable subset E of Ω is called τ–*invariant* if $\mu(\tau^{-1}(E) \,\Delta\, E) = 0$, where Δ stands for the symmetric difference. A τ–invariant measure μ is called *ergodic* (with respect to τ) if for any τ–invariant measurable set $E \subset \Omega$ either $\mu(E) = 0$ or $\mu(E) = 1$. When Ω is a metrizable compact space, τ will be called *uniquely ergodic* if it has only one invariant Borel probability measure μ. In this instance μ is necessarily ergodic with respect to τ, see [13, Proposition 4.1.8].

We will use the following result from [13, Proposition 4.1.15].

Proposition A.2. *Let Ω be a metrizable compact space and $\tau : \Omega \to \Omega$ a continuous map. If for every continuous function φ belonging to a dense set in the space $\mathrm{C}(\Omega)$ the time averages $(1/n)\sum_{k=0}^{n-1} \varphi(\tau^k(\omega))$ converge uniformly to a constant, then τ is uniquely ergodic.*

By applying Proposition A.2, we show

Proposition A.3. *Let Ω and τ be a compact metric space and an isometry on it, respectively. If τ has a dense orbit, then it is uniquely ergodic.*

Proof. Let $\varphi \in \mathrm{C}(\Omega)$. In view of Proposition A.2 it suffices to show that the functions

$$\varphi_n(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} \varphi(\tau^k(\omega))$$

uniformly converge to a constant. It is easy to see that the functions φ_n are equi–bounded by $\|\varphi\|_\infty$, which is finite since Ω is compact and φ is continuous. Moreover, they are equi–continuous, because a continuity modulus for φ plays the same role for each of the φ_n, since τ preserves the distance. By Ascoli–Arzelà Theorem we infer that φ_n uniformly converge to a function ψ which is τ–invariant, i.e. constant on the orbits of f. Since there is a dense orbit by hypothesis and ψ is continuous, we conclude that ψ is constant, as it was to be proved. $\qquad\square$

Note that the previous result applies, in particular, to minimal maps, for which all the orbits are dense.

Let $\mathbb{T}^1$ be the one–dimensional flat torus endowed with the flat Riemannian metric, still denoted by $|\cdot|$, induced by the Euclidean metric on $\mathbb{R}$. We define a distance d on $\mathbb{T}^\infty := \Pi_{j=1}^{+\infty}\mathbb{T}^1$ via

$$d(\omega,\omega') = \sum_{n=1}^{+\infty} \frac{1}{2^n}|\omega_n - \omega'_n| \qquad \omega = (\omega_n)_n,\ \omega' = (\omega'_n)_n \text{ in } \mathbb{T}^\infty. \qquad (32)$$

By Tychonoff Theorem, $\mathbb{T}^\infty$ is a compact metric space with respect to d. We consider on $\mathbb{T}^\infty$ the product probability measure $\mu := \Pi_{j=1}^{+\infty}\mathcal{L}^1 \llcorner \mathbb{T}^1$. For every $m \in \mathbb{N}$ we denote by $\pi_m : \mathbb{T}^\infty \to \mathbb{T}^m$ the projection on the first m–components, and by $\mu_m :=$

$\pi_{m\sharp}\,\mu$ the push–forward on $\mathbb{T}^m$ of the measure μ, i.e. the probability measure given by

$$\mu_m(E) = \mu\left(\pi_m^{-1}(E)\right) \qquad \text{for every Borel set } E \subseteq \mathbb{T}^m.$$

We endow $\mathbb{T}^m$ with the distance d_m defined as

$$d_m(\omega,\omega') = \sum_{j=1}^{m} \frac{1}{2^j}|\omega_j - \omega'_j|, \qquad \omega,\ \omega' \in \mathbb{T}^m.$$

Given a sequence $(\lambda_n)_n$ of vectors in $\mathbb{R}^N$, we consider the group of translation $(\tau_x)_{x\in\mathbb{R}^N}$ defined as

$$(\tau_x\omega)_j \equiv \omega_j + \langle \lambda_j, x\rangle \pmod 1 \qquad \text{for every } j \in \mathbb{N}, \tag{33}$$

Note that μ is invariant with respect to $(\tau_x)_{x\in\mathbb{R}^N}$. We denote, for any $x \in \mathbb{R}^N$, by $\tau_{x|\mathbb{T}^m} : \mathbb{T}^m \to \mathbb{T}^m$ the translation τ_x restricted to the first m components, i.e.

$$(\tau_{x|\mathbb{T}^m}\,\omega)_j \equiv \omega_j + \langle \lambda_j, x\rangle \pmod 1 \qquad \text{for every } j \in \{1,\ldots,m\},$$

for each $\omega = (\omega_1,\ldots,\omega_m) \in \mathbb{T}^m$. Clearly μ_m is invariant with respect to $(\tau_{x|\mathbb{T}^m})_{x\in\mathbb{R}^N}$.

Motivated by the next result, we are specially interested to the case where the sequence $(\lambda_n)_n$ in $\mathbb{R}^N$ is *rationally independent*, i.e. when every finite combination of elements of the sequence with rational coefficients is zero if and only all the coefficients vanish.

The following holds

Proposition A.4. *Let $(\lambda_n)_n$ be a countable family of rationally independent vectors in $\mathbb{R}^N$. Then there exists $\hat{x} \in \mathbb{R}^N$ such that the translations $\tau_{\hat{x}}$ and $\tau_{\hat{x}|\mathbb{T}^m}$ are minimal on $\mathbb{T}^\infty$ and on $\mathbb{T}^m$ for every $m \in \mathbb{N}$, respectively. In particular, μ and μ_m are uniquely ergodic with respect to $\tau_{\hat{x}}$ and $\tau_{\hat{x}|\mathbb{T}^m}$, respectively.*

We will exploit in the proof the following known fact, see [13, Proposition 1.4.1].

Proposition A.5. *Let $\gamma = (\gamma_1,\ldots,\gamma_m)$ be a vector of $\mathbb{R}^m$ and let T_γ be the translation on the torus $\mathbb{T}^m$ defined as*

$$T_\gamma(\omega_1,\ldots,\omega_m) \equiv (\omega_1 + \gamma_1,\ldots,\omega_m + \gamma_m) \pmod 1.$$

Then $\mathbb{T}_\gamma$ is minimal if and only if $\sum_{j=1}^{m} k_j\gamma_j \notin \mathbb{Z}$ for any choice of $(k_1,\ldots,k_m)$ in $\mathbb{Z}^m \setminus \{0\}$.

Proof of Proposition A.4. Let us consider the countable set

$$\mathcal{I} := \{\, \mathbb{k} = (k_n)_n \in \mathbb{Z}^{\mathbb{N}} \ : \ k_j \neq 0 \text{ for a finite and positive number of indices } j \,\}.$$

For every $\mathbb{k} \in \mathcal{I}$, we define

$$V_{\mathbb{k}} := \{x \in \mathbb{R}^N \ : \ \sum_i k_i \langle \lambda_i, x\rangle \notin \mathbb{Z}\}$$

since $\sum_i k_i\lambda_i \neq 0$, this set is open and dense in $\mathbb{R}^N$. Baire's Theorem then implies that $V := \cap_{\mathbb{k}\in\mathcal{I}} V_{\mathbb{k}}$ is dense, in particular is non void. Pick $\hat{x} \in V$. The minimality of $\tau_{\hat{x}|\mathbb{T}^m}$ in $\mathbb{T}^m$ for every $m \in \mathbb{N}$ follows from Proposition A.5.

Let us show that $\tau_{\hat{x}}$ is minimal in $\mathbb{T}^\infty$, i.e.

$$\text{orb}(\omega) \cap B_r(\omega') \neq \emptyset \quad \text{for any } \omega,\ \omega' \text{ in } \mathbb{T}^\infty, \text{ any } r > 0.$$

Let $m \in \mathbb{N}$ be large enough to have $\sum_{j=m+1}^{\infty} 1/2^j < r/2$. Since $\tau_{\hat{x}|\mathbb{T}^m}$ is minimal on $\mathbb{T}^m$, there exists an integer $k \in \mathbb{Z}$ such that

$$\sum_{j=1}^{m} \frac{1}{2^j} \, \|(\tau_{\hat{x}}^k(\omega))_j - \omega_j'\| < r/2.$$

Hence

$$d(\tau_{\hat{x}}^k(\omega), \omega') = \sum_{j=1}^{\infty} \frac{1}{2^j} \, \|(\tau_{\hat{x}}^k(\omega))_j - \omega_j'\| \leq \sum_{j=1}^{m} \frac{1}{2^j} \, \|(\tau_{\hat{x}}^k(\omega))_j - \omega_j'\| + \sum_{j=m+1}^{\infty} \frac{1}{2^j} < r.$$

The remainder of the assertion is a straightforward consequence of Proposition A.3. $\qquad\square$

We summarize what we have proved so far in the next statement.

Theorem A.6. *Let $(\lambda_n)_n$ be a countable family of rationally independent vectors in $\mathbb{R}^N$, d the distance on $\mathbb{T}^\infty$ defined via (32), $(\tau_x)_{x\in\mathbb{R}^N}$ the group of translations on $\mathbb{T}^\infty$ defined according to (33), and μ the product probability measure defined as $\mu := \Pi_{j=1}^{+\infty} \mathcal{L}^1 \llcorner \mathbb{T}^1$. Then $(\mathbb{T}^\infty, d)$ is a compact metric space, in particular separable, and $(\tau_x)_{x\in\mathbb{R}^N}$ is ergodic with respect to μ.*

We proceed to show that given an almost–periodic function f on $\mathbb{R}^N$, a sequence of rationally independent vectors $(\lambda_n)_n$ can be chosen in such a way that f is a specific realization of a random variable on $\mathbb{T}^\infty$ with respect to the group of translations $(\tau_x)_{x\in\mathbb{R}^N}$ defined via (33); in addition such random variable can be taken continuous. In the sequel, we will denote by $\mathbf{0}$ the element of $\mathbb{T}^\infty$ all of whose components are equal to 0.

Proposition A.7. *Let f be an almost periodic function in $\mathbb{R}^N$. There exist a sequence of rationally independent vectors $(\lambda_n)_n$ in $\mathbb{R}^N$, inducing a dynamical system $(\tau_x)_{x\in\mathbb{R}^N}$ on $\mathbb{T}^\infty$ via (33), and a continuous function $\underline{f} : \mathbb{T}^\infty \to \mathbb{R}$ such that $f(x) = \underline{f}(\tau_x\mathbf{0})$.*

Proof. In what follows, we will use some known facts about almost–periodic functions, see [21]. For every $\lambda \in \mathbb{R}^N$, let us set

$$a_\lambda := \lim_{R \to +\infty} \fint_{B_R} f(x)\, e^{-2\pi i \langle \lambda, x \rangle} \, \mathrm{d}x$$

and $\Lambda := \{\, \lambda \in \mathbb{R}^N \, : \, a_\lambda \neq 0 \,\}$. Since f is almost periodic, the set Λ is countable, and we will write $\Lambda = (\widetilde{\lambda}_n)_n$. For every $n \in \mathbb{N}$, we define

$$f_n(x) = \sum_{k=1}^{n} a_{\widetilde{\lambda}_k}\, e^{2\pi i \langle \widetilde{\lambda}_k, x \rangle}$$

It is well known that f_n converge uniformly to f in $\mathbb{R}^N$. We now want to write f as limit of a totally convergent series. To this purpose, we choose an increasing sequence of integers $(\widetilde{k}_n)_n$ in such a way that $\|f_{\widetilde{k}_n} - f\|_{L^\infty(\mathbb{R}^N)} \leq 1/2^{n+2}$, and we set $g_1(\cdot) = f_{\widetilde{k}_1}(\cdot)$ and, for $n \geq 2$,

$$g_n(x) := f_{\widetilde{k}_n}(x) - f_{\widetilde{k}_{n-1}}(x) = \sum_{j=\widetilde{k}_{n-1}+1}^{\widetilde{k}_n} a_{\widetilde{\lambda}_j}\, e^{2\pi i \langle \widetilde{\lambda}_j, x \rangle} \qquad x \in \mathbb{R}^N.$$

Clearly $f(x) = \sum_{n=1}^{\infty} g_n(x)$. Furthermore, $\|g_n\|_{L^\infty(\mathbb{R}^N)} \leq C/2^n$ for every $n \in \mathbb{N}$, where C is a constant greater than $1 + 2\|f_{\widetilde{k}_1}\|_{L^\infty(\mathbb{R}^N)}$. From $(\widetilde{\lambda}_n)_n$ we extract a sequence $(\lambda_n)_n$ of vectors rationally independent in such a way that each $\widetilde{\lambda}_n$ is a rational linear combination of $\lambda_1, \dots, \lambda_n$. By expressing each $\widetilde{\lambda}_j$ in g_n in terms of its rational finite linear combination via elements of $(\lambda_n)_n$, we derive that

$$g_n(x) = G_n\left(\langle \lambda_1, x \rangle, \dots, \langle \lambda_{k_n}, x \rangle\right), \quad x \in \mathbb{R}^N,$$

where $(k_n)_n$ is a non decreasing sequence of indexes with $k_n \leq \widetilde{k}_n$, and $G_n(\omega_1, \dots, \omega_{k_n})$ is a continuous function from $\mathbb{T}^{k_n}$ to $\mathbb{C}$. For every n, we define a continuous function $\underline{g}_n$ on $\mathbb{T}^\infty$ by setting

$$\underline{g}_n(\omega) = G_n \circ \pi_{k_n}(\omega), \qquad \omega \in \mathbb{T}^\infty.$$

Let $(\tau_x)_{x \in \mathbb{R}^N}$ be the group of translations on $\mathbb{T}^\infty$ associated with the vectors $(\lambda_n)_n$ via (33). Note that $\underline{g}_n(\tau_x \mathbf{0}) = g_n(x)$ for every $x \in \mathbb{R}^N$. Since $\{ \tau_x(\mathbf{0}) : x \in \mathbb{R}^N \}$ is dense in $\mathbb{T}^\infty$ by Proposition A.4 and $\underline{g}_n$ is continuous on $\mathbb{T}^\infty$, we derive that $\|\underline{g}_n\|_{L^\infty(\mathbb{T}^\infty)} \leq C/2^n$. This yields that the series

$$\sum_{n=1}^{+\infty} \underline{g}_n(\omega), \qquad \omega \in \mathbb{T}^\infty$$

uniformly converges to a continuous function $\underline{f} : \mathbb{T}^\infty \to \mathbb{C}$, in particular

$$\underline{f}(\tau_x \mathbf{0}) = \sum_{n=1}^{+\infty} \underline{g}_n(\tau_x \mathbf{0}) = \sum_{n=1}^{\infty} g_n(x) = f(x) \qquad \text{for every } x \in \mathbb{R}^N.$$

The fact that $\underline{f}(\mathbb{T}^\infty) \subset \mathbb{R}$ finally follows by noticing that the continuous function $\underline{f}$ takes real values on $\{ \tau_x(\mathbf{0}) : x \in \mathbb{R}^N \}$, which is dense in $\mathbb{T}^\infty$. $\qquad \square$

The last step consists in extending the previous result to functions that additionally depend on p.

Proposition A.8. *Let $H_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying the following assumptions:*

(A1) *for every $p \in \mathbb{R}^N$ the function $H_0(\cdot, p)$ is almost–periodic in $\mathbb{R}^N$;*

(A2) *for every $R > 0$ there exists a modulus η_R such that*

$$|H_0(x, p) - H_0(x, q)| \leq \eta_R(|p - q|) \qquad \text{for every } x \in \mathbb{R}^N \text{ and } p, q \in B_R.$$

Then there exists a continuous $\underline{H} : \mathbb{T}^\infty \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\underline{H}(\tau_x \mathbf{0}, p) = H_0(x, p) \qquad \text{for every } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $(\tau_x)_{x \in \mathbb{R}^N}$ is the group of translations on $\mathbb{T}^\infty$ defined according to (33) for a suitable chosen sequence of rationally independent vectors $(\lambda_n)_n$ in $\mathbb{R}^N$.

Proof. For every λ and p in $\mathbb{R}^N$ let us set

$$a_\lambda(p) := \lim_{R \to +\infty} \fint_{B_R} H_0(x, p) \, e^{-2\pi i \langle \lambda, x \rangle} \, \mathrm{d}x.$$

The fact that a_λ is continuous on $\mathbb{R}^N$ for every fixed λ, follows from the estimate

$$|a_\lambda(p) - a_\lambda(q)| \leq \eta_R(|p - q|),$$

which holds for every $R > 0$, $p, q \in B_R$. Let $(p_k)_k$ be a dense sequence in $\mathbb{R}^N$ and set

$$\Lambda := \bigcup_{k \in \mathbb{N}} \{ \lambda \in \mathbb{R}^N \, : \, a_\lambda(p_k) \neq 0 \}.$$

The almost–periodic character of $H_0(\cdot, p)$ implies that Λ is countable, so we will write $\Lambda = (\widetilde{\lambda}_n)_n$. From the continuity of a_λ we deduce

$$a_\lambda(\cdot) \equiv 0 \qquad \text{for every } \lambda \notin \Lambda.$$

From $(\widetilde{\lambda}_n)_n$ we extract a sequence $(\lambda_n)_n$ of rationally independent vectors in such a way that each $\widetilde{\lambda}_n$ is a rational linear combination of $\lambda_1, \dots, \lambda_n$. Let $(\tau_x)_{x \in \mathbb{R}^N}$ be the group of translations on $\mathbb{T}^\infty$ associated to the vectors $(\lambda_n)_n$ via (33). In view of Proposition A.7, for every $p \in \mathbb{R}^N$ there exists a continuous function $\underline{H}(\cdot, p) : \mathbb{T}^\infty \to \mathbb{R}$ such that

$$\underline{H}(\tau_x \mathbf{0}, p) = H_0(x, p) \qquad \text{for every } x \in \mathbb{R}^N.$$

From this we get that, for every $\omega \in \{ \tau_x(\mathbf{0}) \, : \, x \in \mathbb{R}^N \}$,

$$|\underline{H}(\omega, p) - \underline{H}(\omega, q)| \leq \eta_R(|p - q|) \qquad \text{for every } p, q \in B_R \text{ and } R > 0.$$

Since $\{ \tau_x(\mathbf{0}) \, : \, x \in \mathbb{R}^N \}$ is dense in $\mathbb{T}^\infty$ and $\underline{H}(\cdot, p)$ is continuous on $\mathbb{T}^\infty$ for every fixed p, we derive that the above inequality holds for every $\omega \in \mathbb{T}^\infty$. Hence $\underline{H}$ is jointly continuous in (ω, p) and the proof is complete. $\qquad \square$

We are now in position to prove Therem A.1.

Proof of Theorem A.1. We recall (see for instance [20]) that a convex function $\psi : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz, and its Lipschitz constant in B_R can be controlled with the supremum of $|\psi|$ on B_{R+2}, for every $R > 0$. In particular the Hamiltonian H satisfies assumption (A2) in Proposition A.8 with $\eta_R(h) := L_R h$, where

$$L_R := \sup\{ |H_0(x, p)| \, : \, x \in \mathbb{R}^N, \, p \in B_{R+2} \},$$

which is finite thanks to (B3). Therefore we can apply Proposition A.8 to find a continuous $\underline{H} : \mathbb{T}^\infty \times \mathbb{R}^N \to \mathbb{R}$ such

$$\underline{H}(\tau_x \mathbf{0}, p) = H_0(x, p) \qquad \text{for every } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $(\tau_x)_{x \in \mathbb{R}^N}$ is the group of translations on $\mathbb{T}^\infty$ associated via (33) to a suitably chosen sequence $(\lambda_n)_n$ of rationally independent vectors of $\mathbb{R}^N$. In particular, $\underline{H}$ satisfies conditions (B2), (B3), (B4) on a dense subset of $\mathbb{T}^\infty \times \mathbb{R}^N$, hence on the whole $\mathbb{T}^\infty \times \mathbb{R}^N$ by the continuity of $\underline{H}$. The assertion readily follows with $\Omega := \mathbb{T}^\infty$ by setting

$$H(x, p, \omega) = H(\tau_x \omega, p) \qquad \text{for every } (x, p, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega,$$

and by choosing $\omega_0 = \mathbf{0}$. $\qquad \square$

References

[1] L. Ambrosio, H. Frid, Multiscale Young measures in almost periodic homogenization and applications. *Preprint* (2005) (available at http://cvgmt.sns.it/papers/ambfri/).

[2] M. Arisawa, Multiscale homogenization for first-order Hamilton-Jacobi equations. Proceedings of the workshop on Nonlinear P.D.E., Saitama University, 1998.

[3] M. Bardi, I. Capuzzo Dolcetta, Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] G. Barles, Solutions de viscositè des équations de Hamilton–Jacobi. Mathmatiques & Applications, 17. Springer–Verlag, Paris, 1994.

[5] G. Buttazzo, M. Giaquinta, S. Hildebrandt, One–dimensional variational problems. An introduction. Oxford Lecture Series in Mathematics and its Applications, **15**. The Clarendon Press, Oxford University Press, New York, 1998.

[6] G. Contreras, R. Iturriaga, Global Minimizers of Autonomous Lagrangians. 22nd Brazilian Mathematics Colloquium, IMPA, Rio de Janeiro, 1999.

[7] A. Davini, Bolza Problems with discontinuous Lagrangians and Lipschitz continuity of the value function. *SIAM J. Control Optim.* **46** (2007), no. 5, 1897–1921.

[8] A. Davini, A. Siconolfi, A generalized dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations. *SIAM J. Math. Anal.*, Vol. 38, , no. 2 (2006), 478–502.

[9] A. Davini, A. Siconolfi, Exact and approximate correctors for stochastic Hamiltonians: the 1–dimensional case. Preprint (2008).

[10] A. Davini, A. Siconolfi, A metric analysis of critical Hamilton–Jacobi equations in the stationary ergodic setting. Preprint (2008).

[11] A. Fathi, A. Siconolfi, PDE aspects of Aubry–Mather theory for continuous convex Hamiltonians. *Calc. Var. Partial Differential Equations* **22**, , no. 2 (2005) 185–228.

[12] H. Ishii, Almost periodic homogenization of Hamilton-Jacobi equations. International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 600–605, World Sci. Publ., River Edge, NJ, 2000.

[13] A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, **54** Cambridge University Press, Cambridge, 1995.

[14] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, Homogenization of differential operators and integral functionals. Translated from the Russian by G.A. Yosifian. Springer-Verlag, Berlin, 1994.

[15] P. L. Lions, Generalized solutions of Hamilton Jacobi equations. Research Notes in Mathematics, 69. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[16] P.L. Lions, G. Papanicolau, S.R.S. Varadhan, Homogenization of Hamilton–Jacobi equations, unpublished preprint (1987).

[17] P.L. Lions, P.E. Souganidis, Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Comm. Pure Appl. Math.* **56** (2003), no. 10, 1501–1524.

[18] I. Molchanov, Theory of random sets. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2005.

[19] F. Rezakhanlou, J. E. Tarver, Homogenization for stochastic Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.* **151** (2000), no. 4, 277–309.

[20] R.T. Rockafellar, Convex Analysis. Princeton Mathematical Series, No. 28. Princeton University Press, 1970.

[21] W. Rudin, Walter Fourier analysis on groups. Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

[22] P. E. Souganidis, Stochastic homogenization of Hamilton-Jacobi equations and some applications. *Asymptot. Anal.* **20** (1999), no. 1, 1–11.

[23] B. Tsirelson, Filtrations of random processes in the light of classification theory. I. A topological zero-one law. *Preprint* (2001) (available at http://arxiv.org/abs/math.PR/0107121/).

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