A GENERALIZED DYNAMICAL APPROACH TO THE LARGE TIME BEHAVIOR OF SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

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Abstract. We consider the Hamilton–Jacobi equation

$$\partial_t u + H(x, Du) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{T}^N$$

where $\mathbb{T}^N$ is the flat N–dimensional torus, and the Hamiltonian $H(x,p)$ is assumed continuous in x and strictly convex and coercive in p. We study the large time behavior of solutions, and we identify the limit through a Lax–type formula. Some convergence results are also given for H solely convex. Our qualitative method is based on the analysis of the dynamical properties of the so called Aubry set, performed in the spirit of [13]. This can be viewed as a generalization of the techniques used in [11] and [18]. Analogous results have been obtained in [4] using PDE methods.

1. Introduction

This paper is about the large time behavior of the equation

$$\partial_t u + H(x, Du) = 0$$

in the flat torus $\mathbb{T}^N$. Here and in the sequel (sub–super) solutions are meant in the viscosity sense (see [2, 3, 14]).

The subject has been extensively investigated, first in [16], and subsequently in [11], [4], [18]. It is therefore well understood that, under suitable assumptions on H,

$$u(t, x) + ct$$

converges uniformly, for t diverging positively, to a solution v of the stationary equation

$$H(x, D\phi) = c \qquad \text{in } \mathbb{T}^N,$$

where c is the so–called critical value of the Hamiltonian, i.e. given by

$$c = \min\{a \,:\, H(x, D\phi) = a \text{ has a subsolution}\}.$$

This is also the unique value of a for which $H(x, D\phi) = a$ admits a solution on the whole torus, see [15], [13]. Any (sub) solution of the previous equation with $a = c$ will be called critical in the sequel.

This problem has been attacked, in the quoted literature, either by means of dynamical techniques or by using viscosity solutions methods.

The dynamical approach, which can be found in [11], [18], requires strong regularity assumptions on the Hamiltonian (C^2–regularity, strict convexity and superlinearity at infinity in the second variable), since it is based on the analysis of the associated Hamiltonian flow. The latter is related to the solution u through the Lax–Oleinik formula. As first pointed out in [11], a crucial role is played by the Aubry set, which consists of accumulation points of the flow and is invariant.

The conditions on H can be considerably relaxed by using pure PDE methods. In [4] the authors are able to prove the convergence assuming H to be just continuous and satisfying a coercivity condition. Moreover they require the Hamiltonian to fulfill a convexity–type inequality, which includes also some nonconvex functions, but not all strictly convex Hamiltonians.

The main contribution of the present paper is to employ generalized dynamical methods to achieve the above convergence result in presence of a weak regularity of H, which is taken continuous, strictly convex and coercive.

Our procedure yields, also in the continuous case, a deeper insight of the convergence phenomenon as well as remarkably simple proofs which avoid any technicality.

The core of our argument is the discovery of some distinguished curves on the torus along which the difference of u and any critical subsolution ϕ enjoys a monotonicity property. This is a generalization of something already proved in [18] for curves of the Hamiltonian flow lying on the Aubry set. The crux is that, of course, no Hamiltonian flow can be in general defined in our setting.

We overcome this difficulty following the ideas of [13], where some aspects of the Aubry–Mather theory are extended to continuous quasiconvex Hamiltonians. Using a nonsymmetric semidistance, denoted by S, suitably related to the c–sublevel set of the Hamiltonian, it is, in particular, defined a generalized (projected) Aubry set, say $\mathcal{A}$, and some relevant properties, holding for the classical Aubry set when H is C^2, are recovered.

Under the additional assumption that H is Lipschitz–continuous with respect to x, it is proved in [13], for instance, that a multivalued dynamics can be defined on $\mathcal{A}$. We make here a further step by showing that, even for continuous H, some dynamical properties are encoded in the structure of the Aubry set. We prove indeed that through any point of $\mathcal{A}$ it passes a curve η defined on $\mathbb{R}$ and satisfying

$$S(\eta(t_1), \eta(t_2)) = \int_{t_1}^{t_2} (L(\eta, \dot{\eta}) + c) \, \mathrm{d}s = -S(\eta(t_2), \eta(t_1)) \quad \text{for any } t_1, t_2 \in \mathbb{R},$$

where L is the Lagrangian function related to H. These are precisely the curves satisfying the monotonicity property previously mentioned.

Beside this, we get the convergence result by exploiting, as in [18], the relaxed semilimits theory and a generalization of the fact, proved in [13], that all critical subsolutions are differentiable on $\mathcal{A}$ and have same gradient.

An advantage of our method is to single out the point where the strict convexity condition on H — or, to be more precise, the C^1–regularity of L, which is an equivalent condition — is employed (see Lemma 5.2). This is an interesting issue. It is in fact well known, as shown in an example in [4], that the simple convexity of H does not ensure, in general, the convergence phenomenon.

However we can prove that such property is actually enough when the equilibrium points form a uniqueness set for the critical equation. This accounts for the fact that a small perturbation, in a convex Hamiltonian, can produce a passage from a convergence to a non convergence situation, see Example 5.9. This generalizes the results of [16], where the Hamiltonian is taken only convex, as well.

We are furthermore able to identify the limit function v through a representation formula, which involves $u(\cdot, 0)$, the Aubry set, and the semidistance S. It is the critical solution coinciding on $\mathcal{A}$ with the maximal critical subsolution not exceeding $u(\cdot, 0)$. This should be compared to the formula given in [11] for Hamiltonians of class C^2 using the Peierls barrier. Our formula has been exploited in [19] to perform a numerical approximation of the Aubry set

The paper is organized as follows: in Section 2 some preliminary material is collected, including the definition of the semidistance S, of the generalized Aubry set, as well as some properties of the critical solutions and of the Lax–Oleinik semigroup. In Section 3 we introduce, through a representation formula, a distinguished critical solution, which will be proved to be the limit of $u(t,x)+ct$ for $t \to +\infty$. Section 4 is devoted to study the dynamical properties of the Aubry set and to single out a class of special curves covering $\mathcal{A}$. The main results are finally proved in Section 5. In the Appendix we show that the usual integral representation formula for the Lax–Oleinik semigroup holds also in the case where H is coercive, but not necessarily superlinear at infinity, and so the Lagrangian L is possibly infinite–valued at some points of $\mathbb{T}^N \times \mathbb{R}^N$.

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2. Assumptions and preliminary results

We write below a list of symbols used throughout this paper.

N	an integer number
$B_R(x_0)$	the closed ball in $\mathbb{R}^N$ centered at x_0 of radius R
B_R	the closed ball in $\mathbb{R}^N$ centered at 0 of radius R
$\langle \cdot\, , \cdot \rangle$	the scalar product in $\mathbb{R}^N$
$\vert \cdot \vert$	the Euclidean norm in $\mathbb{R}^N$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{T}^N$	the N–dimensional flat torus
$\mathrm{C}(\mathbb{T}^N)$	the space of real–valued continuous functions on $\mathbb{T}^N$
$\mathrm{Lip}(\mathbb{T}^N)$	the space of real–valued Lipschitz–continuous functions on $\mathbb{T}^N$

A subset of $\mathbb{R}^k$ is called *negligible* if its k–dimensional Lebesgue measure is equal to zero. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^k$ if it holds up to a negligible subset of $\mathbb{R}^k$. Given a measurable function $\varphi : \mathbb{T}^N \to \mathbb{R}$, its L^∞–norm on $\mathbb{T}^N$ will be denoted by $\|\varphi\|_\infty$. We will write $\varphi_n \rightrightarrows \varphi$ on $\mathbb{T}^N$ to mean that the sequence of functions $(\varphi_n)_n$ uniformly converges to φ on $\mathbb{T}^N$.

By modulus we mean a nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, vanishing and continuous at 0. Given a closed convex subset Z of $\mathbb{R}^k$, and $p_0 \in Z$, we define the normal cone of Z at p_0, in symbols $N_Z(p_0)$, as the set $\{q \in \mathbb{R}^N : \langle q, p_0\rangle = \max_{p\in Z}\langle q, p\rangle\}$.

We endow the flat torus $\mathbb{T}^N$ with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^N$. We recall that $\mathbb{T}^N$ can be viewed as the quotient space $\mathbb{R}^N/\,\mathbb{Z}^N$, obtained by identifying all points $x, y \in \mathbb{R}^N$ such that $x - y \in \mathbb{Z}^N$.

With the term *curve*, without any further specification, we refer to a Lipschitz–continuous function from some given interval $[a, b]$ to $\mathbb{T}^N$. The space of all such curves is denoted by $\mathrm{Lip}([a, b], \mathbb{T}^N)$, while $\mathrm{Lip}_{x,y}([a, b], \mathbb{T}^N)$ stands for the family of curves γ joining x to y, i.e. such that $\gamma(a) = x$ and $\gamma(b) = y$, for any fixed x, y in $\mathbb{T}^N$. We denote by $W^{1,1}([a, b], \mathbb{T}^N)$ the space of absolutely continuous curves defined in $[a,b]$. Given a curve γ defined on some interval $[a,b]$, a curve γ' defined on $[a',b']$ will be called a *reparametrization of* γ if there exists an order preserving Lipschitz–continuous map $f:[a',b']\to[a,b]$ surjective and such that $\gamma'=\gamma\circ f$. The Euclidean length of a curve γ is denoted by $\ell(\gamma)$.

Unless otherwise specified, the term (sub, super) solution to some PDE equation is understood in the viscosity sense. Given a continuous function g defined in $\mathbb{R}^k$ and $x_0\in\mathbb{R}^k$, we denote by $D^+g(x_0)$ (resp. $D^-g(x_0)$) the superdifferential (resp. the subdifferential) of g at x_0, i.e. the (possibly empty) set made up by the differentials of viscosity test function from above (resp. from below) of g at x_0. Note that, in the case where g is convex, D^-g coincides with the usual subdifferential of convex analysis. When g is defined on $\mathbb{R}^m\times\mathbb{R}^k$ and $(x_0,p_0)\in\mathbb{R}^m\times\mathbb{R}^k$, we will denote by $D_p^-g(x_0,p_0)$ the subdifferential of the function $g(x_0,\cdot)$ at p_0. For a function $g:\mathbb{R}^k\to(-\infty,+\infty]$, we denote by $\mathrm{dom}(g)$ its effective domain, i.e. the subset of $\mathbb{R}^k$ where g is finite valued.

We deal with an Hamiltonian H, defined on the cotangent bundle $T^*\mathbb{T}^N$, identified to $\mathbb{T}^N\times\mathbb{R}^N$, satisfying the following set of assumptions:

(H1) $H:\mathbb{T}^N\times\mathbb{R}^N\to\mathbb{R}$ is continuous;

(H2) $p\mapsto H(x,p)$ is convex on $\mathbb{R}^N$ for any $x\in\mathbb{T}^N$;

(H3) $\displaystyle\lim_{|p|\to+\infty}\left(\inf_{x\in\mathbb{T}^N}H(x,p)\right)=+\infty$;

(H4) the set of minimizers of $p\mapsto H(x,p)$ has empty interior, for any $x\in\mathbb{T}^N$.

To obtain our general convergence result (see in particular Proposition 5.3, which will constitute a crucial step for that), we will moreover assume:

(H2)$'$ $p\mapsto H(x,p)$ is strictly convex on $\mathbb{R}^N$ for any $x\in\mathbb{T}^N$.

Notice that condition (H4) is certainly satisfied when (H2)$'$ holds true, since, in this case, the set of minimizers of $H(x,\cdot)$ reduces to a point, for any $x\in\mathbb{T}^N$.

Remark 2.1. Exploiting the subdifferentiability properties of the function $p\mapsto H(x,p)$, for any fixed x, we see that the Lipschitz constant of such a function in B_R, for any $R>0$, can be estimated, uniformly with respect to x, in terms of R and of $\max\{H(x,p)\,:\,(x,p)\in\mathbb{T}^N\times B_{2R}\}$, see e.g. [17, Proposition 2.2.6].

Remark 2.2. The problem we are dealing with can equivalently formulated in $\mathbb{R}^N$, instead of $\mathbb{T}^N$, with $\mathbb{Z}^N$–periodicity conditions.

We consider the family of Hamilton–Jacobi equations

$$H(x,D\phi)=a \qquad \text{on } \mathbb{T}^N, \tag{1}$$

with a real parameter, and set

$$c:=\inf\{a\in\mathbb{R}\,:\,\text{equation (1) has a subsolution}\}.$$

This is called the *critical value* of the Hamiltonian H and is characterized by the property of being the unique value for a such that equation (1) admits (at least) one solution (see e.g. [15], [13]). A solution (resp. supersolution, subsolution) of

$$H(x,D\phi)=c \qquad \text{in } \mathbb{T}^N. \tag{2}$$

will be qualified as critical in the sequel. Thanks to hypothesis (H3), all subsolutions of (1) are Lipschitz–continuous. Moreover, by the convexity assumption, there is a complete equivalence between the notions of (viscosity) subsolution and a.e. subsolution (see [2]).

Following [13], we carry out the study of properties of subsolutions to (1) by means of the semidistances S_a defined on $\mathbb{T}^N \times \mathbb{T}^N$, for $a \geq c$, as follows:

$$S_a(x,y) = \inf \left\{ \int_0^1 \sigma_a(\gamma(s), \dot\gamma(s))\, \mathrm{d}s \;:\; \gamma \in \mathrm{Lip}_{x,y}([0,1], \mathbb{T}^N) \right\}, \tag{3}$$

where $\sigma_a(x,q)$ is the support function of the a–sublevel of H, namely

$$\sigma_a(x,q) := \sup \{ \langle q, p \rangle \;:\; H(x,p) \leq a \}. \tag{4}$$

The function $\sigma_a(x,q)$ is convex in q and upper semicontinuous in x (and even continuous in all points x where the set $\{p \in \mathbb{R}^N \;:\; H(x,p) \leq a\}$ has nonempty interior or reduces to a point), while S_a satisfies the following properties:

$$\begin{aligned} S_a(x,y) &\leq S_a(x,z) + S_a(z,y) \\ S_a(x,y) &\leq b_a |x-y| \end{aligned}$$

for all $x, y, z \in \mathbb{T}^N$ and for some positive constant b_a. The following properties hold (see [13]):

Proposition 2.3. *Given $a \geq c$, we have:*

(i) For any $y \in \mathbb{T}^N$, the functions $S_a(y, \cdot)$ and $-S_a(\cdot, y)$ are both subsolutions of (1).

(ii) A function ϕ is a subsolution of (1) if and only if

$$\phi(x) - \phi(y) \leq S_a(y,x) \qquad \textit{for all } x, y \in \mathbb{T}^N.$$

To ease notations, in the sequel we will write S, σ in place of S_c, σ_c, respectively.

In the analysis of the behavior of critical subsolutions, a special role is played by a set $\mathcal{A}$, which has been called in [13] the *(projected) Aubry set*, defined as the collection of points $y \in \mathbb{T}^N$ such that

$$\inf \left\{ \int_0^1 \sigma(\gamma, \dot\gamma)\, \mathrm{d}s \;:\; \gamma \in \mathrm{Lip}_{y,y}([0,1], \mathbb{T}^N),\ \ell(\gamma) \geq \delta \right\} = 0 \quad \text{for some } \delta > 0,$$

or, equivalently (cf. [13, Lemma 5.1]),

$$\inf \left\{ \int_0^1 \sigma(\gamma, \dot\gamma)\, \mathrm{d}s \;:\; \gamma \in \mathrm{Lip}_{y,y}([0,1], \mathbb{T}^N),\ \ell(\gamma) \geq \delta \right\} = 0 \quad \text{for any } \delta > 0.$$

The set $\mathcal{A}$ is closed and nonempty (cf. [13, Corollaries 5.7 and 5.9]). In the next theorem we outline the main properties linking $\mathcal{A}$ to equation (2) (see [13]).

Theorem 2.4.

(i) If ϕ and w are a subsolution and a supersolution of (2) respectively and $\phi \leq w$ on $\mathcal{A}$, then $\phi \leq w$ on $\mathbb{T}^N$. In particular, if two solutions of (2) coincide on $\mathcal{A}$, then they coincide on $\mathbb{T}^N$.

(ii) If w_0 is a function defined on $C \subset \mathcal{A}$ such that

$$w_0(x) - w_0(y) \leq S(y,x) \quad \textit{for every } x, y \in C,$$

then the function

$$w(x) := \min_{y \in C} \big(w_0(y) + S(y,x) \big) \tag{5}$$

is the maximal critical subsolution of (2) equaling w_0 on C, and a critical solution as well.

(iii) If we furthermore set $C = \mathcal{A}$ in (5), then w is the unique critical solution equaling w_0 on $\mathcal{A}$.

We call $y \in \mathbb{T}^N$ an *equilibrium point* if $\min_p H(y,p) = c$. The collection of all such points will be denoted by $\mathcal{E}$. The set $\mathcal{E}$ is a (possibly empty) closed subset of $\mathcal{A}$ (cf. [13, Lemma 5.2]). This property depends on the fact that the c–sublevel $\{p : H(y,p) \leq c\}$ is non–void and has empty interior when $y \in \mathcal{E}$ (the latter is a consequence of (H4), and this is actually the unique point where such condition is used). It is apparent that $c \geq \max_{x\in\mathbb{T}^N} \min_{p\in\mathbb{R}^N} H(x,p)$; we point out that $\mathcal{E}$ is nonempty if and only if the previous formula holds with an equality. In this case, $\mathcal{E}$ is made up by the points x where the maximum is attained.

Let us now focus our attention on the Cauchy problem

$$\begin{cases} \partial_t u + H(x, Du) = 0 & \text{in } (0,+\infty)\times\mathbb{T}^N \\ u(0,x) = u_0(x) & \text{on } \mathbb{T}^N, \end{cases} \tag{6}$$

where u_0 is a continuous initial datum. The following result holds (see e.g. [6]):

Theorem 2.5. *Assume H satisfies assumptions (H1), (H2), (H3), (H4). Then the Cauchy problem (6) admits a unique uniformly continuous solution $u(t,x)$ on $\mathbb{R}_+ \times \mathbb{T}^N$, for any $u_0 \in \mathrm{C}(\mathbb{T}^N)$. If, moreover, the initial datum $u_0 \in \mathrm{Lip}(\mathbb{T}^N)$, then $u(t,x)$ is Lipschitz–continuous on $\mathbb{R}_+ \times \mathbb{T}^N$ and satisfies*

$$\|Du\|_\infty \leq M, \quad \|\partial_t u\|_\infty \leq \text{ess sup}\{|H| : |p| \leq M\}$$

for any positive constant M such that

$$M > \|Du_0\|_\infty, \quad \inf\{H : |p| > M\} > \sup\{|H| : |p| \leq \|Du_0\|_\infty\}. \tag{7}$$

In view of the previous theorem, we can define, for any $t > 0$, a nonlinear operator $\mathcal{S}(t)$ on $\mathrm{C}(\mathbb{T}^N)$ by setting $\mathcal{S}(t)\phi := u(t,\cdot)$ for every $\phi \in \mathrm{C}(\mathbb{T}^N)$, where $u(t,x)$ denotes the unique solution of the Cauchy problem (6) with $u_0 = \phi$. The family of operators $\big(\mathcal{S}(t)\big)_{t>0}$ forms a semigroup, whose main properties are summarized below.

Proposition 2.6.

(i) (Semigroup Property) *For any $t, s > 0$ we have $\mathcal{S}(t+s) = \mathcal{S}(t)\circ\mathcal{S}(s)$.*

(ii) (Monotonicity Property) *For every $\phi, \psi \in \mathrm{C}(\mathbb{T}^N)$ and each $t > 0$ we have*

$$\phi \leq \psi \;\Rightarrow\; \mathcal{S}(t)\phi \leq \mathcal{S}(t)\psi.$$

(iii) For any $a \in \mathbb{R}$ and $\phi \in \mathrm{C}(\mathbb{T}^N)$, we have $\mathcal{S}(t)(\phi + a) = \mathcal{S}(t)\phi + a$.

(iv) (Non–expansiveness) *For each $t > 0$, the map $\mathcal{S}(t)$ is non–expansive, i.e.*

$$\|\mathcal{S}(t)\phi - \mathcal{S}(t)\psi\|_\infty \leq \|\phi - \psi\|_\infty \quad \text{for every } \phi, \psi \in \mathrm{C}(\mathbb{T}^N).$$

(v) For every $\phi \in \mathrm{C}(\mathbb{T}^N)$, we have $\lim_{t\to 0} \mathcal{S}(t)\phi = \phi$.

We define the *Fenchel transform* $L : \mathbb{T}^N \times \mathbb{R}^N \to (-\infty, +\infty]$ of H via

$$L(x,q) := \sup_{p\in\mathbb{R}^N} \{\langle p, q\rangle - H(x,p)\}. \tag{8}$$

The function L is called the Lagrangian related to the Hamiltonian H. We record for later use:

Proposition 2.7. *Let H satisfy assumptions (H1), (H2), (H3). Then the following properties hold for the Lagrangian L:*

(i) $L(x,q)$ is lower semicontinuous on $\mathbb{T}^N \times \mathbb{R}^N$, and convex in q for any fixed $x \in \mathbb{T}^N$.

(ii) L is continuous on $\mathrm{int}(\mathrm{dom}L) =: \Omega$.

If , in addition, H satisfies (H2)′ then:

(iii) for every $(x,q) \in \Omega$, L is differentiable with respect to q, and $(x,q) \mapsto D_q L(x,q)$ is continuous on Ω.

(iv) If (x,q) is such that the supremum in the definition of $L(x,q)$ is a maximum then (x,q) belongs to Ω.

We refer to the Appendix for the proof.

Each operator $\mathcal{S}(t)$ can be represented through the following integral formula

$$\big(\mathcal{S}(t)\phi\big)(x) = \inf\left\{\phi(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \;:\; \gamma \in W^{1,1}([0,t],\mathbb{T}^N),\ \gamma(t)=x\right\} \tag{9}$$

for any $\phi \in \mathrm{C}(\mathbb{T}^N)$. The family of operators $\big(\mathcal{S}(t)\big)_{t>0}$ is called the Lax–Oleinik semigroup.

Remark 2.8. When $\phi \in \mathrm{Lip}(\mathbb{T}^N)$ and L is finite–valued, the validity of (9) can be seen, for instance, by combining [10, Theorem 1.1] with Theorem 2.5. This is the case when H is uniformly superlinear in p. The infimum in (9) is then a minimum by classical results of the Calculus of Variations (see e.g. [5]), and all minimizers are Lipschitz–continuous (cf. [1] for some results on this topic).

We present in the Appendix a proof of (9) for $\phi \in \mathrm{C}(\mathbb{T}^N)$ and general L, possibly infinite–valued in some subset of $\mathbb{T}^N \times \mathbb{R}^N$, and we show the existence of minimizers in this case too.

We will use the following Tonelli–type semicontinuity theorem (see e.g. [5, Theorem 3.6]) in the proof of Propositions 4.12 and A.6.

Theorem 2.9. *Let J be a bounded interval of $\mathbb{R}$, and let $F : \mathbb{R}^N \times \mathbb{R}^N \to (-\infty,+\infty]$ be a function satisfying the following conditions:*

(i) F is lower semicontinuous;

(ii) $F(x,\cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \mathbb{R}^N$;

(iii) F is bounded from below by a constant.

Then the functional

$$\mathcal{F}(\gamma) := \int_J F(\gamma(s),\dot\gamma(s))\,\mathrm{d}s$$

is sequentially weakly lower semicontinuous in $W^{1,1}(J,\mathbb{R}^N)$, i.e. if $(\gamma_k)_k$ converges weakly in $W^{1,1}(J,\mathbb{R}^N)$ to γ, then

$$\mathcal{F}(\gamma) \le \liminf_{k\to+\infty} \mathcal{F}(\gamma_k). \tag{10}$$

Equivalently, we can say that (10) holds if $(\gamma_k)_k$ converges uniformly to γ and the measures $\nu_k(E) := \int_E |\dot\gamma_k|\,\mathrm{d}s$ are equiabsolutely continuous on J with respect to the Lebesgue measure.

3. A distinguished critical solution

Before attacking the convergence problem, we try to guess what the asymptotic limit of $\mathcal{S}(t)u_0 + ct$ should be like. We start by providing a Lax–type formula which involves the initial datum u_0, the Aubry set and the semidistance S, and we show that this defines a critical solution, more precisely the one whose trace on $\mathcal{A}$ coincides with that of the maximal critical subsolution not exceeding u_0. It furthermore generalizes the one given in (5).

Theorem 3.1. *Let $w_0 : \mathbb{T}^N \to \mathbb{R}$ be any function bounded from below. Set*

$$v(x) := \inf_{y\in\mathcal{A}} \left(S(y,x) + \inf_{z\in\mathbb{T}^N} (w_0(z) + S(z,y)) \right) \quad \text{for every } x \in \mathbb{T}^N. \tag{11}$$

Then

- *(i)* $\inf_{y\in\mathbb{T}^N} (S(y,\cdot) + w_0(y)) =: v_0$ *is the maximal critical subsolution not exceeding w_0 on $\mathbb{T}^N$.*
- *(ii) The function v is the critical solution equaling v_0 on $\mathcal{A}$.*
- *(iii) If the inequality $w_0(y) - w_0(x) \le S(x,y)$ holds for all x, $y \in \mathbb{T}^N$, then $v = \min_{y\in\mathcal{A}}(w_0(y) + S(y,\cdot))$ on $\mathbb{T}^N$, and $v_0 = w_0$ on $\mathcal{A}$.*

We show separately, in the next lemma, the relevant fact on which the proof of Theorem 3.1 relies.

Lemma 3.2. *Let C be a subset of $\mathbb{T}^N$ and $w_0 : C \to \mathbb{R}$ be any function bounded from below. Then*

$$w(x) := \inf_{z\in C} \big(w_0(z) + S(z,x)\big).$$

is the maximal subsolution of (2) not exceeding w_0 on C. The function w is moreover a critical solution in $\mathbb{T}^N \setminus \overline{C}$, and in the whole $\mathbb{T}^N$ whenever $C \subset \mathcal{A}$.

Proof. It is easy to check, exploiting the very definition of w, that $w \le w_0$ on C and $w(x) - w(y) \le S(y,x)$ for every $x, y \in \mathbb{T}^N$. The latter inequality implies that w is a critical subsolution by Proposition 2.3. If ϕ is any critical subsolution with $\phi \le w_0$ on C then, taking into account that $\phi(x) - \phi(y) \le S(y,x)$ for every $x, y \in \mathbb{T}^N$, we get

$$\phi(x) \le \min_{z\in C} \big(\phi(z) + S(z,x)\big) \le w(x) \qquad \text{for every } x \in \mathbb{T}^N,$$

which gives the maximality of w. Such a property also implies that w is a supersolution of (2) in $\mathbb{T}^N \setminus \overline{C}$ through a standard argument (see, e.g., the proof of Proposition 3.2 in [13]). If furthermore $C \subset \mathcal{A}$, then by Theorem 2.4 (ii) $w = \min_{z\in C} \big(w(z) + S(z,x)\big)$, and so it is a critical solution in $\mathbb{T}^N$. □

Proof of Theorem 3.1. Item (i) comes directly from Lemma 3.2 with $C = \mathbb{T}^N$, (ii) is therefore a consequence of Theorem 2.4 (iii). Item (iii) can be finally deduced from Theorem 2.4 (ii). □

The proof that the function given in formula (11), with $w_0 = u_0$, actually coincides with the asymptotic limit of $\mathcal{S}(t)u_0 + ct$ is, of course, the main goal of our analysis, and will be attained in the subsequent sections. The remainder of the present one is devoted, instead, to some preliminary remarks which give support to our guess and which cast some light for our further analysis.

We start by noticing that, given a solution w of (2) and a general initial datum

$u_0 \in \mathrm{C}(\mathbb{T}^N)$, there exist, since $\mathbb{T}^N$ is compact, some constants α, β such that

$$w + \alpha \le u_0 \le w + \beta \quad \text{on } \mathbb{T}^N.$$

This implies, in view of the relation $\mathcal{S}(t)w = w - ct$, which holds for every $t > 0$, and the Monotonicity Property of the semigroup $\big(\mathcal{S}(t)\big)_{t>0}$,

$$w + \alpha - ct \le \mathcal{S}(t)u_0 \le w + \beta - ct \quad \text{on } T^N$$

for any $t > 0$, or, in other terms

$$w + \alpha \le \mathcal{S}(t)u_0 + ct \le w + \beta \quad \text{on } \mathbb{T}^N. \tag{12}$$

Since the family of functions $\big(\mathcal{S}(t)u_0 + ct\big)_{t>0}$ is equicontinuous (in view of Theorem 2.5), and equibounded thanks to (12), we can define the relaxed semilimits

$$\underline{u}(x) = \limsup_{t\to+\infty}{}^{*} \, (\mathcal{S}(t)u_0)(x) + ct \quad := \quad \sup \left\{ \limsup_{n\to+\infty} (\mathcal{S}(t_n)u_0)(x_n) + ct_n \right\} \tag{13}$$

$$\overline{u}(x) = \liminf_{t\to+\infty}{}_{*} \, (\mathcal{S}(t)u_0)(x) + ct \quad := \quad \inf \left\{ \liminf_{n\to+\infty} (\mathcal{S}(t_n)u_0)(x_n) + ct_n \right\} \tag{14}$$

where the supremum and the infimum in (13) and (14) respectively are taken for all sequences $(x_n)_n$ converging to x and all diverging sequences $(t_n)_n$. Moreover, thanks to the uniform continuity of the function $(\mathcal{S}(t)u_0)(x)$ on $\mathbb{R}_+ \times \mathbb{T}^N$ (cf. Theorem 2.5), the sequences $(x_n)_n$ may be chosen identically equal to x, so that the following identities hold true:

$$\begin{aligned} \underline{u}(x) &= \sup \{ \psi(x) \,:\, \psi \in \omega_{\mathcal{S}}(u_0) \} \\ \overline{u}(x) &= \inf \{ \psi(x) \,:\, \psi \in \omega_{\mathcal{S}}(u_0) \} \end{aligned}$$

where

$$\omega_{\mathcal{S}}(u_0) := \left\{ \psi \in \mathrm{C}(\mathbb{T}^N) \,:\, \psi = \lim_{n\to+\infty} \mathcal{S}(t_n)u_0 + ct_n \text{ for some diverging sequence } (t_n)_n \right\}.$$

We have (cf. proof of Theorem 1 in [16]):

Theorem 3.3. *The functions $\underline{u}$ and $\overline{u}$ defined by (13) and (14) are a subsolution and a supersolution of equation (2), respectively.*

We proceed to establish the asymptotic convergence of $\mathcal{S}(t)u_0 + ct$ to the function v given in (11) with $w_0 = u_0$, provided u_0 is a critical sub or supersolution.

Theorem 3.4. *Let $u_0 \in \mathrm{C}(\mathbb{T}^N)$ be either a subsolution or a supersolution of (2). Then $\mathcal{S}(t)u_0 + ct$ uniformly converges, as t goes to $+\infty$, to the critical solution v defined by (11) with $w_0 = u_0$.*

Proof. Let us first assume u_0 to be a subsolution of (2). By Theorem 3.1 (iii), v is the maximal critical subsolution satisfying $v = u_0$ on $\mathcal{A}$, hence $v \ge u_0$ on $\mathbb{T}^N$. As $u_0 - ct$ and $v - ct$ are a subsolution and a supersolution of (2) respectively, the Comparison Principle yields

$$u_0 - ct \le \mathcal{S}(t)u_0 \le v - ct \qquad \text{on } \mathbb{T}^N,$$

and consequently, since $v = u_0$ on $\mathcal{A}$, we get

$$u_0 = S(t)u_0 + ct = v \qquad \text{on } \mathcal{A}.$$

for every $t > 0$. It follows that $v = \underline{u} = \overline{u}$ on $\mathcal{A}$, and we finally deduce from Theorem 2.4 (i) $v = \underline{u} = \overline{u}$ on $\mathbb{T}^N$. This proves the assertion when u_0 is a critical subsolution.

Let us now assume u_0 to be a supersolution of (2). Let v_0 be the maximal critical subsolution not exceeding u_0 on $\mathbb{T}^N$, i.e.

$$v_0 = \min_{y \in \mathbb{T}^N} \left(S(y, \cdot) + u_0(y) \right).$$

The maximality of v_0, combined with the fact that u_0 is a critical supersolution, implies that v_0 is a critical solution as well, so that the identity $v = v_0$ on $\mathbb{T}^N$ holds true. Arguing as in the first part of the proof, we therefore obtain

$$v \leq \mathcal{S}(t)u_0 + ct \leq u_0 \qquad \text{on } \mathbb{T}^N,$$

for every $t > 0$. This entails $v \leq \overline{u} \leq \underline{u} \leq u_0$ on $\mathbb{T}^N$. From the fact that $\underline{u}$ is a critical subsolution, and from the maximality property of $v_0 = v$, we get $\underline{u} \leq v$, and so $v = \overline{u} = \underline{u}$ on $\mathbb{T}^N$. □

We deduce from Theorem 3.4:

Proposition 3.5. *Assume $u_0 \in \mathrm{C}(\mathbb{T}^N)$, and let v be the function defined by (11) with $w_0 = u_0$. Then the relaxed semilimits $\underline{u}$ and $\overline{u}$, defined by (13) and (14) respectively, satisfy*

$$v(x) \leq \overline{u}(x) \leq \underline{u}(x) \qquad \textit{for every } x \in \mathbb{T}^N. \tag{15}$$

Proof. Set $v_0 = \min_{y \in \mathbb{T}^N} (S(y, \cdot) + u_0(y))$. It is apparent that $v_0 \leq u_0$ on $\mathbb{T}^N$, hence, by the Monotonicity Property of the semigroup $\big(\mathcal{S}(t)\big)_{t>0}$, we obtain $\mathcal{S}(t)v_0 + ct \leq \mathcal{S}(t)u_0 + ct$ on $\mathbb{T}^N$, and (15) follows in view of Theorem 3.4 and Theorem 3.1. □

Proposition 3.5, Theorem 2.4, and the fact that v is a critical solution and $\underline{u}$ a critical subsolution imply that the convergence result we aim at is proved as soon as the equality $v = \underline{u}$ is obtained on $\mathcal{A}$. This suggests, in the end, that what really matters in our analysis is the asymptotic behavior of $\mathcal{S}(t)u_0 + ct$ on $\mathcal{A}$.

4. Dynamical properties of the projected Aubry set

Here we define a family of curves, called critical, fully covering the Aubry set, which will play an important role in the convergence result of the next section. We will furthermore investigate the behavior of critical subsolutions on such curves. Throughout the section, conditions (H1), (H2), (H3), (H4) are assumed.

Definition 4.1. A curve γ defined on an interval J is called *critical* if

$$S(\gamma(t_1), \gamma(t_2)) = \int_{t_1}^{t_2} \left(L(\gamma, \dot{\gamma}) + c \right) \mathrm{d}s = -S(\gamma(t_2), \gamma(t_1))$$

for every t_1, t_2 in J with $t_2 \geq t_1$.

Lemma 4.2. *Any critical curve is contained in the Aubry set.*

Proof. Let γ be a critical curve, which we first assume to be nonconstant, defined in some interval J. Given t_1, t_2 in J with $t_2 \geq t_1$ and $\gamma(t_1) \neq \gamma(t_2)$, we can find two sequences of curves $\gamma_n^1 \in \mathrm{Lip}_{\gamma(t_1),\gamma(t_2)}([0,1], \mathbb{T}^N)$ and $\gamma_n^2 \in \mathrm{Lip}_{\gamma(t_2),\gamma(t_1)}([0,1], \mathbb{T}^N)$ which approximate the semidistance S of their end points up to $1/n$, for any n. The cycles γ_n, obtained by juxtaposition of γ_n^1 and γ_n^2, and change of parametrization to $[0,1]$, are of length $\ell(\gamma_n) \geq 2|\gamma(t_2) - \gamma(t_1)|$ and satisfy by Definition 4.1

$$\lim_n \int_0^1 \sigma(\gamma_n(s), \dot\gamma_n(s))\,\mathrm{d}s = 0$$

which shows that $\gamma(t_1)$, $\gamma(t_2)$ are in $\mathcal{A}$. If, on the contrary, the support of γ is reduced to a point, say x_0, we find

$$\int_J (L(x_0, 0) + c)\,\mathrm{d}s = 0,$$

which implies $x_0 \in \mathcal{E} \subset \mathcal{A}$. □

A further step in the analysis is carried out by picking up a special parametrization for curves on the torus. To do this, we use the Lagrangian function related to H.

Definition 4.3. A curve γ defined on an interval J is said to have a *Lagrangian parametrization* if

$$L(\gamma(t), \dot\gamma(t)) + c = \sigma(\gamma(t), \dot\gamma(t)) \qquad \text{for a.e. } t \in J. \tag{16}$$

The definition of the semidistance S and the inequality $L(x,q) \geq \sigma(x,q) - c$, which holds for every x and q, imply:

Proposition 4.4. *Any critical curve has a Lagrangian parametrization.*

More generally, the following reparametrization lemma holds.

Proposition 4.5. *Any curve with closure of the support disjoint from $\mathcal{E}$ admits a Lagrangian reparametrization. If, in addition, the curve is defined on a bounded interval, the same holds true for its Lagrangian reparametrization.*

Proof. The first step is to show the existence of an u.s.c. (resp. l.s.c.) function $\lambda(x,q)$ (resp. $\underline{\lambda}(x,q)$) defined in $(\mathbb{T}^N \setminus \mathcal{E}) \times (\mathbb{R}^N \setminus \{0\})$ such that the equality

$$L(x, \lambda(x,q)q) = \lambda(x,q)\sigma(x,q) - c \tag{17}$$

and the similar one obtained by replacing $\lambda(\cdot,\cdot)$ by $\underline{\lambda}(\cdot,\cdot)$ hold true.

Given $(x,q) \in (\mathbb{T}^N \setminus \mathcal{E}) \times (\mathbb{R}^N \setminus \{0\})$, and denoted $\{p \,:\, H(x,p) \leq c\}$ by Z, we have that $q \in N_Z(p_0)$ for some p_0 with $H(x,p_0) = c$, therefore

$$\lambda q \in D_p^- H(x, p_0) \quad \text{for some } \lambda > 0 \tag{18}$$

in force of Theorem 23.7 of [17]. Consequently the set of nonnegative λ satisfying (17) in place of $\lambda(x,q)$, denoted by $F(x,q)$, is nonempty, see Theorem A.2. It is moreover a compact subset of $(0,+\infty)$. We see, in fact, that, for λ large, relation (18) is impossible, when $H(x,p_0) = c$, since Z is compact and $H(x,\cdot)$ locally Lipschitz–continuous. This shows that $F(x,q)$ is bounded from above. It is also closed thanks to the continuity of $\sigma(x,\cdot)$ and $L(x,\cdot)$, respectively, and the inequality

$$L(x, \lambda q) \geq \sigma(x, \lambda q) - c \quad \text{for every } \lambda \geq 0.$$

Moreover, $0 \notin F(x,q)$ because $x \notin \mathcal{E}$, and consequently $L(x,0) = -\max_p H(x,p) < -c$. We then define

$$\lambda(x,q) = \max_{F(x,q)} \lambda \quad , \quad \underline{\lambda}(x,q) = \min_{F(x,q)} \lambda$$

and we see that these functions, for (x,q) varying in $(\mathbb{T}^N \setminus \mathcal{E}) \times (\mathbb{R}^N \setminus \{0\})$, are u.s.c. and l.s.c., respectively.

The assertion is finally obtained arguing as in [13] Proposition 7.4. $\square$

Remark 4.6. A notion of Lagrangian parametrization can be given at any level $a > c$, by replacing in (16) c and σ by a and σ_a, respectively. Proposition 4.5 can be accordingly generalized providing Lagrangian reparametrizations for any curve, without the requirement of empty intersection with $\mathcal{E}$. Such a restriction comes, in fact, from the necessity of avoiding that a p_0 satisfying $H(x_0, p_0) = c$, for some x_0, is a minimizer of $p \mapsto H(x_0, p)$. This possibility is actually ruled out for a p_0 with $H(x_0, p_0) = a$ when $a > c$.

Exploiting the previous remark, we can provide, in a sense, a generalization of Proposition 4.5. This result will be used in the proof of Proposition 5.5.

Lemma 4.7. *Let $\gamma \in \mathrm{Lip}([0,1], \mathbb{T}^N)$. For any $T > 0$ we set*

$$[\gamma]_T := \{\, \xi \in \mathrm{Lip}([0,T], \mathbb{T}^N) \,:\, \xi \text{ is a reparametrization of } \gamma \,\}.$$

Then

$$\int_0^1 \sigma(\gamma, \dot\gamma)\,\mathrm{d}s = \inf\left\{ \int_0^T \left(L(\xi, \dot\xi) + c \right)\,\mathrm{d}s \;:\; \xi \in [\gamma]_T,\ T > 0 \right\}. \tag{19}$$

Proof. It is apparent that the left–hand side term of (19) is not greater than that in the right–hand side one. To prove the converse inequality, we select a decreasing sequence $(\delta_n)_n$ with $\delta_n \downarrow 0$. Since $\sigma(x,q) = \inf_n \sigma_{c+\delta_n}(x,q)$ for every $(x,q) \in \mathbb{T}^N \times \mathbb{R}^N$, by the monotone convergence theorem we get

$$\int_0^1 \sigma(\gamma, \dot\gamma)\,\mathrm{d}s = \inf_n \int_0^1 \sigma_{c+\delta_n}(\gamma, \dot\gamma)\,\mathrm{d}s. \tag{20}$$

Taking into account Remark 4.6, we have a Lagrangian reparametrization γ_n of γ at level $a = c + \delta_n$, for any n, defined in some interval $[0, T_n]$, with $T_n > 0$, such that

$$\begin{aligned} \int_0^1 \sigma_{c+\delta_n}(\gamma, \dot\gamma)\,\mathrm{d}s &= \int_0^{T_n} \sigma_{c+\delta_n}(\gamma_n, \dot\gamma_n)\,\mathrm{d}s = \int_0^{T_n} \left(L(\gamma_n, \dot\gamma_n) + c + \delta_n \right)\,\mathrm{d}s \\ &\geq \int_0^{T_n} \left(L(\gamma_n, \dot\gamma_n) + c \right)\,\mathrm{d}s. \end{aligned}$$

The assertion therefore follows from (20). $\square$

The main result we aim at, in this section, is the following:

Theorem 4.8. *Through any point of $\mathcal{A}$ it passes a critical curve defined on the whole $\mathbb{R}$.*

We start by a lemma, then prove a local version of Theorem 4.8, and thereafter get the full result by using Zorn's lemma.

Lemma 4.9. *There exists a real number $R > 0$ such that*

$$\{ q \in \mathbb{R}^N \,:\, L(x,q) + c = \sigma(x,q) \text{ for some } x \in \mathbb{T}^N \,\} \subseteq B_R.$$

Proof. We can take R as the Lipschitz constant of the function $p \mapsto H(x,p)$ for $x \in \mathbb{T}^N$ and p satisfying $H(x,p) = c$. To see that this quantity is actually well defined, note that the condition on (x,p) singles out a compact set in $\mathbb{T}^N \times \mathbb{R}^N$ in force of the coercivity assumption (H3), and take into account Remark 2.1.

If $q \in \mathbb{R}^N$, $x_0 \in \mathbb{T}^N$ are such that $L(x_0,q) + c = \sigma(x_0,q)$, then $q \in D_p^- H(x_0,p_0)$ for some p_0 with $H(x_0,p_0) = c$, and so $|q| \le R$. □

Lemma 4.10. *For any $y \in \mathcal{A}$, there exists $\delta \in (0,+\infty]$ and a critical curve η, which is defined in $(-\delta,\delta)$ and satisfies $\eta(0) = y$.*

Proof. If $y \in \mathcal{E}$, we simply set $\eta(t) = y$, for every $t \in \mathbb{R}$. By the definition of equilibrium point, we have

$$L(y,0) + c = \max_{p \in \mathbb{R}^N} -H(x,p) + c = 0 \tag{21}$$

for every $t \in \mathbb{R}$, which shows that η is indeed a critical curve. If $y \in \mathcal{A} \setminus \mathcal{E}$, we exploit Lemma 9.4 of [13] to see that there exists a curve γ contained in $\mathcal{A}$, and defined in some neighborhood J of $t = 0$, such that $\gamma(0) = y$ and

$$S(\gamma(t_1),\gamma(t_2)) = \int_{t_1}^{t_2} \sigma(\gamma,\dot\gamma)\,\mathrm{d}s = -S(\gamma(t_2),\gamma(t_1))$$

for every t_1, $t_2 \in J$ with $t_2 > t_1$. Note that this result does not require the Lipschitz continuity of H in x, which was assumed in that paper, and therefore holds also in our present setting.

Because of the local character of the construction, we can assume that γ stays away from $\mathcal{A}$. We thus consider a Lagrangian reparametrization of γ, which does exist in force of Proposition 4.5, to get the required curve. □

Proposition 4.11. *Let $y \in \mathcal{A}$. Then there exists a critical curve η defined on $\mathbb{R}$ with $\eta(0) = y$.*

Proof. In view of Lemma 4.10, we may assume that $y \in \mathcal{A} \setminus \mathcal{E}$. We denote by $\mathcal{C}$ the set of pairs (T,η), where $T \in (0,+\infty]$, and η is a critical curve defined on $(-T,T)$ and equaling y at 0. We give an order relation in $\mathcal{C}$ by defining

$$(T,\eta) \preceq (T',\eta') \quad \text{if} \quad T \le T' \ \text{ and } \ {\eta'}_{|(-T,T)} = \eta.$$

The set $\mathcal{C}$ is nonempty by Lemma 4.10 . To prove that $\mathcal{C}$ is inductively ordered, we take a nonempty chain $\{(T_i,\eta_i)\}$, with i in some set of indices I, and observe that an upper bound $(\hat T,\hat\eta) \in \mathcal{C}$ can be defined through

$$\hat T := \sup_i T_i \qquad \text{and} \qquad \hat\eta(t) := \eta_i(t) \qquad \text{if } t \in (-T_i,T_i), \text{ for every } i \in I.$$

Zorn's Lemma hence provides the existence of a maximal element (T_y,η_y) in $\mathcal{C}$. We claim that $T_y = +\infty$. If, in fact, this were not the case, and $T_y < +\infty$, then the curve η_y should have limit (belonging to $\mathcal{A}$) for t going to $\pm T_y$, in view of Lemma 4.9. It would then be possible to extend η_y to some interval $(-T_y-\delta, T_y+\delta)$ for a suitable $\delta > 0$, by applying Lemma 4.10 to these limit points. This would violate the maximality of (T_y,η_y). □

We denote by $\mathcal{K}$ the family of all maximal critical curves, and by $\mathcal{K}(y)$ the subset of $\mathcal{K}$ made up by those equaling y at $t = 0$, for each $y \in \mathcal{A}$.

We proceed to prove a compactness property for $\mathcal{K}$.

Proposition 4.12. *$\mathcal{K}$ is a compact metric space with respect to the local uniform convergence on $\mathbb{R}$.*

Proof. Let $(\eta_k)_k$ be a sequence in $\mathcal{K}$. The curves η_k are uniformly bounded by the compactness of $\mathbb{T}^N$, and equiLipschitz continuous by Lemma 4.9, hence we can apply Ascoli–Arzelà Theorem to infer the existence of a subsequence (not relabeled) which converges locally uniformly to some curve η defined on $\mathbb{R}$. The limit curve η is contained in $\mathcal{A}$, as the Aubry set is closed, and clearly satisfies

$$S(\eta(t_1), \eta(t_2)) = -S(\eta(t_2), \eta(t_1)) \tag{22}$$

for every t_1, t_2 in $\mathbb{R}$. If, in addition, $t_2 > t_1$, we have

$$S(\eta_k(t_1), \eta_k(t_2)) = \int_{t_1}^{t_2} (L(\eta_k(s), \dot{\eta}_k(s)) + c) \, \mathrm{d}s.$$

for every k, and we therefore deduce, thanks to Theorem 2.9,

$$S(\eta(t_1), \eta(t_2)) = \lim_{k \to +\infty} \int_{t_1}^{t_2} (L(\eta_k(s), \dot{\eta}_k(s)) + c) \, \mathrm{d}s \geq \int_{t_1}^{t_2} (L(\eta(s), \dot{\eta}(s)) + c) \, \mathrm{d}s.$$

Since the converse inequality is apparent, we get in the end

$$S(\eta(t_1), \eta(t_2)) = \int_{t_1}^{t_2} (L(\eta(s), \dot{\eta}(s)) + c) \, \mathrm{d}s. \tag{23}$$

Relations (22), (23) show that $\eta \in \mathcal{K}$. □

Given $\eta \in \mathcal{K}$, we denote by $\omega(\eta)$ the set of its ω–limits, i.e. of the points x_0 satisfying

$$x_0 = \lim_k \eta(s_k) \quad \text{with } s_k \to +\infty \text{ as } k \to +\infty \tag{24}$$

We deduce from Proposition 4.12 that through any point x_0 of $\omega(\eta)$ there passes a critical curve entirely lying in $\omega(\eta)$. If, in fact, (24) holds, then $\{\eta(s_k + \cdot)\}_k$ converges locally uniformly, up to a subsequence, to a curve γ, which equals x_0 at 0, and is contained in $\omega(\eta)$.

Remark 4.13. We can describe more precisely $\omega(\eta)$ if the sequence s_k, appearing in (24), is increasing and such that $s_{k+1} - s_k$ converges to a finite limit, necessarily nonnegative, say T, and $\{\eta(s_k + \cdot)\}_k$ converges locally uniformly to a curve γ.

In this case $\omega(\eta)$ coincides with the support of γ, which is a cycle of period T because of the relations

$$\gamma(t + T) = \lim_k \eta(s_k + T + t) = \lim_k \eta(s_{k+1} + t) = \gamma(t)$$

which hold for any t. If, in fact, $y_0 := \lim_n \eta(t_n)$ belongs to $\omega(\eta)$, with $(t_n)_n$ diverging sequence, then we can select, for any n, an index $k_n \in \mathbb{N}$ satisfying $s_{k_n} \leq t_n < s_{k_n+1}$. The sequence $t_n - s_{k_n}$ is therefore bounded and so convergent, up to a subsequence, to some $t_0 \in [0, T]$. It then follows that $y_0 = \gamma(t_0)$.

If in particular $T = 0$, then γ reduces to a point, which must be the support of a critical curve, and consequently belongs to $\mathcal{E}$.

We know from [13] that, if H is Lipschitz–continuous in x, all critical subsolutions are strictly differentiable at any point of the Aubry set, and have the same derivative. This implies that they coincide, up to an additive constant, on every rectifiable subset of $\mathcal{A}$. These results are based upon some semiconcavity estimates which, in turn, depend essentially on the Lipschitz character of the Hamiltonian in x, that we do not have here. We can nevertheless find something similar, in our setting, looking at the behavior of the critical subsolutions on curves of $\mathcal{K}$.

Theorem 4.14. *Let $\eta \in \mathcal{K}$. Then all critical subsolutions coincide on $\eta(\mathbb{R})$, up to an additive constant. There exists, in addition, a negligible set $\Sigma \subset \mathbb{R}$ such that, for any critical subsolution ϕ, the map $\phi \circ \eta$ is differentiable on $\mathbb{R} \setminus \Sigma$ and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} (\phi \circ \eta)(t_0) = \sigma(\eta(t_0), \dot{\eta}(t_0)) \qquad \textit{whenever } t_0 \in \mathbb{R} \setminus \Sigma. \tag{25}$$

We show first an auxiliary lemma, on which the proof of Theorem 4.14 is based.

Proposition 4.15. *Let $\eta \in \mathcal{K}$. Then there exists a negligible set $\Sigma \subset \mathbb{R}$ such that the functions $\eta(\cdot)$, $S(\eta(t_0), \eta(\cdot))$ and $-S(\eta(\cdot), \eta(t_0))$ are differentiable at any t_0 in $\mathbb{R} \setminus \Sigma$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\eta(t_0), \eta(t))\Big|_{t=t_0} = -\frac{\mathrm{d}}{\mathrm{d}t} S(\eta(t), \eta(t_0))\Big|_{t=t_0} = \sigma(\eta(t_0), \dot{\eta}(t_0)). \tag{26}$$

Proof. Let Σ be a negligible subset of $\mathbb{R}$ such that every $t_0 \in \mathbb{R} \setminus \Sigma$ is a differentiability point for $\eta(\cdot)$ and a Lebesgue point for the function $\sigma(\eta(\cdot), \dot{\eta}(\cdot))$. The existence of such a set is guaranteed by Rademacher and Lebesgue differentiability theorems. As the curve η is critical, we have

$$\frac{S(\eta(t_0), \eta(t))}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^{t} \sigma(\eta(s), \dot{\eta}(s))\, \mathrm{d}s \qquad \text{for every } t > t_0.$$

Since t_0 is a Lebesgue point of $\sigma(\eta(\cdot), \dot{\eta}(\cdot))$, we derive

$$\lim_{t \to t_0^+} \frac{S(\eta(t_0), \eta(t))}{t - t_0} = \sigma(\eta(t_0), \dot{\eta}(t_0))$$

for every $t_0 \in \mathbb{R} \setminus \Sigma$. A similar limit relation for $t \to t_0^-$ can be deduced analogously. □

Proof of Theorem 4.14. Let Σ and ϕ be the subset of $\mathbb{R}$ given by Proposition 4.15 and a critical subsolution, respectively. By Proposition 2.3, we have

$$-S(\eta(t), \eta(t_0)) \le \phi(\eta(t)) - \phi(\eta(t_0)) \le S(\eta(t_0), \eta(t)) \qquad \text{for every } t, t_0 \in \mathbb{R},$$

hence we get (25), for $t_0 \in \mathbb{R} \setminus \Sigma$, in view of Proposition 4.15. This fully proves the assertion. □

We point out two consequences of the previous theorem that we will use in the next section, and that we judge of independent interest, as well.

Proposition 4.16. *Two critical subsolutions coinciding on $\mathcal{M} := \bigcup_{\eta \in \mathcal{K}} \omega(\eta)$, must also coincide on $\mathcal{A}$.*

Proof. Let ϕ_1, ϕ_2 be two critical subsolutions coinciding on $\mathcal{M}$. Take y and η in $\mathcal{A}$ and in $\mathcal{K}(y)$, respectively. Let $(t_n)_n$ be a diverging sequence such that $\lim_n \eta(t_n) = x \in \mathcal{M}$. As $S(y,\cdot)$ is a critical subsolution (cf. Proposition 2.3), Theorem 4.14 yields

$$\phi_i(y) = \phi_i(\eta(0)) - S(y,\eta(0)) = \phi_i(\eta(t_n)) - S(y,\eta(t_n))$$

for every $n \in \mathbb{N}$, $i \in \{1,2\}$. Sending n to $+\infty$, we get

$$\begin{aligned} \phi_1(y) &= \lim_{n\to+\infty} \phi_1(\eta(t_n)) - S(y,\eta(t_n)) = \phi_1(x) - S(y,x) = \phi_2(x) - S(y,x) \\ &= \lim_{n\to+\infty} \phi_2(\eta(t_n)) - S(y,\eta(t_n)) = \phi_2(y), \end{aligned}$$

whence the assertion as y is an arbitrary point of $\mathcal{A}$. □

Remark 4.17. As the curve $\eta(t) := y$, for every $t \in \mathbb{R}$, is critical whenever $y \in \mathcal{E}$, it is apparent from the definitions that the set $\mathcal{E}$ is always contained in $\mathcal{M}$.

Proposition 4.18. *The set $\mathcal{M}$ is an uniqueness set for (2), i.e. two solutions of (2) coinciding on $\mathcal{M}$, coincide on the whole torus too.*

Proof. The assertion comes from the previous proposition and from the property of being $\mathcal{A}$ a uniqueness set for (2), as established in Theorem 2.4. □

5. Convergence to steady states

We are now ready to prove our main convergence result. Throughout this section we will assume, without any loss of generality, $c = 0$. We also assume H to satisfy conditions (H1), (H2)$'$, (H3). We recall that $u_0 \in C(\mathbb{T}^N)$ is the initial datum of the Cauchy problem (6) and that $\omega_{\mathcal{S}}(u_0)$ denotes the family of the uniform limits of $\mathcal{S}(t_n)u_0$, for some diverging sequence $(t_n)_n$. We start by establishing some monotonicity properties for the function $\mathcal{S}(t)\psi - \phi$ on the curves of $\mathcal{K}$, where ψ is any continuous function and ϕ any critical subsolution. The next result is the analogous of Lemma 3.1 in [18].

Proposition 5.1. *Let $\eta \in \mathcal{K}$. Then the map $t \mapsto \big(\mathcal{S}(t)\psi\big)(\eta(t)) - \phi(\eta(t))$ is nonincreasing on $\mathbb{R}_+$ for any $\psi \in C(\mathbb{T}^N)$, and any critical subsolution ϕ.*

Proof. Let t_1, t_2 in $\mathbb{R}_+$ with $t_2 \geq t_1$. Taking into account Theorem 4.14 and the integral representation formula for the Lax–Oleinik semigroup, we get

$$\big(\mathcal{S}(t_2)\psi\big)(\eta(t_2)) - \big(\mathcal{S}(t_1)\psi\big)(\eta(t_1)) \leq \int_{t_1}^{t_2} L(\eta(s),\dot\eta(s))\,\mathrm{d}s = \phi(\eta(t_2)) - \phi(\eta(t_1)),$$

which proves the assertion. □

We proceed to prove that a strict monotonicity property actually holds on the critical curves under appropriate assumptions. This result relies on a lemma, that we demonstrate first, which estimates the modification of the line integral of the Lagrangian on a critical curve, when the Lagrangian parametrization is suitably perturbed. We emphasize that, for this, we essentially use the differentiability of L in q and the continuity of $D_qL(x,q)$ in $\mathrm{int}(\mathrm{dom}L)$, a property that is equivalent, for a continuous Hamiltonian, to the strict convexity of H in the second variable (cf. [7]). These results are key tools for the forthcoming convergence theorem.

Lemma 5.2. *There is a modulus $\omega(\cdot)$ such that, if η is any curve in $\mathcal{K}$ and λ is suitably close to 1, we have*

$$\int_{t_1}^{t_2} L(\eta_\lambda, \dot\eta_\lambda)\,\mathrm{d}s \le S(\eta_\lambda(t_1), \eta_\lambda(t_2)) + |\lambda - 1|\,\omega(|\lambda-1|)(t_2 - t_1)$$

for every t_1, t_2 with $t_2 > t_1$, where $\eta_\lambda(t) := \eta(\lambda t)$ for all $t \in \mathbb{R}$.

Proof. We claim that $K := \{(x,q) \in \mathcal{A} \times \mathbb{R}^N \,:\, L(x,q) = \sigma(x,q)\,\}$ is a compact subset of int(dom$L$). It is in fact closed by the lower and upper semicontinuity of L and σ, respectively, bounded by Lemma 4.9 and contained in int(dom$L$) thanks to Proposition 2.7 (iv). There thus exists $\delta > 0$ such that the set $K_\delta := \{(x, \lambda q) : (x,q) \in K,\, |\lambda - 1| \le \delta\,\}$ is compactly contained in int(dom$L$).

Let us now fix λ in $(1-\delta, 1+\delta)$, and denote by θ a continuity modulus for the function $(x,q) \mapsto D_q L(x,q)$ in K_δ. For a.e. $s \in \mathbb{R}$ we have

$$\begin{aligned} (\eta(\lambda s), \dot\eta(\lambda s)) &\in K, && (27)\\ \langle D_q\, L(\eta(\lambda s), \dot\eta(\lambda s)), \dot\eta(\lambda s)\rangle &= \sigma(\eta(\lambda s), \dot\eta(\lambda s)), && (28) \end{aligned}$$

where the first relation comes from the very definition of critical curve, and the second one holds in view of Theorem A.2. Let s be such that (27) and (28) hold. The application of the mean value theorem to the function $\mu \mapsto L(\eta(\lambda s), \mu\eta(\lambda s))$ in the interval with end points 1 and λ yields

$$L(\eta(\lambda s), \lambda\, \dot\eta(\lambda s)) - L(\eta(\lambda s), \dot\eta(\lambda s)) = (\lambda - 1)\langle D_q\, L(\eta(\lambda s), \mu_0\, \dot\eta(\lambda s)), \dot\eta(\lambda s)\rangle.$$

where μ_0 is a suitable constant between λ and 1. By using (27), (28), and the definition of $\theta(\cdot)$, we derive from this identity

$$L(\eta(\lambda s), \lambda\, \dot\eta(\lambda s)) \le \lambda\, \sigma(\eta(\lambda s), \dot\eta(\lambda s)) + R|\lambda - 1|\,\theta(|\lambda-1|R),$$

where R is the positive constant provided by Lemma 4.9. We now exploit the previous estimate and the fact that η is a critical curve, to get for any t_1, t_2 in $\mathbb{R}$ with $t_2 > t_1$

$$\begin{aligned} \int_{t_1}^{t_2} L(\eta_\lambda, \dot\eta_\lambda)\,\mathrm{d}s &= \int_{t_1}^{t_2} L(\eta(\lambda s), \lambda\dot\eta(\lambda s))\,\mathrm{d}s \\ &\le \int_{t_1}^{t_2} \lambda\, \sigma(\eta(\lambda s), \dot\eta(\lambda s))\,\mathrm{d}s + (t_2 - t_1)|\lambda - 1|\, R\, \theta(R\,|\lambda - 1|) \\ &= S(\eta_\lambda(t_1), \eta_\lambda(t_2)) + (t_2 - t_1)|\lambda - 1|\, R\, \theta(R\,|\lambda - 1|). \end{aligned}$$

□

Proposition 5.3. *Let $\eta \in \mathcal{K}$, $\psi \in C(\mathbb{T}^N)$ and ϕ be a critical subsolution. Let us assume $D^+\big((\psi - \phi)\circ\eta\big)(0) \setminus \{0\} \neq \emptyset$ (recall that D^+ indicates the superdifferential), then*

$$(\mathcal{S}(t)\psi)\,(\eta(t)) - \phi(\eta(t)) < \psi(\eta(0)) - \phi(\eta(0)) \qquad \textit{for every } t > 0. \qquad (29)$$

Proof. We fix $t > 0$. Inequality (29) will be proved for $\phi := -S(\cdot, \eta(t))$, which is enough to get the full result, in view of Theorem 4.14. We also assume, without any loss of generality in view of Proposition 2.6 (iii), that $\psi(\eta(0)) - \phi(\eta(0)) = 0$. We are thus lead to show that the left–hand side term of (29) is strictly negative. To this aim, we take into account the integral formula for the Lax–Oleinik semigroup, given in Section 2, to get, for λ close to 1 and η_λ defined as in Lemma 5.2,

$$(\mathcal{S}(t)\psi)\,(\eta(t)) - \phi(\eta(t)) = (\mathcal{S}(t)\psi)\,(\eta(t)) \le \int_{(1/\lambda-1)t}^{t/\lambda} L(\eta_\lambda, \dot\eta_\lambda)\,\mathrm{d}s + \psi\big(\,\eta((1-\lambda)t)\,\big),$$

whence, by Lemma 5.2,

$$(\mathcal{S}(t)\psi)\,(\eta(t)) - \phi(\eta(t)) \le \psi\big(\,\eta((1-\lambda)t)\,\big) - \phi\big(\,\eta((1-\lambda)t)\,\big) + t|\lambda-1|\,\omega(|\lambda-1|).$$

If $m \neq 0$ is an element of $D^+\big((\psi-\phi)\circ\eta\big)(0)$, we therefore have

$$(\mathcal{S}(t)\psi)\,(\eta(t)) - \phi(\eta(t)) \le m\,\big((1-\lambda)t\big) + o\big((1-\lambda)t\big) + t|\lambda-1|\,\omega(|\lambda-1|),$$

where $o(\cdot)$ satisfies $\lim_{\lambda\to1} \frac{o\left((1-\lambda)t\right)}{1-\lambda} = 0$. A suitable choice of λ close to 1 makes thus the left–hand side term of the previous formula strictly negative, and consequently proves the assertion, for the arbitrariness of t. $\square$

We combine the information gathered in Propositions 5.1 and 5.3 with some properties of the Lax–Oleinik semigroup to get:

Proposition 5.4. *Let ϕ be a critical subsolution, and $\psi \in \omega_{\mathcal{S}}(u_0)$. For any $x_0 \in \mathcal{M}$ there exists a curve $\gamma \in \mathcal{K}(x_0)$ such that the function $t \mapsto \psi(\gamma(t)) - \phi(\gamma(t))$ is constant on $\mathbb{R}$.*

Proof. Let $(s_k)_k$ and $(t_k)_k$ be two diverging sequences, and η a curve of $\mathcal{K}$ such that $x_0 = \lim_k \eta(s_k)$, and ψ is the uniform limit of $\mathcal{S}(t_k)u_0$ in $\mathbb{T}^N$. We can assume that the curve γ, defined by $\gamma(t) = \lim_k \eta(t + s_k)$, for any t, is the local uniform limit of the sequence $\eta(s_k + \cdot)$ in $\mathbb{R}$, and so $\gamma \in \mathcal{K}$. We assume, in addition, that $t_k - s_k \to +\infty$, as $k \to +\infty$, and that $\mathcal{S}(t_k - s_k)u_0$ uniformly converges to some $\psi_1 \in \omega_{\mathcal{S}}(u_0)$. The non–expansiveness of the Lax–Oleinik semigroup implies

$$\|\mathcal{S}(t_k)u_0 - \mathcal{S}(s_k)\psi_1\|_\infty = \|\mathcal{S}(s_k + t_k - s_k)u_0 - \mathcal{S}(s_k)\psi_1\|_\infty \le \|\mathcal{S}(t_k - s_k)u_0 - \psi_1\|_\infty,$$

which entails $\mathcal{S}(s_k)\psi_1 \rightrightarrows \psi$ in $\mathbb{T}^N$. We know from Proposition 5.1 that the function

$$s \mapsto (\mathcal{S}(s)\psi_1)\,(\eta(s)) - \phi(\eta(s))$$

is nonincreasing in $\mathbb{R}_+$, hence it admits a limit, denoted by l, as $s \to +\infty$. Such a limit is furthermore finite, since it is greater or equal than $-\|\overline{u} - \phi\|_\infty$. Given $t > 0$, we have

$$l = \lim_{k\to+\infty} (\mathcal{S}(s_k + t)\psi_1)\,(\eta(s_k + t)) - \phi(\eta(s_k + t)) = (\mathcal{S}(t)\psi)\,(\gamma(t)) - \phi(\gamma(t)).$$

The function $t \mapsto (\mathcal{S}(t)\psi)\,(\gamma(t)) - \phi(\gamma(t))$ is therefore constant on $\mathbb{R}_+$. From this we deduce, by applying Proposition 5.3 to the curve $\gamma(s + \cdot) \in \mathcal{K}$, for any fixed s, that $D^+\big((\psi - \phi)\circ\gamma\big)(s) \setminus \{0\} = \emptyset$ for any $s \in \mathbb{R}$. This implies that $\psi - \phi$ is constant on γ. $\square$

The previous proposition shows that any function ψ in $\omega_{\mathcal{S}}(u_0)$ coincides, on any given critical curve γ lying in $\mathcal{M}$, with some critical subsolution ϕ. Such a critical subsolution may *a priori* depend on the curve γ and on ψ. We proceed to show, on the contrary, that ϕ is uniquely determined and coincides with the function v defined by (11), putting u_0 in place of w_0. In force of Proposition 5.4, it will be enough to prove the following fact.

Proposition 5.5. *Given $\eta \in \mathcal{K}$, $\psi \in \omega_{\mathcal{S}}(u_0)$, $\varepsilon > 0$, there exists $\tau \in \mathbb{R}$ such that*

$$|v(\eta(\tau)) - \psi(\eta(\tau))| < \varepsilon$$

where v is the critical solution defined by (11) with $w_0 = u_0$.

Proof. Since the curve η is contained in $\mathcal{A}$, and in view of Theorem 3.1 (ii), we have

$$v(\eta(0)) = \min_{z \in \mathbb{T}^N} \Big(u_0(z) + S(z, \eta(0)) \Big),$$

hence $v(\eta(0)) = u_0(z_0) + S(z_0, \eta(0))$, for some $z_0 \in \mathbb{T}^N$. We choose a curve $\gamma \in \mathrm{Lip}_{z_0,\eta(0)}([0,1], \mathbb{T}^N)$ such that

$$v(\eta(0)) + \varepsilon/2 = u_0(z_0) + S(z_0, \eta(0)) + \varepsilon/2 > u_0(z_0) + \int_0^1 \sigma(\gamma, \dot\gamma) \, \mathrm{d}s.$$

We, thereafter, take into account Lemma 4.7 and the integral representation formula for the Lax–Oleinik semigroup, to get

$$v(\eta(0)) + \varepsilon/2 > u_0(z_0) + \int_0^T L(\gamma_T, \dot\gamma_T) \, \mathrm{d}s \geq \big(\mathcal{S}(T) u_0\big)(\eta(0)),$$

where γ_T is a suitable reparametrization of γ on $[0, T]$, for some $T > 0$. Let now $(\tau_n)_n$ be a diverging sequence with $\mathcal{S}(\tau_n) u_0 \rightrightarrows \psi$, we have

$$\|\mathcal{S}(\tau_n) u_0 - \psi\|_\infty < \varepsilon/2 \qquad \text{and} \qquad \tau_n - T > 0 \qquad \text{for } n \text{ sufficiently large.}$$

Pick such an n and set $\tau = \tau_n - T$, then use the above inequalities and Theorem 4.14 to obtain

$$\begin{aligned} \psi(\eta(\tau)) - \varepsilon/2 &< \big(\mathcal{S}(\tau_n) u_0\big)(\eta(\tau)) = \big(\mathcal{S}(\tau)\mathcal{S}(T) u_0\big)(\eta(\tau)) \\ &\leq \big(\mathcal{S}(T) u_0\big)(\eta(0)) + \int_0^\tau L(\eta, \dot\eta) \, \mathrm{d}s \\ &< \varepsilon/2 + v(\eta(0)) + \int_0^\tau L(\eta, \dot\eta) \, \mathrm{d}s = \varepsilon/2 + v(\eta(\tau)). \end{aligned}$$

This gives the assertion since $\psi(\eta(\tau)) - v(\eta(\tau)) \geq 0$ by Proposition 3.5. □

We directly derive from Propositions 5.4 and 5.5:

Theorem 5.6. *Any function in $\omega_{\mathcal{S}}(u_0)$ coincides with v on $\mathcal{M}$, where v is the critical subsolution defined by (11), with u_0 in place of w_0.*

We finally prove our main result.

Theorem 5.7. *Let H satisfy conditions (H1), (H2)′, (H3) and $u_0 \in \mathrm{C}(\mathbb{T}^N)$. Then $\mathcal{S}(t) u_0$ uniformly converges to v on $\mathbb{T}^N$ as t goes to $+\infty$, where v is the critical solution given by formula (11) with $w_0 = u_0$.*

Proof. Theorem 5.6 implies that v and $\underline{u}$ coincide on $\mathcal{M}$, they therefore coincide on $\mathcal{A}$ thanks to Proposition 4.16. The comparison principle given in Theorem 2.4 tells hence us that $\underline{u} \leq v$ on the whole torus, since $\underline{u}$ is a critical subsolution and v a critical solution. The assertion is at last obtained thanks to Proposition 3.5. $\square$

We stress that the only point, in the present section (actually, in the whole paper), where the strict convexity assumption is directly employed is Lemma 5.2. It is, more precisely, used the global continuity of $D_q L(x, q)$ in $\mathrm{int}(\mathrm{dom} L)$, a property that is equivalent to the strict convexity of H in p, as previously noticed. As a matter of fact, we do not exploit such a condition in its full strength. The existence of a continuity modulus for $D_q L(x, q)$ in a neighborhood of the image of the map $t \mapsto (\eta, \dot{\eta})$, for each $\eta \in \mathcal{K}$, might be sufficient.

Yet, since the stationary curve $\gamma(\cdot) = y$ belongs to $\mathcal{K}$ whenever $y \in \mathcal{E}$, Proposition 5.5 — which has been proved without exploiting the strict convexity assumption (H2)$'$ — directly implies that any function of $\omega_{\mathcal{S}}(u_0)$ coincides with v on $\mathcal{E}$. Hence, whenever $\mathcal{E}$ is a uniqueness set for the critical equation (2), the same argument of Theorem 5.7 gives the convergence result bypassing Proposition 5.4, which instead relies on Lemma 5.2. This happens, for instance, when $\mathcal{M} = \mathcal{E}$. We can therefore state:

Theorem 5.8. *Let H satisfy conditions (H1), (H2), (H3), (H4) and $u_0 \in \mathrm{C}(\mathbb{T}^N)$. Then any function in $\omega_{\mathcal{S}}(u_0)$ coincides with v on $\mathcal{E}$, where v is the critical subsolution defined by (11), with u_0 in place of w_0. In particular, $\mathcal{S}(t)u_0$ uniformly converges to v on $\mathbb{T}^N$, as t goes to $+\infty$, when $\mathcal{M} = \mathcal{E}$, and, more generally, whenever $\mathcal{E}$ is a uniqueness set for the critical equation (2).*

Note that the previous theorem includes the results of [16], where the Hamiltonian under investigation was assumed only convex and with the Aubry set consisting of equilibria.

The next one–dimensional example deals with a family of Hamiltonians, depending on a parameter $\alpha \in \mathbb{R}$, which satisfy assumptions (H1), (H2), (H3), (H4). It is shown that a suitable initial datum for the time–dependent equation can be selected in such a way that the convergence to a steady state does not take place whenever the Hamiltonian under consideration does not satisfy the assumptions of Theorem 5.8. It can be viewed as a development of the example given in [4, Section 5].

Example 5.9. Consider the $\mathbb{Z}$–periodic Hamiltonian

$$H(x, p) = |p| - f(x)$$

defined in $\mathbb{R}$ (cf. Remark 2.2), where f is a continuous periodic potential with $f \not\equiv 0$, $f \geq 0$ and $\min_{\mathbb{R}} f = 0$. The effective Hamiltonian $\overline{H}(\alpha)$, i.e. the critical value of $H(x, p + \alpha)$, is given, for any $\alpha \in \mathbb{R}$, by

$$\overline{H}(\alpha) = \max\{0, |\alpha| - \int_0^1 f \, \mathrm{d}s\}$$

see [15]. It is not difficult to check that, for $\alpha \in \overline{H}^{-1}(0)$, the set of equilibria $\mathcal{E}(\alpha)$, relative $H(x, p + \alpha)$, coincides with $f^{-1}(0)$ and is a uniqueness set for the corresponding critical equation, while $\mathcal{A}(\alpha) = \mathcal{M}(\alpha) = \mathbb{R}$ and $\mathcal{E}(\alpha) = \emptyset$ as soon as α lies outside the flat part.

Given $\alpha \notin \overline{H}^{-1}(0)$, we define in $[0, +\infty) \times \mathbb{R}$ the function

$$w(t, x) = u_0(x - \operatorname{sgn}\alpha\ t) + \operatorname{sgn}\alpha \int_0^x f\,\mathrm{d}s - \left(\operatorname{sgn}\alpha \int_0^1 f\,\mathrm{d}s\right) x,$$

where sgn indicates the sign function, and u_0 is a C^1 nonconstant periodic function satisfying

$$\operatorname{sgn}\left\{u_0'(x) + \operatorname{sgn}\alpha \left(|\alpha| - \int_0^1 f\,\mathrm{d}s\right)\right\} = \operatorname{sgn}\alpha \qquad \text{for all } x \in \mathbb{R}. \tag{30}$$

Note that relation (30) implies that

$$\operatorname{sgn}\left(u_0'(x) + \operatorname{sgn}\alpha\, f(y) - \operatorname{sgn}\alpha \int_0^1 f\,\mathrm{d}s + \alpha\right) = \operatorname{sgn}\alpha \qquad \text{for all } x, y \in \mathbb{R} \tag{31}$$

as f is nonnegative. The function $w(t, \cdot)$ is periodic in $\mathbb{R}$ for any t, as easily seen. By taking into account (31), a direct calculation shows that

$$\partial_t w(t, x) + |\partial_x w(t, x) + \alpha| - f(x) - |\alpha| + \int_0^1 f\,\mathrm{d}s = 0$$

for every $(t, x) \in (0, +\infty) \times \mathbb{R}$. Hence w is a periodic C^1–solution of the time–dependent equation

$$\partial_t u + H(x, \partial_x u + \alpha) - \overline{H}(\alpha) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R},$$

but it does not converge to any steady state for $t \to +\infty$. Note that H is not strictly convex in the second argument and that $\mathcal{E}(\alpha) = \emptyset$.

Such a construction is clearly not possible when $\alpha \in \overline{H}^{-1}(0)$, or, in other terms, when $|\alpha| \le \int_0^1 f\,\mathrm{d}s$, because condition (30) implies, in this case, that u_0' does not change sign on $\mathbb{R}$, in contrast with u_0 being nonconstant and periodic.

Appendix A

We consider an Hamiltonian $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfying conditions (H1), (H2), (H3), and the corresponding Lagrangian $L : \mathbb{T}^N \times \mathbb{R}^N \to (-\infty, +\infty]$ defined through the Fenchel transform (8). As H is assumed coercive but not superlinear, the Lagrangian L is not finite–valued in general. Our aim is to give first a proof of Proposition 2.7, and afterward to show the validity of the integral representation formula (9) for the Lax–Oleinik semigroup. We start by recalling some basic facts of convex analysis, and by giving a characterization of the interior of $\mathrm{dom}(L)$, where $\mathrm{dom}(L) := \{(x, q) \in \mathbb{T}^N \times \mathbb{R}^N : L(x, q) < +\infty\}$.

Theorem A.1. *Let $f : \mathbb{R}^N \to (-\infty, +\infty]$ be a convex function with $f \not\equiv +\infty$. Then $D^- f(q)$ is a nonempty bounded set if and only if $q \in \mathrm{int}(\mathrm{dom} f)$, and it is empty for $q \notin \mathrm{dom} f$.*

We refer to [17, Theorem 23.4] for a proof. Note that $L(x, \cdot)$ turns out to be convex and lower semicontinuous on $\mathbb{R}^N$, as supremum of continuous convex functions, for any $x \in \mathbb{T}^N$. Moreover $L(x, \cdot) \not\equiv +\infty$ (cf. [17, Theorem 12.2]). We have (cf. [17, Theorem 23.5]):

Theorem A.2. *Let $x \in \mathbb{T}^N$ and $p, q \in \mathbb{R}^N$. The following conditions are equivalent:*

(a) $H(x,p) + L(x,q) \le \langle p, q\rangle$;
(b) $H(x,p) + L(x,q) = \langle p, q\rangle$;
(c) $q \in D_p^- H(x,p)$;
(d) the function $\langle \cdot\,, q\rangle - L(x,\cdot)$ *achieves its maximum at* p;
(e) $p \in D_q^- L(x,q)$;
(f) the function $\langle p, \cdot\,\rangle - L(x,\cdot)$ *achieves its maximum at* q.

Theorem A.3.

(i) For any $x \in \mathbb{T}^N$, $\mathrm{dom}(L(x,\cdot))$ *has a nonempty interior.*
(ii) $\mathrm{int}(\mathrm{dom}L) = \bigcup_{x\in\mathbb{T}^N} \{x\} \times \mathrm{int}(\mathrm{dom}L(x,\cdot))$.

Proof. According to Corollary 13.4.2 in [17], the assertion in (i) holds true if and only if there are no lines along which $H(x,\cdot)$ is (finite and) affine. Such a condition is actually a consequence of the coercivity assumption (H3).

To ease notations, let us temporarily denote by $\widetilde{\Omega}$ the set at the right hand–side of the equality in item (ii). It is apparent that $\mathrm{int}(\mathrm{dom}L) \subset \widetilde{\Omega}$.

To prove the converse inclusion, we assume by contradiction the existence of $(x_0, q_0) \in \widetilde{\Omega} \setminus \mathrm{int}(\mathrm{dom}L)$. According to Theorem A.1, this implies that $D_q^- L(x_0, q_0)$ is nonempty and bounded, and, moreover, that there is a sequence $((x_n, q_n))_n$ converging to (x_0, q_0), with $D_q^- L(x_n, q_n)$ either empty or unbounded. In any case, we may find a sequence $(p_n)_n$ such that $|p_n| \to +\infty$ and

$$\langle p_0, q_n\rangle - H(x_n, p_0) \le \langle p_n, q_n\rangle - H(x_n, p_n), \tag{32}$$

where p_0 is any fixed element of $D_q^- L(x_0, q_0)$. Since the function $p \mapsto \langle p, q_n\rangle - H(x_n, q_n)$ is concave for any $n \in \mathbb{N}$, we see that (32) is still satisfied by putting any convex combination of p_n and p_0 in place of p_n, in particular it holds for some sequence $(\overline{p}_n)_n$ with $|\overline{p}_n - p_0| = r$, where r is an arbitrarily chosen positive constant. Up to subsequences, we can assume that $(\overline{p}_n)_n$ converges to some $\overline{p}$. Sending n to $+\infty$, we obtain

$$L(x_0, q_0) = \langle p_0, q_0\rangle - H(x_0, p_0) \le \langle \overline{p}, q_0\rangle - H(x_0, \overline{p}),$$

which implies that $\overline{p} \in D_q^- L(x_0, q_0)$ by Theorem A.2. This is in contrast with $D_q^- L(x_0, q_0)$ being bounded, because $|\overline{p} - p_0| = r$ and r is arbitrarily large. $\square$

The argument used for the proof of item (ii) in the previous theorem also gives (compare to Remark 2.1):

Corollary A.4. *The set–valued map* $(x,q) \mapsto D_q^- L(x,q)$ *is locally uniformly bounded in* $\mathrm{int}(\mathrm{dom}L)$.

Proof of Proposition 2.7.

(i) The lower semicontinuous and the convex character of L have been already pointed out at the beginning of the Appendix.

(ii) By item (i), we just need to show that L is upper semicontinuous in Ω. Hence, let $(x_0, q_0) \in \Omega$ be the limit of some sequence $\big((x_n, q_n)\big)_n$ contained in Ω. Given $p_n \in D_q^- L(x_n, q_n)$, we have that $(p_n)_n$ is bounded by Corollary A.4, and so convergent, up to a subsequence, to some p_0. Thanks to Theorem A.2 we know that

$$L(x_n, q_n) = \langle p_n, q_n\rangle - H(x_n, p_n),$$

and by sending n to infinity we get

$$\limsup_{n\to+\infty} L(x_n,q_n) = \lim_{n\to+\infty} \langle p_n, q_n\rangle - H(x_n,p_n) = \langle p_0,q_0\rangle - H(x_0,\ p_0) \le L(x_0,q_0),$$

which proves the claim.

(iii) Fix $x \in \mathbb{T}^N$. The C^1 regularity of the function $L(x,\cdot)$ in $\mathrm{int}(\mathrm{dom}L(x,\cdot))$ is equivalent to the strict convexity of $H(x,\cdot)$ on $\mathbb{R}^N$ (cf. [7]). In particular,

$$L(x,q) = \langle D_qL(x,q), q\rangle - H(x, D_qL(x,q)) \qquad \text{for all } (x,q)\in\Omega,$$

and $p = D_qL(x,q)$ is the unique maximizer of the function $\langle\,\cdot\ ,q\rangle - H(x,\cdot)$. To prove the continuity of $D_pL(x,q)$ in Ω, it suffices to show, by Corollary A.4, that $(x_n,q_n)\to(x_0,q_0)$ in Ω and $D_qL(x_n,q_n)\to\overline{p}$ in $\mathbb{R}^N$ imply $\overline{p} = D_qL(x_0,q_0)$. This actually follows from the continuity of L in Ω, since we can pass to the limit in the equality $L(x_n,q_n) = \langle D_qL(x_n,q_n),q_n\rangle - H(x_n, D_qL(x_n,q_n))$ to obtain $L(x_0,q_0) = \langle\overline{p},q_0\rangle - H(x_0,\overline{p})$, which gives $\overline{p} = D_pL(x_0,q_0)$ by what previously remarked.

(iv) If the set of maximizers of $p\mapsto\langle p,q\rangle - H(x,p)$ is nonempty, then it reduces to a singleton by the strict convexity of H with respect to p. The assertion thus follows from Theorem A.1 and Theorem A.3 (ii). $\square$

Let us now define, for each $n\in\mathbb{N}$,

$$H_n(x,p) := H(x,p) + \max\{|p|^2 - n^2, 0\} \qquad \text{for every } (x,p)\in\mathbb{T}^N\times\mathbb{R}^N,$$

and denote by L_n the Fenchel transform of H_n. Note that $(H_n)_n$ is a decreasing sequence of superlinear Hamiltonians, satisfying assumptions (H1), (H2), (H3), uniformly converging to H on compact subset of $\mathbb{T}^N\times\mathbb{R}^N$. This, in turn, implies that $(L_n)_n$ is an increasing sequence of Lagrangians, defined and continuous on $\mathbb{T}^N\times\mathbb{R}^N$, converging pointwise to L on $\mathbb{T}^N\times\mathbb{R}^N$, and uniformly superlinear at infinity in q, as well (see e.g. [7]).

Theorem A.5. *The representation formula (9) holds for every $\phi\in\mathrm{C}(\mathbb{T}^N)$, $t>0$.*

The proof of the theorem is based on a Γ–convergence result (cf. [9]) that we show first. For this, we employ a classical sequential weak compactness criterion in $W^{1,1}$ (see for instance Theorem 2.13 of [5]), which is in turn a consequence of the Dunford–Pettis Theorem (cf. Theorem 2.11 in [5]).

Proposition A.6. *For any fixed $x\in T^N$ and $t>0$, denote by $X_t(x)$ the space*

$$\{\gamma\in W^{1,1}([0,t],\mathbb{T}^N),\ \gamma(t)=x\}$$

endowed with the strong topology of $L^1([0,t],\mathbb{T}^N)$. For any $\phi\in\mathrm{C}(\mathbb{T}^N)$, let us set

$$\begin{aligned}\mathbb{L}^t(\gamma) &:= \phi(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s,\\ \mathbb{L}^t_n(\gamma) &:= \phi(\gamma(0)) + \int_0^t L_n(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.\end{aligned}$$

Then the functionals $\mathbb{L}^t_n$ Γ–converge to $\mathbb{L}^t$ on $X_t(x)$. Moreover

$$\min_{\gamma\in X_t(x)} \mathbb{L}^t(\gamma) = \lim_{n\to+\infty}\ \min_{\gamma\in X_t(x)} \mathbb{L}^t_n(\gamma).$$

Proof. We first set

$$\Theta(t) := \inf_{x\in\mathbb{T}^N}\left(\inf_{|q|\geq t} L_1(x,q)\right) \qquad \text{for every } t\geq 0.$$

and observe that

$$\lim_{t\to+\infty}\frac{\Theta(t)}{t} = +\infty, \qquad \Theta(|q|)\leq L_n(x,q)\leq L(x,q) \tag{33}$$

for any $n\in\mathbb{N}$ and $(x,q)\in\mathbb{T}^N\times\mathbb{R}^N$. We claim that the functionals $\mathbb{L}^t_n$ and $\mathbb{L}^t$ are lower semicontinuous on $X_t(x)$. In fact, any sequence $(\gamma_n)_n$ in $X_t(x)$ with $\lim_n \mathbb{L}^t(\gamma_n) < +\infty$ also satisfies $\sup_n\int_0^t \Theta(\dot\gamma_n)\,\mathrm{d}s < +\infty$ by (33), and this in turn implies that $(\gamma_n)_n$ is weakly convergent in $W^{1,1}([0,t],\mathbb{T}^N)$, up to subsequences (cf. Theorem 2.13 of [5]). This shows the sequential lower semicontinuity of $\mathbb{L}^t$ in $X_t(x)$, in view of Theorem 2.9; the lower semicontinuity follows as $X_t(x)$ is a metric space. The same argument gives the claim for each $\mathbb{L}^t_n$.

The Γ–convergence result is then assured by [9, Proposition 5.4], since $(\mathbb{L}^t_n)_n$ is, in addition, an increasing sequence of functionals converging pointwise to $\mathbb{L}^t$ on $X_t(x)$. To prove the asserted convergence of the minima, we remark that the set

$$K_t(x) := \{\,\gamma\in X_t(x)\,:\,\int_0^t \Theta(|\dot\gamma|)\,\mathrm{d}s \leq \|\phi\|_\infty + \Bbbk\, t\,\},$$

with $\Bbbk := \sup_{y\in\mathbb{T}^N} L(y,0)$, is sequentially weakly compact in $W^{1,1}([0,t],\mathbb{T}^N)$, hence compact in $X_t(x)$ because the weak convergence implies the uniform convergence (cf. [5, Theorem 2.13]). Notice also that

$$\int_0^t \Theta(|\dot\gamma_x|)\,\mathrm{d}s \leq \mathbb{L}^t_n(\gamma_x)\leq \mathbb{L}^t(\gamma_x)\leq \|\phi\|_\infty + \Bbbk\, t$$

for any n, where γ_x denotes the curve in $X_t(x)$ constantly equal to x. Consequently $K_t(x)$ is nonempty and

$$\inf\{\,\mathbb{L}^t_n(\gamma)\,:\,\gamma\in X_t(x)\,\} = \min\{\,\mathbb{L}^t_n(\gamma)\,:\,\gamma\in K_t(x)\,\}$$

for each n, so the assertion follows in view of [9, Theorem 7.4]. □

Proof of Theorem A.5.

We first notice that it is enough to show the assertion for $\phi\in\mathrm{Lip}(\mathbb{T}^N)$. The general case of a continuous initial datum may be in fact recovered by density, thanks to the non–expansiveness property of the Lax–Oleinik semigroup.

We denote by $\mathcal{S}_n(t)$ the semigroup associated to the Cauchy Problem (6), with H_n in place of H. Since $\phi\in\mathrm{Lip}(\mathbb{T}^N)$, we have by Theorem 2.5

$$\mathcal{S}(t)\phi = \mathcal{S}_n(t)\phi \tag{34}$$

for n sufficiently large. By Remark 2.8 each $\mathcal{S}_n(t)\phi$ admits an integral representation of the form (9), with L_n in place of L. This fact can be equivalently expressed, using the symbols introduced in Proposition A.6, by

$$\big(\mathcal{S}_n(t)\phi\big)(x) = \min_{\gamma\in X_t(x)} \mathbb{L}^t_n(\gamma)$$

for every $x\in\mathbb{T}^N$ and $t>0$. In view of Proposition A.6, the assertion follows by sending n to $+\infty$ in (34). □

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