THE SCHRÖDIGER EQUATION IN INHOMOGENOUS MEDIA: LONG TIME EVOLUTION OF OSCILLATION AND CONCENTRATION EFFECTS

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Abstract

The *quantum-classical correspondence principle* roughly states that quantum systems behave according to classical mechanics in the high-frequency limit. The Schrödinger flow on a Riemannian manifold (M, g) (that is, the unitary group $e^{it\Delta}$ generated by the Laplace-Beltrami operator Δ on $L^2(M)$) is one the of simplest quantum mechanical systems. Its classical counterpart is simply the geodesic flow ϕ_t on T^*M.

In order to relate the high-frequency properties of $e^{it\Delta}$ to the geodesic flow, one tries to determine the limiting behavior of position densities $|\psi_h(t,\cdot)|^2$ associated to solutions to the Schrödinger equation

$$i\partial_t \psi_h(t,x) + \frac{1}{2}\Delta\psi_h(t,x) = 0 \qquad (t,x) \in \mathbf{R} \times M, \tag{1}$$

issued from a sequence of highly oscillating initial data $\psi_h|_{t=0} = u_h$, as the characteristic length of the oscillations h tends to zero. One expects that in this limit the dynamics of $|\psi_h(t,\cdot)|^2$ are somehow related to the geodesic flow.

Usually, it is preferable to consider instead of $|\psi_h(t,\cdot)|^2$ the so-called *Wigner distribution* W^h of ψ_h, defined on the classical phase space T^*M, that describes simultaneously the distribution of the energy of $e^{it\Delta/2}u_h$ in physical and frequency space. It turns out that, in the limit $h \to 0$, the dynamics of W^h is related to the geodesic flow ϕ_t. However, precise results heavily depend on the time scales considered.

Two extreme cases that have been extensively studied are:

The semiclassical limit. This corresponds to taking *small times* t of the order of h. As $h \to 0$, the rescaled distributions $W^h(ht,\cdot)$ propagate following classical transport along the geodesic flow of M.
Eigenfunction limits. It consists in taking as initial data eigenfunctions of $-\Delta$. If (ψ_{λ_k}) is a sequence of normalized eigenfunctions $-\Delta\psi_{\lambda_k} = \lambda_k\psi_{\lambda_k}$ with $\lambda_k \to \infty$, then the corresponding solutions to the Schrödinger equation (1) are $e^{it\Delta/2}\psi_{\lambda_k} = e^{-it\lambda_k/2}\psi_{\lambda_k}$. Since $\left|e^{it\Delta/2}\psi_{\lambda_k}\right|^2 = |\psi_{\lambda_k}|^2$, the Wigner distributions are time-independent and the results obtained are *uniform in time.* Moreover, they strongly depend on fine dynamical properties of the geodesic flow.

In this talk, we shall study limits of W^h at intermediate time scales $t \sim 1$. We shall present rather precise results on their structure in the case that the geodesic flow is *completely integrable.* These limits inherit features from both the semiclassical and eigenfunction limits. In particular, they combine a classical averaging process together with a careful analysis of the concentration of (u_h) on the set of *resonant frequiencies.*