Integrali

Calcolare i seguenti integrali indefiniti:

1 $\displaystyle\int \frac{\ln x}{x(1+\ln x)}\, dx$

2 $\displaystyle\int \frac{\sqrt{x}+1}{x^{3/2}-1}\, dx$

3 $\displaystyle\int \frac{\mathrm{e}^x+2}{\mathrm{e}^x(\mathrm{e}^x+1)}\, dx$

4 $\displaystyle\int x^2 \operatorname{arcsen} x\, dx$

5 $\displaystyle\int \operatorname{arcsen} \sqrt[3]{2x-1}\, dx$

6 $\displaystyle\int \operatorname{arctg} \frac{2}{x}\, dx$

7 $\displaystyle\int \frac{\operatorname{sen}(2x)-\cos x}{\operatorname{sen} x + 9 \operatorname{sen}^3 x}\, dx$

8 $\displaystyle\int \operatorname{arctg} \frac{x}{x^2-2}\, dx$

9 $\displaystyle\int e^{2x} \ln(e^{2x}-2e^x+2)\, dx$

10 $\displaystyle\int \frac{dx}{(1+x)\sqrt{|x^2-1|}}$

11 $\displaystyle\int \frac{2\cos x + 6 - 3\sin^2 x}{(2\cos x + \sin x + 2)\, \sin^2 x}\, dx$

12 $\displaystyle\int \frac{\cos x \sin x}{(1+\cos^2 x)(1+\cos x)}\, dx$

13 $\displaystyle\int \frac{1}{\tan x - \sin x}\, dx$

14 $\displaystyle\int \sqrt{\frac{3-x}{x}}\, \frac{dx}{1+x}$

15 $\displaystyle\int x^3 \sqrt{x^2-4}\, dx$

16 $\displaystyle\int \frac{1}{x\sqrt{2x-1}}\, dx$

17 $\displaystyle\int \frac{1-\sqrt[3]{1+x}}{\sqrt{1+x}+\sqrt[3]{1+x}}\, dx$

18 $\displaystyle\int \frac{e^x(e^x-1)}{e^{2x}-1}\, dx$

Calcolare i seguenti integrali definiti:

19 $\displaystyle\int_{e^{-1/3}}^{\sqrt{e}} \frac{dx}{x(2\ln x|\ln x| - \ln x - 1)}$

20 $\displaystyle\int_0^{\pi/2} \frac{\cos t}{(1+\sin t)(2+\sin t)}\, dt$

21 $\displaystyle\int_0^1 \frac{\mathrm{e}^t+\mathrm{e}^{-t}}{\mathrm{e}^t+1}\, dt$

22 $\displaystyle\int_{\pi/4}^{\pi/2} \sin 2t\, \ln(\sin t)\, dt$

23 $\displaystyle\int_0^1 \frac{x^2}{4-x^2}\, dx$

24 $\displaystyle\int_{-1}^1 \frac{x^2}{(2+x)(2-|x|)}\, dx$

25 $\displaystyle\int_1^3 \ln \frac{20x}{2x^2+3}\, dx$

26 $\displaystyle\int_{-2}^{+2} x^2 \arctan \sqrt{\frac{2+x}{2-x}}\, dx$

27 $\displaystyle\int_{\pi/2}^{\pi} \frac{\cos x + \sqrt{\sin x - \sin^3 x}}{1+\sqrt{\sin x}}\, dx$

28 $\displaystyle\int_0^{\pi/2} \frac{\cos t}{(1+\sin t)(2+\sin t)}\, dt$

29 $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin t}\, dt$

30 $\displaystyle\int_{3\cosh 1}^{3} \frac{1}{\sqrt{t^2-9}}\, dt$

31 $\displaystyle\int_1^3 \sqrt{3+|x^2-4|}\, dx$

32 $\displaystyle\int_3^5 \frac{\log(x^2-4)}{x^3}\, dx$

33 $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2}{3}\pi} \frac{dx}{\sin^2 x + \sin x}$

34 $\displaystyle\int_0^{\sqrt{3}} \frac{x^3}{\sqrt{4-x^2}}\,dx$

35 (∗) Calcolare, se esiste, $\displaystyle\lim_{x\to 0} \frac{1}{x^2}\int_0^{x^{2/3}} \frac{\operatorname{sen} t}{t}\,dt.$

1 Risposte ad alcuni esercizi

1: $\ln\left(\dfrac{x}{|1+\ln x|}\right)+c\,.$

2:

$$\frac{4}{3}\ln|\sqrt{x}-1|+\frac{1}{3}\ln(x+\sqrt{x}+1)+\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sqrt{x}+1}{\sqrt{3}}\right)+c\,.$$

3: $\ln(1+e^{-x})-2e^{-x}+c\,.$

4: Integrando per parti si ha

$$\int x^2\operatorname{arcsen} x\,dx=\frac{1}{3}\left(x^3\operatorname{arcsen} x-\int\frac{x^3}{\sqrt{1-x^2}}\,dx\right).$$

Vediamo l'ultimo integrale: ponendo $x^2=t$, da cui $2x\,dx=dt$, si ottiene

$$\int\frac{x^3}{\sqrt{1-x^2}}\,dx=\frac{1}{2}\int\frac{t}{\sqrt{1-t}}\,dt\,,$$

da cui, con la sostituzione $\sqrt{1-t}=s$ (da cui $t=1-s^2$, $dt=-2s\,ds$),

$$\int\frac{x^3}{\sqrt{1-x^2}}\,dx=\int(s^2-1)\,ds=\frac{s^3}{3}-s+c\,.$$

In definitiva

$$\begin{aligned}\int x^2\operatorname{arcsen} x\,dx&=\frac{1}{3}\left(x^3\operatorname{arcsen} x+\sqrt{1-x^2}-\frac{1}{3}\big(1-x^2\big)^{3/2}\right)\\&=\frac{1}{9}\left(3x^3\operatorname{arcsen} x+(x^2+2)\sqrt{1-x^2}\right).\end{aligned}$$

In alternativa si poteva porre $x=\operatorname{sen} t$, con $-\dfrac{\pi}{2}<t<\dfrac{\pi}{2}$, da cui $\sqrt{1-x^2}=\cos t$, $dx=\cos t\,dt$, e quindi

$$\begin{aligned}\int\frac{x^3}{\sqrt{1-x^2}}\,dx=\int\operatorname{sen}^3 t\,dt&=\int\operatorname{sen} t\,dt-\int\cos^2 t\,\operatorname{sen} t\,dt\\&=-\cos t+\frac{\cos^3 t}{3}+c\,,\end{aligned}$$

dove $\cos t=\sqrt{1-x^2}$.

5: con la sostituzione $\sqrt[3]{2x-1}=t$ (da cui $x=\dfrac{t^3+1}{2}$, $dx=\dfrac{3}{2}t^2\,dt$), si ottiene

$$\begin{aligned}\int\operatorname{arcsen}\sqrt[3]{2x-1}\,dx&=\frac{3}{2}\int t^2\operatorname{arcsen} t\,dt\\&=[\text{per quanto visto prima}]\\&=\frac{1}{6}\left(3t^3\operatorname{arcsen} t+\big(t^2+2\big)\sqrt{1-t^2}\right)\\&=\frac{1}{6}\Big((6x-3)\operatorname{arcsen}\sqrt[3]{2x-1}+\\&\qquad+\big((2x-1)^{2/3}+2\big)\sqrt{1-(2x-1)^{2/3}}\Big).\end{aligned}$$

6: Integrando per parti, si ha

$$\begin{aligned}&\int\operatorname{arctg}\frac{2}{x}\,dx=x\operatorname{arctg}\frac{2}{x}+\int\frac{2x}{x^2+4}\,dx=\\&=x\operatorname{arctg}\frac{2}{x}+\int\frac{d(x^2+4)}{x^2+4}=x\operatorname{arctg}\frac{2}{x}+\ln(x^2+4)+c\,.\end{aligned}$$

7:

$$\begin{aligned}&\int\frac{\operatorname{sen}(2x)-\cos x}{\operatorname{sen} x+9\operatorname{sen}^3 x}\,dx=\int\frac{2\operatorname{sen} x-1}{\operatorname{sen} x+9\operatorname{sen}^3 x}\cos x\,dx=\\&\Big[\text{sost. } \operatorname{sen} x=t\,,\ \cos x\,dx=dt\Big]=\int\frac{2t-1}{t\,(1+9t^2)}\,dt=\\&\Big[\text{scomponendo in fratti semplici}\Big]\\&=\int\left(\frac{A}{t}+\frac{18Bt}{1+9t^2}+\frac{C}{1+9t^2}\right)dt=\\&=A\ln|t|+B\ln(1+9t^2)+\frac{C}{3}\operatorname{arctg}(3t)+c\,.\end{aligned}$$

Poiché si trova subito $A=-1$, $B=\dfrac{1}{2}$, $C=2$, si ottiene:

$$\begin{aligned}&\int\frac{\operatorname{sen}(2x)-\cos x}{\operatorname{sen} x+9\operatorname{sen}^3 x}\,dx\\&=-\ln|\operatorname{sen} x|+\frac{1}{2}\ln(1+9\operatorname{sen}^2 x)+\frac{2}{3}\operatorname{arctg}(3\operatorname{sen} x)+c\,.\end{aligned}$$

9: $\dfrac{e^{2x}}{2}\big(\ln(e^{2x}-2e^x+2)-1\big)-e^x+2\operatorname{arctg}(e^x-1)+c\,;$

26: Calcoliamo prima l'integrale indefinito:

$$\int f(x)\,dx=\int\ln(20x)\,dx-\int\ln(2x^2+3)\,dx\,.$$

I due integrali si calcolano per parti:

$$\int\ln(20x)\,dx=x\,\ln(20x)-\int\frac{x}{x}\,dx=x\,\big(\ln(20x)-1\big)+c\,;$$

$$\begin{aligned}
\int \ln(2x^2+3)\,dx &= x\,\ln(2x^2+3) - \int \frac{4x^2}{2x^2+3}\,dx \\
&= x\,\ln(2x^2+3) - \int \frac{(4x^2+6)-6}{2x^2+3}\,dx \\
&= x\,\ln(2x^2+3) - 2\int dx + 6\int \frac{dx}{2x^2+3} \\
&= x\,\ln(2x^2+3) - 2x + 2\int \frac{dx}{\left(\sqrt{\frac{2}{3}}\,x\right)^2+1} \\
&= x\,\ln(2x^2+3) - 2x + \sqrt{6}\,\mathrm{arctg}\left(\sqrt{\frac{2}{3}}\,x\right) + c\,.
\end{aligned}$$

Quindi

$$\int f(x)\,dx = x\,\ln\frac{20x}{2x^2+3} + x - \sqrt{6}\,\mathrm{arctg}\left(\sqrt{\frac{2}{3}}\,x\right) + c\,.$$

L'integrale vale pertanto:

$$\begin{aligned}
\int_1^3 f(x)\,dx &= \left[x\,\ln\frac{20x}{2x^2+3} + x - \sqrt{6}\,\mathrm{arctg}\left(\sqrt{\frac{2}{3}}\,x\right)\right]_1^3 \\
&= \ln\frac{2000}{343} + 2 + \sqrt{6}\left(\mathrm{arctg}\left(\sqrt{\frac{2}{3}}\right) - \mathrm{arctg}\,\sqrt{6}\right).
\end{aligned}$$