Global existence for parabolic problems involving the p-Laplacian and a critical gradient term

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Abstract.

We study existence and regularity of solutions for nonlinear parabolic problems whose model is

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \beta(u)|\nabla u|^p + f & \text{in } \Omega\times]0,\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,\infty[\\ u(x,0) = u_0, & \text{in } \Omega \end{cases} \tag{1}$$

where $p > 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set; as far as the function β is concerned, we make no assumption on its sign; instead, we consider three possibilities of growth for β, which essentially are: (1) constant, (2) polynomial and (3) exponential. In each case, we assume appropriate hypotheses on the data f and u_0, depending on the growth of β, and prove that a solution u exists such that an exponential function of u belongs to the natural Sobolev "energy" space. Since the solutions may well be unbounded, one cannot use sub/supersolution methods. However we show that, under slightly stronger assumptions on the data, the solution that we find is bounded. Our existence results, in the cases (2) and (3) above, rely on new logarithmic Sobolev inequalities.

1 Introduction.

In this article we study different classes of nonlinear parabolic problems with lower order terms depending on the gradient of the solution, with natural growth. All classes are modeled by problem (1) above, where Ω is a bounded open set in $\mathbb{R}^N$ and $p > 1$. We will focus on three different classes, depending on the growth of the real function $\beta(s)$, which will be supposed to be continuous. Basically, the growths on $\beta(s)$ studied in this article are: (1) constant (or, more generally, bounded), (2) polynomial, and (3) exponential. Different assumptions on f and u_0 will be needed in order to get solutions in these three situations. More general growths, for instance $\beta(s) \sim \exp(\exp(s))$, could be studied with similar methods, but for simplicity we confine ourselves to the three cases above. These higher growths have been studied in [13], when $p = 2$.

We point out, however, that $\beta(s)$ could be "well signed" (which means that $\beta(s)$ has the opposite sign as s), and in this case its growth has no influence as far as existence is concerned (see [24], [14]). To be more precise, the difference among the three cases described above is a consequence of the growth of the part of the function $\beta(s)$ whose sign is "bad". If we denote $\beta_1 = -(\beta^-)\chi_{[0,+\infty[}+(\beta^+)\chi_{]-\infty,0]}$ and $\beta_2 = (\beta^+)\chi_{[0,+\infty[}-(\beta^-)\chi_{]-\infty,0]}$, and decompose the function as

$$\beta(s) = \beta_1(s) + \beta_2(s)\,,$$

then these two terms do not play the same role: $\beta_1(s)$ is the part of $\beta(s)$ where its sign is "good", while $\beta_2(s)$ has a "bad" sign. Therefore, on the one hand, we will not impose any limitation on the growth of $\beta_1(s)$, but, on the other hand, we have to impose certain limitations to that of $\beta_2(s)$ in order to obtain a solution of the problem. Let us illustrate this point with an example: if $\beta(s) = -s^3 + s^2$, then $\beta_2 = \beta\chi_{[0,1]}$ and we are in the situation of β_2 bounded; when the function is the opposite, that is $\beta(s) = s^3 - s^2$, we have to consider to be in the polynomial case.

We point out that the growth of the function β_2 induces some summability assumptions on f and u_0 to obtain a weak solution. In other words, given β_2, data must belong to suitable Orlicz spaces related to β_2 in order to prove an existence result. It seems intuitive that the faster β_2 grows, the smaller the spaces to which the data belong (i.e., the stronger the assumptions on the data) have to be, and vice versa. This actually is what we find (compare the assumptions in Theorems 2.1, 2.2 and 2.3).

The main novelty of the present paper (as the previous ones [12] and [13] that deal with the particular case $p = 2$) with respect to some other related papers, as in [17] and [9], is that the function β_2, which describes the growth of the reaction term with respect to u, can be unbounded. Moreover, we get distributional solutions under this unboundedness hypothesis, without assuming existence of sub and super-solutions as in the previous known literature ([5], [25], [18]), so that we may handle more general data. The case of bounded β_2 is considered here for the sake of completeness: the existence and regularity results are comparable with those obtained in [17] and [9]. However, it is worth noting that we give here a detailed definition of weak solution, which allows to study further regularity properties (for instance, boundedness) when one makes stronger assumptions on the data. Furthermore, we also provide a precise meaning to the initial datum.

In order to explain the existence and regularity results, and the assumptions on the data, let us consider problem (1) with $p = 2$ and $f, u_0 \geq 0$, for a general continuous function $\beta : [0, +\infty[\to [0, +\infty[$ satisfying $\lim_{s\to+\infty} \beta(s) \in]0, +\infty]$. If one performs the Cole-Hopf change of variable

$$v = \Psi(u) = \int_0^u \exp\Big(\int_0^s \beta(\sigma)\,d\sigma\Big)\,ds,$$

one obtains the semilinear problem

$$\begin{cases} v_t - \Delta v = f(x,t)g(v), & \text{in} \quad \Omega\times]0,\infty[; \\ v(x,t) = 0, & \text{on} \quad \partial\Omega\times]0,\infty[; \\ v(x,0) = v_0(x) := \Psi(u_0), & \text{in} \quad \Omega\,, \end{cases}$$

where $g(v) = \exp\Big(\int_0^u \beta(s)\,ds\Big) = \Psi'(\Psi^{-1}(v))$ has a linear or slightly superlinear growth in the sense that

$$\lim_{s\to+\infty} \frac{g(s)}{s} = \lim_{s\to+\infty} \beta(s) \in]0, +\infty]$$

and

$$\int^{+\infty} \frac{1}{g(s)}\,ds = +\infty,$$

as can easily be checked by a change of variable. For instance, if we start from equation

$$u_t - \Delta u = (e^u + 1)|\nabla u|^2 + f$$

and apply the change of variable $v = \exp\big(\exp(s) - 1\big) - 1$, then we obtain the equation

$$v_t - \Delta v = f(v+1)\big(\log(v+1) + 1\big).$$

Let us point out that also for $p \neq 2$, in the model case (1), one can perform (as in the case $p = 2$) a simple change of unknown which makes the gradient term disappear, leading to a semilinear problem. However, in this process, the parabolic operator changes. For instance, if we start from the equation

$$u_t - \Delta_p u = (e^{u/(p-1)} + 1)\,|\nabla u|^p + f(x,t)\,,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and we set

$$v = (p-1)\left(e^{e^{u/(p-1)}-1} - 1\right),$$

then we obtain the following equation for v:

$$(\varphi(v))^{p-2}\, v_t - \Delta_p v = f(x,t)\,(\varphi(v))^{p-1}\,, \tag{2}$$

where

$$\varphi(v) = \Big(1 + \frac{v}{p-1}\Big)\Big(1 + \log\Big(1 + \frac{v}{p-1}\Big)\Big).$$

The presence of the logarithm in the definition of $\varphi(v)$ causes some extra difficulties that do not occur when $p = 2$. For instance, to obtain a priori estimates in (2) a standard logarithmic Sobolev inequality like

$$\int_\Omega |w|^p \log|w|^p \le \varepsilon \int_\Omega |\nabla w|^p + \Big(\int_\Omega |w|^p\Big) \log\Big(\int_\Omega |w|^p\Big) + C(\varepsilon)\Big(\int_\Omega |w|^p\Big)$$

is not sufficient to applu Gronwall's lemma, so new inequalities are required (see Proposition 5.1 and Proposition 6.1 below).

When the operator has a more general structure, an explicit change of variable is not possible. Nevertheless, the use of convenient exponential test functions allows us to absorb the gradient term by the principal part, obtaining the same estimates one would get with the change of unknown in the model case. More precisely, let us suppose for simplicity $f \geq 0$, $\beta_1 \equiv 0$ and $\beta = \beta_2$ and let us consider the following functions

$$\gamma(s) = \int_0^s \beta(\sigma)d\sigma\,, \qquad \Psi(s) = \int_0^s e^{\frac{\gamma(\sigma)}{p-1}}\,d\sigma\,.$$

Working on a convenient sequence of approximating problems with solutions u_n and using the exponential test function $\Psi(u_n)e^{\gamma(u_n)}$ in each of them, we are able to cancel the gradient term with one of the integral terms given by the diffusion operator, but we then we have to handle the term $\iint_{Q_T} f\Psi(u_n)e^{\gamma(u_n)}$. Hence, we have to investigate the growth of $e^{\gamma(u_n)}$ with respect to $\Psi(u_n)$. We can see that the case $\beta(s)$ bounded gives $e^{\gamma(s)} \sim (\Psi(s))^{p-1}$ when s goes to $+\infty$, while $\beta(s) = s^\lambda$ ($\lambda > 0$) gives $e^{\gamma(s)} \sim \big[\Psi(s)(\log\Psi(s))^\alpha\big]^{p-1}$ $(0 < \alpha < 1)$; finally, $\beta(s) = e^s$ gives $e^{\gamma(s)} \sim \big[\Psi(s)\log\Psi(s)\big]^{p-1}$. Moreover, we have to prove similar estimates as far as the integral term stemming from the time derivative is concerned. Due to the behaviour of $e^{\gamma(s)}$ at $+\infty$, the case of unbounded growth for β requires the use of logarithmic Sobolev inequalities (similar to those in [19], [2] and [8]) in order to apply a suitable nonlinear version of Gronwall's lemma for differential inequalities. Here we prove two logarithmic Sobolev inequalities (see Sections 5 and 6 below), that lead quite naturally to the different hypotheses on the data $f(x,t)$ and $u_0(x)$.

Let us point out that the case $p \neq 2$ induces also some technical difficulties in the proof of boundedness of solutions under the classical additional hypotheses $f \in L^r(0,T; L^q(\Omega))$, $q > \frac{N}{p}\, r'$, and u_0 bounded (see Section 9). Indeed, in the case $p = 2$, we can quite easily achieve known estimates involving the measure of the level set of the solutions and then apply the method of [21] (see [13] for the details). On the contrary, when $p \neq 2$, this is not straightforward and the cases $1 < p < 2$ and $p > 2$ have to be treated separately (see Section 9 below).

The plan of the paper is the following. In Section 2 we will give the precise assumptions and state the main results. In Section 3 we will define the approximate problems, state the a priori estimates that we want to obtain, and recall some tools to prove them. In the Sections which follow we will prove the a priori estimates under the assumption that $\beta_2(s)$ is bounded (Section 4), or grows like a power (Section 5), or has an exponential growth (Section 6), respectively. In each Section, suitable hypotheses on the data $f(x,t)$ and $u_0(x)$ will be made. In Section 7 we will prove strong convergence of $\{u_n\}$ and their gradients $\{\nabla u_n\}$. Section 8 is devoted to conclude the proof of the main existence results. Finally, Section 9 will be devoted to L^∞ estimates under stronger assumptions on the data.

2 Assumptions and statement of the main results.

Let Ω be a bounded, open set in $\mathbb{R}^N$. We will denote by Q the cylinder $\Omega\times]0,\infty[$, while, for $t > 0$, we will denote by Q_t the cylinder $\Omega\times]0,t[$. Sometimes we will also use the notation $Q_{\tau,t}$ to designate the cylinder $\Omega\times]\tau,t[$.

The symbols $L^q(\Omega)$, $L^r(0,T;L^q(\Omega))$, will denote the usual Lebesgue spaces, see for instance [16]. We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space of measurable functions having weak derivative in $L^q(\Omega)$ and zero trace on $\partial\Omega$. If $T > 0$, the spaces $L^r(0,T;W_0^{1,q}(\Omega))$ have obvious meaning, see again [16].

Moreover, we will denote by $W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$. Here q' is Hölder's conjugate exponent of $q > 1$, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$. Finally, if $1 \leq q < N$, we will denote by $q^* = Nq/(N-q)$ its Sobolev conjugate exponent.

For the sake of brevity, instead of writing "$u(x,t) \in L^r(0,\tau;W_0^{1,q}(\Omega))$ for every $\tau > 0$", we shall write $u(x,t) \in L^r_{\mathrm{loc}}([0,\infty);W_0^{1,q}(\Omega))$. Similarly, we shall write $u \in L^q_{\mathrm{loc}}(\overline{Q})$ instead of $u \in L^q(Q_\tau)$ for every $\tau > 0$.

The general problem we are going to study is

$$(P) \qquad \begin{cases} u_t - \operatorname{div}(a(x,t,u,\nabla u)) = b(x,t,u,\nabla u) + f, & \text{in } Q \\ u(x,0) = 0 & \text{on } \partial\Omega\times]0,\infty[\\ u(x,0) = u_0 & \text{in } \Omega \end{cases}$$

when the following assumptions are made.

- The function $a : \Omega\times]0,\infty[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function such that:

A1) There exists a positive constant Λ_1 such that

$$\Lambda_1|\xi|^p \leq a(x,t,s,\xi)\cdot\xi$$

for almost every $(x,t) \in Q$, for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$.

A2) For all $\xi,\eta \in \mathbb{R}^N$, $\xi \neq \eta$, for all $s \in \mathbb{R}$ and almost all (x,t) we have

$$[a(x,t,s,\xi) - a(x,t,s,\eta)]\cdot(\xi-\eta) > 0\,.$$

A3) There exists $\Lambda_2 > 0$ such that, for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, and almost all $(x,t) \in Q$ one has

$$|a(x,t,s,\xi)| \leq \Lambda_2\,|\xi|^{p-1}\,,$$

- The function $b : \Omega\times]0,\infty[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function for which

B) There exist continuous non negative functions $\beta_1,\beta_2 : \mathbb{R}\to\mathbb{R}$ such that

$$-\beta_1(s)|\xi|^p \leq b(x,t,s,\xi)\,\mathrm{sign}\ s \leq \beta_2(s)|\xi|^p$$

for all $(x,t,s,\xi) \in Q \times \mathbb{R} \times \mathbb{R}^N$. Note that this implies

$$|b(x,t,s,\xi)| \leq \max\{\beta_1(s)\,,\,\beta_2(s)\}\,|\xi|^p\,.$$

No other assumptions will be made on $\beta_1(s)$, while special emphasis will be placed on the assumptions on $\beta_2(s)$. Three cases will be studied in detail, as they will require different assumptions on the data $f(x,t)$ and $u_0(x)$, and will provide different regularity for the solution $u(x,t)$ of problem (P). More precisely we will focus on the following three important cases:

C1)
$$\beta_2(s) \equiv M > 0\,;$$

C2)
$$\beta_2(s) = M(|s|^\lambda + 1)\,, \qquad \lambda > 0;$$

C3)
$$\beta_2(s) = M\,e^{\delta|s|}\,.$$

We define the following auxiliary functions:

$$\gamma(s) = \frac{1}{\Lambda_1}\left|\int_0^s \beta_2(\sigma)d\sigma\right|, \qquad \Psi(s) = \int_0^s e^{\frac{\gamma(\sigma)}{p-1}}\,d\sigma\,, \qquad \Phi(s) = \int_0^s \Psi(\sigma)e^{\gamma(\sigma)}\,d\sigma\,. \tag{3}$$

As far as the data are concerned, we require the following assumption on the initial datum u_0:

D)
$$\int_\Omega \Phi(u_0)\,dx < \infty\,,$$

With respect to the source datum $f(x,t)$, the assumptions will be stated in each case separately, depending on the assumptions on β_2.

Next, we will explain which sense we give to a weak solution of our problems (P).

Definition 2.1 *By a weak solution to problem* (P) *we mean a measurable function* $u : Q \to \mathbb{R}$ *satisfying the following conditions:*

(1) $\Psi(u) \in L^p_{\mathrm{loc}}([0,\infty[\,;\,W_0^{1,p}(\Omega))$.

(2) $\Phi(u) \in C([0,\infty[\,;\,L^1(\Omega))$.

(3) $b(x,t,u,\nabla u)\,,\ b(x,t,u,\nabla u)\,e^{\gamma(u)}\,\Psi(u)$ *and* $f\,e^{\gamma(u)}\,\Psi(u)$ *belong to* $L^1_{\mathrm{loc}}(\overline{Q})$.

(4) *For every* $\tau > 0$ *and every* $v \in L^p(0,\tau\,;\,W_0^{1,p}(\Omega)) \cap L^\infty(Q_\tau)$ *such that its distributional derivative with respect to time* v_t *belongs to* $L^{p'}(0,\tau\,;\,W^{-1,p'}(\Omega))$*, the following equality holds:*

$$\int_\Omega u(\tau)\,v(\tau)\,dx - \int_\Omega u_0\,v(0)\,dx - \int_0^\tau \langle v_t(t), u(t)\rangle\,dt + \iint_{Q_\tau} a(x,t,u,\nabla u)\cdot\nabla v\,dx\,dt$$
$$= \iint_{Q_\tau} b(x,t,u,\nabla u)\,v\,dx\,dt + \iint_{Q_\tau} f\,v\,dx\,dt\,. \tag{4}$$

(5) *For every* $\tau > 0$ *and every locally Lipschitz continuous function* $h : \mathbb{R} \to \mathbb{R}$ *such that* $h(0) = 0$, $|h'(s)| \leq M_1\,\Psi'(s)^p$ *and* $|h(s)| \leq M_1(1 + e^{\gamma(s)}|\Psi(s)|)$ *the following equality holds:*

$$\int_\Omega H(u(\tau))\,dx - \int_\Omega H(u_0)\,dx + \iint_{Q_\tau} h'(u)\,a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt$$
$$= \iint_{Q_\tau} b(x,t,u,\nabla u)\,h(u)\,dx\,dt + \iint_{Q_\tau} f\,h(u)\,dx\,dt\,, \tag{5}$$

where $H(s) = \int_0^s h(\sigma)\,d\sigma$.

Remark 2.1 We point out that every term in (4) and (5) is well defined; this is a consequence of the following observations.

a) By condition **(1)**, and since $|u| \le |\Psi(u)|$ and $|\nabla u| \le |\nabla \Psi(u)|$, we have

$$u \in L^p_{\rm loc}([0,\infty[\,;\, W_0^{1,p}(\Omega))\,.$$

b) Condition **(2)** implies, on the one hand, that

$$u \in C([0,\infty[\,;\, L^\rho(\Omega))\,, \qquad \text{for all } 1 \le \rho < \infty\,,$$

since the function Φ is, at least, of exponential type. On the other hand, it follows from the connections between Φ and Ψ (see Lemmata 4.1, 5.1, 6.1) that

$$\Psi(u) \in C([0,\infty[\,;\, L^q(\Omega))\,, \qquad \text{for all } 1 \le q < p\,.$$

The value $q = p$ can always be attained when $p \ge 2$, and for any p under hypothesis **C1)**.

c) It is well known (see, for instance, [23]) that if $v \in L^p_{\rm loc}([0,\infty[\,;\, W_0^{1,p}(\Omega)) \cap L^\infty_{\rm loc}(\overline{Q})$ is such that $v_t \in L^{p'}_{\rm loc}([0,\infty[\,;\, W^{-1,p'}(\Omega))$, then $v \in C([0,\infty[\,;\, L^\rho(\Omega))$ for every $1 \le \rho < \infty$. Moreover,

$$v(t) \in L^\infty(\Omega) \qquad \text{for all } t \ge 0\,,$$

since one can take a sequence $t_n \to t$ such that $\|v(t_n)\|_\infty \le c$ and $v(t_n) \to v(t)$ a.e. in Ω.

d) It follows from $|h(s)| \le M_1(1 + e^{\gamma(s)}|\Psi(s)|)$ that

$$|H(s)| \le \int_0^{|s|} |h(\sigma)|\,d\sigma \le M_1(|s| + \Phi(s)) \le c\,(\Phi(s)+1)\,.$$

Therefore, assumption **(2)** in Definition 2.1 implies $H(u) \in C([0,\infty[; L^1(\Omega))$, therefore $H(u(\tau))$ in (5) has a meaning.

Remark 2.2 It is clear that a weak solution of problem (P) is also a solution in the sense of distributions.

Remark 2.3 In the recent paper [1], the following equation is studied

$$u_t - \Delta u = \beta(u)|\nabla u|^2 + f(x,t)\,,$$

with Cauchy-Dirichlet boundary conditions. The authors prove that there exist infinitely many distributional solutions $u \in L^2_{\rm loc}([0,\infty[; W_0^{1,2}(\Omega))$ to this problem, which are related (via a change of variable) to semilinear problems with measure data. However, only one of these solutions is a weak solution in the sense of Definition 2.1. Therefore, in this special case, the definition of weak solutions ensures uniqueness.

The main results of the paper are the following

Theorem 2.1 *Assume that* **A1)**, **A2)**, **A3)**, **B)** *and* **D)** *hold true, with*

C1) $$\beta_2(s) \equiv M > 0\,.$$

If $f(x,t)$ satisfies

$$f \in L^r_{\rm loc}([0,\infty[\,;\, L^q(\Omega))\,, \qquad \textit{with } \; q \ge \frac{N}{p} r' \; \textit{ and } \; 1 < r < \infty\,, \tag{6}$$

then there exists a weak solution u for problem (P) *such that*

$$\Psi(u) \in C([0,\infty[\,;\, L^p(\Omega))\,.$$

Remark 2.4 One can check, by adapting the proof, that the result of the previous Theorem also holds true in the case where the datum f satisfies a limit case in (6), i.e. $f \in L^\infty_{\rm loc}([0,\infty[; L^{N/p}(\Omega))$, provided the following condition is verified: for every $T, \varepsilon > 0$ there exist two functions $f_1^{(T,\varepsilon)}(x,t)$ and $f_2^{(T,\varepsilon)}(x,t)$ such that $f = f_1^{(T,\varepsilon)} + f_2^{(T,\varepsilon)}$, $f_1^{(T,\varepsilon)} \in L^\infty(Q_T)$ and $\|f_2^{(T,\varepsilon)}\|_{L^\infty(0,T;L^{N/p}(\Omega))} \leq \varepsilon$. This is true, for instance, if $f(x,t) = f(x) \in L^{N/p}(\Omega)$ or if $f \in C([0,\infty[; L^{N/p}(\Omega))$.

Theorem 2.2 *Assume that* $1 < p < N$ *and that* **A1)**, **A2)**, **A3)**, **B)** *and* **D)** *hold true, with*

C2) $$\beta_2(s) = M(|s|^\lambda + 1)\,, \qquad \lambda, M > 0\,.$$

If $f(x,t)$ *satisfies*

$$\int_0^T \|f(t)\|_q^r \left(\log^* \|f(t)\|_q\right)^{\frac{\lambda}{\lambda+1}[r(p-1)-(p-2)]} dt < \infty\,,$$

$$\text{with } \; q \geq \frac{N}{p} \max\left\{ r'\,,\, 1 + (p-1)\lambda \right\} \; \text{ and } \; 1 < r < \infty\,, \tag{7}$$

for every $T > 0$*, then there exists a weak solution* u *for problem* (P) *such that*

$$\Psi(u) \in C([0,\infty[\,; L^p(\Omega))\,, \quad \text{if } p \geq 2\,,$$
$$\Psi(u) \in C([0,\infty[\,; L^\sigma(\Omega))\,, \quad \text{for every } \sigma < p, \quad \text{if } 1 < p < 2\,.$$

Theorem 2.3 *Assume that* **A1)**, **A2)**, **A3)**, **B)** *and* **D)** *hold true, with*

C3) $$\beta_2(s) = M\, e^{\delta|s|}\,, \qquad M, \delta > 0\,.$$

If $f(x,t)$ *satisfies*

$$\int_0^T \|f(t)\|_\varphi \left(\log^* \|f(t)\|_\varphi\right)\left(\log^*\left(\log^* \|f(t)\|_\varphi\right)\right) dt < \infty\,, \tag{8}$$

for every $T > 0$*, where* $\|.\|_\varphi$ *denotes the Orlicz norm (see Section 6 below for the definition) corresponding to an N-function* $\varphi(s) \sim \exp(\exp(s))$ *for* $s \to \infty$ *, then there exists a weak solution* u *for problem* (P) *such that*

$$\Psi(u) \in C([0,\infty[\,; L^p(\Omega))\,, \quad \text{if } p \geq 2\,,$$
$$\Psi(u) \in C([0,\infty[\,; L^\sigma(\Omega))\,, \quad \text{for every } \sigma < p, \quad \text{if } 1 < p < 2\,.$$

We explicitly observe that the functions Φ and Ψ which appear in the statements above depend on β_2, therefore they change from theorem to theorem.

If we assume a slightly stronger hypothesis on the data i.e.

D′) $$u_0 \in L^\infty(\Omega)\,,$$

F) $$f \in L^r(0,T; L^q(\Omega))\,, \qquad \frac{1}{r} + \frac{N}{pq} < 1\,,$$

then we will show that every weak solution of problem (P) is bounded. We point out that **F)** is the same assumption used in [3] to prove the boundedness of solutions. We will state and prove the statement in the most general setting, that is, under the assumption **C3)** on the first order term.

Theorem 2.4 *If* **A1)**, **A2)**, **A3)**, **B)**, **C3)**, **D′)** *and* **F)** *hold true. Then every weak solution of problem* (P) *(in the sense of Definition 2.1) is essentially bounded in* Q_T*, for every* $T > 0$.

For $k>0$, we will denote by T_k the usual truncation at level $\pm k$, and by G_k its complement, i.e.

$$T_k s=\max\{-k,\min\{k,s\}\}\,,\qquad G_k s=s-T_k s\,. \tag{9}$$

Finally, we will sometimes write $a(u,\nabla u)$ and $b(u,\nabla u)$ instead of $a(x,t,u(x,t),\nabla u(x,t))$ and $b(x,t,u(x,t),\nabla u(x,t))$, respectively. Similarly, we will sometimes omit writing dx and dt in the integrals, when no confusion may arise.

3 Approximate problems

For $n\in\mathbb{N}$, let us consider the problem

$$\begin{cases} (u_n)_t-\mathrm{div}a(x,t,u_n,\nabla u_n)=T_n b(x,t,u_n,\nabla u_n)+T_n f & \text{in } Q\\[2mm] u_n(x,t)=0 & \text{on } \partial\Omega\times]0,\infty[\\[2mm] u_n(x,0)=u_{0,n} & \text{in } \Omega\end{cases} \tag{10}$$

where $u_{0,n}$ belongs to $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ and satisfies

$$\Phi(u_{0,n})\to\Phi(u_0)\quad\text{strongly in } L^1(\Omega),\qquad \lim_{n\to\infty}\frac{1}{n}\|u_{0,n}\|_{W_0^{1,p}(\Omega)}=0\,. \tag{11}$$

This last requirement will be used in the proof of the convergence of the gradients (see Section 7).

Problem (10) admits at least one solution $u_n\in L^\infty_{\mathrm{loc}}(\overline{Q})\cap L^p_{\mathrm{loc}}([0,\infty[\,;\,W_0^{1,p}(\Omega))$ (see [23]).

We will fix an arbitrary time $T>0$, and we will look for a priori estimates in $Q_T=\Omega\times]0,T[$. We will prove, in each of the cases **C1)**, **C2)** and **C3)**, that, under the assumptions of Theorems 2.1, 2.2 and 2.3, respectively, the following estimates hold.

For every $T>0$, there exists a positive constant $C(T)$ such that

$$\int_\Omega \Phi(u_n(x,\tau))\,dx\le C(T)\qquad\text{for every }\ \tau\in[0,T] \tag{12}$$

$$\iint_{Q_T}|\nabla\Psi(u_n)|^p\,dx\,dt\le C(T)\,, \tag{13}$$

$$\iint_{Q_T}\big(|f(x,t)|+|T_n b(x,t,u_n,\nabla u_n)|\big)\,e^{\gamma(u_n)}\,|\Psi(u_n)|\,dx\,dt\le C(T) \tag{14}$$

$$\iint_{Q_T}\big|T_n b(x,t,u_n,\nabla u_n)\big|\,dx\,dt\le C(T) \tag{15}$$

for all $n\in\mathbb{N}$. Moreover,

$$\lim_{k\to+\infty}\iint_{Q_T\cap\{|u_n|>k\}}|T_n b(x,t,u_n,\nabla u_n)|\,dx\,dt=0\,,\qquad\text{uniformly in } n\in\mathbb{N}. \tag{16}$$

The following result (which is a variant of a result contained in [6]) will be used throughout the paper to obtain the main a priori estimates.

Proposition 3.1 *Assume that u_n is a bounded weak solution of* (10)*. If ψ is a locally Lipschitz-continuous and increasing function such that $\psi(0)=0$, then for a.e. $t>0$ one has*

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi(u_n(x,t))\,dx&+\Lambda_1\int_\Omega e^{\gamma(u_n(x,t))}\psi'(u_n(x,t))|\nabla u_n(x,t)|^p\,dx\\ &+\int_{\{u_n\,b(u_n,\nabla u_n)<0\}}|T_n b(x,t,u_n(x,t),\nabla u_n(x,t))|\,e^{\gamma(u_n(x,t))}\,|\psi(u_n(x,t))|\,dx\\ &\qquad\le\int_\Omega|f(x,t)|\,e^{\gamma(u_n(x,t))}|\psi(u_n(x,t))|\,dx\,,\end{aligned}$$

where $\gamma(s)$ *is defined by* (3)*, and* $\phi(s) = \int_0^s e^{\gamma(\sigma)}\psi(\sigma)\,d\sigma$*. Therefore*

$$\sup_{\tau\in[0,T]} \int_\Omega \phi(u_n(\tau))\,dx + \Lambda_1 \iint_{Q_T} e^{\gamma(u_n)}\psi'(u_n)|\nabla u_n|^p\,dx\,dt$$
$$+ \iint_{\{u_n\, b(u_n,\nabla u_n)<0\}} |T_n b(x,t,u_n,\nabla u_n)|\, e^{\gamma(u_n)}|\,\psi(u_n)|\,dx\,dt$$
$$\le 2\iint_{Q_T} |f|\,e^{\gamma(u_n)}|\psi(u_n)|\,dx\,dt + 2\int_\Omega \phi(u_{0,n})\,dx\,.$$

Proof: We multiply the equation by $e^{\gamma(u_n)}\psi(u_n)$. We obtain:

$$\frac{d}{dt}\int_\Omega \phi(u_n)\,dx + \int_\Omega |\nabla u_n|^p e^{\gamma(u_n)}|\psi(u_n)|\beta_2(u_n)\,dx + \Lambda_1\int_\Omega |\nabla u_n|^p e^{\gamma(u_n)}\psi'(u_n)\,dx$$
$$= \int_\Omega T_n b(u_n,\nabla u_n)\,e^{\gamma(u_n)}\psi(u_n)\,dx + \int_\Omega T_n f\,e^{\gamma(u_n)}\,\psi(u_n)\,dx\,,$$

and then we observe that, by assumption **B)**,

$$\int_\Omega T_n b(u_n,\nabla u_n)\,e^{\gamma(u_n)}\psi(u_n)\,dx$$
$$\le \int_{\{u_n\, b(u_n,\nabla u_n)\ge 0\}} \beta_2(u_n)|\nabla u_n|^p e^{\gamma(u_n)}|\psi(u_n)|\,dx - \int_{\{u_n\, b(u_n,\nabla u_n)<0\}} |T_n b(u_n,\nabla u_n)|\,e^{\gamma(u_n)}|\psi(u_n)|\,dx\,.$$

■

A simple numerical inequality that we will use in the proofs of the logarithmic Sobolev inequalities is the following (see [11] for the proof).

Lemma 3.1 *Let* $A\,:\,[0,\infty[\to[0,\infty[$ *be an increasing, continuous function satisfying the so-called* Δ_2*–condition: there exist positive constants* t_0 *and* K *such that*

$$A(2t) \le K\,A(t) \qquad \text{for every } t\ge t_0.$$

Then there exists a positive constant c *satisfying*

$$x\,A(\log^* y) \le c\Big(x\,A(\log^* x) + y\Big)\,,$$

for all $x,y>0$*.*

Let us finally recall the following interpolation results due to Gagliardo-Nirenberg (see [15]):

Lemma 3.2 *Let* $v(x)$ *be a function in* $W_0^{1,p}(\Omega)$*,* $p\ge 1$*. Then, for every* σ *satisfying*

$$\begin{cases} p\le\sigma\le p^*\,, & \text{if } p<N\,;\\ p\le\sigma<\infty\,, & \text{if } p=N\,;\\ p\le\sigma\le\infty\,, & \text{if } p>N\,;\end{cases} \tag{17}$$

one has

$$\|v\|_{L^\sigma(\Omega)} \le C(N,p)\,\|\nabla v\|^{\alpha}_{L^p(\Omega;\mathbb{R}^N)}\,\|v\|^{1-\alpha}_{L^p(\Omega)}\,,$$

where

$$\alpha = \frac{N\,(\sigma-p)}{\sigma\,p}\,.$$

Lemma 3.3 *Let $v(x,t)$ be a function such that*

$$v \in L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))\,,$$

with $p \geq 1$. Then $v \in L^\rho(0,T;L^\sigma(\Omega))$, for all σ as in (17) and for all $\rho \in [p,\infty]$ satisfying

$$\frac{N}{\sigma} + \frac{p}{\rho} = \frac{N}{p}\,,$$

and the following estimate holds

$$\int_0^T \|v(t)\|^\rho_{L^\sigma(\Omega)}\,dt \leq C(N,p)\,\|v\|^{\rho-p}_{L^\infty(0,T;L^p(\Omega))}\int_0^T \|\nabla v(t)\|^p_{L^p(\Omega;\mathbb{R}^N)}\,dt\,.$$

4 A priori estimates: the case of constant β_2.

In this Section we will assume that the function β_2 which appears in hypothesis **B)** is given by

C1) $$\beta_2(s) \equiv M\,.$$

In this case the following result holds.

Lemma 4.1 *Assume that* **C1)** *holds, and that the functions γ, Ψ and Φ are defined by* (3). *Then there exist positive constants M_1, M_2, M_3 such that*

$$\begin{gathered} M_1\,|\Psi(s)| \leq \exp\Big(\frac{\gamma(s)}{p-1}\Big) \leq M_2\,\big(1+|\Psi(s)|\big)\,, \\ \Phi(s) \geq M_1|\Psi(s)|^p - M_3\,, \\ \beta_2(s)e^{\gamma(s)}|\Psi(s)| \leq M_2\,\Psi'(s)^p\,, \end{gathered} \tag{18}$$

for every $s \in \mathbb{R}$.

Proof: It is a consequence of De l'Hôpital's rule. Indeed, by applying it several times, we get

$$\lim_{s\to+\infty}\frac{e^{\frac{\gamma(s)}{p-1}}}{1+|\Psi(s)|} = \frac{M}{\Lambda_1(p-1)}\,, \quad \lim_{s\to+\infty}\frac{\Phi(s)}{|\Psi(s)|^p} = \frac{1}{p}\Big(\frac{M}{\Lambda_1(p-1)}\Big)^{p-2}\,,$$
$$\lim_{s\to+\infty}\frac{\beta_2(s)e^{\gamma(s)}|\Psi(s)|}{\Psi'(s)^p} = \Lambda_1(p-1) \qquad \blacksquare$$

Proposition 4.1 *Assume that the same hypotheses of Theorem 2.1 hold true and let $\{u_n\}$ be a sequence of solutions of problems* (10). *Then the estimates* (12)–(16) *hold true. Moreover, one has*

$$\int_\Omega |\Psi(u_n(x,\tau))|^p\,dx \leq C(T) \qquad \textit{for every } \tau \in [0,T] \textit{ and for every } n \in \mathbb{N}\,. \tag{19}$$

Proof: Let us begin by observing that $pq = Nr'$ can be assumed in (6). Indeed, if $pq > Nr'$ occurs, the value r may be replaced by a smaller one satisfying the equality and then we may apply the usual inclusions between Lebesgue's spaces.

Applying Proposition 3.1, with $\psi = \Psi$, in the approximating problem, it yields

$$\frac{d}{dt}\int_\Omega \Phi(u_n)\,dx + \Lambda_1\int_\Omega e^{\gamma(u_n)}\Psi'(u_n)|\nabla u_n|^p\,dx + \int_{\{u_n\,b_n(u_n,\nabla u_n)<0\}} |T_n b(u_n,\nabla u_n)|\,e^{\gamma(u_n)}|\Psi(u_n)|\,dx$$
$$\leq \int_\Omega |f|e^{\gamma(u_n)}|\Psi(u_n)|\,dx. \tag{20}$$

Having in mind hypothesis **C1)** and the definition of $\Psi(s)$, and dropping a nonnegative term, this inequality becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u_n)\,dx + \int_\Omega |\nabla\Psi(u_n)|^p\,dx &\le c\int_\Omega |f|\,|\Psi(u_n)|\,(1+|\Psi(u_n)|)^{p-1}\,dx \\ &\le c\int_\Omega |f|\,dx + c\int_\Omega |f|\,|\Psi(u_n)|^p\,dx\,,\end{aligned}\tag{21}$$

by Young's inequality. The last term in (21) is estimated by the Hölder, Gagliardo-Nirenberg and Young inequalities. Indeed,

$$\begin{aligned}\int_\Omega |f|\,|\Psi(u_n)|^p\,dx &\le \|f(\cdot,t)\|_q\,\|\Psi(u_n(\cdot,t))\|^p_{pq'} \\ &\le c\,\|f(\cdot,t)\|_q\|\nabla\Psi(u_n(\cdot,t))\|_p^{N/q}\,\|\Psi(u_n(\cdot,t))\|_p^{(pq-N)/q} \\ &\le \varepsilon\|\nabla\Psi(u_n(\cdot,t))\|_p^p + c(\varepsilon)\,\|f(\cdot,t)\|_q^r\,\|\Psi(u_n(\cdot,t))\|_p^p\,.\end{aligned}\tag{22}$$

Taking now $\varepsilon = 1/2$ and going back to (21), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u_n)\,dx + \frac{1}{2}\|\nabla\Psi(u_n(\cdot,t))\|_p^p &\le c\|f(\cdot,t)\|_1 + c\|f(\cdot,t)\|_q^r\|\Psi(u_n(\cdot,t))\|_p^p \\ &\le c\|f(\cdot,t)\|_1 + c\|f(\cdot,t)\|_q^r\,\Big(1+\int_\Omega \Phi(u_n)\,dx\Big)\,,\end{aligned}\tag{23}$$

by (18). Therefore, setting $\xi_n(t) = \int_\Omega \Phi(u_n(x,t))\,dx$, we get an inequality of the form

$$\xi_n'(t) \le c\|f(\cdot,t)\|_1 + c\|f(\cdot,t)\|_q^r\,(1+\xi_n(t)) \le \Upsilon(t)\,[1+\xi_n(t)]\,,$$

where $\Upsilon(t)\in L^1(0,T)$. Therefore, it follows that

$$\log(1+\xi_n(t)) - \log(1+\xi_n(0)) \le C(T)\,,$$

which implies an estimate on $\xi_n(t)$. Going back to (23), this yields the desired estimate (12); now (19) follows from (18), and integrating in (23) we obtain (13). Moreover, observe that the right-hand side in (20) has been estimated by the one in (23), so that, integrating, it gives

$$\iint_{Q_T} |f(x,t)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| \le C(T)\,.$$

In order to complete the proof of (14), we need to check that there exists a positive constant $C(T)$ satisfying

$$\iint_{Q_T} |T_n b(x,t,u_n,\nabla u_n)|\;e^{\gamma(u_n)}\,|\Psi(u_n)|\,dx\,dt \le C(T)\,,\tag{24}$$

for all $n\in\mathbb{N}$. Denoting for brevity $b_n(u_n) = T_n b(x,t,u_n,\nabla u_n)$, (24) is a consequence of the following computations:

$$\begin{aligned}\iint_{Q_T} |b_n(u_n)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| &= \iint_{\{u_nb_n(u_n)<0\}} |b_n(u_n)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| + \iint_{\{u_nb_n(u_n)\ge 0\}} |b_n(u_n)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| \\ &\le \iint_{Q_T} |f|\,e^{\gamma(u_n)}\,|\Psi(u_n)| + \iint_{Q_T} \beta_2(u_n)\,e^{\gamma(u_n)}\,|\Psi(u_n)|\,|\nabla u_n|^p \\ &\le c\Big[\iint_{Q_T} |f|\,|\Psi(u_n)|^p\,dx + \iint_{Q_T} \Psi'(u_n)^p\,|\nabla u_n|^p + \iint_{Q_T} |f|\Big]\,,\end{aligned}$$

where we have applied (20), **B)** and Lemma 4.1. Now, the last integral is bounded by (13), while the integral $\iint_{Q_T} |f|\, |\Psi(u_n)|^p$ can be estimated using the same calculations as in (22). Therefore, (24) follows.

Finally, (15) and (16) are straightforward consequences of (14).

5 A priori estimates: the case of polinomial growth.

In this Section we will assume that $1 < p < N$, and that the function β_2 which appears in hypothesis **B)** is given by

C2) $$\beta_2(s) = M\,(|s|^\lambda + 1)\,,$$

and let us define

$$\theta = \frac{\lambda}{\lambda + 1}\,.$$

In this case the following result holds.

Lemma 5.1 *Assume that* **C2)** *holds, and that the functions γ, Ψ and Φ are defined by* (3). *Then there exist positive constants M_1, M_2, M_3 such that*

$$M_1\,|\Psi(s)|\,(\log^*|\Psi(s)|)^\theta \le \exp\Big(\frac{\gamma(s)}{p-1}\Big) \le M_2\,\big(1 + |\Psi(s)|\,(\log^*|\Psi(s)|)^\theta\big)\,, \tag{25}$$

$$\Phi(s) \ge M_1|\Psi(s)|^p\,(\log^*|\Psi(s)|)^{\theta(p-2)} - M_3\,, \tag{26}$$

$$\beta_2(s)e^{\gamma(s)}|\Psi(s)| \le M_2\Psi'(s)^p\,,$$

for every $s \in \mathbb{R}$.

Proof: It suffices to observe that, by a repeated use of De L'Hôpital's rule, one has

$$\lim_{s\to+\infty} \frac{\exp\big(\frac{\gamma(s)}{p-1}\big)}{\Psi(s)\,(\log^*\Psi(s))^\theta} = \left(\frac{M\,(\lambda+1)^\lambda}{\Lambda_1(p-1)}\right)^{\frac{1}{\lambda+1}};$$

$$\lim_{s\to+\infty} \frac{\Phi(s)}{(\Psi(s))^p\,(\log^*\Psi(s))^{\theta\,(p-2)}} = \frac{1}{p}\lim_{s\to+\infty}\left(\frac{\exp\big(\frac{\gamma(s)}{p-1}\big)}{\Psi(s)\,(\log^*\Psi(s))^\theta}\right)^{p-2} = \frac{1}{p}\left(\frac{M\,(\lambda+1)^\lambda}{\Lambda_1(p-1)}\right)^{\frac{p-2}{\lambda+1}},$$

$$\lim_{s\to+\infty} \frac{\beta_2(s)\,e^{\gamma(s)}\,\Psi(s)}{\Psi'(s)^p} = \Lambda_1(p-1)\,. \qquad \blacksquare$$

The next instrument we will need is a logarithmic Sobolev inequality which will be used in the main a priori estimates.

Proposition 5.1 *For each $0 < \delta \le 1$ and each $\alpha \in \mathbb{R}$, there exists a positive constant $\overline{C}$ (depending on N, p, $\operatorname{meas}\Omega$, α, δ) such that, for every $\varepsilon > 0$ and every $u \in W_0^{1,p}(\Omega)$,*

$$\begin{aligned}
&\int_\Omega |u(x)|^p(\log^*|u(x)|)^{\alpha+\delta}\,dx \le \\
&\quad\le \overline{C}\left[\varepsilon\int_\Omega |\nabla u(x)|^p\,dx + (\log^* 1/\varepsilon)^\delta\Big(\int_\Omega |u(x)|^p(\log^*|u(x)|)^\alpha\,dx\Big)\right.\\
&\qquad\left. + \Big(\int_\Omega |u(x)|^p(\log^*|u(x)|)^\alpha\,dx\Big)\Big(\log^*\int_\Omega |u(x)|^p(\log^*|u(x)|)^\alpha\,dx\Big)^\delta + 1\right],
\end{aligned}$$

where $\log^* s = \max\{1, \log s\}$.

Proof: It is enough to prove the above inequality for $0 < \varepsilon \le 1/e$. If $1 < p < N$, then we consider a convex and increasing function $\Gamma : [0, +\infty[\to [0, +\infty[$ satisfying

$$\Gamma(s) \sim \exp((p^* - p)s^{1/\delta})s^{-\alpha/\delta} \qquad \text{as } s \to +\infty\,.$$

When $p \ge N$, we may consider, instead of p^*, any value greater than p. We point out that

$$\Gamma\left((\log|s|)^\delta\right) \sim |s|^{p^*-p}(\log|s|)^{-\alpha}\,, \qquad \Gamma^{-1}(t) \sim \left(\frac{\log|t|}{p^*-p}\right)^\delta \qquad \text{for } s \to \infty. \tag{27}$$

Note that we may always assume $\Gamma(1) > 1$. We begin by applying Jensen's inequality with the weight function $|u|^p(\log|u|)^\alpha$ in the set $\{|u| > e\}$ to obtain

$$1 < \Gamma(1) \le \Gamma\left(\frac{\displaystyle\int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha(\log|u|)^\delta\,dx}{\displaystyle\int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha\,dx}\right) \le \frac{\displaystyle\int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha\Gamma\left((\log|u|)^\delta\right)\,dx}{\displaystyle\int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha\,dx}\,.$$

Denoting

$$J = \int_{\{|u|>e\}} |u|^p(\log|u|)^{\alpha+\delta}\,dx \quad \text{and} \quad I = \int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha\,dx\,,$$

this inequality becomes

$$J \le I\,\Gamma^{-1}\left(\frac{1}{I}\int_{\{|u|>e\}} |u|^p(\log|u|)^\alpha\Gamma\big((\log|u|)^\delta\big)\,dx\right).$$

Note that the argument of Γ^{-1} is greater than $\Gamma(1) > 1$, therefore using (27) we can write

$$\begin{aligned}
J &\le C\,I\left(\log\left(\frac{1}{I}\int_{\{|u|>e\}} |u|^p\,(\log|u|)^\alpha\,\Gamma\big((\log|u|)^\delta\big)\,dx\right)\right)^\delta \\
&\le CI\left[\log\left(C\int_{\{|u|>e\}} |u|^{p^*}\,dx\right) - \log I\right]^\delta \\
&\le C\left[I^{1/\delta}\log\left(C\int_\Omega |u|^{p^*}\,dx\right) + 1\right]^\delta,
\end{aligned}$$

since $I^{1/\delta}\log I$ is bounded from below. Hence,

$$J \le C\left[I^{1/\delta}\log\left(C\varepsilon\|u\|_{p^*}^p\right) + I^{1/\delta}\log(1/\varepsilon) + 1\right]^\delta\,. \tag{28}$$

On the other hand, we may apply Lemma 3.1, obtaining

$$I^{1/\delta}\log\left(C\varepsilon\|u\|_{p^*}^p\right) \le C\left(I^{1/\delta}\log^* I + \varepsilon^{1/\delta}\|u\|_{p^*}^{p/\delta}\right)\,. \tag{29}$$

Thus, it follows from (28), (29) and Sobolev's inequality that

$$\begin{aligned}
J \quad &\le C\left[I^{1/\delta}\log^* I + \varepsilon^{1/\delta}\|\nabla u\|_p^{p/\delta} + I^{1/\delta}\log^*(1/\varepsilon) + 1\right]^\delta \\
&\le C\left[I\left(\log^* I\right)^\delta + \varepsilon\|\nabla u\|_p^p + I\left(\log^*(1/\varepsilon)\right)^\delta + 1\right]\,.
\end{aligned}$$

Therefore, on account of $\int_\Omega |u|^p(\log(1+|u|))^{\alpha+\delta}\,dx \le J + C$, the above estimate on J implies the desired result. ∎

Proposition 5.2 *Assume that the assumptions of Theorem 2.2 hold, and that $\{u_n\}$ is a sequence of solutions of the approximate problems* (10). *Let the functions γ, Ψ, Φ be defined as in* (3). *Then the estimates* (12)–(16) *hold true. Moreover, one has*

$$\int_\Omega |\Psi(u_n(x,\tau))|^p \, dx \le C(T) \qquad \textit{for every } \tau \in [0,T], \textit{ if } p \ge 2; \tag{30}$$

$$\int_\Omega |\Psi(u_n(x,\tau))|^\sigma \, dx \le C(\sigma,T) \quad \textit{for every } \sigma < p, \textit{ for every } \tau \in [0,T], \textit{ if } 1 < p < 2, \tag{31}$$

for all $n \in \mathbb{N}$.

Proof: As in the proof of Proposition 4.1, applying Proposition 3.1, with $\psi = \Psi$, and using inequality (25), one obtains

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Phi(u_n)\,dx + \Lambda_1 \int_\Omega |\nabla\Psi(u_n)|^p\,dx &\le \int_\Omega |f|\,|\Psi(u_n)|\,e^{\gamma(u_n)} \\
&\le c\int_\Omega |f|\,|\Psi(u_n)|\left(1+|\Psi(u_n)|\,(\log^*|\Psi(u_n)|)^\theta\right)^{p-1}dx \\
&\le c\int_\Omega |f|\,dx + c\int_\Omega |f|\,|\Psi(u_n)|^p\,(\log^*|\Psi(u_n)|)^{\theta(p-1)}\,dx\,.
\end{aligned} \tag{32}$$

From now on, in order to minimize notation we set

$$v = |\Psi(u_n)|\,.$$

Using the standard inclusions among Lebesgue spaces in sets of finite measure, we can assume that the pair (p,q) satisfies

$$qp = Nr'\,, \qquad r' \ge 1 + (p-1)\,\lambda\,.$$

Therefore, using the inequality

$$\frac{1}{q} + \frac{1}{r} + \frac{p}{p^* r'} = 1\,,$$

Hölder's and Sobolev's inequalities and Proposition 5.1 with

$$\alpha = \theta\,(p-2) = \frac{\lambda(p-2)}{\lambda+1}\,, \qquad \delta = \theta\,r\left[\frac{p-2}{r'}+1\right] = \theta\,[r(p-1)-(p-2)] \le 1\,,$$

one has, for every positive η and ε,

$$\begin{aligned}
\int_\Omega |f|\,v^p\,(\log^* v)^{\theta(p-1)}\,dx &= \int_\Omega |f|\,v^{p/r'}\,v^{p/r}\,(\log^* v)^{\theta(p-1)}\,dx \\
&\le \|f(t)\|_q \left(\int_\Omega v^{p^*}\,dx\right)^{\frac{p}{p^* r'}} \left(\int_\Omega v^p\,(\log^* v)^{\theta r(p-1)}\,dx\right)^{\frac{1}{r}} \\
&\le \eta\left(\int_\Omega v^{p^*}\,dx\right)^{\frac{p}{p^*}} + c(\eta)\,\|f(t)\|_q^r \int_\Omega v^p\,(\log^* v)^{\theta r(p-1)}\,dx \\
\le c\,\eta\int_\Omega |\nabla v|^p\,dx + c(\eta)\,\overline{C}\,\|f(t)\|_q^r &\Bigg[\varepsilon\int_\Omega |\nabla v|^p\,dx + \left(\log^*\frac{1}{\varepsilon}\right)^\delta \int_\Omega v^p\,(\log^* v)^\alpha\,dx \\
&+ \left(\int_\Omega v^p\,(\log^* v)^\alpha\,dx\right)\left(\log^*\int_\Omega v^p\,(\log^* v)^\alpha\,dx\right)^\delta + 1\Bigg]\,.
\end{aligned}$$

We now choose

$$\eta = \frac{\Lambda_1}{2c}\,, \qquad \varepsilon = \varepsilon(t) = \frac{\Lambda_1}{2\,\overline{C}\,c(\eta)\,\|f(t)\|_q^r}\,,$$

so that the two terms containing $\int_\Omega |\nabla v|^p \, dx$ can be absorbed by the corresponding term in the l.h.s. of (32). Moreover inequality (26) can be written as

$$v^p \left(\log^* v\right)^\alpha \leq c \left(\Phi(u_n) + 1\right),$$

therefore

$$\frac{d}{dt} \int_\Omega \Phi(u_n(t)) \, dx \leq c \Bigg[\|f(t)\|_1 + \|f(t)\|_q^r + \|f(t)\|_q^r \big(\log^* \|f(t)\|_q \big)^\delta \int_\Omega \Phi(u_n(t)) \, dx$$
$$+ \|f(t)\|_q^r \int_\Omega \Phi(u_n(t)) \, dx \bigg(\log^* \int_\Omega \Phi(u_n(t)) \, dx \bigg)^\delta \Bigg].$$

Setting

$$\xi_n(t) = \int_\Omega \Phi(u_n(t)) \, dx$$

and using assumption (7) on f, we have proved that

$$\xi_n'(t) \leq \Upsilon(t) \left(1 + H(\xi_n(t))\right), \tag{33}$$

where $\Upsilon(t)$ is a positive, integrable function on $]0, T[$, while $H(s)$ is a positive function such that

$$\int_0^{+\infty} \frac{ds}{1 + H(s)} = \infty,$$

since $\delta \leq 1$. Therefore, if we define

$$G(s) = \int_0^s \frac{d\sigma}{1 + H(\sigma)},$$

it follows from (33) that

$$G(\xi_n(t)) - G(\xi_n(0)) \leq C(T),$$

which implies an estimate on $\xi_n(t)$, since, by the assumption on the initial data u_{0n}, the initial value $\xi_n(0)$ is uniformly bounded. The estimate on $\xi_n(t)$ immediately implies (12) and (13). Estimates (30) and (31) follow at once from (12) and inequality (26). The proof of (14), (15) and (16) can be done exactly as in the case of constant β_2 in the previous Section. ■

6 A priori estimates: the case of exponential growth

In this Section we will assume that the function β_2 which appears in hypothesis **B)** is given by

C3) $$\beta_2(s) = M \, e^{\delta |s|}.$$

In this case the following result holds.

Lemma 6.1 *Assume that* **C3)** *holds, and that the functions* γ, Ψ *and* Φ *are defined by* (3). *Then there exist positive constants* M_1, M_2, M_3 *such that*

$$M_1 \, |\Psi(s)| \, \log^* |\Psi(s)| \leq \exp\left(\frac{\gamma(s)}{p-1}\right) \leq M_2 \left(1 + |\Psi(s)| \, \log^* |\Psi(s)|\right), \tag{34}$$

$$\Phi(s) \geq M_1 \, |\Psi(s)|^p \left(\log^* |\Psi(s)|\right)^{p-2} - M_3, \tag{35}$$

$$\beta_2(s) e^{\gamma(s)} |\Psi(s)| \leq M_2 \, \Psi'(s)^p,$$

for every $s \in \mathbb{R}$.

Proof: As in the corresponding Lemma of the previous Section, it suffices to repeatedly use De L'Hôpital's rule to obtain

$$\lim_{s\to+\infty} \frac{\exp\left(\frac{\gamma(s)}{p-1}\right)}{\Psi(s)\,\log^*\Psi(s)} = \delta\,;$$

$$\lim_{s\to+\infty} \frac{\Phi(s)}{(\Psi(s))^p\,(\log^*\Psi(s))^{p-2}} = \frac{1}{p}\lim_{s\to+\infty}\left(\frac{\exp\left(\frac{\gamma(s)}{p-1}\right)}{\Psi(s)\,\log^*\Psi(s)}\right)^{p-2} = \frac{\delta^{p-2}}{p}\,;$$

$$\lim_{s\to+\infty} \frac{\beta_2(s)\,e^{\gamma(s)}\,\Psi(s)}{\Psi'(s)^p} = \Lambda_1(p-1)\,. \qquad \blacksquare$$

Then, as before, in order to obtain a priori estimates, we will need a new logarithmic Sobolev inequality, since Proposition 5.1 is not sufficient in this exponential case.

We point out that the inequality in Proposition 5.1 can be written for $\alpha = p-2$ and $\delta = 1$ as

$$\begin{aligned}&\int_\Omega |v|^p(\log^*|v|)^{p-2}A(\log^*|v|)\,dx\\ &\le c\Bigg[\varepsilon\int_\Omega|\nabla v|^p\,dx + A(\log^*\frac{1}{\varepsilon})\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\\ &\quad+\Big(\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\Big)A\Big(\log^*\Big(\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\Big)\Big)+1\Bigg],\end{aligned}$$

with $A(s) = s$.

To solve our problem under hypothesis **C3)**, we need the above inequality for

$$A(s) = s\,\log^* s\,.$$

Note that A satisfies

$$A(t+s) \le c(A(t)+A(s)) \qquad \text{for every } s,t>0, \tag{36}$$

$$A(\lambda s) \le k(\lambda)A(s) \qquad \text{for every } s,\lambda>0. \tag{37}$$

Actually, one could prove a family of logarithmic inequalities for a general $A(s)$ satisfying (36) and (37) (in the same spirit as in [2]). One could use such general inequalities to deal with more general growths for the function $\beta_2(s)$, for instance $\beta_2(s)\sim\exp(\exp s)$, as it is done, in the case $p=2$, in [13] (Section 5) and in [11].

Proposition 6.1 *There exists a constant $C = C(p,N,\operatorname{meas}\Omega) > 0$ such that, for every $\varepsilon>0$ and every function $v\in W_0^{1,p}(\Omega)$, the following inequality holds:*

$$\begin{aligned}&\int_\Omega |v|^p(\log^*|v|)^{p-2}A(\log^*|v|)\,dx\\ &\le C\Bigg[\varepsilon\int_\Omega|\nabla v|^p\,dx + A(\log^*\frac{1}{\varepsilon})\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\\ &\quad+\Big(\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\Big)A\Big(\log^*\Big(\int_\Omega|v|^p(\log^*|v|)^{p-2}\,dx\Big)\Big)+1\Bigg].\end{aligned}$$

Proof: It is enough to prove the inequality for $\varepsilon\le\frac{1}{e}$. Moreover, to minimize notation, assume that $v\ge0$. We can consider a convex, increasing function $\Gamma(s)\ :\ [0,+\infty)\to[0,+\infty)$ such that

$$\Gamma(s)\sim e^{(p^*-p)\frac{s}{\log s}}\left(\frac{\log s}{s}\right)^{p-2} \qquad \text{for } s\to+\infty$$

(if $p \geq N$, replace p^* with any number $q > p$). Then it is easy to check that

$$\Gamma(A(\log s)) \leq \frac{s^{p^*-p}}{(\log s)^{p-2}} \qquad \text{for large } s\,.$$

Moreover one can always assume that $\Gamma(1) \geq e$. Assume also that the set $E = \{x \in \Omega \,:\, v(x) > e\}$ has positive measure, and define

$$I = \int\limits_E v^p (\log v)^{p-2}\, dx\,, \qquad J = \int\limits_E v^p (\log v)^{p-2} A(\log v)\, dx\,.$$

Then, by Jensen's inequality, one obtains

$$e \leq \Gamma(1) \leq \Gamma\Big(\frac{J}{I}\Big) \leq \frac{1}{I} \int\limits_E v^p (\log v)^{p-2} \Gamma(A(\log v))\, dx \leq \frac{c}{I} \int\limits_\Omega v^{p^*}\, dx\,.$$

Therefore

$$J \leq I\, \Gamma^{-1}\Big(\frac{c}{I} \int\limits_\Omega v^{p^*}\, dx\Big)\,.$$

One easily checks that

$$\Gamma^{-1}(t) \sim (\log t)\,(\log\log t) = A(\log t) \qquad \text{for } t \to +\infty,$$

so that $\Gamma^{-1}(t) \leq cA(\log t)$ for all $t \geq e$. Thus, using property (36) and Sobolev's inequality, we get

$$\begin{aligned}
J &\leq c\, I\, A\Big(\log\Big(\frac{c}{I} \int\limits_\Omega v^{p^*}\, dx\Big)\Big) \\
&\leq c\, I\, A\Big(\frac{p^*}{p}\, \log\big(c\,\varepsilon\, \|\nabla v\|_p^p\big) + \log\frac{1}{\varepsilon} - \log I\Big) \\
&\leq c\, I\, \Big[A\Big(\Big(\log\big(c\,\varepsilon\, \|\nabla v\|_p^p\big)\Big)^+\Big) + A\Big(\log\frac{1}{\varepsilon}\Big) + A((-\log I)^+)\Big]\,.
\end{aligned}$$

In order to estimate the product $I\, A\Big(\Big(\log\big(c\,\varepsilon\, \|\nabla v\|_p^p\big)\Big)^+\Big)$ which appears in the last inequality, one applies Lemma 3.1 yielding

$$I\, A\Big(\Big(\log\big(c\,\varepsilon\, \|\nabla v\|_p^p\big)\Big)^+\Big) \leq c\, \Big(I\, A(\log^* I) + \varepsilon\, \|\nabla v\|_p^p\Big)\,.$$

Thus we have obtained

$$J \leq c\, \Big[\varepsilon \int\limits_\Omega |\nabla v|^p\, dx + I\, A\big(\log^* \frac{1}{\varepsilon}\big) + I\, A(\log^* I) + I\, A\big((-\log I)^+\big)\Big]\,.$$

Since

$$I\, A\big((-\log I)^+\big) \leq c\Big(I\, A(\log^* I) + 1\Big)\,,$$

it follows that

$$J \leq c\, \Big[\varepsilon \int\limits_\Omega |\nabla v|^p\, dx + I\, A\big(\log^* \frac{1}{\varepsilon}\big) + I\, A(\log^* I) + 1\Big]\,.$$

Now we recall that

$$\int\limits_\Omega v^p (\log v)^{p-2} A(\log v)\, dx \leq J + e^p \,\mathrm{meas}\,\Omega$$

and we obtain the desired inequality. ∎

We need to introduce some definitions concerning Orlicz spaces; we will refer to [20] for a more detailed presentation. Let us recall that a function $\varphi(s) : [0,+\infty[\to [0,+\infty[$ is called an N-function if it admits the representation

$$\varphi(s) = \int_0^s p(t)\,dt$$

where $p(t)$ is right continuous for $t \geq 0$, positive for $t > 0$, nondecreasing and satisfying $p(0) = 0$ and $p(\infty) = \infty$. If φ is an N-function, we call Orlicz space associated to φ, denoted by $L_\varphi(\Omega)$, the class of those measurable real functions u, defined on Ω, for which the norm

$$\|u\|_{L_\varphi(\Omega)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \varphi\Big(\frac{|u|}{\lambda}\Big)\ dx \leq 1\right\}$$

is finite. It is clear that N-functions which are asymptotically equivalent near infinity generate the same Orlicz spaces. The following inequality always holds true:

$$\|u\|_{L_\varphi(\Omega)} \leq 1 + \int_\Omega \varphi(|u(x)|)\,dx \tag{38}$$

We will sometimes write $\|u\|_\varphi$ instead of $\|u\|_{L_\varphi(\Omega)}$.

Let φ and $\tilde\varphi$ be two N-functions of class C^1. We say that they are conjugate if $\varphi' = (\tilde\varphi')^{-1}$. For instance, the functions $\varphi(s) = s^p/p$ and $\tilde\varphi(s) = s^{p'}/p'$, with $p, p' > 1$ and $1/p + 1/p' = 1$, are conjugate N-functions. Moreover, as in the case of Lebesgue's spaces, if φ and $\tilde\varphi$ are two conjugate N-functions, the following Hölder inequality holds:

$$\int_\Omega uv\,dx \leq 2\|u\|_{L_\varphi(\Omega)}\|v\|_{L_{\tilde\varphi}(\Omega)}\,, \tag{39}$$

for all $u \in L_\varphi(\Omega)$, $v \in L_{\tilde\varphi}(\Omega)$. Evolution Orlicz spaces $L_\psi(0,T;L_{\tilde\varphi}(\Omega))$ can be defined in an obvious way.

Proposition 6.2 *Assume that the hypotheses of Theorem 2.3 hold, and that $\{u_n\}$ is a sequence of bounded solutions of the approximate problems* (10)*. Let the functions γ, Ψ, Φ be defined as in* (3)*. Then the estimates* (12)–(16) *hold true. Moreover, one has*

$$\int_\Omega |\Psi(u_n(x,\tau))|^p\,dx \leq C(T) \qquad \textit{for every } \tau \in [0,T],\ \textit{if } p \geq 2; \tag{40}$$

$$\int_\Omega |\Psi(u_n(x,\tau))|^\sigma\,dx \leq C(\sigma,T) \quad \textit{for every } \sigma < p,\ \textit{for every } \tau \in [0,T],\ \textit{if } 1 < p < 2, \tag{41}$$

for all $n \in \mathbb{N}$.

Proof: As in the previous Section, we use Proposition 3.1 with $\psi = \Psi$; then by (34), one obtains (again we set $v = v_n = |\Psi(u_n)|$ for brevity)

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u_n)\,dx + \Lambda_1\int_\Omega |\nabla v|^p\,dx &\leq \int_\Omega |f|\,|\Psi(u_n)|\,e^{\gamma(u_n)} \leq c\int_\Omega |f|\,v\,\big(1 + v\,\log^* v\big)^{p-1}\,dx \\ &\leq c\int_\Omega |f|\,dx + c\int_\Omega |f|\,v^p\ (\log^* v)^{p-1}\,dx\,, \end{aligned}\tag{42}$$

We use the generalized Hölder-Orlicz inequality with the pair of conjugate N-functions

$$\phi(s) = \int_0^s \log\big(1 + \log(1+\sigma)\big)\,d\sigma \qquad \tilde\phi(s) = \int_0^s \big(e^{(e^\sigma - 1)} - 1\big)\,d\sigma\,.$$

Note that, for $s \to +\infty$,

$$\phi(s) \sim s \log\log s\,, \qquad \frac{\tilde{\phi}(s)}{\varphi(s)} \to 0\,,$$

where $\varphi(s) \sim \exp(\exp(s))$ is the same function appearing in assumption (8). Then using (39), (38) and Proposition 6.1, we obtain

$$\begin{aligned}\int\limits_{\Omega} |f|\, v^p (\log^* v)^{p-1}\, dx &\le 2\, \|f(t)\|_{\tilde{\phi}}\, \|v^p (\log^* v)^{p-1}\|_{\phi} \le c\, \|f(t)\|_{\varphi} \left[1 + \int\limits_{\Omega} \phi\big(v^p (\log^* v)^{p-1}\big)\, dx\right] \\ &\le c\, \|f(t)\|_{\varphi} \left[1 + \int\limits_{\Omega} v^p (\log^* v)^{p-1} (\log^* \log^* v)\, dx\right] \\ &\le c\, \|f(t)\|_{\varphi} \left[\varepsilon \int\limits_{\Omega} |\nabla v|^p\, dx + A(\log^* \frac{1}{\varepsilon}) \int\limits_{\Omega} v^p (\log^* v)^{p-2}\, dx \right. \\ &\qquad \left. + \left(\int\limits_{\Omega} v^p (\log^* v)^{p-2}\, dx\right) A\left(\log^* \left(\int\limits_{\Omega} v^p (\log^* v)^{p-2}\, dx\right)\right) + 1\right]. \end{aligned} \tag{43}$$

Then, if we choose

$$\varepsilon = \varepsilon(t) = \frac{\eta}{\|f(t)\|_{\varphi}}\,,$$

with η small enough, setting

$$\xi_n(t) = \int\limits_{\Omega} \Phi(u_n(t))\, dx\,,$$

and recalling inequality (35), from (42) and (43) we obtain

$$\begin{aligned}\xi_n'(t) \le c \Big[&\|f(t)\|_1 + \|f(t)\|_{\varphi} + \|f(t)\|_{\varphi} (\log^* \|f(t)\|_{\varphi})\, (\log^* \log^* \|f(t)\|_{\varphi})\, \big(1 + \xi_n(t)\big) \\ &+ \xi_n(t)\, (\log^* \xi_n(t))\, (\log^* \log^* \xi_n(t)) + 1 \Big]\,.\end{aligned}$$

Using assumption (8) and **D)** on the data f and u_0, this implies an inequality of the form

$$\xi_n'(t) \le \Upsilon(t)\, \big(1 + H(\xi_n(t))\big)\,, \qquad \xi_n(0) \le C\,,$$

where $\Upsilon(t)$ is an integrable function on $]0, T[$, and

$$\int\limits_0^{+\infty} \frac{ds}{1 + H(s)} < \infty\,.$$

This proves estimates (12), (13), (40) and (41). The proof of (14), (15) and (16) can be done exactly as in the previous Sections. ∎

7 Convergence of the gradients

In this section we will consider a sequence $\{u_n\}_n$ of solutions to problems (10) and we will see that (up to a subsequence) this sequence converges almost everywhere to a function u and the sequence $\{\nabla u_n\}_n$ of its gradients converges almost everywhere to ∇u. The three cases **Ci's)** will be consider together.

We already have proved in the previous Sections some estimates on the approximate solutions u_n. For fixed $T > 0$, we know that $\Psi(u_n)$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$, and moreover the gradient term is bounded in $L^1(Q_T)$. Therefore, using the equation (10), the time derivative $(u_n)_t$ is bounded in $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)$. Hence, we may apply the compactness results contained in [27] and then use a diagonal argument to obtain the following result.

Proposition 7.1 *If $\{u_n\}_n$ is a sequence of solutions of the approximate problems* (10)*, then there exists a subsequence, still denoted by $\{u_n\}_n$, and a function $u \in L^1_{\mathrm{loc}}(\overline{Q})$ such that*

$$u_n \to u \quad \textit{a.e. in } Q \textit{ and strongly in } L^1(Q_T), \textit{ for every } T > 0.$$

The convergence of the gradients $\{\nabla u_n\}_n$ is more cumbersome and some preliminar notation is required.

We fix $T > 0$. We begin by introducing a suitable regularization with respect to time (see [22], [24], [26]). For every $\nu \in \mathbb{N}$, we define $(T_k u)_\nu$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{1}{\nu}[(T_k u)_\nu]_t + (T_k u)_\nu = T_k u; \\ \\ (T_k u)_\nu(0) = T_k u_{0,\nu}\,, \end{cases}$$

where $T_k u_{0,\nu}$ are the truncations of the same initial data $u_{0,\nu}$ used for the approximate problems (10). Then, by the assumptions (11) on $u_{0,\nu}$, one has (see [22]):

$$(T_k u)_\nu \in L^p(0,T;W_0^{1,p}(\Omega)) \qquad ((T_k u)_\nu)_t \in L^p(0,T;W_0^{1,p}(\Omega))\,,$$

$$\|(T_k u)_\nu\|_{L^\infty(Q_T)} \le \|T_k u\|_{L^\infty(Q_T)} \le k,$$

and as ν goes to infinity

$$(T_k u)_\nu \to T_k u \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)).$$

From now on, $\omega(\nu)$ will denote a quantity which goes to zero as ν goes to infinity, $\omega(n,\nu)$ will denote a quantity which goes to zero as first n and then ν go to infinity, while $\omega^\nu(n)$ will denote a quantity which goes to zero as n goes to infinity, for every fixed ν.

Proposition 7.2 *Assume that* **A1)***,* **A2)***,* **A3)***,* **B)***,* **D)** *hold, and also that one of the hypotheses* **C1)***,* **C2)***, or* **C3)** *is satisfied, together the corresponding assumption on f:* (6)*,* (7) *or* (8)*, respectively. Let $\{u_n\}$ be a sequence of solutions of the approximate problems* (10) *which converges to u. Then, for every $k > 0$, one has*

$$\nabla T_k u_n \to \nabla T_k u \qquad \textit{strongly in } L^p(Q_T;\mathbb{R}^N).$$

In particular, up to the extraction of a subsequence, ∇u_n converges to ∇u almost everywhere in Q.

Proof: The proof follows the lines introduced in [24], [14], [9] and [10].

We multiply problems (10) by $e^{\gamma(u_n)-\gamma(T_k u_n)}\,\varphi\big((T_k u_n - (T_k u)_\nu)^+\big)$, where

$$\varphi(s) = \varphi_\mu(s) = e^{\mu s} - 1\,,$$

and μ is a positive number that will be conveniently chosen hereafter. Note that, since $|(T_k u)_\nu| \le k$, this function is zero in the set where $u_n < -k$.

One obtains

$$\boxed{\mathrm{A}} \qquad \int_0^T \langle (u_n)_t\,,\, e^{\gamma(u_n)-\gamma(T_k u_n)}\,\varphi\big((T_k u_n - (T_k u)_\nu)^+\big)\rangle\,dt$$

$$\boxed{\mathrm{B}} \qquad + \frac{1}{\Lambda_1} \iint\limits_{\{u_n > k\}} e^{\gamma(u_n)-\gamma(k)}\,\beta_2(u_n)\,a(u_n,\nabla u_n)\cdot\nabla u_n\,\varphi\big((k - (T_k u)_\nu)^+\big)$$

$$\boxed{\mathrm{C}} \qquad + \iint\limits_{Q_T} e^{\gamma(u_n)-\gamma(T_k u_n)}\,a(u_n,\nabla u_n)\cdot\nabla(T_k u_n - (T_k u)_\nu)^+ \varphi'((T_k u_n - (T_k u)_\nu)^+)$$

$$= \iint\limits_{Q_T} e^{\gamma(u_n)-\gamma(T_k u_n)}\,b(u_n,\nabla u_n)\,\varphi((T_k u_n - (T_k u)_\nu)^+) \qquad \boxed{\mathrm{D}}$$

$$+ \iint\limits_{Q_T} T_n f(x,t)\,e^{\gamma(u_n)-\gamma(T_k u_n)}\,\varphi((T_k u_n - (T_k u)_\nu)^+)\,. \qquad \boxed{\mathrm{E}}$$

We start analyzing the integrals which appear in the previous equality:

Integral $\boxed{\text{A}}$: We wish to show that

$$\boxed{\text{A}} \geq \omega(n,\nu)\,. \tag{44}$$

To this end, for $\sigma > 0$, we define $u_{n,\sigma}$ as the solution of

$$\begin{cases} \frac{1}{\sigma}(u_{n,\sigma})_t + u_{n,\sigma} = u_n; \\ \\ u_{n,\sigma}(0) = u_{0,n}. \end{cases}$$

Then the functions $u_{n,\sigma}$ satisfy the same properties shown above for $(T_k u)_\nu$. Moreover

$$(u_{n,\sigma})_t \to (u_n)_t \qquad \text{strongly in } L^{p'}(0,T;W^{-1,p'}(\Omega)), \text{ as } \sigma \to \infty.$$

Let us define the function $v_\sigma = v_{\nu,n,\sigma} = \varphi\left((T_k u_{n,\sigma} - (T_k u)_\nu)^+\right) e^{\gamma(u_{n,\sigma})-\gamma(T_k u_{n,\sigma})}$, so that

$$\begin{aligned}
\int_0^T \langle (u_n)_t, e^{\gamma(u_n)-\gamma(T_k u_n)} \varphi\left((T_k u_n - (T_k u)_\nu)^+\right)\rangle &= \lim_{\sigma\to\infty} \iint_{Q_T} (u_{n,\sigma})_t v_\sigma \\
&= \lim_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t}(T_k u_{n,\sigma} + G_k u_{n,\sigma})\; v_\sigma \\
&\geq \liminf_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t} T_k u_{n,\sigma}\; \varphi\left((T_k u_{n,\sigma} - (T_k u)_\nu)^+\right) + \liminf_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t} G_k u_{n,\sigma}\; v_\sigma \\
&\geq \liminf_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t}(T_k u_{n,\sigma} - (T_k u)_\nu)\; \varphi\left((T_k u_{n,\sigma} - (T_k u)_\nu)^+\right) \\
&\quad + \liminf_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t}(T_k u)_\nu\; \varphi\left((T_k u_{n,\sigma} - (T_k u)_\nu)^+\right) + \liminf_{\sigma\to\infty} \iint_{Q_T} \frac{\partial}{\partial t} G_k u_{n,\sigma}\; v_\sigma \\
&= \liminf_{\sigma\to\infty} I^1_\sigma + \liminf_{\sigma\to\infty} I^2_\sigma + \liminf_{\sigma\to\infty} I^3_\sigma
\end{aligned}$$

(here we have used the fact that the term $\frac{\partial}{\partial t} T_k u_{n,\sigma}$ is zero where $|u_{n,\sigma}| > k$. If we set $\phi(s) = \int_0^s \varphi(\sigma)\,d\sigma$, we obtain

$$\begin{aligned}
I^1_\sigma &= \int_\Omega \phi((T_k u_{n,\sigma}(T) - (T_k u)_\nu(T))^+) - \int_\Omega \phi((T_k u_{n,\sigma}(0) - (T_k u)_\nu(0))^+) \\
&\geq -\int_\Omega \phi((T_k u_{0,n} - T_k u_{0,\nu})^+) = \omega^\nu(n) + \omega(\nu).
\end{aligned}$$

On the other hand

$$\begin{aligned}
I^2_\sigma &= \nu \iint_{Q_T} (T_k u - (T_k u)_\nu)\; \varphi\left((T_k u_{n,\sigma} - (T_k u)_\nu)^+\right) \\
&= \nu \iint_{Q_T} (T_k u - (T_k u)_\nu)\; \varphi\left((T_k u_n - (T_k u)_\nu)^+\right) + \omega^{\nu,n}(\sigma) \\
&= \nu \iint_{Q_T} (T_k u - (T_k u)_\nu)\; \varphi\left((T_k u - (T_k u)_\nu)^+\right) + \omega^{\nu,n}(\sigma) + \omega^\nu(n) \\
&\geq \omega^{\nu,n}(\sigma) + \omega^\nu(n),
\end{aligned}$$

where $\omega^{\nu,n}(\sigma)$ denotes a quantity which tends to zero as $\sigma \to \infty$, for every fixed ν and n. If we set $H_k(s) := \int_0^s e^{\gamma(\tau + k \text{ sign } (\tau)) - \gamma(k \text{ sign } (\tau))} d\tau$, the term I_σ^3 can be estimated as follows

$$
\begin{aligned}
I_\sigma^3 &= \iint_{Q_T} \frac{\partial}{\partial t} H_k(G_k u_{n,\sigma})\, \varphi\big((T_k u_{n,\sigma} - (T_k u)_\nu)^+\big) \\
&= \int_\Omega H_k(G_k u_{n,\sigma}(T))\, \varphi\big((T_k u_{n,\sigma}(T) - (T_k u)_\nu(T))^+\big) \\
&\quad - \int_\Omega H_k(G_k u_{n,\sigma}(0))\, \varphi\big((T_k u_{0,n} - T_k u_{0,\nu})^+\big) \\
&\quad - \iint_{Q_T} H_k(G_k u_{n,\sigma})\, \frac{\partial}{\partial t} \varphi\big((T_k u_{n,\sigma} - (T_k u)_\nu)^+\big) \\
&= I_\sigma^{3,1} + I_\sigma^{3,2} + I_\sigma^{3,3}.
\end{aligned}
$$

We note that $I_\sigma^{3,1} \ge 0$. Indeed, one has $|(T_k u)_\nu| \le k$, thus in the set where $G_k u_{n,\sigma}(T)$ is different from zero, that is, the set where $|u_{n,\sigma}(T)| > k$, the function $(T_k u_{n,\sigma}(T) - (T_k u)_\nu(T))^+$ (which is nonnegative) is different from zero only where $u_{n,\sigma}(T) > k$, so that $H_k(G_k u_{n,\sigma}(T)) = H_k(u_{n,\sigma}(T) - k) \ge 0$. To analyze the remaining terms, we will need the following easy estimate

$$
|H_k(G_k s)| \le \eta \Phi(s) + c(\eta), \tag{45}
$$

for every $\eta > 0$, $s \in \mathbb{R}$. Then one has

$$
I_\sigma^{3,2} = -\int_\Omega H_k(G_k u_0)\, \varphi\big((T_k u_0 - T_k u_{0,\nu})^+\big) + \omega^\nu(n) = \omega^\nu(n) + \omega(\nu).
$$

Finally

$$
\begin{aligned}
I_\sigma^{3,3} &= \iint_{Q_T} H_k^+(G_k u_{n,\sigma})\, \frac{\partial}{\partial t}(T_k u)_\nu\, \varphi'\big((T_k u_{n,\sigma} - (T_k u)_\nu)^+\big) \\
&= \nu \iint_{Q_T} H_k^+(G_k u_{n,\sigma})\, (T_k u - (T_k u)_\nu)\, \varphi'\big((T_k u_{n,\sigma} - (T_k u)_\nu)^+\big) \\
&= \nu \iint_{Q_T} H_k^+(G_k u_n)\, (T_k u - (T_k u)_\nu)\, \varphi'\big((T_k u_n - (T_k u)_\nu)^+\big) + \omega^{\nu,n}(\sigma)\,,
\end{aligned}
$$

since $H_k^+(G_k u_{n,\sigma})$ converges to $H_k^+(G_k u_n)$ in $L^1(Q_T)$ as σ goes to infinity. Observe also that $H_k^+(G_k u_n)$ converges to $H_k^+(G_k u)$ in $L^1(Q_T)$, as n goes to infinity, due to (45) and to the fact that the sequence $\{\Phi(u_n)\}$ is bounded in $L^1(Q_T)$. Therefore,

$$
I_\sigma^{3,3} = \nu \iint_{Q_T} H_k^+(G_k u)\, (T_k u - (T_k u)_\nu)\, \varphi'\big((T_k u - (T_k u)_\nu)^+\big) + \omega^{\nu,n}(\sigma) + \omega^\nu(n)\,,
$$

from where we deduce that $I_\sigma^{3,3} = \omega^{\nu,n}(\sigma) + \omega^\nu(n) + \omega(\nu)$. Putting all these estimates together, we conclude the proof of (44).

Integral $\boxed{\text{B}}$: One obviously has, by **A1)**:

$$
\boxed{\text{B}} \ge \iint_{\{u_n > k\}} e^{\gamma(u_n) - \gamma(k)}\, \beta_2(u_n)\, |\nabla u_n|^p\, \varphi\big(k - (T_k u)_\nu\big)\,.
$$

Integral $\boxed{\mathrm{C}}$:

$$\begin{aligned}
\boxed{\mathrm{C}} &= \iint\limits_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\cdot\nabla(u_n-(T_ku)_\nu)^+\varphi'((u_n-(T_ku)_\nu)^+) \\
&\quad - \iint\limits_{\{u_n>k\}} e^{\gamma(u_n)-\gamma(k)}\, a(u_n,\nabla u_n)\cdot\nabla(T_ku)_\nu\,\varphi'(k-(T_ku)_\nu) \\
&= \iint\limits_{\{|u_n|\le k\}} \Big(a(u_n,\nabla u_n)-a(u_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(u_n-(T_ku)_\nu)^+\,\varphi'((u_n-(T_ku)_\nu)^+) \qquad \boxed{\mathrm{C1}} \\
&\quad + \iint\limits_{\{|u_n|\le k\}} a(u_n,\nabla(T_ku)_\nu)\cdot\nabla(u_n-(T_ku)_\nu)^+\,\varphi'((u_n-(T_ku)_\nu)^+) \qquad \boxed{\mathrm{C2}} \\
&\quad - \iint\limits_{\{u_n>k\}} e^{\gamma(u_n)-\gamma(k)}\, a(u_n,\nabla u_n)\cdot\nabla(T_ku)_\nu\,\varphi'(k-(T_ku)_\nu) \qquad \boxed{\mathrm{C3}}
\end{aligned}$$

Then we can write

$$\begin{aligned}
\boxed{\mathrm{C2}} &= \iint\limits_{\{|u_n|\le k,|u|\neq k\}} a(u_n,\nabla(T_ku)_\nu)\cdot\nabla(u_n-(T_ku)_\nu)^+\,\varphi'((u_n-(T_ku)_\nu)^+) \\
&\quad + \iint\limits_{\{|u_n|\le k,|u|=k\}} a(u_n,\nabla(T_ku)_\nu)\cdot\nabla(u_n-(T_ku)_\nu)^+\,\varphi'((u_n-(T_ku)_\nu)^+) \\
&\quad = \omega(n,\nu) + \iint\limits_{\{|u_n|\le k,|u|=k\}} a(u_n,\nabla(T_ku)_\nu)\cdot\nabla(u_n-(T_ku)_\nu)^+\,\varphi'((u_n-(T_ku)_\nu)^+)\,.
\end{aligned}$$

Here we have used the weak convergence of $\nabla T_k u_n$ to $\nabla T_k u$ in $L^p(Q_T;\mathbb{R}^N)$, the strong convergence of $\nabla(T_ku)_\nu$ to $\nabla T_k u$ in the same space (which in turn implies the weak convergence of $a(u,\nabla(T_ku)_\nu)$ to $a(u,\nabla T_k u)$ in $L^{p'}(Q_T;\mathbb{R}^N)$, and finally the convergence of $\chi_{\{|u_n|\le k\}}\chi_{\{|u|\neq k\}}$ to $\chi_{\{|u|<k\}}$ almost everywhere. Finally, as far as the last integral of the above formula is concerned, using assumption **(A3)** on a, one has

$$\left|\boxed{\mathrm{C2}}\right| \le \omega(n,\nu)+c(k)\Big(\iint\limits_{Q_T}|\nabla(T_ku_n-(T_ku)_\nu)|^p\Big)^{1/p}\Big(\iint\limits_{\{|u|=k\}}|\nabla(T_ku)_\nu)|^p\Big)^{1/p'} \le \omega(n,\nu)+c(k)\omega(\nu)\,.$$

On the other hand, using assumption **(A3)**, the definition of Ψ, Hölder's inequality and the estimate on $|\nabla\Psi(u_n)|$ in $L^p(Q_T)$, one easily obtains

$$\begin{aligned}
\left|\boxed{\mathrm{C3}}\right| &\le c(k)\,\Lambda_2 \iint\limits_{\{u_n>k\}} e^{\gamma(u_n)}|\nabla u_n|^{p-1}|\nabla(T_ku)_\nu| \le c(k)\,\Lambda_2 \iint\limits_{\{u_n>k\}} |\nabla\Psi(u_n)|^{p-1}|\nabla(T_ku)_\nu| \\
&\le c(k)\,\Lambda_2\Big(\iint\limits_{Q_T}|\nabla\Psi(u_n)|^p\Big)^{(p-1)/p}\Big(\iint\limits_{\{u_n>k\}}|\nabla(T_ku)_\nu|^p\Big)^{1/p} \le c\Big(\iint\limits_{\{u_n>k\}}|\nabla(T_ku)_\nu|^p\Big)^{1/p} \\
&\le c\Big(\iint\limits_{\{u_n>k\,,\,u\neq k\}}|\nabla(T_ku)_\nu|^p + \iint\limits_{\{u=k\}}|\nabla(T_ku)_\nu|^p\Big)^{1/p} = \omega(n,\nu)\,.
\end{aligned}$$

Therefore

$$\boxed{\mathrm{C}} \ge \boxed{\mathrm{C1}} + \omega(n,\nu)\,.$$

Integral $\boxed{\text{D}}$:

$$\iint_{Q_T} e^{\gamma(u_n)-\gamma(T_k u_n)}\, b(u_n,\nabla u_n)\,\varphi((T_k u_n-(T_k u)_\nu)^+)$$

$$\le \iint_{\{b(u_n,\nabla u_n)\ge 0\}} e^{\gamma(u_n)-\gamma(T_k u_n)}\, b(u_n,\nabla u_n)\,\varphi((T_k u_n-(T_k u)_\nu)^+)$$

$$\le \iint_{\{b(u_n,\nabla u_n)\ge 0\,,\,u_n>k\}} e^{\gamma(u_n)-\gamma(k)}\,\beta_2(u_n)\,|\nabla u_n|^p\,\varphi(k-(T_k u)_\nu)$$

$$+\iint_{\{b(u_n,\nabla u_n)\ge 0\,,\,|u_n|\le k\}} b(u_n,\nabla u_n)\,\varphi((u_n-(T_k u)_\nu)^+)$$

$$\le \boxed{\text{B}} + c(k)\iint_{\{|u_n|\le k\}} |\nabla u_n|^p\,\varphi((u_n-(T_k u)_n u)^+)\,.$$

Integral $\boxed{\text{E}}$: In order to estimate this term, we only have to observe that $\varphi((T_k u_n-(T_k u)_\nu)^+)$ is bounded by a constant not depending on n or ν, and then it yields

$$\boxed{\text{E}} \le \iint_{Q_T} |f(x,t)|\,e^{\gamma(u_n)}\,\varphi((T_k u_n-(T_k u)_\nu)^+) = \omega(n,\nu)\,,$$

as a consequence of the following result.

Lemma 7.1 *The sequence* $\left\{|f(x,t)|\,e^{\gamma(u_n)}\right\}_n$ *converges to* $|f(x,t)|\,e^{\gamma(u)}$ *in* $L^1(Q_T)$.

Proof: Since we already know that $\left\{|f(x,t)|\,e^{\gamma(u_n)}\right\}$ converges to $|f(x,t)|\,e^{\gamma(u)}$ a.e. in Q_T, because of Vitali's Theorem, we only have to prove that this sequence is equi-integrable. From the a priori estimate (14), we have obtained in the three cases that there exists a positive constant C satisfying

$$\iint_{Q_T} |f(x,t)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| \le C \qquad \text{for all } n\in\mathbb{N}\,.$$

Thus,

$$\iint_{\{|u_n|>k\}} |f(x,t)|\,e^{\gamma(u_n)} \le \frac{1}{|\Psi(k)|}\iint_{Q_T} |f(x,t)|\,e^{\gamma(u_n)}\,|\Psi(u_n)| \le \frac{C}{|\Psi(k)|}\,,$$

and it follows that

$$\lim_{k\to\infty}\iint_{\{|u_n|>k\}} |f(x,t)|\,e^{\gamma(u_n)} = 0\,,\ \text{uniformly in } n.$$

Hence, given $\varepsilon>0$, we may find k large enough such that $\iint_{\{|u_n|>k\}} |f(x,t)|\,e^{\gamma(u_n)} \le \frac{\varepsilon}{2}$. If we fix such a k, then for every measurable set $E\subset Q_T$, we have

$$\iint_E |f(x,t)|\,e^{\gamma(u_n)} \le c(k)\iint_{E\cap\{|u_n|\le k\}} |f(x,t)| + \iint_{\{|u_n|>k\}} |f(x,t)|\,e^{\gamma(u_n)} \le c(k)\iint_E |f(x,t)| + \frac{\varepsilon}{2}\,.$$

Now the integrability of $f(x,t)$ implies the equi-integrability of our sequence. ∎

End of the Proof of Proposition 7.2: Putting all the estimates together, we have shown that

$$\boxed{\text{C1}} \le c(k)\iint_{\{|u_n|\le k\}} |\nabla u_n|^p\,\varphi((u_n-(T_k u)_\nu)^+) + \omega(n,\nu)$$

$$\le \frac{c(k)}{\Lambda_1}\iint_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\cdot\nabla u_n\,\varphi((u_n-(T_k u)_\nu)^+) + \omega(n,\nu)\,.$$

Now we observe that

$$\iint\limits_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\cdot\nabla u_n\,\varphi((u_n-(T_ku)_\nu)^+)$$

$$= \iint\limits_{\{|u_n|\le k\}} \Big(a(u_n,\nabla u_n)-a(u_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(u_n-(T_ku)_\nu)\,\varphi((u_n-(T_ku)_\nu)^+)$$

$$+ \iint\limits_{\{|u_n|\le k\}} a(u_n,\nabla(T_ku)_\nu)\cdot\nabla(u_n-(T_ku)_\nu)\,\varphi((u_n-(T_ku)_\nu)^+)$$

$$+ \iint\limits_{\{|u_n|\le k\}} a(u_n,\nabla u_n)\cdot\nabla(T_ku)_\nu\,\varphi((u_n-(T_ku)_\nu)^+)\,.$$

The last two integrals can be treated similarly to term $\boxed{\text{C2}}$ above, therefore

$$\boxed{\text{C1}} \le \frac{c(k)}{\Lambda_1}\iint\limits_{\{|u_n|\le k\}} \Big(a(u_n,\nabla u_n)-a(u_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(u_n-(T_ku)_\nu)\,\varphi((u_n-(T_ku)_\nu)^+) + \omega(n,\nu)$$

This is where we use the function φ defined in (7). Indeed, $\varphi(s)$ satisfies

$$\varphi(s)\le\frac{\varphi'(s)}{\mu}\qquad\text{for every } s\ge 0,$$

therefore, if we choose μ such that $\mu>\frac{2\,c(k)}{\Lambda_1}$, and recall the definition of $\boxed{\text{C1}}$, we obtain

$$\boxed{\text{C1}}\le\frac{1}{2}\boxed{\text{C1}}+\omega(n,\nu)\,,$$

from which immediately follows

$$\iint\limits_{\{|u_n|\le k\}} \Big(a(u_n,\nabla u_n)-a(u_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(u_n-(T_ku)_\nu)^+=\omega(n,\nu)\,. \tag{46}$$

Similarly, using $-e^{\gamma(u_n)-\gamma(T_ku_n)}\,\varphi\big((T_ku_n-(T_ku)_\nu)^-\big)$ as test function in (10), one obtains

$$\iint\limits_{\{|u_n|\le k\}} \Big(a(u_n,\nabla u_n)-a(u_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(u_n-(T_ku)_\nu)^-=\omega(n,\nu)\,,$$

which, together with (46), implies

$$\iint\limits_{\{|u_n|\le k\}} \Big(a(T_ku_n,\nabla T_ku_n)-a(T_ku_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(T_ku_n-(T_ku)_\nu)=\omega(n,\nu)\,. \tag{47}$$

On the other hand,

$$\iint\limits_{\{|u_n|>k\}} \Big(a(T_ku_n,\nabla T_ku_n)-a(T_ku_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(T_ku_n-(T_ku)_\nu)$$

$$= \iint\limits_{\{|u_n|>k\}} a(T_ku_n,\nabla(T_ku)_\nu)\cdot\nabla(T_ku)_\nu=\omega(n,\nu)\,,$$

which, together with (47), gives

$$\iint\limits_{Q_T} \Big(a(T_ku_n,\nabla T_ku_n)-a(T_ku_n,\nabla(T_ku)_\nu)\Big)\cdot\nabla(T_ku_n-(T_ku)_\nu)=\omega(n,\nu)\,.$$

Therefore

$$
\begin{aligned}
&\iint_{Q_T} \Big(a(T_k u_n, \nabla T_k u_n) - a(T_k u_n, \nabla T_k u)\Big) \cdot \nabla(T_k u_n - T_k u) \\
&= \iint_{Q_T} \Big(a(T_k u_n, \nabla T_k u_n) - a(T_k u_n, \nabla (T_k u)_\nu)\Big) \cdot \nabla(T_k u_n - (T_k u)_\nu) \\
&\quad + \iint_{Q_T} a(T_k u_n, \nabla T_k u_n) \cdot \nabla((T_k u)_\nu - T_k u) - \iint_{Q_T} a(T_k u_n, \nabla T_k u) \cdot \nabla(T_k u_n - T_k u) \\
&\quad + \iint_{Q_T} a(T_k u_n, \nabla (T_k u)_\nu) \cdot \nabla(T_k u_n - (T_k u)_\nu) \\
&= \omega(n,\nu)\,.
\end{aligned}
$$

Since the left-hand side does not depend on ν, this means

$$
\iint_{Q_T} \Big(a(T_k u_n, \nabla T_k u_n) - a(T_k u_n, \nabla T_k u)\Big) \cdot \nabla(T_k u_n - T_k u) \xrightarrow{n} 0\,.
$$

By a well-known lemma by Browder ([7], p.27, see also [4], p. 190) the last convergence implies the strong convergence of the truncations:

$$
\nabla T_k u_n \xrightarrow{n} \nabla T_k u \qquad \text{strongly in } L^p(Q_T;\mathbb{R}^N), \text{ for every } k>0.
$$

■

8 Proof of the existence theorems

This Section is devoted to prove the main existence theorems, namely Theorems 2.1, 2.2 and 2.3. Since these Theorems essentially differ from each other only in the proof of the a priori estimates (which was carried over in the previous Sections), they will be proved together.

Up to now we have extracted a subsequence of approximate solutions, still denoted by $\{u_n\}_n$, such that

$$
u_n \to u \quad \text{a.e. in } Q, \text{ strongly in } L^1(Q_T) \text{ and weakly in } L^p(0,T;W_0^{1,p}(\Omega))\,, \tag{48}
$$

$$
\nabla u_n \to \nabla u \quad \text{a.e. in } Q\,,
$$

$$
\nabla T_k u_n \to \nabla T_k u \quad \text{strongly in } L^p(Q_T;\mathbb{R}^N), \text{for every } k>0\,, \tag{49}
$$

$$
a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \quad \text{a.e. in } Q \text{ and weakly in } L^{p'}(Q_T;\mathbb{R}^N)\,, \tag{50}
$$

$$
T_n\, b(x,t,u_n,\nabla u_n) \to b(x,t,u,\nabla u) \quad \text{a.e. in } Q\,. \tag{51}
$$

$$
\Psi(u_n) \rightharpoonup \Psi(u) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega))\,,
$$

for every $T>0$. Note that (51) implies, by Fatou's Lemma and estimate (14), that

$$
b(x,t,u,\nabla u)\, e^{\gamma(u)}\, \Psi(u)\,, \qquad f\, e^{\gamma(u)}\, \Psi(u) \in L^1_{\mathrm{loc}}(\overline{Q})\,.
$$

From now on we will assume that $\{u_n\}_n$ is a subsequence satisfying the previous properties.

In this Section we will finish the proof of our main theorems by showing three points. First we will see that the sequence $\{T_n b(x,t,u_n,\nabla u_n)\}_n$ converges in $L^1(Q_T)$ to $b(x,t,u,\nabla u)$ for every $T>0$, then that the initial datum has sense and finally that the limit function u is a weak solution of our problem.

8.1 Convergence of the gradient term

Proposition 8.1

$$T_nb(x,t,u_n,\nabla u_n) \to b(x,t,u,\nabla u) \quad \textit{strongly in } L^1(Q_T)\,, \textit{ for every } T>0. \tag{52}$$

Proof: By (51), we only have to prove that the sequence $\{T_nb(x,t,u_n,\nabla u_n)\}_n$ is equi-integrable and then apply Vitali's Theorem.

Let E be a measurable subset of Q_T and take $\varepsilon>0$. Since

$$\iint\limits_E |T_nb(x,t,u_n,\nabla u_n)| \le \iint\limits_{E\cap\{|u_n|\le k\}} |T_nb(x,t,u_n,\nabla u_n)| + \iint\limits_{\{|u_n|>k\}} |T_nb(x,t,u_n,\nabla u_n)|\;, \tag{53}$$

the result follows from estimating the right hand side. To estimate the second integral in the right hand side, using (16)) one can choose a $k>0$ such that

$$\iint\limits_{\{|u_n|>k\}} T_nb(x,t,u_n,\nabla u_n)\,dx\,dt < \frac{\varepsilon}{2}$$

for all $n\in\mathbb{N}$. On the other hand, we point out that hypothesis **B)** implies

$$|T_nb(x,t,u_n,\nabla u_n)| \le [\beta_1(u_n)+\beta_2(u_n)]\,|\nabla u_n|^p\,,$$

so that

$$\iint\limits_{E\cap\{|u_n|\le k\}} |T_nb(x,t,u_n,\nabla u_n)| \le \max_{|s|\le k}[\beta_1(s)+\beta_2(s)] \iint\limits_E |\nabla T_ku_n|^p\,.$$

Hence, it follows from (53) that

$$\iint\limits_E |T_nb(x,t,u_n,\nabla u_n)| \le c\iint\limits_E |\nabla T_ku_n|^p + \frac{\varepsilon}{2}\,.$$

By Proposition 7.2, the sequence $\{|\nabla T_ku_n|^p\}_n$ is equi-integrable and it implies that the sequence $\{T_nb(x,t,u_n,\nabla u_n)\}_n$ is also equi-integrable. ∎

8.2 Giving sense to the initial datum

The next result was proved (when $p=2$) in [10], Proposition 6.4. The generalization to $p\neq 2$ is straightforward.

Proposition 8.2 *Let $v_n \in L^1\big(0,T;W_0^{1,p}(\Omega)\big)\cap C\big([0,T];L^p(\Omega)\big)$ be a sequence of solutions to problems*

$$\begin{cases} (v_n)_t - \operatorname{div} a(x,t,v_n,\nabla v_n) = g_n, & \textit{in} \quad Q_T;\\ v_n(x,0)=v_{0,n} & \textit{in} \quad \Omega, \end{cases}$$

such that

$$g_n \to g \;\textit{ in } L^1(Q_T), \qquad v_{0,n}\to v_0 \;\textit{ in } L^1(\Omega),$$
$$\nabla T_kv_n \to \nabla T_kv \;\textit{ in } L^p(Q_T;\mathbb{R}^N),\textit{ for every } k>0,$$
$$\nabla v_n \;\textit{ bounded in } L^p(Q_T;\mathbb{R}^N).$$

Then $v_n\to v$ in $C\big([0,T];L^1(\Omega)\big)$.

Applying the previous result with $g_n = T_nb(u_n,\nabla u_n)+T_nf$, we obtain

$$u_n \to u \qquad \text{in } C\big([0,T];L^1(\Omega)\big). \tag{54}$$

Note that this convergence and the a priori estimates for u_n imply that

$$\int\limits_\Omega \Phi(u(x,t))\,dx \le C(T) \qquad \text{for every } t\in[0,T].$$

8.3 End of the proof of the existence results.

We now wish to show that u satisfies the equality (4) for every $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $v_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ (which, we recall, implies that, for every t, $v(t)$ is well defined as an element of $L^\infty(\Omega)$). Using the weak formulation of problems (10), one has

$$\int_\Omega u_n(\tau)\,v(\tau) - \int_\Omega u_{0,n}\,v(0) - \int_0^\tau \langle v_t(t)\,,\,u_n(t)\rangle\,dt + \iint_{Q_\tau} a(u_n,\nabla u_n)\cdot\nabla v$$
$$= \iint_{Q_\tau} T_n b(u_n,\nabla u_n)\,v + \iint_{Q_\tau} T_n f\,v\,.$$

Using the convergences (11), (48), (50), (51), (52) and (54), one can take the limit as $n \to \infty$ in each term of the last equality, obtaining (4).

We want to show that (5) holds. Indeed, let $h(s) : \mathbb{R} \to \mathbb{R}$ be a function as in Definition 2.1 **(5)**. Then one can use $h(T_k u_n)$ as test function in the approximate problems (10), obtaining

$$\int_\Omega H_k(u_n(\tau)) - \int_\Omega H_k(u_{0,n}) + \iint_{Q_\tau} a(u_n,\nabla u_n)\cdot\nabla T_k u_n\, h'(T_k u_n)$$
$$= \iint_{Q_\tau} \big(T_n b(u_n,\nabla u_n) + T_n f\big)\,h(T_k u_n)\,, \quad (55)$$

where

$$H_k(s) = \int_0^s h(T_k\sigma)\,d\sigma\,.$$

Note that $|H_k(s)| \le c(k)\,(|s|+1)$. Using the convergences (54), (11), (49), (50) and (52), one can easily take the limit as $n \to +\infty$ in equality (55), obtaining

$$\int_\Omega H_k(u(\tau)) - \int_\Omega H_k(u_0) + \iint_{Q_\tau} a(u,\nabla u)\cdot\nabla T_k u\, h'(T_k u) = \iint_{Q_\tau} \big(b(u,\nabla u) + f\big)\,h(T_k u)\,. \quad (56)$$

Then we let k go to infinity. Note that, under our assumptions on h,

$$|H_k(u)| \le M_1 \left| \int_0^u \big(1 + e^{\gamma(T_k s)}|\Psi(T_k s)|\big)\,ds \right| \le c\,\big(1 + \Phi(u)\big)\,,$$
$$|a(u,\nabla u)\cdot\nabla T_k u\, h'(T_k u)| \le \Lambda_2\, M_1 |\nabla u|^p \Psi'(u)^p = \Lambda_2\, M_1 |\nabla\Psi(u)|^p\,,$$
$$\big|\big(b(u,\nabla u) + f\big)\,h(T_k u)\big| \le M_1\big(|b(u,\nabla u)| + |f|\big)\,\big(1 + e^{\gamma(u)}|\Psi(u)|\big)\,,$$

therefore we can apply the dominated convergence theorem to every integral in (56), and therefore (5) is proved.

Finally, we will show that $\Phi(u) \in C\big([0,\infty[;L^1(\Omega)\big)$: if we choose $h(s) = e^{\gamma(s)}\Psi(s)$ in (5), one obtains

$$\int_\Omega \Phi(u(t)) - \int_\Omega \Phi(u(\tau)) + \iint_{Q_{\tau,t}} \Psi'(u)^p\, a(u,\nabla u)\cdot\nabla u$$
$$+ \iint_{Q_{\tau,t}} \frac{\beta_2(u)}{\Lambda_1}|\Psi(u)|e^{\gamma(u)}\, a(u,\nabla u)\cdot\nabla u = \iint_{Q_{\tau,t}} b(u,\nabla u)\Psi(u)e^{\gamma(u)} + \iint_{Q_{\tau,t}} f\Psi(u)e^{\gamma(u)}\,,$$

for every $\tau, t \ge 0$. From this it follows that the function $\xi(t) = \int_\Omega \Phi(u(t))\,dx$ is continuous on $[0,\infty[$. Consider a sequence t_n in $[0,\infty[$ converging to t. Then $u(t_n) \to u(t)$ in $L^1(\Omega)$, and, up to subsequences, $u(t_n) \to u(t)$ a.e in Ω. Thus, it yields

$$\Phi(u(t_n)) \to \Phi(u(t)) \qquad \text{a.e in } \Omega\,.$$

On the other hand, one has $\|\Phi(u(.,t_n)\|_1 \to \|\Phi(u(.,t)\|_1$. Therefore we have proved that $\Phi(u(.,t_n)) \to \Phi(u(.,t))$ in $L^1(\Omega)$, as desired. ■

9 Proof of the L^∞ estimates

This Section is devoted to the proof of the boundedness result stated in Theorem 2.4. First of all, for $T > 0$ and $t \in [0,T]$ let us define the sets

$$A_k = \big\{(x,t) \in Q_T \ : \ |\Psi(u(x,t))| \geq k\big\}\,, \qquad A_k(t) = \big\{x \in \Omega \ : \ |\Psi(u(x,t))| \geq k\big\}\,.$$

Let us take

$$h(u) = h_k(u) = e^{\gamma(u)} G_k\Psi(u)$$

as test function in (5), where γ and Ψ are defined as in (3), $G_k s$ is defined as in (9), while

$$k \geq \max\{1, \|\Psi(u_0)\|_\infty\}\,.$$

Then, after simplifying the gradient term as in Proposition 3.1, one obtains:

$$\begin{aligned} \max_{\tau\in[0,T]} \int_\Omega H_k(u(x,\tau)) + \iint_{Q_T} |\nabla G_k\Psi(u)|^p &\leq c\left(\int_\Omega H_k(u_0(x)) + \iint_{Q_T} f\, e^{\gamma(u)}\, G_k\Psi(u)\right) \\ &\leq c \iint_{Q_T} |f|\, e^{\gamma(u)}\, |G_k\Psi(u)|\,, \end{aligned} \tag{57}$$

where

$$H_k(s) = \int_0^s e^{\gamma(\sigma)}\, G_k\Psi(\sigma)\, d\sigma\,.$$

We now have to consider separately the two cases $p \geq 2$ and $1 < p < 2$. Let us consider the case $p \geq 2$. In this case one has

$$H_k(s) \geq c\,|G_k\Psi(s)|^p \tag{58}$$

for some positive constant c. Indeed, using (34),

$$\begin{aligned} H_k(s) = H_k(|s|) = &\int_0^{|s|} e^{\frac{\gamma(\sigma)}{p-1}}\, e^{\frac{p-2}{p-1}\gamma(\sigma)}\, G_k\Psi(\sigma)\, d\sigma \\ &\geq c\int_0^{|s|} \Psi(\sigma)^{p-2}\, (\log^*\Psi(\sigma))^{p-2}\, e^{\frac{\gamma(\sigma)}{p-1}}\, G_k\Psi(\sigma)\, d\sigma \\ &\geq c\int_0^{|s|} \big(G_k\Psi(\sigma)\big)^{p-1}\,\big(G_k\Psi(\sigma)\big)'\, d\sigma = c\,|G_k\Psi(s)|^p\,. \end{aligned}$$

On the other hand, we wish to show that, for every $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$e^{\gamma(s)}\,|G_k\Psi(s)| \leq c(\varepsilon)\left(|G_k\Psi(s)|^{p+\varepsilon} + k^p\right), \tag{59}$$

for every $k \geq 1$ and every $s \in \mathbb{R}$. Indeed, using the second inequality in (34), one has, for $\delta = \frac{\varepsilon}{p+\varepsilon} < \varepsilon$,

$$e^{\gamma(s)} \leq c(\varepsilon)\,|\Psi(s)|^{p-1+\delta} \leq c(\varepsilon)\left(|G_k\Psi(s)|^{p-1+\delta} + k^{p-1+\delta}\right)$$

for every s such that $|\Psi(s)| > 1$; therefore

$$e^{\gamma(s)}\,|G_k\Psi(s)| \leq c(\varepsilon)\left(|G_k\Psi(s)|^{p+\delta} + k^{p-1+\delta}\,|G_k\Psi(s)|\right), \quad s \in \mathbb{R}.$$

By Young's inequality we have

$$k^{p-1+\delta}\,|G_k\Psi(s)| \le c(\varepsilon)\left(k^p + |G_k\Psi(s)|^{\frac{p}{1-\delta}}\right) = c(\varepsilon)\left(k^p + |G_k\Psi(s)|^{p+\varepsilon}\right),$$

and so

$$e^{\gamma(s)}\,|G_k\Psi(s)| \le c(\varepsilon)\left(|G_k\Psi(s)|^{p+\delta} + |G_k\Psi(s)|^{p+\varepsilon} + k^p\right),$$

which implies (59), since $|G_k\Psi(s)|^{p+\delta} \le |G_k\Psi(s)|^{p+\varepsilon} + 1$. Therefore, going back to (57), if we set

$$w_k = |G_k\Psi(u)|,$$

we have shown that

$$\max_{\tau\in[0,T]}\int_\Omega w_k(x,\tau)^p\,dx + \iint_{Q_T}|\nabla w_k|^p \le c(\varepsilon)\iint_{A_k}|f|\left(|w_k|^{p+\varepsilon} + k^p\right), \tag{60}$$

where ε will be chosen as in (64) below.

We now proceed to estimate the right hand side of this inequality. Namely, we will prove the following claim: *There exist positive constants η and c satisfying*

$$\max_{\tau\in[0,T]}\int_\Omega w_k(x,\tau)^p\,dx + \iint_{Q_T}|\nabla w_k|^p \le c\,k^p\mu(k)^{\frac{1+\eta}{\rho}p}, \tag{61}$$

where

$$\mu(k) = \int_0^T |A_k(t)|^{\rho/\sigma}\,dt, \qquad \textit{with } \rho,\sigma > 1 \textit{ such that } \quad \frac{N}{\sigma} + \frac{p}{\rho} = \frac{N}{p}. \tag{62}$$

Once this claim is proved, we can apply Theorem 6.1 of Chapter II in [21] to obtain the boundedness of $\Psi(u)$, and a fortiori of u.

The idea to prove (61) is, roughly speaking, that the term $\iint_{A_k}|f|\,w_k^{p+\varepsilon}$ in (60) may be absorbed by the left-hand side. Indeed, recall that, by assumption **F)**, (r,q) satisfies $\frac{1}{r} + \frac{N}{p\,q} < 1$, therefore we may find $\nu > 1$ such that the exponents

$$q_1 = \frac{q}{\nu'}, \qquad r_1 = \frac{r}{\nu'}$$

still satisfy

$$\frac{1}{r_1} + \frac{N}{p\,q_1} < 1. \tag{63}$$

Then Young's inequality implies

$$|f|w_k^\varepsilon \le |f|^{\nu'} + w_k^{\varepsilon\nu},$$

and so

$$\iint_{Q_T}|f|w_k^{p+\varepsilon} \le \iint_{Q_T}|f|^{\nu'}w_k^p + \iint_{Q_T}w_k^{p+\varepsilon\nu} = I_1 + I_2.$$

As a consequence of Hölder's inequality and of (63), we have that

$$\iint_{Q_T}|f|^{\nu'}w_k^p \le \|f\|^{\nu'}_{\nu' q_1,\nu' r_1}\|w_k\|^p_{p\,q_1',p\,r_1'} \le c\,\|f\|^{\nu'}_{q,r}\|w_k\|^p_{\sigma,\rho},$$

with (σ,ρ) satisfying $\dfrac{N}{\sigma} + \dfrac{p}{\rho} = \dfrac{N}{p}$. Thus, if $\|f\|_{q,r}$ is small enough, we may absorb I_1 by the left hand side of (60); otherwise, one can divide the interval $]0,T[$ in a finite number of subintervals such that in each one of them the above norm is small enough.

A similar manipulation, although a little bit involved since f does not appear, can be done to absorb I_2. Consider a pair (σ_1,ρ_1) such that $\dfrac{N}{\sigma_1} + \dfrac{p}{\rho_1} = \dfrac{N}{p}$ and apply Hölder's inequality to get

$$\iint_{Q_T}w_k^{p+\varepsilon\nu} \le \|w_k\|^{\varepsilon\nu}_{\sigma_1,\rho_1}\|w_k\|^p_{\frac{p\sigma_1}{\sigma_1-\varepsilon\nu},\frac{p\rho_1}{\rho_1-\varepsilon\nu}} \le c\,\|w_k\|^p_{\frac{p\sigma_1}{\sigma_1-\varepsilon\nu},\frac{p\rho_1}{\rho_1-\varepsilon\nu}}.$$

Indeed, since $f\, e^{\gamma(u)}\,\Psi(u) \in L^1(Q_T)$, it follows from (57) and (58) that w_k is bounded in the spaces $L^\infty(0,T;L^p(\Omega))$ and $L^p(0,T;W_0^{1,p}(\Omega))$ uniformly in k. Therefore, applying Lemma 3.3, we deduce that the norm $\|w_k\|_{\sigma_1,\rho_1}$ is bounded by a constant depending on T but not on k.

Note that we may choose ε small enough (depending on ν) to have

$$\frac{N}{p\sigma_1/(\sigma_1-\varepsilon\nu)} + \frac{p}{p\rho_1/(\rho_1-\varepsilon\nu)} = \frac{N}{p}\Big(1-\frac{\varepsilon\nu}{\sigma_1}\Big) + \Big(1-\frac{\varepsilon\nu}{\rho_1}\Big) > \frac{N}{p}\,. \tag{64}$$

Then using Hölder's inequality again, it results that

$$\|w_k\|^p_{\frac{p\sigma_1}{\sigma_1-\varepsilon\nu},\frac{p\rho_1}{\rho_1-\varepsilon\nu}} \le \omega(T)\|w_k\|^p_{\sigma,\rho}$$

where (σ,ρ) satisfies $\dfrac{N}{\sigma}+\dfrac{p}{\rho}=\dfrac{N}{p}$ and $\omega(T)$ is a quantity which tends to 0, as T goes to 0. Therefore, if $\omega(T)$ is small enough, then the term I_2 can be absorbed by the left hand side of (70) choosing T small enough, otherwise, as before, one can split the interval $]0,T[$ in a finite number of subintervals.

Finally we analyse the integral $\displaystyle\iint_{A_k}|f|$. Having in mind the assumption **F)**, one can choose $\eta>0$ such that

$$1+\eta = \frac{p}{N}\,\Big(\frac{N}{q'p}+\frac{1}{r'}\Big)\,.$$

Then the exponents

$$\rho = r'p\,(1+\eta)\,, \qquad \sigma = q'p\,(1+\eta)\,,$$

satisfy the equality in (62). Then

$$\iint_{A_k}|f| \le \|f\|_{q,r}\Big(\int_0^T |A_k(t)|^{r'/q'}\,dt\Big)^{1/r'} = c\Big(\int_0^T |A_k(t)|^{\rho/\sigma}\,dt\Big)^{p(1+\eta)/\rho} = c\,\mu(k)^{p(1+\eta)/\rho}\,.$$

Summing up, we have proved the claim (61), and so the L^∞ estimate in the case $p\ge 2$.

Let us turn our attention to the case $1<p<2$. Now we are not able to follow the same arguments as above, since (58) is not true in this case. Instead of defining w_k as above, we now define

$$w_k = e^{\gamma(u)\frac{p-2}{p(p-1)}}\,(G_k\Psi(u))^{\frac{2}{p}}\,;$$

our next aim is to transform inequality (57) in terms of this new w_k.

Left-hand side.- Since

$$\begin{aligned} H_k(s) = H_k(|s|) &= \int_0^s e^{\gamma(\sigma)\frac{p-2}{p-1}}\,e^{\frac{\gamma(\sigma)}{p-1}}\,G_k\Psi(\sigma)\,d\sigma \\ &\ge e^{\gamma(s)\frac{p-2}{p-1}}\int_0^s \Psi'(\sigma)\,G_k\Psi(\sigma)\,d\sigma = \frac{1}{2}e^{\gamma(s)\frac{p-2}{p-1}}\,(G_k\Psi(s))^2\,, \end{aligned}$$

it follows that

$$H_k(u) \ge \frac{1}{2}w_k^p\,. \tag{65}$$

As far as the second term is concerned, we will next see that

$$|\nabla G_k\Psi(u)|^p \ge c|\nabla w_k|^p\,. \tag{66}$$

First note that both members vanish when $|\Psi(u)|\le k$; thus we only have to consider $(x,t)\in A_k$. Fix one of these points. Then

$$\begin{aligned} |\nabla w_k| &\le \frac{2-p}{\Lambda_1 p(p-1)}\,\beta_2(u)\,e^{\gamma(u)\frac{p-2}{p(p-1)}}\,(G_k\Psi(u))^{2/p}\,|\nabla u| + \frac{2}{p}\,e^{\gamma(u)\frac{p-2}{p(p-1)}}\,\big|G_k\Psi(u)\big|^{(2-p)/p}\,e^{\frac{\gamma(u)}{p-1}}\,|\nabla u| \\ &\le c\,\beta_2(u)\,e^{\gamma(u)\frac{p-2}{p(p-1)}}\,\Psi(u)^{2/p}\,|\nabla u| + c\,e^{\gamma(u)\frac{2}{p}}\,|\Psi(u)|^{(2-p)/p}\,|\nabla u| \\ &= I_1+I_2\,. \end{aligned} \tag{67}$$

As a consequence of Lemma 6.1 and of De L'Hôpital's rule, it yields

$$\lim_{s\to\infty}\frac{\beta_2(s)\,e^{\gamma(s)\frac{p-2}{p(p-1)}}\,\Psi(s)^{2/p}}{e^{\gamma(s)/(p-1)}}=\lim_{s\to\infty}\left(\frac{\Psi(u)}{e^{\gamma(s)/(p-1)}}\right)^{\frac{2-p}{p}}=0\,;$$

and, on the other hand,

$$\lim_{s\to\infty}\frac{e^{2\gamma(s)/p}\,\Psi(s)^{(2-p)/p}}{e^{\gamma(s)/(p-1)}}=\lim_{s\to\infty}\left(\frac{\Psi(u)}{e^{\gamma(s)/(p-1)}}\right)^{\frac{2-p}{p}}=0\,.$$

Therefore, there exist positive constants c_1, c_2 satisfying

$$I_i\le c_i\,e^{\frac{\gamma(u)}{p-1}}\,|\nabla u|\,,\qquad i=1,2\,.$$

Going back to (67), we obtain

$$|\nabla w_k|\le c\,e^{\frac{\gamma(u)}{p-1}}\,|\nabla u|\,,$$

from where (66) follows.

Right-hand side.- Next we will prove the following claim: for every $\varepsilon>0$ there exists $c(\varepsilon)>0$ satisfying

$$e^{\gamma(u)}\,|G_k\Psi(u)|\le c(\varepsilon)\big(w_k^{p+\varepsilon}+k^p\big)\chi_{A_k}\,. \tag{68}$$

Let $\delta>1$ satisfy $\varepsilon=p\,(\delta-1)$, and apply Young's inequality to obtain

$$\begin{aligned} e^{\gamma(u)}\,|G_k\Psi(u)| &= e^{\gamma(u)\frac{p-2}{2(p-1)}}\,|G_k\Psi(u)|\,e^{\gamma(u)\frac{p}{2(p-1)}}\\ &\le e^{\gamma(u)\frac{(p-2)\delta}{p-1}}\,|G_k\Psi(u)|^{2\delta}+e^{\gamma(u)\frac{p\,\delta}{(p-1)(2\delta-1)}}\le w_k^{p+\varepsilon}+\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-\varepsilon'}, \end{aligned}\tag{69}$$

where $\varepsilon'=p\,(\delta-1)/(2\delta-1)$. Having in mind the inequalities (34) and $|\Psi(u)|>k\ge1$, and performing easy manipulations in the last term of (69), it yields

$$\begin{aligned}\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-\varepsilon'}&=\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-2}\left(e^{\frac{\gamma(u)}{p-1}}\right)^{2-\varepsilon'}\le c(\varepsilon)\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-2}\Psi(u)^2\\ &\le c(\varepsilon)\,e^{\gamma(u)\frac{p-2}{p-1}}\big(|\Psi(u)|-k\big)^2+c(\varepsilon)\,k^2\,e^{\gamma(u)\frac{p-2}{p-1}}\,.\end{aligned}$$

By noting that $e^{\gamma(u)\frac{p-2}{p-1}}\big(|\Psi(u)|-k\big)^2=w_k^p$ and $e^{\gamma(u)\frac{p-2}{p-1}}\le c\,|\Psi(u)|^{p-2}\le c\,k^{p-2}$ (recall that $p-2<0$), we deduce that

$$\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-\varepsilon'}\le c(\varepsilon)\,(w_k^p+k^p)\,.$$

Moreover, since $w_k^p\le w_k^{p+\varepsilon}+1\le w_k^{p+\varepsilon}+k^p$, it follows that

$$\left(e^{\frac{\gamma(u)}{p-1}}\right)^{p-\varepsilon'}\le c(\varepsilon)\,(w_k^{p+\varepsilon}+k^p)\,.$$

Hence, from here, (69) implies (68). On account of (65), (66) and (68), the inequality (57) becomes

$$\max_{\tau\in[0,T]}\int_\Omega w_k(x,\tau)^p\,dx+\iint_{Q_T}|\nabla w_k|^p\le c(\varepsilon)\iint_{A_k}|f|\,\big(w_k^{p+\varepsilon}+k^p\big)\,. \tag{70}$$

From here, one obtains the claim (61) exactly as in the case $p\ge2$. However, that inequality is now not enough to obtain the L^∞-estimate directly, since now $w_k\ne|G_k(\Psi(u))|$ and so we cannot apply Theorem 6.1 in [21]. Hence, we are going to estimate also the left hand side of (70), proving that, for every $\varepsilon>0$, there exists a constant $c(\varepsilon)>0$ such that

$$\|w_k\|_{\sigma,\rho}^p\ge c(\varepsilon)(h-k)^2h^{(1+\varepsilon)(p-2)}\mu(h)^{p/\rho}\qquad\text{for all }\ h>k\ge1\,, \tag{71}$$

for (ρ,σ) as in (62). Indeed, on account of hypothesis **C3)**, and recalling that the exponent of h is negative and the function $s \to s^{\frac{(1+\varepsilon)(p-2)}{p}}(s-k)^{\frac{2}{p}}$ is increasing, one obtains

$$\|w_k\|_{\rho,\sigma}^p = \left[\int_0^T \left(\int_{A_k(t)} \left(e^{\gamma(u)\frac{p-2}{p(p-1)}}|G_k\Psi(u)|^{2/p}\right)^\sigma\right)^{\frac{\rho}{\sigma}}\right]^{\frac{p}{\rho}}$$

$$\geq c(\varepsilon)\left[\int_0^T \left(\int_{A_k(t)} \left(|\Psi(u)|^{\frac{(1+\varepsilon)(p-2)}{p}}|G_k\Psi(u)|^{2/p}\right)^\sigma\right)^{\frac{\rho}{\sigma}}\right]^{\frac{p}{\rho}}$$

$$\geq c(\varepsilon)h^{(1+\varepsilon)(p-2)}(h-k)^2\left[\int_0^T |A_h(t)|^{\frac{\rho}{\sigma}}\right]^{\frac{p}{\rho}} = c(\varepsilon)h^{(1+\varepsilon)(p-2)}(h-k)^2\mu(h)^{\frac{p}{\rho}}\,.$$

Having in mind Lemma 3.3, it follows from (61) and (71) that

$$\mu(h)^{1/\rho} \leq c(\varepsilon)\frac{h^{(1+\varepsilon)(2-p)/p}}{(h-k)^{2/p}}k\mu(k)^{(1+\eta)/\rho} \qquad \text{for all } \ h > k \geq 1\,. \tag{72}$$

Our purpose is to deduce from this inequality that $\mu(k) = 0$ for k large enough; this straightforwardly implies our L^∞-estimate.

First observe that $\Psi(u) \in C([0,\infty[: L^1(\Omega))$ implies $|A_k(t)| \leq \frac{c}{k}$ for all $t \in [0,T]$ and so

$$\mu(k) = \int_0^T |A_k(t)|^{\rho/\sigma}\,dt \leq \frac{c}{k^{\rho/\sigma}}\,.$$

Thus, we have obtained

$$\mu(k)^{1/\rho} \leq \frac{c}{k^\lambda}\,, \qquad \text{with } \lambda > 0.$$

We now take $\varepsilon > 0$ small enough to satisfy $\lambda > \frac{\varepsilon(2-p)}{p\eta}$, and consider M such that

$$\mu(M)^{1/\rho} \leq c(\varepsilon)^{-1/\eta}\,M^{-\varepsilon\frac{2-p}{p\eta}}\,b^{-1/\eta^2}\,, \tag{73}$$

where $b = 2^{2/p} > 1$. If we denote $k_n = M\,(2-2^{-n})$, then, by (72), we have

$$\mu(k_{n+1})^{1/\rho} \leq c(\varepsilon)\,\frac{k_n\,k_{n+1}^{(1+\varepsilon)(2-p)/p}}{(k_{n+1}-k_n)^{2/p}}\,\mu(k_n)^{(1+\eta)/\rho}\,.$$

It follows that

$$\mu(k_{n+1})^{1/\rho} \leq c\,M^{\varepsilon(2-p)/p}\,b^n\,\mu(k_n)^{(1+\eta)/\rho}\,.$$

Since (73) holds true, we may apply the result in [21], Lemma 5.6, Chapter II, to obtain that $\mu(2M) = 0$. ∎

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