Existence of bounded solutions for nonlinear elliptic equations in unbounded domains

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Abstract

In this paper we study the existence of bounded weak solutionsfor some nonlinear Dirichlet problems in unbounded domains. The principal part of the operator behaves like the p-laplacian operator, and the lower order terms, which depend on the solution u and its gradient ∇u, have a power growth of order $p-1$ with respect to these variables, while they are bounded in the x variable. The source term belongs to a Lebesgue space with a prescribed asymptotic behaviour at infinity.

Key words and phrases: existence, nonlinear elliptic equations, p-Laplacian, unbounded domains, L^∞-estimate, homogeneous lower order terms.

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1 Introduction

In this paper we deal with existence of bounded weak solutions of some nonlinear elliptic problems in a possibly unbounded open subset Ω of $\mathbf{R}^N$. More precisely, we refer to the problem

$$\begin{cases} -\Delta_p(u) + \nu_0 |u|^{p-2}u + \gamma(x)\cdot \nabla u |\nabla u|^{p-2} = g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega)\,, \end{cases} \tag{1}$$

as a model case. Here $p \geq 2$, $\nu_0 > 0$, $\gamma(x) \in (L^\infty(\Omega))^N$, $-\Delta_p(u) = -\text{div}\,(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator and $g \in L^q_{\text{loc}}(\Omega)$, $q > N/p$, with a prescribed asymptotic behavior at infinity. The term $\gamma(x)\cdot\nabla u|\nabla u|^{p-2}$ may be replaced by any function $b(x,\nabla u)$ such that

$$|b(x,\nabla u)| \leq d\,|\nabla u|^{p-1}$$

for some positive constant d. Let us point out that, even if Ω is a bounded open set, the operator associated to the problem is, in general, not coercive.

Many authors have considered problem (1) in bounded domains since the earliest paper [15] by Stampacchia, where $|\gamma(x)|$ is supposed to belong to $L^N(\Omega)$, with a sufficiently small L^N-norm. We refer for example to [1], [2], [7], [11] for the linear case

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($p = 2$) and to [5] for the nonlinear case. In particular in [2] the authors also deal with unbounded domains and with a function $\gamma(x)$ not necessarily small in the L^N-norm. Similar hypotheses on $\gamma(x)$ and Ω (*i.e.*, $|\gamma(x)| \in L^s(\Omega) \cap L^\infty(\Omega)$ for some $1 \le s \le +\infty$, Ω unbounded) are made in [6], where the principal part is a quasi-linear operator and more general growth conditions on the lower order term in the gradient are allowed.

It is well known that L^p spaces in domains having infinite measure are not comparable and therefore the hypothesis $|\gamma(x)| \in L^\infty(\Omega)$ does not imply any summability property.

The main existence result for the case of bounded coefficients in domains having infinite measure is due to P.-L. Lions (see also [3] for further results). This result is strictly confined to the framework of linear operators ($p = 2$) because of the techniques used there (duality argument, linear interpolation).

We point out that we are interested in solutions having finite energy, that is, solutions in $W_0^{1,p}(\Omega)$. If we just look for distributional solutions in $W_{\mathrm{loc}}^{1,p}(\Omega)$, we can drop several assumptions on the operator (in particular the assumption $p \ge 2$) and on the datum. This is the subject of the last section of this paper.

The strategy we use consists in approximating the domain Ω by bounded sets Ω_n, and solving a more regular problem in Ω_n. The main difficulty that arises is to get a uniform (with respect to n) L^p-estimate on the solutions u_n of the approximate problems. Such an estimate is obtained in a two-step process: first we give an a priori L^∞-estimate for the solutions. To obtain an L^p-estimate outside of compact subsets of Ω, we use a method of sub and supersolutions, together with a comparison result. Since we need to find explicit supersolutions, we are forced to require some differentiability assumptions on the operator, which would not be necessary otherwise. However we point out that several intermediate results do not use such strong assumptions.

Once an L^p-estimate on the solution is obtained, a simple use of Young's inequality will give a uniform estimate of the $W_0^{1,p}$-norms and therefore the weak convergence, up to a subsequence, to a function $u \in W_0^{1,p}(\Omega)$, which is shown to be a solution of problem (1). The limit process does not present any real difficulty since the principal part is a strongly monotone operator, which gives a strong local convergence of the gradients in the L^p-norm.

The plan of the paper is as follows. In Section 2 we give the precise assumptions, the notations, and state the main result. In order to get uniform L^p-estimates for the approximate solutions u_n we first need a uniform L^∞-estimate, which will be proved in Section 3. In the next Section 4 we prove a comparison principle (which also holds for domains of infinite measure) for solutions to equation (1). We remark that this is the only step where the assumption $p \ge 2$ is required. Since our proof relies heavily on such a comparison result, we are currently unable to treat the case where the principal part depends directly on u. In Section 5 we construct a fixed supersolution of the approximate problems in the exterior domain $\Omega_n \setminus B$, where B is a fixed ball centered at the origin. This will give the required L^p-regularity in $\Omega \setminus B$ and consequently in Ω. Section 6 is devoted to the passage to the limit, which concludes the proof of the existence theorem. Section 7 deals with the case $p = 2$, with much more general hypotheses on the datum g. The method used here relies on the reduction to the linear case and on the estimates proved by P.-L. Lions in [12] for solutions of linear equations in unbounded domains. Finally, in Section 8, we show that, if we limit ourselves to look for distributional solutions in $W_{\mathrm{loc}}^{1,p}(\Omega)$, we can obtain an existence theorem for a much more general operator (in particular, we may take $p > 1$), under the only hypothesis $g \in L^q(\Omega)$, with q large enough (but finite). The boundary condition is assumed in the sense that there exists $\delta > 1$ such that $|u|^\delta$ belongs to $W_0^{1,p}(\Omega)$. However this does not allow to conclude that $u \in W_0^{1,p}(\Omega)$.

The case where the first-order term of (1) has a "natural" growth of order p (instead of $p-1$) with respect to the gradient has been studied in [4]. While it is clear that in the case of bounded domains terms with growth p are more difficult to handle than terms with growth $p-1$, in the case of domains having infinite measure the respective difficulties are not comparable. Indeed the case of order $p-1$ is not simpler than the case of order p, since $|\nabla u|$ may be less than 1 on a set of infinite measure.

The problem when the principal part is a general Leray-Lions type operator (just monotone in the gradient variable) and the datum g is in $L^{\frac{p}{p-1}}(\Omega)$, or in $L^{\frac{Np}{Np-N+p}}(\Omega)$ $(p < N)$ (which are more natural spaces) is, as far as we know, still open.

2 Notations, assumptions and main result

Let Ω be an open subset of $\mathbf{R}^N$, possibly unbounded. We consider the following nonlinear problem

$$\begin{cases} A(u) := -\operatorname{div}(a(x, \nabla u)) + b(x, \nabla u) + c(x, u) = g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{P}$$

All the functions $a(x, \xi) = (a_1(x, \xi), \ldots, a_N(x, \xi)) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, $b(x, \xi) : \Omega \times \mathbf{R}^N \to \mathbf{R}$, $c(x, s) : \Omega \times \mathbf{R} \to \mathbf{R}$ are always assumed to be Carathéodory functions, *i.e.*, they are continuous with respect to the second variable, for almost every $x \in \Omega$, and measurable with respect to x for every fixed ξ, or s, respectively.

Moreover, in order to prove the main result we will require the following hypotheses:

$$p \geq 2\,;$$

(A1) $a(x, \xi)$ is differentiable with respect to ξ for almost every $x \in \Omega$, and weakly differentiable with respect to x for every $\xi \in \mathbf{R}^N$; moreover there exist positive constants M_1 and M_2 such that, for almost every $x \in \Omega$ and every $\xi \in \mathbf{R}^N$,

$$|a_\xi(x, \xi)| \leq M_1 |\xi|^{p-2} \tag{2}$$

and

$$|a_x(x, \xi)| \leq M_2 |\xi|^{p-1}\,; \tag{3}$$

(A2) there exists a constant $\beta > 0$ such that

$$|a(x, \xi)| \leq \beta |\xi|^{p-1}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^N$;

(A3) there exists a constant $\nu > 0$ such that

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) \geq \nu|\xi - \eta|^p$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbf{R}^N$.

Let us observe that from (A2) and (A3) it follows that

(A4) $a(x,\xi) \cdot \xi \geq \nu|\xi|^p$ for almost every $x \in \Omega$ and every $\xi \in \mathbf{R}^N$.

We remark that the differentiability assumptions (A1) are only needed for the results of Section 5, while the strong monotonicity (A3) is used for the comparison principle (Section 4).

Moreover we assume the further conditions on the remaining terms:

(B1) there exists a constant $d > 0$ such that for almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^N$

$$|b(x,\xi)| \leq d|\xi|^{p-1};$$

(B2) b is a locally Lipschitz function with respect to ξ uniformly with respect to x, *i.e.*, for every $k > 0$ there exists $L(k) > 0$ such that for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbf{R}^N$, with $|\xi| \leq k$, $|\eta| \leq k$,

$$|b(x,\xi) - b(x,\eta)| \leq L(k)|\xi - \eta|;$$

(C1) there exists a constant $\Lambda > 0$ such that for almost every $x \in \Omega$ and for every $s \in \mathbf{R}$

$$|c(x,s)| \leq \Lambda|s|^{p-1};$$

(C2) there exists a constant $\nu_0 > 0$ such that for almost every $x \in \Omega$ and for every $s, t \in \mathbf{R}$

$$[c(x,s) - c(x,t)]\,(s - t) \geq \nu_0|s - t|^p.$$

Let us observe that from (C1) and (C2) it follows that

(C3) for almost every $x \in \Omega$ and for every $s \in \mathbf{R}$

$$c(x,s)s \geq \nu_0|s|^p.$$

We point out that assumptions (B2) and (C2) are only required for the comparison principle (Section 4). In the rest of the paper only assumptions (B1), (C1), (C3) are needed on the lower order terms.

Remark 2.1 Typical examples of operators satisfying the previous assumptions are

$$A(u) = -\Delta_p(u) + \nu_0|u|^{p-2}u + \gamma(x) \cdot \nabla u|\nabla u|^{p-2},$$

where $\gamma(x) \in (L^\infty(\Omega))^N$, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator, and

$$A(u) = -\Delta_p(u) + \nu_0|u|^{p-2}u + \gamma(x)|\nabla u|^{p-1},$$

with $\gamma(x) \in L^\infty(\Omega)$.

As far as the right-hand side of (P) is concerned, we assume that:

(G1) There exists $R_0 > 0$ such that $g(x)\Big|_{\Omega\cap B_{R_0}} \in L^q(\Omega \cap B_{R_0})$ for some $q > \max\left\{\frac{N}{p}, 1\right\}$ (here and in the sequel B_{R_0} denotes the ball with center 0 and radius R_0);

(G2) there exist $K > 0$, $r > N/p'$ (with $1/p + 1/p' = 1$) such that, for almost every $x \in \Omega$, with $|x| \geq R_0$ (same R_0 as in (G1)),

$$|g(x)| \leq \frac{K}{|x|^r}.$$

Let us remark that condition (G2) implies that $g \in L^{p'}(\Omega \setminus B_{R_0})$.

We are interested in finding weak solutions of problem (P), i.e. solutions of (P) in the sense of distributions.

The main result in this paper is the following.

Theorem 2.2 *Assume that $p \geq 2$, and that the assumptions* (A1)–(A3), (B1), (B2), (C1), (C2), (G1), (G2) *hold. Then there exists a bounded weak solution u for problem* (P).

The proof of the main theorem will be carried out in sections 5 and 6, by approximating our problem by more regular ones on bounded domains. In Sections 3 and 4 we will prove some preliminary results (L^∞ estimates and a comparison theorem) which will be applied to the approximate problems.

We now introduce some notation and results which will be useful in the sequel. For $k > 0$, we will denote by $G_k(s)$ the function

$$G_k(s) = \begin{cases} s - k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ k + s & \text{if } s < -k. \end{cases}$$

If v is a measurable function on Ω, and $k > 0$, we denote by A_k the set

$$A_k = A_k(v) = \{x \in \Omega : |v(x)| > k\}.$$

We state some classical results which will be used in the next sections.

Lemma 2.3 (Stampacchia, see [15]). *Let φ be a nonnegative, nonincreasing function defined on the half line $[k_0, \infty)$. Suppose that there exist positive constants A, γ, β, with $\beta > 1$, such that*

$$\varphi(h) \leq \frac{A}{(h-k)^\gamma}\varphi(k)^\beta$$

for every $h > k \geq k_0$. Then $\varphi(k) = 0$ for every $k \geq k_1$, where

$$k_1 = k_0 + A^{1/\gamma} 2^{\beta/(\beta-1)} \varphi(k_0)^{(\beta-1)/\gamma}.$$

Lemma 2.4 (Gagliardo and Nirenberg, see [14]). *Let v be a function in $W_0^{1,r}(\Omega) \cap L^m(\Omega)$, with $r \geq 1$, $m \geq 1$. Then there exists a positive constant C, depending only on N, r and m, such that*

$$\|v\|_{L^\sigma(\Omega)} \leq C \, \|\nabla v\|^{\theta}_{(L^r(\Omega))^N} \, \|v\|^{1-\theta}_{L^m(\Omega)} \,, \tag{4}$$

for every θ and σ satisfying

$$0 \leq \theta \leq 1 \,, \quad 1 \leq \sigma < +\infty \,, \qquad \frac{1}{\sigma} = \theta \left(\frac{1}{r} - \frac{1}{N} \right) + \frac{1-\theta}{m} \,. \tag{5}$$

Lemma 2.5 (Poincaré's and Sobolev's inequalities, see [7], Chapters 7.7 and 7.8). *Let Ω be an open subset of $\mathbf{R}^N$ with finite Lebesgue measure. Then, for every p such that $1 \leq p < +\infty$, the following Poincaré inequality holds:*

$$\|u\|_{L^p(\Omega)} \leq \left(\frac{|\Omega|}{\omega_N} \right)^{1/N} \|\nabla u\|_{L^p(\Omega)} \tag{6}$$

for every $u \in W_0^{1,p}(\Omega)$, where ω_N is the measure of the unit ball in $\mathbf{R}^N$. Furthermore, there exists a constant $C = C(N,p)$ such that, for any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \qquad \text{for } p < N,$$

$$\|u\|_{L^\infty(\Omega)} \leq C |\Omega|^{1/N - 1/p} \|\nabla u\|_{L^p(\Omega)} \qquad \text{for } p > N,$$

where $p^* = Np/(N-p)$.

Remark 2.6 By an approximation argument, the same inequalities hold if one replaces $|\Omega|$ with $|\{x \in \Omega \ : \ u(x) \neq 0\}|$.

3 L^∞-estimate

In this section we show that every solution of problem (P) is bounded, and we obtain an *a priori* L^∞-estimate for the solutions. We point out that the estimate also holds for domains having infinite measure, that the result of this section is valid for every $p > 1$, and that only very weak hypotheses on the operator are assumed. In the case of unbounded domains, this result seems new also if the first order term $b(x, \nabla u)$ is absent.

Proposition 3.1 *Let Ω be an open subset of $\mathbf{R}^N$. We assume the hypotheses* (A4), (B1), (C3) *and*

$$g = g_1 + g_2 \,, \quad \text{with } g_1 \in L^\infty(\Omega) \text{ and } g_2 \in L^q(\Omega),\ q > \max\left\{ \frac{N}{p}, 1 \right\},\ p > 1. \tag{7}$$

Let u be a weak solution of

$$\begin{cases} A(u) = g(x) & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \,. \end{cases} \tag{8}$$

Then u is essentially bounded in Ω and

$$\|u\|_{L^\infty(\Omega)} \le D\,, \tag{9}$$

where D depends on $N, p, q, d, \nu, \nu_0, \|g_1\|_{L^\infty(\Omega)}$ and $\|g_2\|_{L^q(\Omega)}$ but not on $\|u\|_{W_0^{1,p}(\Omega)}$.

Remark 3.2 We point out that hypotheses (G1) and (G2) imply (7).

Proof of Proposition 3.1. For simplicity we use the notations $\|v\|_r$ and $\|\nabla v\|_r$ instead of $\|v\|_{L^r(\Omega)}$ and $\|\nabla v\|_{(L^r(\Omega))^n}$ respectively.
Step 1. We first show that every solution u of (8) is bounded, and moreover that, if k_0 satisfies

$$\nu_0 k_0^{p-1} \ge \|g_1\|_\infty \tag{10}$$

and

$$d\left(\frac{|A_{k_0}(u)|}{\omega_N}\right)^{1/N} \le \frac{\nu}{2}\,, \tag{11}$$

where ω_N is the measure of the unit ball in $\mathbf{R}^N$, then

$$\|u\|_\infty \le c = c\left(k_0, |A_{k_0}(u)|\right)\,. \tag{12}$$

Actually we will prove in Step 2 that is possible to choose k_0 independent of the norm of u in $W_0^{1,p}(\Omega)$ such that (10) and (11) hold. We take $G_k(u)$, with $k \ge k_0$, as test function in (8). From (A4), (C3) and (B1) we obtain

$$\begin{aligned} \nu \int |\nabla G_k(u)|^p + \nu_0 \int |u|^{p-1}|G_k(u)| & \\ \le\ d\int |\nabla G_k(u)|^{p-1}|G_k(u)| + \int |g_1|\,|G_k(u)| + \int |g_2|\,|G_k(u)|\,. & \end{aligned} \tag{13}$$

If k_0 satisfies (10), since $|u| > k_0$ on the set where $G_k(u) \ne 0$, the second integral of the right-hand side of (13) is less than or equal to the last integral of the left-hand side. Moreover, by Hölder's and Poincaré's inequalities,

$$\begin{aligned} d\int |\nabla G_k(u)|^{p-1}|G_k(u)| &\le d\left[\int |\nabla G_k(u)|^p\right]^{\frac{1}{p'}} \left[\int |G_k(u)|^p\right]^{\frac{1}{p}} \\ &\le d\left(\frac{|A_{k_0}|}{\omega_N}\right)^{\frac{1}{N}} \int |\nabla G_k(u)|^p\,. \end{aligned}$$

Therefore, if (11) holds, from (13) we get

$$\frac{\nu}{2}\int |\nabla G_k(u)|^p \le \int |g_2|\,|G_k(u)| \le \|g_2\|_q\,\|G_k(u)\|_{q'}\,. \tag{14}$$

We now deal separately with the three cases $p < N$, $p = N$, $p > N$.

Case $p < N$. As usual, we denote by $p^* = \frac{Np}{N-p}$ Sobolev's conjugate exponent of p. Using Sobolev's and Hölder's inequalities, since $q' < p^*$, from (14) we obtain

$$c_1\|G_k(u)\|_{p^*}^p \le \|\nabla G_k(u)\|_p^p \le \frac{2}{\nu}\,\|g_2\|_q\|G_k(u)\|_{q'} \le \frac{2}{\nu}\,\|g_2\|_q\|G_k(u)\|_{p^*}|A_k(u)|^{\frac{1}{q'}-\frac{1}{p^*}}\,,$$

where $c_1 = c_1(N,p)$ is the reciprocal of the Sobolev constant. Therefore, if $h > k$ and $G_k(u) \neq 0$ (if $G_k(u) = 0$ the result is trivial)

$$c_1(h-k)^{p-1}|A_h(u)|^{\frac{p-1}{p^*}} \leq \frac{2}{\nu}\left\|g_2\right\|_q |A_k(u)|^{\frac{1}{q'}-\frac{1}{p^*}},$$

from which it follows

$$|A_h(u)| \leq c_2\left(N,p,\nu,\left\|g_2\right\|_q\right)\frac{|A_k(u)|^{\frac{p^*-q'}{q'(p-1)}}}{(h-k)^{p^*}}.$$

Since

$$\frac{p^*-q'}{q'(p-1)} > 1$$

for $q > N/p$, we can apply Lemma 2.3 with $\varphi(k) = |A_k(u)|$ to obtain (12).

Case $p = N$. Let $r < N$ be such that $r^* = rN/(N-r) > q'$. From (14) we obtain

$$\begin{aligned}
\|\nabla G_k(u)\|_N^N &\leq \frac{2}{\nu}\left\|g_2\right\|_q \left\|G_k(u)\right\|_{r^*} |A_k(u)|^{\frac{1}{q'}-\frac{1}{r^*}}\\
&\leq \frac{2c_3(N,r)}{\nu}\left\|g_2\right\|_q \left\|\nabla G_k(u)\right\|_r |A_k(u)|^{\frac{1}{q'}-\frac{1}{r^*}}\\
&\leq \frac{2c_3(N,r)}{\nu}\left\|g_2\right\|_q \left\|\nabla G_k(u)\right\|_N |A_k(u)|^{\frac{1}{q'}}.
\end{aligned}$$

Therefore, by Lemma 2.5,

$$\begin{aligned}
|A_h(u)|^{\frac{N-1}{N}}|A_k(u)|^{\frac{1-N}{N}}(h-k)^{N-1} &\leq |A_k(u)|^{\frac{1-N}{N}}\|G_k(u)\|_N^{N-1} \leq c_4(N)\|\nabla G_k(u)\|_N^{N-1}\\
&\leq c_5\left(N,r,\nu,\left\|g_2\right\|_q\right)|A_k(u)|^{\frac{1}{q'}},
\end{aligned}$$

from which we conclude again using Lemma 2.3.

Case $p > N$. This is the easiest case, since it suffices to take $k = k_0$, so that, by (14) and Sobolev's inequality,

$$\begin{aligned}
\|\nabla G_{k_0}(u)\|_p^p &\leq \frac{2}{\nu}\left\|g_2\right\|_q \left\|G_{k_0}(u)\right\|_\infty |A_{k_0}(u)|^{\frac{1}{q'}}\\
&\leq \frac{2}{\nu c_1(N,p)}\left\|g_2\right\|_q \left\|\nabla G_{k_0}(u)\right\|_p |A_{k_0}(u)|^{\frac{1}{N}-\frac{1}{p}+\frac{1}{q'}}.
\end{aligned}$$

Therefore an estimate for $\left\|\nabla G_{k_0}(u)\right\|_p$ follows immediately in terms of $|A_{k_0}(u)|$. By Sobolev's imbedding, (12) is proved, and Step 1 is completed.

Step 2. In view of Step 1, it suffices to show that there exists some $k_0 > 0$ (depending only on the data of the problem) such that (11) holds, and to do this we only need to prove that there exists $\delta = \delta(p,q,d,\nu,\nu_0) > 1$, $k_1 = k_1(p,\nu_0,\left\|g_1\right\|_\infty)$ and $C_1 = C_1(N,p,q,d,\nu,\nu_0,\left\|g_2\right\|_q)$ such that

$$\left\|G_{k_1}(u)\right\|_\delta \leq C_1. \tag{15}$$

To show this, we use $|G_{k_1}(u)|^\alpha G_{k_1}(u)$ as test function in (8), where $\alpha > 0$ and $k_1 > 0$ will be chosen later. For the sake of brevity we set $v = G_{k_1}(u)$. We obtain, using Young's and Hölder's inequalities,

$$\nu(\alpha+1)\int|\nabla v|^p|v|^\alpha + \nu_0\int|u|^{p-1}|v|^{\alpha+1} \le d\int|\nabla v|^{p-1}|v|^{\alpha+1} + \int(g_1+g_2)|v|^{\alpha+1} \le$$

$$\le \frac{\nu(\alpha+1)}{p'}\int|\nabla v|^p|v|^\alpha + \frac{d^p}{p(\nu(\alpha+1))^{p-1}}\int|v|^{p+\alpha} + \|g_1\|_\infty\int|v|^{\alpha+1} + \|g_2\|_q\left(\int|v|^{(\alpha+1)q'}\right)^{\frac{1}{q'}}.$$

Choosing α such that

$$\frac{d^p}{p(\nu(\alpha+1))^{p-1}} < \frac{\nu_0}{4} \tag{16}$$

and k_1 such that

$$\frac{\nu_0}{4}k_1^{p-1} \ge \|g_1\|_\infty, \tag{17}$$

and defining

$$w = |v|^{\frac{\alpha+p}{p}},$$

we obtain

$$c_1(\alpha,\nu,p)\int|\nabla w|^p + \frac{\nu_0}{2}\int w^p \le \|g_2\|_q\left[\int w^{\frac{pq'(\alpha+1)}{\alpha+p}}\right]^{\frac{1}{q'}}. \tag{18}$$

We wish to apply Lemma 2.4 with $\sigma = \frac{pq'(\alpha+1)}{\alpha+p}$, $r = m = p$. To show that the hypotheses of Lemma 2.4 are satisfied, we have to prove that

$$p \le \frac{pq'(\alpha+1)}{\alpha+p} \le p^* \qquad \text{if } p < N \tag{19}$$

and

$$p \le \frac{pq'(\alpha+1)}{\alpha+p} \qquad \text{if } p \ge N. \tag{20}$$

The first inequality of (19), and of course (20), are true if we assume the further condition

$$\alpha \ge \frac{p-q'}{q'-1}. \tag{21}$$

The second inequality of (19) is always true for $q > N/p$. Therefore, from (18) and Lemma 2.4 we obtain

$$\begin{aligned} c_1\int|\nabla w|^p + \frac{\nu_0}{2}\int w^p &\le c_2(N,p,\alpha)\|g_2\|_q\left[\int|\nabla w|^p\right]^{\frac{\theta(\alpha+1)}{\alpha+p}}\left[\int w^p\right]^{\frac{(1-\theta)(\alpha+1)}{\alpha+p}} \\ &\le c_2(N,p,\alpha)\|g_2\|_q\left(\varepsilon\int|\nabla w|^p + \varepsilon\int w^p + c_3(\varepsilon)\right), \end{aligned} \tag{22}$$

where ε is any positive number. By choosing it small enough, we get an estimate for the left-hand side of (22):

$$\int|\nabla w|^p + \int w^p \le c_4\left(N,p,q,\|g_2\|_q,\nu,\nu_0,\alpha\right)$$

and this implies (15) with $\delta = \alpha + p$.

■

4 Comparison principle

The aim of this section is to obtain a comparison result for solutions of equation (P).

In order to prove Theorem 2.2, we will apply this result to each of the approximate problems defined in next section (see (P_n), (27) and (28)), for which the following global Lipschitz continuity property holds.

(B2′) There exists a constant L such that for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbf{R}^N$

$$|b(x,\xi) - b(x,\eta)| \le L|\xi - \eta|.$$

Theorem 4.1 *Let Ω an open subset of $\mathbf{R}^N$. Let us assume that $p \ge 2$,* (A2), (A3), (B1), (B2′), (C1), (C2). *Let $u, v \in W^{1,p}(\Omega)$ such that*

$$\begin{cases} A(u) \le A(v) & \text{in } \mathcal{D}'(\Omega), \\ u \le v\,, & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Then $u \le v$ almost everywhere in Ω.

Remark 4.2 Under the hypotheses of Theorem 4.1 the solution of problem (P) is unique.

Remark 4.3 Let us observe that the same result holds if (B2) is assumed instead of (B2′), provided u and v belong to $W^{1,\infty}(\Omega)$. This is true, for instance, in the case where the domain Ω and the function $a(x,\xi)$ which defines the principal part of the operator are smooth enough (see for instance [8] for a $W^{1,\infty}$-regularity result).

Proof of Theorem 4.1. Let $\lambda = \sup(u - v)$, and suppose by contradiction that $\lambda > 0$. We confine ourselves to the case $\lambda < \infty$, since otherwise the proof can be easily adapted. Let us take as a test function in (23) the positive function $w = (u - v - h)_+ = \max\{u - v - h, 0\}$, with h such that $\lambda/2 < h < \lambda$. By (A3), (B2′), (C2), if we consider the set of finite measure $E_h = \{x \in \Omega : u - v \ge h, \nabla u \ne \nabla v\}$, then

$$\begin{aligned} \nu \int_{E_h} |\nabla w|^p + \nu_0 \int_{E_h} (u-v)^{p-1} w &\le \int_{E_h} |b(x,\nabla u) - b(x,\nabla v)|\, w \le L \int_{E_h} |\nabla w| w \le \\ &\le \frac{\nu}{2} \int_{E_h} |\nabla w|^p + c_1 \int_{E_h} w^{p'}\,, \end{aligned}$$

where $c_1 = c_1(p, L, \nu)$. Then we get

$$\frac{\nu}{2} \int_{E_h} |\nabla w|^p + \nu_0 \int_{E_h} (u-v)^{p-1} w \le c_1 \int_{E_h} w^{p'}\,. \tag{24}$$

Let us first consider the case $p > 2$. Then, since $p' < p$, for every $\varepsilon > 0$ we have

$$w^{p'} \le \varepsilon w + c_2(\varepsilon) w^p\,. \tag{25}$$

On the other hand, by the choice of h, on the set E_h we have

$$(u-v)^{p-1} \ge \left(\frac{\lambda}{2}\right)^{p-1}\,,$$

and choosing $\varepsilon = \dfrac{\nu_0}{c_1}\left(\dfrac{\lambda}{2}\right)^{p-1}$ in (25), from (24) we get

$$\int_{E_h} |\nabla w|^p \le c_3 \int_{E_h} w^p \,, \tag{26}$$

where $c_3 = c_3(p, \nu, \nu_0, \lambda, L)$. In the simpler case $p = 2$, (26) follows directly from (24). We now consider the following two cases.

Case I: $p < N$. By Sobolev's embedding theorem and Hölder's inequality, we obtain

$$\left[\int_{E_h} w^{p^*}\right]^{\frac{p}{p^*}} \le c_4 \left[\int_{E_h} w^{p^*}\right]^{\frac{p}{p^*}} |E_h|^{1-\frac{p}{p^*}} \ ,$$

where c_4 depends on c_3, N and p. This implies that $|E_h| \ge c_5 > 0$, with c_5 independent on h. Then we have a contradiction since as $h \to \sup(u - v)$ we have $|E_h| \to |\{x : u(x) - v(x) = \sup(u - v), \nabla u(x) \neq \nabla v(x)\}| = 0$.

Case II: $p \ge N$. Fix $r < N$ such that $r^* = \frac{Nr}{N-r} > p$. Then

$$\begin{aligned}
\left[\int_{E_h} w^{r^*}\right]^{\frac{p}{r^*}} &\le c_6(N, r) \left[\int_{E_h} |\nabla w|^r\right]^{\frac{p}{r}} \le c_6 |E_h|^{\frac{p-r}{r}} \int_{E_h} |\nabla w|^p \\
&\le c_3 c_6 |E_h|^{\frac{p-r}{r}} \int_{E_h} w^p \le c_3 c_6 \left[\int_{E_h} w^{r^*}\right]^{\frac{p}{r^*}} |E_h|^{\frac{p}{N}} \,.
\end{aligned}$$

Since c_3 and c_6 do not depend on h, we can conclude as in Case I. ∎

5 Approximate problems and $W_0^{1,p}$-estimates

We consider the following approximate problems

$$\begin{cases} A_n(u_n) = g(x) & \text{in } \Omega_n, \\ u_n \in W_0^{1,p}(\Omega_n), \end{cases} \tag{P$_n$}$$

where $\{\Omega_n\}$ is an increasing sequence of bounded open sets invading Ω (for instance $\Omega_n = \Omega \cap B_n$) and the operator A_n is defined by:

$$A_n(v) := -\text{div}\,(a(x, \nabla v)) + b_n(x, \nabla v) + c(x, v), \tag{27}$$

with

$$b_n(x, \xi) = \begin{cases} b(x, \xi) & \text{if } |\xi| < n, \\ b\left(x, \dfrac{\xi}{|\xi|} n\right) & \text{if } |\xi| \ge n. \end{cases} \tag{28}$$

Let us remark that, by (B2), for every fixed $n \in \mathbf{N}$ the function $b_n(x, \cdot)$ satisfies condition (B2′). The existence of a solution of (P$_n$) is well known (see for instance [9]).

We are interested now in $W_0^{1,p}$-estimates for u_n, independent of n.

Proposition 5.1 *Let u_n be a solution of* (P_n)*. Under the hypotheses of Theorem 2.2 there exists a positive constant M, independent of n such that*

$$\|u_n\|_{W_0^{1,p}(\Omega_n)} \leq M.$$

Proof. We note that by taking u_n as test function in (P_n) and using Young's inequality, it will be sufficient to prove a uniform L^p-estimate on u_n. The result is an immediate consequence of Proposition 3.1 if Ω has finite measure. The only difficult case is when Ω has infinite measure. By Proposition 3.1 we just need an L^p-estimate in $\Omega_n \setminus B_{R_1}$, where $R_1 \geq R_0$ (see hypothesis (G2)). In order to do this, we look for a supersolution of the equation

$$A_n(z) = g \quad \text{in} \quad \Omega_n \setminus \overline{B}_{R_1}$$

for n sufficiently large (*i.e.*, n such that $\Omega_n \setminus \overline{B}_{R_1} \neq \emptyset$). We consider

$$z(x) = \frac{H}{|x|^q},$$

where H and q will be chosen later. It is easy to check that by (2) and (3) we have

$$-\mathrm{div}\,(a(x, \nabla z)) \geq -\tilde{c}\,\frac{H^{p-1}}{|x|^{(q+1)(p-1)}},$$

where $\tilde{c} = \tilde{c}(N, p, q, M_1, M_2)$ is a positive constant. Therefore, by (B1) and (C3),

$$A_n(z) \geq \frac{H^{p-1}}{|x|^{q(p-1)}} \left\{ \nu_0 - \frac{\tilde{c} + d\, q^{p-1}}{|x|^{p-1}} \right\}. \tag{29}$$

If we choose R_1 large enough (depending on N, ν_0, q, p, d, M_1, M_2), then, for every $|x| > R_1$, we have

$$A_n(z) \geq \frac{\nu_0}{2} \frac{H^{p-1}}{|x|^{q(p-1)}}.$$

By (G2), if we choose q, H such that

$$q(p-1) = r, \qquad \frac{\nu_0}{2} H^{p-1} > K,$$

then the function $z(x)$ satisfies $A_n(z) \geq g$ in $\Omega \setminus \overline{B}_{R_1}$. Moreover, by Proposition 3.1 there exists a positive constant D independent of n such that

$$\|u_n\|_\infty \leq D.$$

If we assume the further condition on H

$$\frac{H}{R_1^q} \geq D,$$

then we get $z \geq u_n$ on ∂B_{R_1}. Since $z(x) \geq u_n(x)$ on $\partial \Omega_n$, we can apply Theorem 4.1 in $\Omega_n \setminus \overline{B}_{R_1}$ to the operator A_n and we obtain $u_n(x) \leq z(x) = H/|x|^q$ in this set. In a similar way one can prove that $u_n(x) \geq -z(x)$. By the choice of $q = r/(p-1)$ we have $q > N/p$ and therefore $z \in L^p(\mathbf{R}^N \setminus \overline{B}_{R_1})$. This implies a uniform estimate for u_n in $L^p(\Omega_n \setminus \overline{B}_{R_1})$ and therefore in $L^p(\Omega_n)$. ∎

6 Proof of the main theorem

By using the results obtained in the previous sections we will now give the proof of Theorem 2.2. We will continue to denote by u_n the zero-extensions of u_n outside of Ω_n.

Proof of Theorem 2.2. By Proposition 5.1 it follows, passing to a subsequence, that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad W_0^{1,p}(\Omega) \tag{30}$$

and

$$u_n \to u \quad \text{strongly} \quad \text{in} \quad L^p_{\text{loc}}(\Omega). \tag{31}$$

Let us take $A \subset\subset \Omega$ and $\psi \in C_0^\infty(\Omega)$, $0 \le \psi \le 1$, $\psi \equiv 1$ on A. If we take n large enough, we have $\operatorname{supp}\psi \subset\subset \Omega_n$. We use $(u_n - u)\psi$ as test function in (P_n) and we obtain

$$\int a(x,\nabla u_n)\cdot\nabla(u_n-u)\psi + \int a(x,\nabla u_n)\cdot\nabla\psi(u_n-u) + \int c(x,u_n)(u_n-u)\psi \le$$

$$\le d\int |\nabla u_n|^{p-1}|u_n-u|\psi + \int |g||u_n-u|\psi.$$

Then

$$\begin{aligned}
&\int [a(x,\nabla u_n) - a(x,\nabla u)]\cdot\nabla(u_n-u)\psi \le \\
&\le \; -\int a(x,\nabla u)\cdot\nabla(u_n-u)\psi - \int a(x,\nabla u_n)\cdot\nabla\psi(u_n-u) \\
&\quad -\int c(x,u_n)(u_n-u)\psi + d\int |\nabla u_n|^{p-1}(u_n-u)\psi + \int |g||u_n-u|\psi\,.
\end{aligned} \tag{32}$$

Each term in the right-hand side of (32) tends to 0 as n goes to ∞, by (A2), (C1), (30) and (31). Therefore, using (A3), one obtains that $\nabla u_n \to \nabla u$ strongly in $L^p_{\text{loc}}(\Omega)$. Thus

$$\int a(x,\nabla u)\cdot\nabla\Phi + \int c(x,u)\Phi + \int b(x,\nabla u)\Phi = \int g\Phi$$

for every $\Phi \in C_0^\infty(\Omega)$ and, by density, for every $\Phi \in W_0^{1,p}(\Omega)$. ■

7 Further results in the case $p = 2$

In this section we show that, in the special case $p = 2$, using the estimates proved by P.-L. Lions in [12] for linear problems, it is possible to obtain an existence result under weaker hypotheses on the datum g. More precisely, we consider the following problem

$$\begin{cases} -\operatorname{div}(a(x,u)\nabla u) + b(x,\nabla u) + c(x,u) = g(x) \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{P$'$}$$

where:

$a(x,s) = \{a_{ij}(x,s)\}_{i,j} : \Omega \times \mathbf{R} \to \mathbf{R}^{N\times N}$ is a Carathéodory matrix, such that

$$|a(x,s)| \le \beta\,, \qquad \sum_{i,j} a_{ij}(x,s)\xi_i\xi_j \ge \nu|\xi|^2\,, \tag{33}$$

for almost every $x \in \Omega$, every $s \in \mathbf{R}$ and every $\xi = (\xi_1, \ldots, \xi_N) \in \mathbf{R}^N$, and β, ν are positive constants;

$b(x, \xi)$ satisfies (B1), $c(x, s)$ satisfies (C1), (C3) (for $p = 2$), while $g(x)$ satisfies

$$g(x) \in L^2(\Omega) + L^q(\Omega)\,, \tag{34}$$

with $q = 1$ if $N = 1$, $1 < q < 2$ if $N = 2$, $q = 2N/(N+2)$ if $N \geq 3$.

Theorem 7.1 *Under the assumptions* $p = 2$, (B1), (C1), (C3), (33), (34), *there exists a weak solution of problem* (P′).

Proof. We consider the approximate problems (P_n) defined as in Section 5. Let $\{u_n\}$ be a sequence of solutions of (P_n). Then u_n is also solution of the *linear* equation

$$\begin{cases} -\operatorname{div}(\tilde{a}_n(x)\nabla u_n) + \tilde{b}_n(x)\cdot\nabla u_n + \tilde{c}_n(x)u_n = g_n(x) \quad \text{in } \Omega_n, \\ u_n \in H_0^1(\Omega_n), \end{cases} \tag{P_n}$$

where

$$\tilde{a}_n(x) = a(x, u_n(x))\,,$$

$$\tilde{b}_n(x) = \begin{cases} \dfrac{b_n(x, \nabla u_n(x))\nabla u_n(x)}{|\nabla u_n(x)|^2} & \text{if } \nabla u_n(x) \neq 0, \\ 0 & \text{if } \nabla u_n(x) = 0, \end{cases}$$

$$\tilde{c}_n(x) = \begin{cases} \dfrac{c_n(x, u_n(x))}{u_n(x)} & \text{if } u_n(x) \neq 0, \\ \nu_0 & \text{if } u_n(x) = 0. \end{cases}$$

It is easy to check that the matrix $\tilde{a}_n$, the vector $\tilde{b}_n$ and the function $\tilde{c}_n$ are uniformly bounded, that $\tilde{a}_n$ is uniformly elliptic, and that $\tilde{c}_n(x) \geq \nu_0 > 0$.

Let us assume for the moment that $g(x)$ belongs to $L^2(\Omega)$. Then it is possible to apply the linear estimate by P.-L. Lions (see (5) in [12]), and obtain that the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$. As observed in Section 5, this implies a uniform estimate in $H_0^1(\Omega)$. On the other hand, if $g(x) \in L^q(\Omega)$ ($q = 1$ if $N = 1$, $1 < q < 2$ if $N = 2$, $q = 2N/(N+2)$ if $N \geq 3$), then, again by P.-L. Lions' theorem one obtains that the sequence $\{u_n\}$ is bounded in $L^q(\Omega)$. Taking u_n as test function in (P_n), and using Sobolev's inequality, one again obtains an estimate in $H_0^1(\Omega)$. In the general case $g(x) \in L^2(\Omega) + L^q(\Omega)$, using the linearity of problem (P_n), we can write $\{u_n\}$ as the sum of two bounded sequences in $H_0^1(\Omega)$. Once the estimate in $H_0^1(\Omega)$ is proved, the existence follows in a standard way. ∎

8 Solutions in $W^{1,p}_{\text{loc}}(\Omega)$

In this Section we will drop the request for a solution to belong to $W^{1,p}_0(\Omega)$ and we will show that it is possible, under much weaker assumptions, to prove the existence of a distributional solution of following problem

$$\begin{cases} A(u) := -\text{div}\,(a(x,\nabla u)) + b(x,\nabla u) + c(x,u) = g(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \text{ and at infinity.} \end{cases} \tag{35}$$

In fact we will prove that there exists $\delta > 1$ such that

$$|u|^\delta \in W^{1,p}_0(\Omega)\,, \tag{36}$$

which gives a meaning to the boundary condition.

Since in this case we do not need to use a comparison principle, or to find super-solutions, we may assume $p > 1$, and drop several assumptions on the operator (in particular (A1), (B2), (C2), (G2), while the strong monotonicity assumption (A3) can be relaxed). Similarly, we do not need to obtain L^∞ estimates, and this leads to still weaker hypotheses on the datum g.

More precisely, we will assume that the operator A defined in (35) satisfies $p > 1$, (A2), (A4), (B1), (C1), (C3),

(A3′) for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbf{R}^N$, with $\xi \neq \eta$,

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) > 0\,.$$

Theorem 8.1 *Let $p > 1$, and assume that A satisfies* (A2), (A3′), (A4), (B1), (C1), (C3)*, and that*

$$g(x) \in L^q(\Omega)\,, \qquad \textit{with } q > \begin{cases} \dfrac{N(\bar\alpha + p)}{N(p-1) + p(\bar\alpha + 1)} & \textit{if } p < N, \\ 1 & \textit{if } p \geq N, \end{cases} \tag{37}$$

where $\bar\alpha$ satisfies

$$\left(\frac{d}{p}\right)^p \left(\frac{p-1}{\nu(\bar\alpha + 1)}\right)^{p-1} = \nu_0\,. \tag{38}$$

Then there exists a solution u of (35) *in the sense of distributions, i.e. $u \in W^{1,p}_{\text{loc}}(\Omega)$ and* (36) *holds for a suitable $\delta = \delta(N,p,\nu,\nu_0,d,q) > 1$.*

Remark 8.2 Let us point out that, with minor modifications to the proof, one can also allow the function $a(x,\nabla u(x))$ which appears in the principal part of the operator to depend also on the value $u(x)$, with the usual growth conditions with respect to u.

Remark 8.3 Condition (37) is always satisfied if $g \in L^q(\Omega)$, with $q > \max\{N/p, 1\}$, but is more general than this one.

Proof of Theorem 8.1. Again we approach problem (35) by approximate problems (P_n) as in Section 5. In order to obtain an *a priori* estimate for the solutions, we follow similar calculations to those used in Step 2 of the Proof of Proposition 3.1. More precisely, if we take $|u_n|^\alpha u_n$ as test function in (P_n), with α such that

$$\alpha > \bar{\alpha}\,, \tag{39}$$

and using (38) and (39) to cancel the term involving $b(x, \nabla u_n)$, we obtain easily (for brevity of notation we omit the index n everywhere)

$$\int \left|\nabla(|u|^{\frac{\alpha+p}{p}})\right|^p + \int |u|^{\alpha+p} \le c_1 \|g\|_q \left[\int |u|^{q'(\alpha+1)}\right]^{\frac{1}{q'}}\,, \tag{40}$$

where $c_1 = c_1(\alpha, \nu, \nu_0, p, d)$. In order to use Gagliardo-Nirenberg's embedding, we will show that, under our assumptions on q, there exists $\alpha > \bar{\alpha}$ such that

$$\alpha + p \le q'(\alpha+1) \le \frac{p^*(\alpha+p)}{p} \qquad \text{if } p < N \tag{41}$$

or

$$\alpha + p \le q'(\alpha+1) \qquad \text{if } p \ge N. \tag{42}$$

Assume for a moment that (41), (42) hold. Then, using Lemma 2.4, and proceeding as in the proof of Proposition 3.1, one easily obtains a bound for the left-hand side of (40). Therefore we have obtained the *a priori* estimate

$$\||u|^\delta\|_{W_0^{1,p}(\Omega)} \le c_2\left(N, p, \alpha, \nu, \nu_0, d, q, \|g\|_q\right)\,, \qquad \text{with } \delta = \frac{\alpha+p}{p} > 1, \tag{43}$$

where α is any number such that (39), (41) and (42) hold. From (43) it follows immediately that u is bounded in $L^p_{\mathrm{loc}}(\Omega)$. This implies that u is also bounded in $W^{1,p}_{\mathrm{loc}}(\Omega)$. Indeed for $\Omega_0 \subset\subset \Omega$ we can take $u\varphi$ as test function in the equation satisfied by u (where φ is a cut-off function such that $\varphi \equiv 1$ on Ω_0), to obtain easily an estimate for $\int_{\Omega_0} |\nabla u|^p$.

Let's go back to the approximating problems (P_n). By the estimates just proved, one can extract a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W^{1,p}_{\mathrm{loc}}(\Omega), \\ u_n \to u & \text{almost everywhere in } \Omega, \\ |u_n|^\delta \rightharpoonup |u|^\delta & \text{weakly in } W_0^{1,p}(\Omega), \end{cases} \tag{44}$$

where $\delta = (\alpha+p)/p$. In order to pass to the limit in the distributional formulation of problem (P_n), the only difficulty is to prove the strong convergence of the gradients ∇u_n in $L^p_{\mathrm{loc}}(\Omega)$. By J.-L. Lions' Lemma and (A3′) (see [10]), it is enough to show that, for every $\varphi \in C_0^\infty(\Omega)$,

$$\lim_{n\to+\infty} \int [a(x, \nabla u_n) - a(x, \nabla u)] \cdot (\nabla u_n - \nabla u)\varphi = 0\,,$$

and this can be shown as in the proof of Theorem 2.2, taking into account (37) and (44).

We only have to show that it is possible to choose $\alpha > \bar{\alpha}$ such that (41) and (42) are satisfied. If $p < N$, condition (41) is the same as

$$y_1(\alpha) = \frac{\alpha + p}{\alpha + 1} \leq q' \leq \frac{p^*(\alpha + p)}{p(\alpha + 1)} = y_2(\alpha)\,. \tag{45}$$

By studying the behaviour of the functions $y_1(\alpha)$ and $y_2(\alpha)$, for $\alpha > \bar{\alpha}$, it is easy to see that condition (45) can be fulfilled for every q satisfying (37). If $p \geq N$, only the function $y_1(\alpha)$ is involved, and since $\lim_{\alpha\to+\infty} y_1(\alpha) = 1$, we only need $q' > 1$, which is equivalent to $q < +\infty$. ■

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