# Existence and regularity results for some elliptic equations with degenerate coercivity

Lucio Boccardo[1] Andrea Dall'Aglio[2] Luigi Orsina[1]

with an Appendix by Raffaele Mammoliti[1]

*Dedicato a Calogero Vinti per il suo settantesimo compleanno*

## 1 Introduction

In this paper we are interested in the study of the following elliptic problem:

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Here $\Omega$ is a bounded, open subset of $\mathbf{R}^N$, with $N > 2$, and $a(x,s) : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (that is, measurable with respect to $x$ for every $s \in \mathbf{R}$, and continuous with respect to $s$ for almost every $x \in \Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta\,, \tag{1.2}$$

for some real number $\theta$ such that

$$0 \le \theta < 1\,, \tag{1.3}$$

[1]Dipartimento di Matematica, Università di Roma I, Piazzale A. Moro 2, 00185, Roma, Italia

[2]Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67a, 50134, Firenze, Italia

for almost every $x \in \Omega$, for every $s \in \mathbf{R}$, where $\alpha$ and $\beta$ are positive constants.

As far as the datum $f$ is concerned, we will assume that it belongs to the Lebesgue space $L^m(\Omega)$, for some $m \geq 1$.

The main problem in dealing with problem (1.1) is the fact that, due to hypothesis (1.2), the differential operator $A(u) = -\mathrm{div}(a(x,u)\nabla u)$, though well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, is not coercive on $H_0^1(\Omega)$ when $u$ is large (see [10] for an explicit proof of this fact). This implies that the classical methods used in order to prove the existence of a solution for problem (1.1) cannot be applied even if the datum $f$ is very regular.

We will prove here the existence of solutions for problem (1.1), under various hypotheses on the datum $f$. To do this, we will approximate problem (1.1) with some nondegenerate problems (which thus have solution), and we will prove some *a priori* estimates on the solutions of these problems. Once this has been accomplished, the linearity of the operator with respect to the gradient, as well as the boundedness and the continuity of $a$, will allow to pass to the limit, thus finding a solution.

The first result considers the case where $f$ has a high summability.

**Theorem 1.1** *Let $f$ be a function in $L^m(\Omega)$, with $m > \frac{N}{2}$. Then there exists a function $u$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ which is weak solution of (1.1) in the sense that*

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla v\,dx = \int_\Omega f\,v\,dx\,, \qquad \forall v \in H_0^1(\Omega)\,. \tag{1.4}$$

**Remark 1.2** The previous theorem is somewhat surprising, since the result does not depend on $\theta$, and coincides with the classical boundedness results for uniformly elliptic operators (see [11]). The main tool of the proof will be an $L^\infty(\Omega)$ *a priori* estimate, which then implies the $H_0^1(\Omega)$ estimate: it is clear indeed that if $u$ is bounded the operator $A$ is uniformly elliptic.

The next result deals with data $f$ which give unbounded solutions in $H_0^1(\Omega)$.

**Theorem 1.3** *Let $f$ be a function in $L^m(\Omega)$, with $m$ such that*

$$\frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2}\,. \tag{1.5}$$

*Then there exists a function $u$ in $H_0^1(\Omega) \cap L^r(\Omega)$, with*

$$r = \frac{Nm(1-\theta)}{N-2m}\,, \tag{1.6}$$

*which is weak solution of (1.1) in the sense of (1.4).*

**Remark 1.4** The main part of the previous result is the fact that the solution $u$ belongs to $H_0^1(\Omega)$, which cannot be directly derived from the equation since $u$ may be unbounded. Again, the proof will be achieved proving *a priori* estimates in $L^r(\Omega)$, which will then be used, as well as the hypotheses on $f$, in order to prove that $|\nabla u|$ belongs to $L^2(\Omega)$. Moreover, we observe that both terms of the weak formulation (1.4) are well defined in this case, the first for the boundedness of $a$, and the second because the hypotheses on $m$ imply that $f$ belongs to $L^{\frac{2N}{N+2}}(\Omega)$, which is the dual of $L^{2^*}(\Omega)$, where $2^* = \frac{2N}{N-2}$ is the Sobolev embedding exponent for $H_0^1(\Omega)$. As a further remark, we note that $r \geq 2^*$ for every possible value of $\theta$ and $m$ satisfying (1.3) and (1.5).

**Example 1.5** Let us consider the following function:

$$u(\rho) = \frac{c}{\rho^{\frac{N}{2}-1}(-\ln\rho)^\beta} - 1\,,$$

in the ball $\Omega = B_{1/2}(0) = \{x \in \mathbf{R}^N \,:\, |x| < 1/2\}$, where

$$\rho = |x|\,, \qquad c = 2^{-\frac{N-2}{2}}(\ln 2)^\beta\,, \qquad \beta > \frac{N+2-\theta(N-2)}{2N(1-\theta)}\,.$$

It is easy to see that $u$ belongs to $H_0^1(\Omega)$ but it does not belong to $W_0^{1,p}(\Omega)$ for any $p > 2$. Moreover, we have

$$-\mathrm{div}\left(\frac{\nabla u(\rho)}{(1+|u(\rho)|)^\theta}\right) = f(\rho)\,,$$

where

$$f(\rho) = \frac{c_1}{\rho^{\frac{N}{2}(1-\theta)+1+\theta}(-\ln\rho)^{\beta(1-\theta)}} + \text{lower order terms.}$$

Such a function $f$ belongs to $L^m(\Omega)$, with $m = \frac{2N}{N+2-\theta(N-2)}$, but not to $L^s(\Omega)$, with $s > m$. In this sense, the result of Theorem 1.3 is sharp.

**Remark 1.6** If $\theta = 0$, the result of the preceding theorems coincide with the classical regularity results for uniformly elliptic equations (see [11] and [4]).

**Remark 1.7** Even in the "classical" case $\theta = 0$, if $f$ belongs to $L^{\frac{N}{2}}(\Omega)$, the solution belongs to a suitable (exponential) Orlicz space (and in particular it is not bounded). We will not deal with this limit case.

If we decrease the summability of $f$, we find solutions which do not in general belong any more to $H_0^1(\Omega)$.

**Theorem 1.8** *Let $f$ be a function in $L^m(\Omega)$, with $m > 1$ such that*

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}\,. \tag{1.7}$$

*Then there exists a function $u$ in $W_0^{1,q}(\Omega)$, with*

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)} < 2\,, \tag{1.8}$$

*which solves (1.1) in the sense of distributions, that is,*

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla\varphi\,dx = \int_\Omega f\,\varphi\,dx\,, \qquad \forall\varphi \in C_0^\infty(\Omega)\,. \tag{1.9}$$

*Moreover, $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$.*

**Remark 1.9** By Sobolev's embedding, we still obtain that the solution $u$ given by the previous theorem belongs to $L^r(\Omega)$ with $r$ as in (1.6): indeed, $q^* = r$, as can be easily calculated. The lower bound for $m$ in (1.7) is due to the fact that $q$ must not be smaller than 1. If $m = 1$, the previous result is not true in general. Indeed, if $\theta = 0$, the value $q$ given by (1.8) is $\frac{N}{N-1}$, and the solutions of problem (1.1) with $\theta = 0$ and $f$ in $L^1(\Omega)$ do not belong to $W_0^{1,\frac{N}{N-1}}(\Omega)$, but to every $W_0^{1,s}(\Omega)$, for every $s < \frac{N}{N-1}$ (see [3] and [9]). See Theorem 1.17 for the precise result if $m = 1$.

**Remark 1.10** One can wonder when it is possible to choose $u$ as test function in (1.1). If $f$ satisfies the hypotheses of Theorem 1.1 or Theorem 1.3, then $u$ belongs to $H_0^1(\Omega)$ and so is an admissible test function. If $f$ is less summable, it is not clear whether $u$ can be chosen as test function: for example, the product $f\,u$ has to belong to $L^1(\Omega)$. In view of Theorem 1.8, this is true if and only if

$$m \geq \frac{N(2-\theta)}{N+2-N\,\theta}\,. \tag{1.10}$$

We will prove in Proposition 2.7 that under hypothesis (1.10) one can take $\varphi = u$ in (1.9). The same condition on $m$ is used in [10] to obtain a uniqueness result for (1.1).

**Remark 1.11** As a consequence of the previous theorems, we also have an existence result for the problem

$$-\mathrm{div}\left(\frac{\nabla u}{\ln(\mathrm{e}+|u|)}\right) = f\,,$$

with zero boundary conditions and $f$ in $L^m(\Omega)$. Since the function $\frac{1}{\ln(\mathrm{e}+|s|)}$ satisfies hypothesis (1.2) for every $\theta > 0$, we have the following:

1) if $m > \frac{N}{2}$ there exists a solution $u$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$;

2) if $\frac{2N}{N+2} < m < \frac{N}{2}$ there exists a solution $u$ in $H_0^1(\Omega) \cap L^r(\Omega)$ for every $r < \frac{Nm}{N-2m}$;

3) if $1 < m \leq \frac{2N}{N+2}$ there exists a solution $u$ in $W_0^{1,q}(\Omega)$, for every $q < \frac{Nm}{N-m}$.

Up to now, we have obtained solutions belonging to some Sobolev space. If we weaken the summability hypotheses on $f$, then the gradient of $u$ (and even $u$ itself) may no longer be in $L^1(\Omega)$. However, it is possible to give a meaning to solution for problem (1.1), using the concept of *entropy solutions* which has been introduced in [2]. In order to give the definition of entropy solution, we define, for $k > 0$, the truncation function

$$T_k(s) = \max\{-k, \min\{k, s\}\}\,, \tag{1.11}$$

and we recall the following result (see [2], Lemma 2.1).

**Proposition 1.12** *Let $u$ be a measurable function such that $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$. Then there exists a unique measurable function $v : \Omega \to \mathbf{R}^N$ such that*

$$v\,\chi_{\{|u|<k\}} = \nabla T_k(u)\,, \quad \textit{almost everywhere in } \Omega,\ \forall k > 0.$$

*If, moreover, $u$ belongs to $W_0^{1,1}(\Omega)$, then $v$ coincides with the standard distributional gradient of $u$.*

**Definition 1.13** Let $u$ be a measurable function such that $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$. We define $\nabla u$, the *weak gradient* of $u$, as the function $v$ given by Proposition 1.12.

**Definition 1.14** Let $f$ be a function in $L^1(\Omega)$. A measurable function $u$ is an *entropy solution* of (1.1) if $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$ and if

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla T_k(u-\varphi)\,dx \le \int_\Omega f\,T_k(u-\varphi)\,dx\,, \tag{1.12}$$

for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

**Remark 1.15** We observe that every term in (1.12) is meaningful. This is clear for the right hand side, while for the left hand side we have

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla T_k(u-\varphi)\,dx = \int_\Omega a(x,u)\,\nabla T_M(u) \cdot \nabla T_k(u-\varphi)\,dx\,,$$

where $M = k + \|\varphi\|_{L^\infty(\Omega)}$. The formulation (1.12), though apparently a very weak one, is actually strong enough to obtain uniqueness results, under additional hypotheses on the operator, if $\theta = 0$ and $f \in L^1(\Omega)$ (see [2]) or if $\theta \in (0,1)$ and $f \in L^m(\Omega)$, with $m$ as in (1.10) (see [10]). We will not consider the problem of uniqueness in the present paper.

Let us recall the definition of Marcinkiewicz spaces, also called weak Lebesgue spaces.

**Definition 1.16** Let $p$ be a positive number. The Marcinkiewicz space $M^p(\Omega)$ is the set of all measurable functions $f : \Omega \to \mathbf{R}$ (where, as usual, we identify functions which differ only on a set of zero Lebesgue measure) such that

$$|\{x \in \Omega : |f(x)| > k\}| \le \frac{c}{k^p}, \quad \text{for every } k > 0, \tag{1.13}$$

for some constant $c > 0$. The norm of $f$ in $M^p(\Omega)$ is defined by

$$\|f\|^p_{M^p(\Omega)} = \inf\{c > 0 \text{ such that (1.13) holds}\}.$$

The alternate name of weak $L^p$ space is due to the fact that, if $\Omega$ has finite measure, then

$$L^p(\Omega) \subset M^p(\Omega) \subset L^{p-\varepsilon}(\Omega) \tag{1.14}$$

for every $p \ge 1$, for every $0 < \varepsilon < p - 1$.

We will prove the following existence result.

**Theorem 1.17** *Let $f$ be a function in $L^m(\Omega)$, with*

$$1 \le m \le \max\left\{\frac{N}{N+1-\theta(N-1)}, 1\right\}. \tag{1.15}$$

*Then there exists an entropy solution $u$ of (1.1), with*

$$u \in M^r(\Omega), \qquad |\nabla u| \in M^q(\Omega), \tag{1.16}$$

*with*

$$r = \frac{Nm(1-\theta)}{N-2m}, \qquad q = \frac{Nm(1-\theta)}{N-m(1+\theta)}.$$

**Remark 1.18** If $0 \le \theta < \frac{1}{N-1}$, then (1.15) becomes $m = 1$ and $q = \frac{N(1-\theta)}{N-1-\theta}$ which is greater than 1. In view of the embeddings between Marcinkiewicz and Lebesgue spaces, we have the $u$ belongs to $W^{1,s}_0(\Omega)$, for every $s < q$. If in particular $\theta = 0$, this is the same result obtained in [3] for elliptic equations with $L^1(\Omega)$ (or measure) data.

If $\frac{1}{N-1} \le \theta < 1$, then the upper bound on $m$ is $\frac{N}{N+1-\theta(N-1)}$, which is the lower bound on $m$ given by Theorem 1.8.

The following is a picture which summarizes the different regularity results obtained in this paper in dependence of $\theta$ and $m$. If $(\theta, m)$ lies in region $A$, then the solution $u$ belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$ (see Theorem 1.1); in region

$B$ the function $u$ is not bounded, though it still belongs in $H_0^1(\Omega)$ (see Theorem 1.3); in the third region $C$ the solution we find is no longer in $H_0^1(\Omega)$, and belongs to some Sobolev space $W_0^{1,q}(\Omega)$, with $q < 2$ (see Theorem 1.8). Finally, in region $D$, we prove the existence of an entropy solution, which does not belong to any Sobolev space (see Theorem 1.17). In the region above the dashed line it is possible to choose $u$ as test function (see Remark 1.10 and Proposition 2.7).

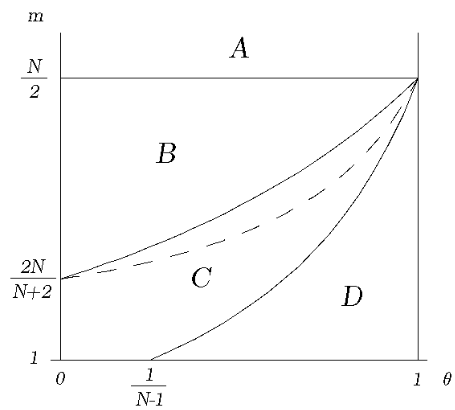


Figure 1

**Remark 1.19** The same results of Theorems 1.1 and 1.3, as far as the part of *a priori* estimates is concerned, have been obtained by A. Alvino, V. Ferone and G. Trombetti by means of symmetrization techniques (see [1]).

**Remark 1.20** The previous results can be extended in order to deal more general, nonlinear equations, such as (for instance)

$$-\mathrm{div}(a(x,u)\,|\nabla u|^{p-2}\,\nabla u) = f\,,$$

with $p > 1$, and $a$ satisfying hypotheses similar to (1.2). Differently from the present (linear) case, the *a priori* estimates are no longer enough in order to pass to the limit in the approximate equations, since the operator is not linear with respect to the gradient. In order to achieve the proof, it is thus necessary a further result of almost everywhere convergence of the gradients of the approximating solutions, as in [12] or in [3].

The study of the nonlinear equation will be the subject of a forthcoming paper.

The plan of this paper is as follows: in the next Section we will give the *a priori* estimates and the proof of Theorems 1.1, 1.3 and 1.8. In the third Section we will prove Theorem 1.17.

# 2 Solutions in Sobolev spaces

The proof of all the existence results will be obtained by approximation. Let $f$ be a function in $L^m(\Omega)$, with $m$ as in the statements of Theorems 1.1, 1.3 and 1.8. Let $\{f_n\}$ be a sequence of functions such that

$$f_n \in L^{\frac{2N}{N+2}}(\Omega)\,, \qquad f_n \to f \quad \text{strongly in } L^m(\Omega), \tag{2.1}$$

and such that

$$\|f_n\|_{L^m(\Omega)} \le \|f\|_{L^m(\Omega)}\,, \qquad \forall n \in \mathbf{N}\,. \tag{2.2}$$

Let us define the following sequence of problems:

$$\begin{cases} -\mathrm{div}(a(x, T_n(u_n))\nabla u_n) = f_n & \text{in } \Omega, \\ \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Since

$$a(x, T_n(s)) \ge \frac{\alpha}{(1+n)^\theta}\,,$$

for almost every $x \in \Omega$ and for every $s$ in $\mathbf{R}$, and since $f_n$ belongs to $H^{-1}(\Omega)$, by well-known results (see [8]) there exists at least a solution $u_n$ in $H_0^1(\Omega)$ of problem (2.3) in the sense that

$$\int_\Omega a(x, T_n(u))\, \nabla u_n \cdot \nabla v\, dx = \int_\Omega f_n\, v\, dx\,, \qquad \forall v \in H_0^1(\Omega)\,. \tag{2.4}$$

We remark that, for every $n$ in $\mathbf{N}$, the function $a(x, T_n(s))$ satisfies condition (1.2).

To prove the $L^\infty(\Omega)$ *a priori* estimate, we will need the following result, whose proof will be given in the Appendix.

**Lemma 2.1** *Let $w$ be a function in $W_0^{1,\sigma}(\Omega)$ such that, for $k$ greater than some $k_0$,*

$$\int_{A_k} |\nabla w|^\sigma \, dx \le c\, k^{\theta\,\sigma} |A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon}\,, \tag{2.5}$$

*where $\varepsilon > 0$, $0 \le \theta < 1$, $\sigma^* = \frac{N\,\sigma}{N-\sigma}$ and*

$$A_k = \{x \in \Omega \,:\, |w(x)| > k\}\,.$$

*Then the norm of $w$ in $L^\infty(\Omega)$ is bounded by a constant which depends on $c$, $\theta$, $\sigma$, $N$, $\varepsilon$, $k_0$, and $|\Omega|$.*

**Lemma 2.2** *Assume that $m > \frac{N}{2}$, let $f$ in $L^m(\Omega)$ and let $u_n$ be a solution of (2.3) in the sense of (2.4), with $f_n = f$ for every $n \in \mathbf{N}$. Then the norms of $u_n$ in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$ are bounded by a constant which depends on $\theta$, $m$, $N$, $\alpha$, $|\Omega|$ and the norm of $f$ in $L^m(\Omega)$.*

**Proof.** Let us start with the estimate in $L^\infty(\Omega)$. Define, for $s$ in $\mathbf{R}$ and for $k > 0$,

$$G_k(s) = (|s| - k)_+ \operatorname{sgn}(s) = s - T_k(s)\,.$$

For $k > 0$, if we take $G_k(u_n)$ as test function in (2.4), and use hypothesis (1.2), we obtain

$$\begin{aligned} \alpha \int_{A_k} \frac{|\nabla u_n|^2}{(1+|u_n|)^\theta}\, dx &\le \int_{A_k} f\, G_k(u_n)\, dx \\ &\le \ \|f\|_{L^m(\Omega)} \left[\int_{A_k} |G_k(u_n)|^{m'}\, dx\right]^{\frac{1}{m'}} \le c \left[\int_{A_k} |G_k(u_n)|^{m'}\, dx\right]^{\frac{1}{m'}}\,, \end{aligned}$$

where $m' = m/(m-1)$ and we have set

$$A_k = \{x \in \Omega \,:\, |u_n| > k\}\,.$$

Therefore, if $\sigma < 2$, we can write, by the Hölder inequality:

$$\begin{aligned} \int_{A_k} |\nabla u_n|^\sigma\, dx &= \int_{A_k} \frac{|\nabla u_n|^\sigma}{(1+|u_n|)^{\frac{\theta\sigma}{2}}} (1+|u_n|)^{\frac{\theta\sigma}{2}}\, dx \\ &\le \left[\int_{A_k} \frac{|\nabla u_n|^2}{(1+|u_n|)^\theta}\, dx\right]^{\frac{\sigma}{2}} \left[\int_{A_k} (1+|u_n|)^{\frac{\theta\sigma}{2-\sigma}}\, dx\right]^{\frac{2-\sigma}{2}} \\ &\le c \left[\int_{A_k} |G_k(u_n)|^{m'}\, dx\right]^{\frac{\sigma}{2m'}} \left[\int_{A_k} (1+|u_n|)^{\frac{\theta\sigma}{2-\sigma}}\, dx\right]^{\frac{2-\sigma}{2}}\,. \end{aligned} \tag{2.6}$$

Let us choose $\sigma$ such that its Sobolev conjugate exponent $\sigma^* = N\sigma/(N-\sigma)$ is equal to $m'$, that is,

$$\sigma = \frac{Nm}{Nm+m-N}\,. \tag{2.7}$$

It is easy to check that the hypotheses on $m$ imply $\sigma < \frac{N}{N-1} < 2$. From (2.6) and Sobolev's embedding theorem we obtain

$$\int_{A_k} |\nabla u_n|^\sigma\,dx \le c\left[\int_{A_k} |\nabla u_n|^\sigma\,dx\right]^{\frac{1}{2}} \left[\int_{A_k} (1+|u_n|)^{\frac{\theta\sigma}{2-\sigma}}\,dx\right]^{\frac{2-\sigma}{2}}\,,$$

and therefore

$$\int_{A_k} |\nabla u_n|^\sigma\,dx \le c\left[\int_{A_k} (1+|u_n|)^{\frac{\theta\sigma}{2-\sigma}}\,dx\right]^{2-\sigma}\,. \tag{2.8}$$

Since if $k \ge 1$, one has on $A_k$ that $1+|u_n| \le 2(k+|G_k(u_n)|)$, we can write

$$\int_{A_k} |\nabla u_n|^\sigma\,dx \le c\left\{k^{\theta\sigma}\,|A_k|^{2-\sigma} + \left[\int_{A_k} |G_k(u_n)|^{\frac{\theta\sigma}{2-\sigma}}\,dx\right]^{2-\sigma}\right\}\,.$$

Since $\theta < 1$ and $m > N/2$, with our choice of $\sigma$ one has

$$\frac{\theta\sigma}{2-\sigma} < \sigma^*\,,$$

and therefore, using Hölder's, Sobolev's and Young's inequalities, one obtains

$$\begin{aligned}
\int_{A_k} |\nabla u_n|^\sigma\,dx &\le c\left\{k^{\theta\sigma}\,|A_k|^{2-\sigma} + \left[\int_{A_k} |G_k(u_n)|^{\sigma^*}\,dx\right]^{\frac{\theta\sigma}{\sigma^*}} |A_k|^{2-\sigma-\frac{\theta\sigma}{\sigma^*}}\right\} \\
&\le c\left\{k^{\theta\sigma}\,|A_k|^{2-\sigma} + \left[\int_{A_k} |\nabla u_n|^\sigma\,dx\right]^{\theta} |A_k|^{2-\sigma-\frac{\theta\sigma}{\sigma^*}}\right\} \\
&\le c\left\{k^{\theta\sigma}\,|A_k|^{2-\sigma} + \delta\int_{A_k} |\nabla u_n|^\sigma\,dx + c(\delta)\,|A_k|^{\frac{(2-\sigma)\sigma^*-\theta\sigma}{\sigma^*(1-\theta)}}\right\}\,.
\end{aligned}$$

If we choose $\delta$ small, we can take the term containing the gradient to the right hand side, obtaining

$$\int_{A_k} |\nabla u_n|^\sigma\,dx \le c\left\{k^{\theta\sigma}\,|A_k|^{2-\sigma} + |A_k|^{\frac{(2-\sigma)\sigma^*-\theta\sigma}{\sigma^*(1-\theta)}}\right\}\,.$$

Since $\sigma < \frac{N}{N-1}$ we have

$$2 - \sigma < \frac{(2-\sigma)\sigma^* - \theta\sigma}{\sigma^*(1-\theta)} \,,$$

so that, for $k \geq 1$, we can write, observing that $|A_k| \leq |\Omega|$,

$$\int_{A_k} |\nabla u_n|^\sigma \, dx \leq c\, k^{\theta\sigma} \, |A_k|^{2-\sigma} \,.$$

Let now $\varepsilon$ be such that $2 - \sigma = \frac{\sigma}{\sigma^*} + \varepsilon$; then it is easy to see that $\varepsilon > 0$. Therefore we can apply Lemma 2.1, with $w = u_n$, and obtain a bound for $u_n$ in $L^\infty(\Omega)$.

The estimate in $H_0^1(\Omega)$ is now very easy. Taking $u_n$ as test function in (2.4), using hypothesis (1.2) one obtains, if $\|u_n\|_{L^\infty(\Omega)} \leq c_1$,

$$\frac{\alpha}{(1+c_1)^\theta} \int_\Omega |\nabla u_n|^2 \, dx \leq \int_\Omega f \, u_n \, dx \,,$$

and the right hand side is trivially bounded since $f$ belongs to $L^1(\Omega)$.

$\blacksquare$

The next result will be used in the proof of Theorem 1.3.

**Lemma 2.3** *Assume that $m$ satisfies (1.5), let $f$ belong to $L^m(\Omega)$, and let $u_n$ be a solution of (2.3) in the sense of (2.4), with $f_n = f$ for every $n \in \mathbf{N}$. Let $r$ be as in (1.6). Then the norms of $u_n$ in $L^r(\Omega)$ and in $H_0^1(\Omega)$ are bounded by a constant which depends on $\theta$, $m$, $N$, $\alpha$, $|\Omega|$ and the norm of $f$ in $L^m(\Omega)$.*

**Proof.** Let us define, for $k \in \mathbf{N}$, the function $\varphi_k(s) = T_1(G_k(s))$. If we use $\varphi_k(u_n)$ as test function in (2.4), we obtain the inequality

$$\alpha \int_{B_k} |\nabla u_n|^2 \, dx \leq (2+k)^\theta \int_{A_k} |f| \, dx \,, \tag{2.9}$$

where we have set

$$A_k = \{x \in \Omega \,:\, |u_n| \geq k\} \,, \quad B_k = \{x \in \Omega \,:\, k \leq |u_n| < k+1\} \,. \tag{2.10}$$

In the following, we will use the same technique of [5].

If $\gamma \geq 1$, then, using the Sobolev inequality and (2.9), we obtain

$$
\begin{aligned}
\left[\int_\Omega |u_n|^{2^*\gamma}\,dx\right]^{\frac{2}{2^*}} &\leq c \int_\Omega |\nabla(|u_n|^\gamma)|^2\,dx = c\int_\Omega |u_n|^{2(\gamma-1)}\,|\nabla u_n|^2\,dx \\
&= c\sum_{k=0}^{+\infty}\int_{B_k}|u_n|^{2(\gamma-1)}\,|\nabla u_n|^2\,dx \leq c\sum_{k=0}^{+\infty}(1+k)^{2(\gamma-1)}\int_{B_k}|\nabla u_n|^2\,dx \\
&\leq c\sum_{k=0}^{+\infty}(1+k)^{2(\gamma-1)}(2+k)^\theta\int_{A_k}|f|\,dx \\
&\leq c\sum_{k=0}^{+\infty}(2+k)^{2\gamma-2+\theta}\sum_{h=k}^{+\infty}\int_{B_h}|f|\,dx\,.
\end{aligned}
$$

Therefore, changing the order of summation, and recalling that

$$
\sum_{k=0}^{h} k^\rho \leq c\,(h+1)^{\rho+1}\,, \tag{2.11}
$$

with $c = c(\rho)$, we have

$$
\begin{aligned}
\left[\int_\Omega |u_n|^{2^*\gamma}\,dx\right]^{\frac{2}{2^*}} &\leq c\sum_{h=0}^{+\infty}\int_{B_h}|f|\,dx\,\sum_{k=0}^{h}(2+k)^{2\gamma-2+\theta} \\
&\leq c\sum_{h=0}^{+\infty}\int_{B_h}|f|\,dx\,(3+h)^{2\gamma-1+\theta} \\
&\leq c\int_\Omega |f|\,(3+|u|)^{2\gamma-1+\theta}\,dx \\
&\leq c\,\|f\|_{L^m(\Omega)}\left(1+\left[\int_\Omega |u_n|^{m'(2\gamma-1+\theta)}\,dx\right]^{\frac{1}{m'}}\right).
\end{aligned} \tag{2.12}
$$

We now choose $\gamma$ such that

$$
m'(2\gamma-1+\theta) = 2^*\gamma\,,
$$

which is equivalent to

$$
\gamma = \frac{(N-2)\,(1-\theta)\,m}{2\,(N-2m)}\,.
$$

This choice of $\gamma$ implies $2^*\gamma = r$, with $r$ as in (1.6). Since $2/2^* > 1/m'$ being $m < N/2$, from (2.12) we obtain

$$\int_\Omega |u_n|^{2^*\gamma}\,dx = \int_\Omega |u_n|^r\,dx \le c\,.$$

Moreover, since $\gamma \ge 1$, the previous calculations imply

$$\int_{\{|u_n|>1\}} |\nabla u_n|^2\,dx \le \int_\Omega |u_n|^{2(\gamma-1)}\,|\nabla u_n|^2\,dx \le c\,.$$

Since (2.9), written for $k = 0$, implies

$$\int_{\{|u_n|\le 1\}} |\nabla u_n|^2\,dx \le c\,\|f\|_{L^1(\Omega)} \le c\,,$$

the last two inequalities yield

$$\int_\Omega |\nabla u_n|^2\,dx \le c\,.$$

To end the proof we only have to prove that $\gamma \ge 1$; it is easy to check that this is equivalent to

$$m \ge \frac{2N}{N+2-\theta(N-2)}\,.$$

■

Under a slightly weaker hypothesis it is possible to obtain a slightly weaker result, in terms of Marcinkiewicz spaces.

We recall that, if $f \in M^p(\Omega)$, and $E \subset \Omega$, the following Hölder's type inequality holds:

$$\int_E |f|\,dx \le \|f\|_{M^p(\Omega)}\,|E|^{1-\frac{1}{p}}\,. \tag{2.13}$$

**Lemma 2.4** *Let $m$ be a real number such that*

$$\frac{2N}{N+2-\theta(N-2)} < m < \frac{N}{2}\,, \tag{2.14}$$

*let $f$ belong to $M^m(\Omega)$. Let $u_n$ be a solution of (2.3) in the sense of (2.4), with $f_n = f$ for every $n \in \mathbf{N}$ (such a solution exists since $f$ belongs to $L^{\frac{2N}{N+2}}(\Omega)$ by the hypotheses on $m$ and by (1.14)). Let $r$ be as in (1.6). Then the norms of $u_n$ in $M^r(\Omega)$ and in $H_0^1(\Omega)$ are bounded by a constant which depends on $\theta$, $m$, $N$, $\alpha$, $|\Omega|$ and the norm of $f$ in $M^m(\Omega)$*

**Proof.** For $k \geq 1$, let us define the function $\psi_k(s) = T_k(G_k(s))$. Then, taking $\psi_k(u_n)$ as test function in (2.4), one obtains, using (1.2),

$$\frac{\alpha}{(1+2k)^\theta} \int_\Omega |\nabla \psi_k(u_n)|^2 \, dx \leq \alpha \int_\Omega \frac{|\nabla \psi_k(u_n)|^2}{(1+|u_n|)^\theta} \, dx \leq \int_{A_k} f \, \psi_k(u_n) \, dx \, .$$

Therefore, using Sobolev's embedding and (2.13) we obtain

$$\begin{aligned} \left[\int_{A_k} |\psi_k(u_n)|^{2^*} \, dx\right]^{\frac{2}{2^*}} &\leq c \int_\Omega |\nabla \psi_k(u_n)|^2 \, dx \leq c \, (1+2k)^\theta \int_{A_k} f \, \psi_k(u_n) \, dx \\ &\leq \; c \, (1+2k)^\theta \, \|f\|_{M^m(\Omega)} \, |A_k|^{\frac{N+2}{2N}\left[1-\frac{2N}{m(N+2)}\right]} \left[\int_{A_k} |\psi_k(u_n)|^{2^*} \, dx\right]^{\frac{1}{2^*}} \\ &\leq \; c \, (1+2k)^\theta \, |A_k|^{\frac{Nm+2m-2N}{2Nm}} \left[\int_{A_k} |\psi_k(u_n)|^{2^*} \, dx\right]^{\frac{1}{2^*}} . \end{aligned}$$

Thus, for $k \geq 1$, one has

$$\left[\int_{A_k} |\psi_k(u)|^{2^*} \, dx\right]^{\frac{1}{2^*}} \leq c \, k^\theta \, |A_k|^{\frac{Nm+2m-2N}{2Nm}} . \tag{2.15}$$

Since

$$\left[\int_{A_k} |\psi_k(u_n)|^{2^*} \, dx\right]^{\frac{1}{2^*}} \geq \left[\int_{A_{2k}} |\psi_k(u_n)|^{2^*} \, dx\right]^{\frac{1}{2^*}} = k \, |A_{2k}|^{\frac{1}{2^*}} ,$$

from (2.15) one obtains

$$|A_{2k}| \leq c \, \frac{|A_k|^{\frac{Nm+2m-2N}{(N-2)m}}}{k^{2^*(1-\theta)}} , \quad \text{for every } k \geq 1 \, . \tag{2.16}$$

Let us define $\rho(k) = k^r \, |A_k|$, where $r$ is as in (1.6). Then (2.16) implies

$$\rho(2k) \leq c \, \rho(k)^\gamma ,$$

where $\gamma = \frac{Nm+2m-2N}{(N-2)m}$ belongs to $(0,1)$. Thus, by induction,

$$\rho(2^n k) \leq c^{\sum_{i=0}^{n-1} \gamma^i} \, \rho(k)^{\gamma^n} \leq c \, \rho(k)^{\gamma^n} \leq c \, (1+\rho(k))$$

for every $n \in \mathbf{N}$ and for every $k \geq 1$. Since $\rho(k)$ is bounded for $k \in [1,2)$, and since every real number $h \geq 1$ can be written in the form $h = 2^n k$, where

$k \in [1,2)$ and $n \in \mathbf{N}$, we have proved that $\rho(h) \leq c$, for every $h \geq 1$. This proves the estimate for $\|u_n\|_{M^r(\Omega)}$.

In order to prove the bound in $H_0^1(\Omega)$, one could use the embeddings (1.14) and Lemma 2.3. However, once the bound in $M^r(\Omega)$ has been proved, an estimate for the gradients follows more naturally from the simple calculation below. From (2.9) we obtain, for $k \geq 1$,

$$\begin{aligned} \int_{B_k} |\nabla u_n|^2 \, dx &\leq c\,(2+k)^\theta \int_{A_k} |f| \, dx \\ &\leq c\,(2+k)^\theta \, \|f\|_{M^m(\Omega)} \, |A_k|^{1-\frac{1}{m}} \,. \end{aligned}$$

Therefore

$$\int_\Omega |\nabla u_n|^2 \, dx = \sum_{k=0}^{+\infty} \int_{B_k} |\nabla u_n|^2 \, dx \leq c \left( 1 + \sum_{k=1}^{+\infty} \frac{(2+k)^\theta}{k^{\frac{r(m-1)}{m}}} \right) .$$

Under hypothesis (2.14), $\frac{r(m-1)}{m} - \theta > 1$, so that the series on the right hand side converges. Hence an estimate for $u_n$ in $H_0^1(\Omega)$ follows. ■

The next result deals with the case in which the sequence $u_n$ is not bounded in $H_0^1(\Omega)$.

**Lemma 2.5** *Assume that $m$ satisfies (1.7), let $\{f_n\}$ be a sequence of functions satisfying (2.1) and (2.2), and let $u_n$ be a solution of (2.3) in the sense of (2.4). Let $r$ be as in (1.6) and let $q$ be as in (1.8). Then the norms of $u_n$ in $L^r(\Omega)$ and in $W_0^{1,q}(\Omega)$ are bounded by a constant which depends on $\theta$, $m$, $N$, $\alpha$, $|\Omega|$ and the norm of $f_n$ in $L^m(\Omega)$. Moreover, for every $k > 0$,*

$$\int_\Omega |\nabla T_k(u_n)|^2 \, dx \leq c\,(1+k)^{1+\theta} \,, \tag{2.17}$$

*with $c$ depending on $\alpha$ and on the norm of $f_n$ in $L^1(\Omega)$.*

**Proof.** We begin with the proof of (2.17). Taking $T_k(u_n)$ as test function in (2.4), and using (1.2), we get

$$\begin{aligned} \frac{\alpha}{(1+k)^\theta} \int_\Omega |\nabla T_k(u_n)|^2 \, dx &\leq \int_\Omega a(x, T_n(u_n)) \, \nabla u_n \cdot \nabla T_k(u_n) \, dx \\ &= \int_\Omega f_n \, T_k(u_n) \, dx \leq k \, \|f_n\|_{L^1(\Omega)} \,, \end{aligned}$$

which then implies (2.17).

Now we turn to the estimates in $W_0^{1,q}(\Omega)$. As in the proof of Lemma 2.3, taking $\varphi_k(u_n)$ as test function in (2.3) we obtain inequality (2.9). From this, if $\lambda$ is a positive number (to be fixed later), we can write

$$
\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda}\,dx &= \sum_{k=0}^{+\infty}\int_{B_k}\frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda}\,dx \\
&\le \sum_{k=0}^{+\infty}\frac{1}{(1+k)^\lambda}\int_{B_k}|\nabla u_n|^2\,dx \le c\sum_{k=0}^{+\infty}(2+k)^\theta(1+k)^{-\lambda}\int_{A_k}|f_n|\,dx \\
&\le c\sum_{k=0}^{+\infty}(1+k)^{\theta-\lambda}\sum_{h=k}^{+\infty}\int_{B_h}|f_n|\,dx\,,
\end{aligned}
$$

where $A_k$ and $B_k$ are the sets defined in (2.10). Changing the order of summation, and using again (2.11), we obtain

$$
\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda}\,dx &\le c\sum_{h=0}^{+\infty}\int_{B_h}|f_n|\,dx\sum_{k=0}^{h}(1+k)^{\theta-\lambda} \\
&\le c\sum_{h=0}^{+\infty}(1+h)^{1+\theta-\lambda}\int_{B_h}|f_n|\,dx \le c\sum_{h=0}^{+\infty}\int_{B_h}|f_n|\,(1+|u_n|)^{1+\theta-\lambda}\,dx \\
&= c\int_\Omega |f_n|\,(1+|u_n|)^{1+\theta-\lambda}\,dx \le c\|f_n\|_{L^m(\Omega)}\left[\int_\Omega (1+|u_n|)^{(1+\theta-\lambda)m'}\,dx\right]^{\frac{1}{m'}} \\
&\le c\left\{1+\left[\int_\Omega |u_n|^{(1+\theta-\lambda)m'}\,dx\right]^{\frac{1}{m'}}\right\}.
\end{aligned}
$$

Let $r$ and $q$ be as in (1.6) and (1.8). Then one can check that $q<2$, and that $r=q^*=Nq/(N-q)$. Therefore, using the Sobolev inequality, we have

$$
\begin{aligned}
\left[\int_\Omega |u_n|^r\,dx\right]^{\frac{q}{r}} &\le c\int_\Omega |\nabla u_n|^q\,dx = c\int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\frac{\lambda q}{2}}}(1+|u_n|)^{\frac{\lambda q}{2}}\,dx \\
&\le c\left[\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda}\,dx\right]^{\frac{q}{2}}\left[\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}}\,dx\right]^{\frac{2-q}{2}} \qquad (2.18)\\
&\le c\left\{1+\left[\int_\Omega |u|^{(1+\theta-\lambda)m'}\,dx\right]^{\frac{q}{2m'}}\right\}\left\{1+\left[\int_\Omega |u_n|^{\frac{\lambda q}{2-q}}\,dx\right]^{\frac{2-q}{2}}\right\}.
\end{aligned}
$$

We now choose $\lambda$ such that $\lambda q/(2-q) = r$, that is,

$$\lambda = \frac{2N - 2m(1+\theta) - Nm(1-\theta)}{N-2m}\,.$$

It is easy to check that this implies

$$(1+\theta-\lambda)m' \le r\,.$$

Therefore the previous calculations and the Hölder inequality imply

$$\left[\int_\Omega |u_n|^r\,dx\right]^{\frac{q}{r}} \le c\left\{1 + \left[\int_\Omega |u_n|^r\,dx\right]^{\frac{(1+\theta)q}{2r}}\right\}. \tag{2.19}$$

Since $\theta < 1$, the last exponent in (2.19) is smaller than $q/r$. Therefore inequality (2.19) implies an estimate for the norm of $u_n$ in $L^r(\Omega)$. Going back to (2.18), this in turn implies an estimate for the norm of $|\nabla u_n|$ in $L^q(\Omega)$. Lemma 2.5 is therefore completely proved. ■

We are now in position to prove Theorems 1.1, 1.3 and 1.8.

**Proof of Theorems 1.1, 1.3 and 1.8.** Let $f_n$ be a sequence of functions satisfying (2.1) and (2.2), with $m$ as in the statements of the theorems; if the hypotheses of Theorems 1.1 or 1.3hold, take $f_n = f$ for every $n$ in $\mathbf{N}$. Let $u_n$ be a sequence of solutions of (2.3). Using the results of Lemmas 2.2, 2.3 and 2.5, we obtain that the sequence $\{u_n\}$ is bounded in the Sobolev and Lebesgue spaces as in the statements of the theorems. Thus, up to a subsequence, it converges weakly to some function $u$ which belongs to the same spaces. Moreover, $u_n$ converges to $u$ almost everywhere in $\Omega$ as a consequence of the Rellich theorem.

Let $\varphi$ be a function in $C_0^\infty(\Omega)$, and take $\varphi$ as test function in (2.4). We obtain

$$\int_\Omega a(x, T_n(u_n))\,\nabla u_n\cdot\nabla\varphi\,dx = \int_\Omega f_n\,\varphi\,dx\,.$$

The right hand side passes to the limit as $n$ tends to infinity since $f_n$ converges (at least) in $L^1(\Omega)$. As for the left hand side, we have

$$a(x, T_n(u_n)) \to a(x,u) \ \ast\text{-weakly in } L^\infty(\Omega) \text{ and almost everywhere in } \Omega,$$

$$\nabla u_n \to \nabla u \text{ weakly in } L^1(\Omega;\mathbf{R}^N),$$

so that

$$\lim_{n\to+\infty} \int_\Omega a(x, T_n(u_n))\, \nabla u_n \cdot \nabla \varphi \, dx = \int_\Omega a(x,u)\, \nabla u \cdot \nabla \varphi \, dx\,.$$

Hence, $u$ is a solution of (1.1) in the sense (1.9). If, moreover, $m$ satisfies the hypotheses of Theorems 1.1 or 1.3, then $u$ belongs to $H_0^1(\Omega)$. Therefore it is possible, by standard density arguments, to extend (1.9) to include also test functions in $H_0^1(\Omega)$, that is, to obtain (1.4).

The fact that $T_k(u)$ belongs to $H_0^1(\Omega)$ if the hypotheses of Theorem 1.8 hold follows easily from (2.17). ∎

**Remark 2.6** Observe that under the hypotheses of Theorem 1.1, since the norms of $u_n$ in $L^\infty(\Omega)$ are bounded by a constant $c$, then the function $u_n$ is a solution of (1.1) if $n > c$, since $T_n(u_n) = u_n$. In other words, in this case there is no need of passing to the limit as $n$ tends to infinity.

We now state and prove the results concerning the possibility of choosing $u$ as test function.

**Proposition 2.7** *Assume that $m$ satisfies (1.10), that is*

$$m \geq \frac{N(2-\theta)}{N+2-N\,\theta}\,.$$

*Let $u$ be a solution of (1.1) found, as in Theorems 1.1, 1.3 or 1.8, by approximation. Then*

$$\int_\Omega a(x,u)\, \nabla u \cdot \nabla u \, dx = \int_\Omega f\, u \, dx\,.$$

Before giving the proof, we need the following "weak lower semicontinuity" result.

**Lemma 2.8** *Let $\{v_n\}$ be a sequence of functions which is weakly convergent to $v$ in $H_0^1(\Omega)$, and let $u_n$ be a sequence of functions which is almost everywhere convergent to some function $u$ in $\Omega$. Then*

$$\int_\Omega a(x,u)\, |\nabla v|^2 \, dx \leq \liminf_{n\to+\infty} \int_\Omega a(x, T_n(u_n))\, |\nabla v_n|^2 \, dx \leq c\,.$$

**Proof.** We start from the inequality

$$0 \leq \int_\Omega a(x, T_n(u_n))\, |\nabla(v_n - v)|^2\, dx\,,$$

which we can rewrite as

$$2\int_\Omega a(x, T_n(u_n))\, \nabla v_n \cdot \nabla v\, dx - \int_\Omega a(x, T_n(u_n))\, |\nabla v|^2\, dx$$
$$\leq \int_\Omega a(x, T_n(u_n))\, |\nabla v_n|^2\, dx\,.$$

Due to the boundedness and the continuity of $a(x,s)$, and on the hypotheses on $v_n$ and $u_n$, the left hand side converges to

$$\int_\Omega a(x,u)\, |\nabla v|^2\, dx\,,$$

so that the result is completely proved. ■

**Proof of Proposition 2.7.** If we are under the hypotheses of Theorems 1.1 or 1.3, the result is trivial since $u$ belongs to $H_0^1(\Omega)$. It remains to deal with the case of solutions not in $H_0^1(\Omega)$. In this case, let $\{u_n\}$ be a sequence of solutions of (2.3) with data $f_n$ satisfying (2.1) and (2.2). By Lemma 2.5, $u_n$ is bounded in $L^r(\Omega)$, with $r$ as in (1.6), and $T_k(u_n)$ is bounded in $H_0^1(\Omega)$. Moreover, it is easy to see that, under our hypotheses on $m$, we have $r \geq m'$.

Taking $T_k(u_n)$ as test function in (2.4), we have

$$\int_\Omega a(x, T_n(u_n))\, |\nabla T_k(u_n)|^2\, dx = \int_\Omega f_n\, T_k(u_n)\, dx \leq c\,.$$

Applying Lemma 2.8 with $v_n = T_k(u_n)$, we thus have

$$\int_\Omega a(x,u)\, |\nabla T_k(u)|^2\, dx \leq \int_\Omega f\, T_k(u)\, dx\,.$$

Letting $k$ tend to infinity, we obtain

$$\int_\Omega a(x,u)\, |\nabla u|^2\, dx \leq \int_\Omega f\, u\, dx \leq c\,. \tag{2.20}$$

Let $\{\varphi_n\}$ be a sequence of functions in $C_0^\infty(\Omega)$ which converges to $T_k(u)$ strongly in $H_0^1(\Omega)$ and $*$-weakly in $L^\infty(\Omega)$, and choose $\varphi_n$ as test function in (1.9). We obtain

$$\int_\Omega a(x,u)\, \nabla u \cdot \nabla \varphi_n\, dx = \int_\Omega f\, \varphi_n\, dx\,.$$

The sequence $a(x,u)\,\nabla u \cdot \nabla \varphi_n$ converges almost everywhere. Moreover, if $E$ is a measurable subset in $\Omega$, we have

$$\int_E a(x,u)\,\nabla u \cdot \nabla \varphi_n \,dx \le \left(\int_E a(x,u)\,|\nabla u|^2\,dx\right)^{\frac12} \left(\int_E a(x,u)\,|\nabla \varphi_n|^2\,dx\right)^{\frac12},$$

so that it is now easy to prove that the sequence $a(x,u)\,\nabla u \cdot \nabla \varphi_n$ is equi-integrable by the hypotheses on a, on $\varphi_n$ and by (2.20). Therefore, using Vitali's theorem, we can pass to the limit to obtain

$$\int_\Omega a(x,u)\,|\nabla T_k(u)|^2\,dx = \int_\Omega f\,T_k(u)\,dx\,.$$

A further limit on $k$ yields the result. ∎

# 3 Solutions not in Sobolev spaces

In this section we are going to consider data $f$ such that the solution does not belong to any Sobolev space, proving Theorem 1.17.

We begin with an *a priori* estimate on functions satisfying a certain inequality.

**Lemma 3.1** *Assume that $m$ satisfies (1.15), and let $f$ in $L^m(\Omega)$. Suppose that $u$ is a measurable function such that $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$, and suppose that $u$ satisfies*

$$\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta}\,dx \le c \int_\Omega |f|\,|T_k(u)|\,dx\,, \tag{3.1}$$

*for every $k > 0$. Then $u$ belongs to $M^r(\Omega)$, with $r$ as in (1.6), and $\nabla u$, the weak gradient of $u$, is such that $|\nabla u|$ belongs to $M^q(\Omega)$, with $q$ as in (1.8).*

Before the proof, we need a technical lemma.

**Lemma 3.2** *Let $u$ be a measurable function in $M^s(\Omega)$ for some $s > 0$, and suppose that there exists a positive constant $\rho$ such that*

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le c\,k^\rho\,, \qquad \forall k > 0\,.$$

*Then $\nabla u$, the weak gradient of $u$, is such that $|\nabla u|$ belongs to $M^p(\Omega)$, with $p = \frac{2s}{\rho+s}$.*

**Proof.** We follow the lines of the proof of [2], Lemma 4.2. Let $\lambda$ be a fixed positive real number. We have, for every $k > 0$,

$$
\begin{aligned}
|\{|\nabla u| > \lambda\}| &= |\{|\nabla u| > \lambda\,, |u| \le k\}| + |\{|\nabla u| > \lambda\,, |u| > k\}| \\
&\le |\{|\nabla u| > \lambda\,, |u| \le k\}| + |\{|u| > k\}|\,.
\end{aligned}
\tag{3.2}
$$

Moreover,

$$
|\{|\nabla u| > \lambda\,, |u| \le k\}| \le \frac{1}{\lambda^2} \int_\Omega |\nabla T_k(u)|^2\,dx \le c\,\frac{k^\rho}{\lambda^2}\,.
$$

By the hypothesis on the Marcinkiewicz regularity of $u$, (3.2) then implies

$$
|\{|\nabla u| > \lambda\}| \le c\,\frac{k^\rho}{\lambda^2} + \frac{c}{k^s}\,,
$$

and this latter inequality holds for every $k > 0$. Minimizing on $k$, we easily get

$$
|\{|\nabla u| > \lambda\}| \le \frac{c}{\lambda^{\frac{2s}{\rho+s}}}\,,
$$

which is the desired result. ■

**Proof of Lemma 3.1.** Starting from (3.1), and applying Hölder inequality, we get

$$
\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta}\,dx \le c\,\|f\|_{L^m(\Omega)} \left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{m'}}\,. \tag{3.3}
$$

We distinguish now among three cases: $m > \frac{2N}{N+2}$, $1 < m \le \frac{2N}{N+2}$, and $m = 1$.

If $m > \frac{2N}{N+2}$, then $m' < 2^*$, and this implies that there exists $\rho < 2$ such that $\rho^* = m'$. We have

$$
\begin{aligned}
\int_\Omega |\nabla T_k(u)|^\rho\,dx &= \int_\Omega \frac{|\nabla T_k(u)|^\rho}{(1+|T_k(u)|)^{\frac{\rho\theta}{2}}}\,(1+|T_k(u)|)^{\frac{\rho\theta}{2}}\,dx \\
&\le \left(\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|T_k(u)|)^\theta}\,dx\right)^{\frac{\rho}{2}} \left(\int_\Omega (1+|T_k(u)|)^{\frac{\rho\theta}{2-\rho}}\,dx\right)^{1-\frac{\rho}{2}},
\end{aligned}
$$

and, recalling the choice of $\rho$ and using (3.3), this implies, by Sobolev's embedding,

$$
\left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{m'}} \le c\,\left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{2m'}} \left(\int_\Omega (1+|T_k(u)|)^{\frac{\rho\theta}{2-\rho}}\,dx\right)^{\frac{2-\rho}{2\rho}}\,.
$$

We suppose now that $\rho$ and $m$ are such that

$$\frac{\rho\theta}{2-\rho} > m' , \tag{3.4}$$

so that, for $k \geq 1$,

$$\int_\Omega (1 + |T_k(u)|)^{\frac{\rho\theta}{2-\rho}}\, dx \leq c\, k^{\frac{\rho\theta}{2-\rho} - m'} \int_\Omega (1 + |T_k(u)|^{m'})\, dx .$$

We thus have

$$\left(\int_\Omega |T_k(u)|^{m'}\, dx\right)^{\frac{1}{2m'}} \leq c\, k^{\frac{\theta}{2} - \frac{(2-\rho)m'}{2\rho}} \left(1 + \int_\Omega |T_k(u)|^{m'}\, dx\right)^{\frac{2-\rho}{2\rho}} .$$

It is easy to see that this latter inequality implies that there exists a positive constant $c$, independent on $k$, such that for every $k \geq 1$ one has

$$\left(\int_\Omega |T_k(u)|^{m'}\, dx\right)^{\frac{1}{2m'}} \leq c\, k^{\frac{\theta}{2} - \frac{(2-\rho)m'}{2\rho}} \left(\int_\Omega |T_k(u)|^{m'}\, dx\right)^{\frac{2-\rho}{2\rho}} .$$

Thus

$$\left(\int_\Omega |T_k(u)|^{m'}\, dx\right)^{\frac{1}{2m'} - \frac{2-\rho}{2\rho}} \leq c\, k^{\frac{\theta}{2} - \frac{(2-\rho)m'}{2\rho}} . \tag{3.5}$$

If $A_k = \{x \in \Omega : |u| > k\}$, and since $|T_k(u)| = k$ on $A_k$, the latter inequality yields

$$k^{\frac{1}{2} - \frac{(2-\rho)m'}{2\rho}} |A_k|^{\frac{1}{2m'} - \frac{2-\rho}{2\rho}} \leq c\, k^{\frac{\theta}{2} - \frac{(2-\rho)m'}{2\rho}} ,$$

so that, after some easy calculations,

$$|A_k| \leq c\, k^{-r} ,$$

where $r$ is as in (1.6). We only need to check that (3.4) holds true. It is easy to see that it is equivalent to

$$m < \frac{N(2-\theta)}{N+2-N\theta} ,$$

which is easily proven to hold if $\frac{2N}{N+2} < m < \frac{N}{N+1-\theta(N-1)}$ (see also Figure 1).

Let us give now the gradient estimate. We start from (3.5), which can be rewritten, recalling the relations between $\rho$ and $m$, as

$$\left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{m'}} \le c\,k^{\frac{\theta N(m-1)+2N-2m-Nm}{N-2m}}\,.$$

From (3.3) we get

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le (1+k)^\theta \left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{m'}}\,,$$

and so, if $k \ge 1$,

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le c\,k^{\frac{\theta m(N-2)+2N-2m-Nm}{N-2m}}\,. \tag{3.6}$$

Applying Lemma 3.2, one obtains that $|\nabla u|$ belongs to $M^q(\Omega)$, with $q$ as in (1.8).

Now we turn to the case $1 < m \le \frac{2N}{N+2}$. Since $m' \ge 2^*$, we can write

$$\begin{aligned}\left(\int_\Omega |T_k(u)|^{m'}\,dx\right)^{\frac{1}{m'}} &= \left(\int_\Omega |T_k(u)|^{2^*}\,|T_k(u)|^{m'-2^*}\,dx\right)^{\frac{1}{m'}} \\ &\le k^{1-\frac{2^*}{m'}} \left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{1}{m'}}\,.\end{aligned} \tag{3.7}$$

Substituting in (3.3), and using Sobolev's embedding, we have

$$\left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \le c\,(1+k)^{1+\theta-\frac{2^*}{m'}} \left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{1}{m'}}\,,$$

so that

$$\left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{2}{2^*}-\frac{1}{m'}} \le c\,(1+k)^{1+\theta-\frac{2^*}{m'}}\,.$$

Thus, since $|T_k(u)| = k$ on $A_k$, we have

$$k^{2-\frac{2^*}{m'}}\,|A_k|^{\frac{2}{2^*}-\frac{1}{m'}} \le c\,(1+k)^{1+\theta-\frac{2^*}{m'}}\,.$$

Since $\frac{2}{2^*} - \frac{1}{m'} = \frac{1-\theta}{r}$, with $r$ as in (1.6), we then have, for $k \ge 1$,

$$|A_k|^{\frac{1-\theta}{r}} \le c\,k^{\theta-1}\,,$$

and this implies that $u$ belongs to $M^r(\Omega)$. Now we consider the estimates on the weak gradient of $u$. Taking again the estimates (3.3) and (3.7) into account, one obtains

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le c\,(1+k)^{1+\theta-\frac{2^*}{m'}}\left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{1}{m'}},$$

which, again by Sobolev's embedding, yields

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le c\,(1+k)^{1+\theta-\frac{2^*}{m'}}\left(\int_\Omega |\nabla T_k(u)|^2\,dx\right)^{\frac{2^*}{2m'}},$$

that is to say, if $k \ge 1$, one re-obtains (3.6), and so the estimate for the weak gradient in $M^q(\Omega)$, with $q$ as in the statement.

Finally, if $m = 1$, from (3.1) we deduce, for $k \ge 1$,

$$\int_\Omega |\nabla T_k(u)|^2\,dx \le c\,k^{1+\theta}.$$

Using Sobolev's embedding and reasoning as before, we have

$$k^2\,|A_k|^{\frac{2}{2^*}} \le \left(\int_\Omega |T_k(u)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \le c\,k^{1+\theta},$$

which then implies

$$|A_k| \le c\,k^{-r}$$

with $r = \frac{N(1-\theta)}{N-1}$, which is the value of $r$ given by (1.6) with $m = 1$. Applying Lemma 3.2, we then get that $|\nabla u|$ belongs to $M^q(\Omega)$, with $q$ as in (1.8) written for $m = 1$. ■

Before the proof of Theorem 1.17, we need another technical result whose proof can be found in [2].

**Lemma 3.3** *Let $\{u_n\}$ be a sequence of measurable functions such that $T_k(u_n)$ is bounded in $H_0^1(\Omega)$ for every $k > 0$. Then there exists a measurable function $u$, with $T_k(u)$ belonging to $H_0^1(\Omega)$ for every $k > 0$, and a subsequence, still denoted by $u_n$, such that*

$$u_n \to u \text{ almost everywhere in } \Omega, \quad T_k(u_n) \to T_k(u) \text{ weakly in } H_0^1(\Omega).$$

**Proof of Theorem 1.17.** Let $m$ as in the statement, let $\{f_n\}$ be a sequence of functions satisfying (2.1) and (2.2), and let $u_n$ be the solutions of (2.3). Reasoning as in the proof of Lemma 2.5, we have that $T_k(u_n)$ is bounded in $H_0^1(\Omega)$ for every $k > 0$. Thus, by Lemma 3.3, there exists a subsequence, still denoted by $u_n$, and a function $u$ such that $u_n$ converges to $u$ almost everywhere in $\Omega$, and $T_k(u_n)$ converges to $T_k(u)$ weakly in $H_0^1(\Omega)$. Moreover, choosing $T_k(u_n)$ as test function in (2.4), we have

$$\int_\Omega a(x, T_n(u_n))\, |\nabla T_k(u_n)|^2\, dx = \int_\Omega f_n\, T_k(u_n)\, dx\,.$$

We then apply Lemma 2.8 to the left hand side, with $v_n = T_k(u_n)$, and find that $u$ is such that

$$\int_\Omega a(x, u)\, |\nabla T_k(u)|^2\, dx \leq \int_\Omega f\, T_k(u)\, dx\,.$$

Using (1.2), we have that $u$ satisfies the hypotheses of Lemma 3.1, and so it belongs to $M^r(\Omega)$, while $|\nabla u|$ belongs to $M^q(\Omega)$.

Now we have to prove that $u$ is an entropy solution of (1.1). Let $\varphi$ be a function in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and choose $T_k(u_n - \varphi)$ as test function in (2.4). We have

$$\int_\Omega a(x, T_n(u_n))\, \nabla u_n \cdot \nabla T_k(u_n - \varphi)\, dx = \int_\Omega f_n\, T_k(u_n - \varphi)\, dx\,.$$

The right hand side easily passes to the limit as $n$ tends to infinity. As for the left hand side, we can write it as

$$\int_\Omega a(x, T_n(u_n))\, |\nabla(u_n - \varphi)|^2\, dx + \int_\Omega a(x, T_n(u_n))\, \nabla\varphi \cdot \nabla T_k(u_n - \varphi)\, dx\,.$$

For the first term we have, observing that it is equal to

$$\int_\Omega a(x, T_n(u_n))\, |\nabla T_k(u_n - \varphi)|^2\, dx\,,$$

and applying Lemma 2.8 with $v_n = T_k(u_n - \varphi)$,

$$\begin{aligned}
&\int_\Omega a(x, u)\, |\nabla T_k(u - \varphi)|^2\, dx \\
&\qquad \leq \liminf_{n \to +\infty} \int_\Omega a(x, T_n(u_n))\, |\nabla T_k(u_n - \varphi)|^2\, dx\,,
\end{aligned}$$

while the second converges to

$$\int_\Omega a(x,u)\,\nabla\varphi\cdot\nabla T_k(u-\varphi)\,dx\,,$$

as $n$ tends to infinity. Putting together the terms, we thus have

$$\int_\Omega a(x,u)\,\nabla u\cdot\nabla T_k(u-\varphi)\,dx\le\int_\Omega f\,T_k(u-\varphi)\,dx\,,$$

for every $\varphi$ in $H_0^1(\Omega)\cap L^\infty(\Omega)$ and so $u$ is an entropy solution of (1.1).

■

**Remark 3.4** We remark explicitly that in the part of the preceding proof concerning the existence of an entropy solution we have never used the fact that $m$ satisfies (1.15). In other words, the solution $u$ obtained in Theorems 1.1, 1.3 and 1.8 is also an entropy solution of (1.1). Observe that, in the cases of the three theorems above, the weak gradient of $u$ is indeed the standard distributional gradient of $u$.

# Appendix

by Raffaele Mammoliti

In this Appendix we give the proof of Lemma 2.1. This result is similar to the result of Lemma 5.3 in Chapter 2 of [7]. This latter result has slightly more general hypotheses, but gives an *a priori* estimate on $u$ in $L^\infty(\Omega)$ depending on the norm of $u$ in $L^1(\Omega)$. The present results gives instead an estimate in $L^\infty(\Omega)$ independent on other norms of $u$.

**Lemma 2.1.** *Let $w$ be a function in $W_0^{1,\sigma}(\Omega)$ such that, for $k$ greater than some $k_0$,*

$$\int_{A_k} |\nabla w|^\sigma \, dx \le c\, k^{\theta\,\sigma} |A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon}\,, \tag{A.1}$$

*where $\varepsilon > 0$, $0 \le \theta < 1$, $\sigma^* = \frac{N\sigma}{N-\sigma}$ and*

$$A_k = \{x \in \Omega \,:\, |w(x)| > k\}\,.$$

*Then the norm of $w$ in $L^\infty(\Omega)$ is bounded by a constant which depends on $c$, $\theta$, $\sigma$, $N$, $\varepsilon$, $k_0$, and $|\Omega|$.*

**Proof.** We use the same technique used in [11]. Applying Sobolev's inequality to the left hand side of (A.1), we get

$$\left(\int_{A_k} |G_k(w)|^{\sigma^*} \, dx\right)^{\frac{\sigma}{\sigma^*}} \le \int_{A_k} |\nabla w|^\sigma \, dx \le c\, k^{\theta\,\sigma} |A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon}\,.$$

Choosing $h > k > 0$, and observing that $G_k(w) \ge h - k$ on $A_h$, we thus have

$$(h-k)^\sigma \, |A_h|^{\frac{\sigma}{\sigma^*}} \le c\, k^{\theta\,\sigma} |A_k|^{\frac{\sigma}{\sigma^*}+\varepsilon}\,,$$

which can be rewritten as

$$|A_h| \le \frac{c}{(h-k)^{\sigma^*}}\, k^{\theta\,\sigma^*} \, |A_k|^{1+\frac{\varepsilon\,\sigma^*}{\sigma}}\,.$$

The result then follows from Lemma A.1 below, applied with $\lambda = \frac{\varepsilon\,\sigma^*}{\sigma}$, $\rho = \sigma^*$ and $\varphi(h) = |A_h|$. ∎

**Lemma A.1.** *Let $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ be a non increasing function such that*

$$\varphi(h) \le \frac{c_0}{(h-k)^\rho}\, k^{\theta\,\rho}\, [\varphi(k)]^{1+\lambda} \qquad \forall h > k > 0\,, \tag{A.2}$$

*for some positive constant $c$, with $\rho > 0$, $0 \le \theta < 1$ and $\lambda > 0$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.*

**Proof.** For $k_0 > 0$, define the increasing sequence

$$k_s = k_0 + d - \frac{d}{2^s} \qquad s \in \mathbf{N}\,,$$

where

$$d^\rho = c_0\, \Lambda\, [\varphi(k_0)]^\lambda\, 2^{(1+\lambda)\,\mu}\,,$$

with $\Lambda$ a positive real number to be chosen later, and $\mu = \frac{\rho}{\lambda} > 0$. We claim that, with such definitions, we have

$$\varphi(k_s) \le \frac{\varphi(k_0)}{2^{s\mu}} \qquad \forall s \in \mathbf{N}\,. \tag{A.3}$$

Indeed, formula (A.3) is trivially satisfied for $s = 0$. Assuming that (A.3) holds for $s > 0$, we have, applying (A.2) with $h = k_{s+1}$ and $k = k_s$, and recalling the definition of $d$,

$$\begin{aligned}
\varphi(k_{s+1}) &\le \frac{c_0}{(k_{s+1}-k_s)^\rho}\, k_s^{\theta\,\rho}\, [\varphi(k_s)]^{1+\lambda} \le \frac{c_0}{d^\rho\left(\frac{1}{2^s}-\frac{1}{2^{s+1}}\right)^\rho}\, k_s^{\theta\,\rho}\, \frac{[\varphi(k_0)]^{1+\lambda}}{2^{s\mu(1+\lambda)}} \\
&= \frac{c_0\, k_s^{\theta\,\rho}\, 2^{\rho(s+1)}}{c_0\,\Lambda\,[\varphi(k_0)]^\lambda\, 2^{\mu(1+\lambda)}}\, \frac{[\varphi(k_0)]^{1+\lambda}}{2^{s\mu(1+\lambda)}} = \frac{k_s^{\theta\,\rho}}{\Lambda}\, \frac{\varphi(k_0)}{2^{(s+1)\,((1+\lambda)\mu-\rho)}} \\
&= \frac{k_s^{\theta\,\rho}}{\Lambda}\, \frac{\varphi(k_0)}{2^{(s+1)\mu}} \le \frac{(k_0+d)^{\theta\,\rho}}{\Lambda}\, \frac{\varphi(k_0)}{2^{(s+1)\mu}}\,.
\end{aligned}$$

Thus, (A.3) will hold true with $s$ replaced by $s+1$ if there exists a positive constant $\Lambda$ such that

$$(k_0+d)^{\theta\,\rho} \le \Lambda \tag{A.4}$$

(recall that $d$ depends on $\Lambda$). If $\theta = 0$ this is trivial. Otherwise, setting

$$\tilde{c}_0 = c_0\, [\varphi(k_0)]^\lambda\, 2^{(1+\lambda)\,\mu}\,,$$

and recalling the definition of $d$, (A.4) can be rewritten as

$$k_0 + \tilde{c}_0^{\frac{1}{\rho}}\,\Lambda^{\frac{1}{\rho}} \leq \Lambda^{\frac{1}{\theta\rho}}\,.$$

As $\Lambda$ tends to infinity, and since $\theta < 1$, the right hand side of the preceding inequality diverges faster than the left hand side. Thus, there exists $\Lambda > 0$ such that (A.4) holds, that is, (A.3) is proved. Passing to the limit as $s$ tends to infinity in (A.3), since $\varphi$ is non increasing and $\mu$ is positive, we get

$$0 \leq \varphi(k^*) \leq \lim_{s\to+\infty} \varphi(k_s) \leq \lim_{s\to+\infty} \frac{\varphi(k_0)}{2^{\mu s}} = 0\,,$$

where $k^* = k_0 + d$. This concludes the proof. ∎

**Acknowledgments.** The present paper contains the unpublished part of the results presented by the first author at the conference. The authors would like to thank Enrique Fernandez Cara for the observation which is contained in Remark 2.6. The authors have been supported by Italian MURST research funds.

# References

[1] A. Alvino, V. Ferone, G. Trombetti, A priori estimates for a class of non uniformly elliptic equations, *on this volume*.

[2] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An $L^1$ theory of existence and uniqueness of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **22** n. 2 (1995), 240–273.

[3] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, **87** (1989), 149–169.

[4] L. Boccardo, D. Giachetti, Alcune osservazioni sulla regolarità delle soluzioni di problemi fortemente non lineari e applicazioni, *Ricerche Mat.*, **24** (1985), 309–323.

[5] L. Boccardo, D. Giachetti, $L^s$-regularity of solutions of some nonlinear elliptic problems, *preprint*.

[6] L. Boccardo, L. Orsina, Existence and regularity of minima for integral functionals non coercive in the energy space, *preprint*.

[7] O. Ladyženskaya, N. Ural'ceva, *Linear and quasilinear elliptic equations*, Academic Press, New York (1968).

[8] J.-L. Lions, *Quelques méthodes de resolution de problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris (1969).

[9] L. Orsina, Solvability of linear and semilinear eigenvalue problems with $L^1$ data, *Rend. Sem. Mat. Univ. Padova*, **90** (1993), 207–238.

[10] A. Porretta, Uniqueness and homogenization for a class of non coercive operators in divergence form, *on this volume*.

[11] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)*, **15** n. 1 (1965), 189–258.

[12] C. Vinti, Convergenza in area, *Atti Sem. Mat. Fis. Modena*, **17** (1968), 29–46.