EXISTENCE AND REGULARITY RESULTS FOR SOME NONLINEAR PARABOLIC EQUATIONS

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Abstract

We prove summability results for the solutions of nonlinear parabolic problems of the form

$$u' + A(u) = f \qquad \text{in } \Omega \times (0,T)\,,$$

with homogeneous Cauchy-Dirichlet boundary conditions, where A is a pseudomonotone, coercive, uniformly elliptic operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ to its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, and the datum f belongs to $L^q(0,T;L^r(\Omega))$. As a consequence of these estimates we also obtain an existence theorem in the case where f does not belong to $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

We also give summability results for data in divergence form.

1 Introduction and statement of the results

Let Ω be a bounded domain in $\boldsymbol{R}^N$, $N \geq 1$. For $T > 0$, let us denote by Q the cylinder $\Omega \times (0,T)$, and by Γ the lateral surface $\partial\Omega \times (0,T)$. We will consider the following nonlinear parabolic Cauchy-Dirichlet problem:

$$\begin{cases} u' - \operatorname{div}(a(x,t,u,\nabla u)) = f(x,t) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

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where $-\mathrm{div}\,(a(x,t,u,\nabla u))$ is a pseudomonotone, coercive, uniformly elliptic operator acting from $L^p(0,T;W_0^{1,p}(\Omega))$ to its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ (p is a real number greater than 1), and f is a measurable function on Q. If f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$, then it is well known (see [8]) that there exists at least a solution u of (1.1), with u in $L^p(0,T;W_0^{1,p}(\Omega))$.

We will be concerned with some existence and regularity results for the solutions of (1.1), depending on the summability of the datum f.

We begin recalling some results in the case $p = 2$ (which includes, for example, the case of operators which are linear with respect to the gradient): if f belongs to $L^r(0,T;L^q(\Omega))$, and if r and q are large enough, that is, if

$$\frac{N}{q} + \frac{2}{r} < 2\,, \qquad r \geq 1\,, \qquad q \geq 1\,,$$

then Aronson and Serrin (see [1]) proved that any solution u of (1.1) belongs to $L^\infty(Q)$.

On the other hand, if

$$a(x,t,u,\nabla u) = A(x,t)\,\nabla u\,,$$

and if r and q satisfy the opposite inequality and an additional one, more precisely if

$$2 < \frac{N}{q} + \frac{2}{r} \leq \min\left\{\frac{N}{r} + 2, \frac{N}{2} + 2\right\}\,, \tag{1.2}$$

then in [7], Chapter III, Theorem 9.1, it is proved that any weak solution (if it exists) belongs to $L^s(Q)$, with s given by

$$s = \frac{(N+2)qr}{Nr - 2q(r-1)}\,, \tag{1.3}$$

and this then implies that such a solution belongs to $L^2(0,T;H_0^1(\Omega))$.

Observe that if $r \geq 2$, then (1.2) becomes

$$2 < \frac{N}{q} + \frac{2}{r} \leq \frac{N}{r} + 2\,,$$

and in this case the function f belongs to the space $L^2(0,T;H^{-1}(\Omega))$, so that there exists at least a solution of (1.1) (which is indeed unique since a does not depend on u). Conversely, if $1 \le r < 2$, the hypotheses on r and q can be rewritten as

$$2 < \frac{N}{q} + \frac{2}{r} \le \frac{N}{2} + 2\,,$$

and in this case the function f is not in general in $L^2(0,T;H^{-1}(\Omega))$. However, one can use the linearity of the operator with respect to the gradient and the *a priori* estimates in order to prove the existence of a solution of (1.1) by means of duality techniques.

The *a priori* estimates in $L^2(0,T;H_0^1(\Omega))$ can be obtained, always in the linear case, by means of different techniques, such as norm estimates using the heat kernel, duality properties and interpolation between Lebesgue spaces ([5]).

If a is of the form

$$a(x,t,u,\nabla u) = A(x,t,u)\,\nabla u\,,$$

where A is a Carathéodory matrix-valued function, bounded and uniformly elliptic, the duality techniques cannot be applied, but the same *a priori* estimates (with the same bounds on r and q, and the same value of s) continue to hold. These estimates, together with the linearity of the operator with respect to the gradient, were used by the authors in order to prove the existence of at least a solution of (1.1) in the case of data not in $L^2(0,T;H^{-1}(\Omega))$ (this case was presented by the first author in [2], where also the results of the present paper were announced).

In this paper we are going to deal with the general case of a nonlinear pseudomonotone operator. Before stating our results, let us make our assumptions more precise.

Let p be a real number, $p > 1$, and let p' be its Hölder conjugate exponent (i.e., $1/p + 1/p' = 1$).

We recall that Ω is a bounded domain in $\boldsymbol{R}^N$, $N \ge 1$, Q denotes the cylinder $\Omega \times (0,T)$, and Γ is the lateral surface $\partial\Omega \times (0,T)$ of Q, with $T > 0$.

Let $a : Q \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a Carathéodory function (i.e., $a(\cdot,\cdot,\sigma,\xi)$ is measurable in Q for every (σ,ξ) in $\boldsymbol{R} \times \boldsymbol{R}^N$, and $a(x,t,\cdot,\cdot)$ is continuous on $\boldsymbol{R} \times \boldsymbol{R}^N$ for almost every (x,t) in Q), such that:

$$a(x,t,\sigma,\xi) \cdot \xi \ge \alpha\,|\xi|^p\,, \tag{1.4}$$

$$|a(x,t,\sigma,\xi)| \le \beta\,[k(x,t) + |\sigma|^{p-1} + |\xi|^{p-1}]\,, \tag{1.5}$$

$$[a(x,t,\sigma,\xi) - a(x,t,\sigma,\eta)]\cdot(\xi-\eta) > 0\,, \tag{1.6}$$

for almost every (x,t) in Q, for every σ in $\boldsymbol{R}$, for every ξ, η in $\boldsymbol{R}^N$, $\xi \neq \eta$, where α and β are positive real numbers, and k is a nonnegative function in $L^{p'}(Q)$.

Let us define the differential operator

$$A(u) = -\mathrm{div}\,(a(x,t,u,\nabla u))\,.$$

Under our assumptions on a, the operator A turns out to be a uniformly elliptic, coercive and pseudomonotone operator acting from $L^p(0,T;W_0^{1,p}(\Omega))$ to its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Moreover (see [8]), for every datum g in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ there exists at least one solution u in $L^p(0,T;W_0^{1,p}(\Omega))$ of the following Cauchy-Dirichlet problem

$$\begin{cases} u' + A(u) = g & \text{in } Q,\\ u(x,0) = 0 & \text{in } \Omega,\\ u(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{1.7}$$

By a solution u of (1.7) we mean a function in $L^p(0,T;W_0^{1,p}(\Omega))$ which satisfies the equation (1.7) in the sense of distributions. The solution u belongs to $C^0([0,T];L^2(\Omega))$ (see for instance [8]), so that the initial datum has a meaning.

Furthermore, if $\psi : \boldsymbol{R} \to \boldsymbol{R}$ is a Lipschitz continuous function such that $\psi(0) = 0$, we can choose $\varphi = \psi(u)$ as test function in order to obtain (recall that $u(x,0) = 0$)

$$\int_\Omega \Psi(u(T))\,dx + \int_Q a(x,t,u,\nabla u)\cdot\nabla(\psi(u)) = \langle g, \psi(u)\rangle\,, \tag{1.8}$$

where

$$\Psi(s) = \int_0^s \psi(\rho)\,d\rho\,,$$

and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $L^p(0,T;W_0^{1,p}(\Omega))$.

Our first result concerns the regularity of solutions of (1.7) in the case of data which belong to the dual space (see Remark 1.2 below).

Theorem 1.1 *Let r and q be real numbers such that*

$$p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r} + p\,, \qquad r \geq p'\,, \qquad q > 1\,, \tag{1.9}$$

Let f be a function in $L^r(0,T;L^q(\Omega))$. Then any solution u of (1.7) in the space $L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^s(Q)$ where

$$s = \frac{qr(N+p) + N(p-2)(qr-q+r)}{Nr - pq(r-1)}\,. \tag{1.10}$$

Moreover, if $p > N$, one can choose $q = 1$ in (1.9).

Remark 1.2 Let us make some comments on the bounds on r and q, and on the value of s. If $p < N$, it is easy to see that (1.9) implies

$$q \geq \frac{Np}{Np - N + p} = (p^*)'\,,$$

where $p^* = Np/(N-p)$ is the Sobolev exponent of p. This fact yields that the datum f always belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$, so that there exists a solution for (1.7). Moreover, since $p < N$, we have that $(p^*)' > 1$, so that the condition $q > 1$ of (1.9) is always satisfied.

If $p = N$, then (1.9) becomes

$$\frac{1}{q} + \frac{1}{r} > 1\,, \qquad q > 1\,, \qquad r \geq \frac{N}{N-1}\,.$$

If $p > N$, the condition

$$\frac{N}{q} + \frac{p}{r} \leq \frac{N}{r} + p$$

is always satisfied for r and q greater or equal to than 1, so that the conditions on r and q become

$$\frac{N}{q} + \frac{p}{r} > p\,, \qquad r \geq p'\,, \qquad q \geq 1\,.$$

If $q = 1$, then r must satisfy

$$p' \leq r < \frac{p}{p-N}\,.$$

One can check that the value of s is greater than $p(N+2)/N$, which is the Gagliardo-Nirenberg embedding exponent for functions that belong to $L^p(0,T;W_0^{1,p}(\Omega))$ and to $L^\infty(0,T;L^2(\Omega))$ (see also (2.3), below). Observe that if $p=2$, we obtain the value of s given by (1.3).

Our next result is not only a regularity theorem, but also an existence result.

Theorem 1.3 *Let r and q be real numbers such that*

$$p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np+2p-N}{2}\,, \quad 1 \leq r < p'\,, \quad q \geq 1\,, \quad (1.11)$$

and let f be a function in $L^r(0,T;L^q(\Omega))$. Then there exists at least one solution u in $L^p(0,T;W_0^{1,p}(\Omega))$ of (1.7); moreover, u belongs to $L^s(Q)$ where s is as in (1.10).

Remark 1.4 In this case, being $r<p'$, the datum f does not belong to the dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$. If $r=1$, then q has to be greater than or equal to 2.

As in the case of Theorem 1.1, if $p<N$ then the first two conditions of (1.11) imply $q>1$. If $p=N$, then $q=1$ if and only if $r=N/(N-1)=p'$, a value that r cannot attain. Thus, q can be equal to 1 if and only if $p>N$.

If $q=1$, then r must satisfy

$$\frac{Np+2p-2N}{Np+2p-3N} \leq r < p'\,.$$

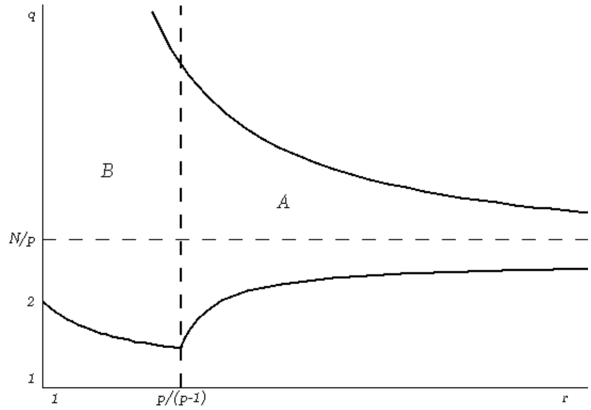
Again, if $1<p<N$ the possible values for s are greater than $p(N+2)/N$.

We will also give a regularity result for solutions of (1.7) in the case of data in divergence form (see Proposition 2.4, below).

A result concerning local estimates for solutions of nonlinear parabolic problems was obtained by Porzio in [10], using different techniques.

The following is a picture which summarizes, in the case $p<N$, the possible values of r and q given by the bounds in Theorem 1.1 and Theorem 1.3. If (r,q) lies in region A, then the solution u belongs to $L^s(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$, with s given by (1.10) (see Theorem 1.1); in region B the datum f is not in the

dual space, and we have existence of a solution in $L^s(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$, with s as in (1.10) (see Theorem 1.3).



The main tool of the proof of both Theorem 1.1 and Theorem 1.3 will be an *a priori* estimate on the solutions of (1.7) with data in the dual space. In the case of Theorem 1.1 this estimate will directly give the result. Under the hypotheses of the existence Theorem 1.3, however, we will need to proceed by approximation: more precisely, we will consider data f_n (belonging to the dual space) which converge to the datum f in $L^r(0,T;L^q(\Omega))$, and take into account a sequence of solutions u_n of the corresponding problems. We will have a uniform estimate on u_n in $L^s(Q)$ and then in $L^p(0,T;W_0^{1,p}(\Omega))$. We would like to pass to the limit in order to find a solution for our problem but, since the equation is nonlinear, the *a priori* estimates are not enough in order to let n tend to infinity: we need a strong convergence result for the sequence $\{\nabla u_n\}$. In order to obtain this result, we will use a theorem proved by the authors in [3], which will allow us to conclude.

The plan of the paper is as follows: in the next section we will prove the *a priori* estimates on the solutions of (1.7) which form the core of the proof of Theorem 1.1 and Theorem 1.3; we will also state and prove an *a priori* estimate for the solutions of (1.7) in the case of data of the form $-\mathrm{div}\,(F)$. The third section will be devoted to the proof of the theorems.

2 A priori estimates

Before stating and proving our results, let us recall the Gagliardo-Nirenberg inequality (see [9], Lecture II).

Lemma 2.1 *Let v be a function in $W_0^{1,p}(\Omega) \cap L^\rho(\Omega)$, with $p \geq 1$, $\rho \geq 1$. Then there exists a positive constant C, depending on N, p, ρ, and σ such that*

$$\|v\|_{L^\sigma(\Omega)} \leq C \, \|\nabla v\|^\theta_{L^p(\Omega;\boldsymbol{R}^N)} \, \|v\|^{1-\theta}_{L^\rho(\Omega)} \,, \tag{2.1}$$

for every θ and σ satisfying

$$0 \leq \theta \leq 1 \,, \quad 1 \leq \sigma < +\infty \,, \qquad \frac{1}{\sigma} = \theta \left(\frac{1}{p} - \frac{1}{N} \right) + \frac{1-\theta}{\rho} \,. \tag{2.2}$$

If $p > N$, one can choose $\sigma = +\infty$ (that is, $1/\sigma = 0$).

An immediate consequence of the previous lemma is the following embedding result:

$$\int_Q |v|^\sigma \leq C \, \|v\|^{\frac{\rho p}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^p \,, \tag{2.3}$$

which holds for every function v in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, with $p \geq 1$, $\rho \geq 1$ and $\sigma = p(N+\rho)/N$ (see, for instance, [6], Proposition 3.1).

We now state the *a priori* estimate we need.

Proposition 2.2 *Let r and q be real numbers such that*

$$p < \frac{N}{q} + \frac{p}{r} \leq \min\left\{ \frac{N}{r} + p, \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np + 2p - N}{2} \right\} , \quad r \geq 1 \,, \tag{2.4}$$

and

$$q > 1 \quad \text{if } p \leq N, \qquad q \geq 1 \quad \text{if } p > N. \tag{2.5}$$

hold. Let f be a function in $L^r(0,T;L^q(\Omega)) \cap L^{p'}(0,T;W^{-1,p'})$. Let u be a solution in $L^p(0,T;W_0^{1,p}(\Omega))$ of (1.7) with datum f, and let s be given by (1.10). Then the norms of u in $L^s(Q)$ and in $L^p(0,T;W_0^{1,p}(\Omega))$ are bounded by a constant which depends on α, N, p, q, r, Q and on the norm of f in $L^r(0,T;L^q(\Omega))$.

Remark 2.3 We observe that (2.4) can be written as

$$p < \frac{N}{q} + \frac{p}{r} \leq \begin{cases} \dfrac{N}{r}\left(1 - \dfrac{p}{2}\right) + \dfrac{Np + 2p - N}{2} & \text{if } 1 \leq r < p', \\ \dfrac{N}{r} + p & \text{if } r \geq p', \end{cases}$$

and therefore Proposition 2.2 gives *a priori* estimates under the assumptions of both Theorem 1.1 and Theorem 1.3.

Proof of Proposition 2.2. In the following, we will denote by c any constant depending only on α, N, p, q, r and $\|f\|_{L^r(0,T;L^q(\Omega))}$. The value of c may vary from line to line.

Let n be a positive integer, and suppose that $r > 1$ and $q > 1$.

For $\gamma \geq 0$ (which will be chosen below), and $\tau \in (0, T)$, we can take $\varphi = |T_n(u)|^{\gamma p} T_n(u) \chi_{(0,\tau)}(t)$ as test function. This is possible since $T_n(u) \in L^\infty(Q) \cap L^p(0, T; W_0^{1,p}(\Omega))$, so that φ belongs to the same space. Using (1.8), and setting

$$\Psi_n(s) = \int_0^s |T_n(t)|^{\gamma p} \, T_n(t) \, dt \,,$$

we obtain

$$\begin{aligned} \int_\Omega \Psi_n(u(\tau)) \, dx + (\gamma p + 1) \int_0^\tau \int_\Omega a(x, t, u, \nabla u) \cdot \nabla T_n(u) \, |T_n(u)|^{\gamma p} \\ \leq \int_0^\tau \int_\Omega |f| \, |T_n(u)|^{\gamma p + 1} \,. \end{aligned} \tag{2.6}$$

Observing that

$$\Psi_n(s) \geq \frac{|T_n(s)|^{\gamma p + 2}}{\gamma p + 2} \,,$$

and recalling the ellipticity assumption (1.4) on a, we obtain from (2.6)

$$\int_\Omega |T_n(u)(\tau)|^{\gamma p + 2} \, dx \leq (\gamma p + 2) \int_0^\tau \int_\Omega |f| \, |T_n(u)|^{\gamma p + 1} \,,$$

and

$$\int_0^\tau \int_\Omega |\nabla(|T_n(u)|^{\gamma + 1})|^p \leq \frac{(\gamma + 1)^p}{\alpha \, (\gamma p + 1)} \int_0^\tau \int_\Omega |f| \, |T_n(u)|^{\gamma p + 1} \,.$$

Taking the supremum of both terms for τ in $(0,T)$ and using Hölder's inequality twice, we obtain

$$
\begin{aligned}
\|T_n(u)\|^{\gamma p+2}_{L^\infty(0,T;L^{\gamma p+2}(\Omega))} + \int_Q |\nabla(|T_n(u)|^{\gamma+1})|^p &\le c \int_Q |f|\,|T_n(u)|^{\gamma p+1} \\
&\le c \int_0^T \|f(t)\|_{L^q(\Omega)} \|T_n(u)(t)\|^{\gamma p+1}_{L^{(\gamma p+1)\,q'}(\Omega)}\,dt \\
&\le c\,\|f\|_{L^r(0,T;L^q(\Omega))} \left[\int_0^T \|T_n(u)(t)\|^{(\gamma p+1)\,r'}_{L^{(\gamma p+1)\,q'}(\Omega)}\,dt\right]^{\frac{1}{r'}} \\
&\le c \left[\int_0^T \|T_n(u)(t)\|^{(\gamma p+1)\,r'}_{L^{(\gamma p+1)\,q'}(\Omega)}\,dt\right]^{\frac{1}{r'}},
\end{aligned}
\tag{2.7}
$$

with c depending also on γ. Observe that both r' and q' are real numbers since $r > 1$ and $q > 1$. Setting

$$
w = |T_n(u)|^{\gamma+1},
$$

formula (2.7) can be rewritten as

$$
\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \le c \left[\int_0^T \|w(t)\|^{\frac{(\gamma p+1)\,r'}{\gamma+1}}_{L^{\frac{(\gamma p+1)\,q'}{\gamma+1}}(\Omega)}\,dt\right]^{\frac{1}{r'}}. \tag{2.8}
$$

Let us define θ such that

$$
\frac{\gamma+1}{(\gamma p+1)\,q'} = \theta\left(\frac{1}{p} - \frac{1}{N}\right) + \frac{(1-\theta)(\gamma+1)}{\gamma p+2}. \tag{2.9}
$$

We will check below that, under our assumptions on r and q, and with our choice of γ (see (2.12) below), θ belongs to $[0,1]$. Thus, by the Gagliardo-Nirenberg inequality (2.1), applied with

$$
\sigma = \frac{(\gamma p+1)q'}{\gamma+1}, \qquad \rho = \frac{\gamma p+2}{\gamma+1},
$$

we have, from (2.8),

$$
\begin{aligned}
&\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \\
&\qquad \le c\,\|w\|^{\frac{\gamma p+1}{\gamma+1}(1-\theta)}_{L^\infty(0,T;L^{\gamma p+2}(\Omega))} \left[\int_0^T \|\nabla w(t)\|^{\frac{\gamma p+1}{\gamma+1}\theta r'}_{L^p(\Omega;\boldsymbol{R}^N)} dt\right]^{\frac{1}{r'}}.
\end{aligned}
\tag{2.10}
$$

If $\theta < 1$, applying the Young inequality with exponents

$$\frac{\gamma p + 2}{(\gamma p + 1)(1-\theta)} \qquad \text{and} \qquad \frac{\gamma p + 2}{1+\theta(\gamma p + 1)}\,,$$

we obtain

$$\begin{aligned} &\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \\ &\quad \le \frac{1}{2}\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + c\left[\int_0^T \|\nabla w(t)\|^{\frac{\gamma p+1}{\gamma+1}\theta r'}_{L^p(\Omega;\boldsymbol{R}^N)} dt\right]^{\frac{\gamma p+2}{r'[1+\theta(\gamma p+1)]}}. \end{aligned} \tag{2.11}$$

Now we choose γ satisfying

$$\frac{\gamma p + 1}{\gamma + 1}\theta r' = p\,, \tag{2.12}$$

and we will check below that, under our assumptions on r and q, γ is non-negative. From (2.11) we then obtain

$$\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \le c\,\|\nabla w\|^{p\left[\frac{\gamma p+2}{r'[1+\theta(\gamma p+1)]}\right]}_{L^p(Q;\boldsymbol{R}^N)}. \tag{2.13}$$

Since, from (2.12),

$$\theta r'(\gamma p + 1) = \gamma p + p\,,$$

we have

$$\frac{\gamma p + 2}{r'[1+\theta(\gamma p+1)]} = \frac{\gamma p + 2}{r' + \gamma p + p} < 1$$

(the latter inequality is equivalent to $r' > 2 - p$ which is true since $2 - p < 1$), and so

$$\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \le c\,\|\nabla w\|^{\delta p}_{L^p(Q;\boldsymbol{R}^N)}\,, \tag{2.14}$$

with $\delta < 1$. If $\theta = 1$, choosing γ as in (2.12), (2.10) becomes (2.14) with $\delta = 1/r' < 1$. Thus from (2.14) immediately follows

$$\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \le c\,,$$

and therefore, by inequality (2.3),

$$\|w\|_{L^\sigma(Q)} \le c\,,$$

where

$$\sigma = p\frac{N + \frac{\gamma p+2}{\gamma+1}}{N}\,.$$

Recalling the definition of w, we thus have proved that

$$\|T_n(u)\|_{L^s(Q)} \le c\,, \tag{2.15}$$

where c depends on α, N, p, q, r, γ, $\|f\|_{L^r(0,T;L^q(\Omega))}$ but not on n, and

$$s = p\frac{N(\gamma+1)+\gamma p+2}{N} = (\gamma p+2)\frac{N+p}{N} + p - 2\,. \tag{2.16}$$

From (2.9) and (2.12) we obtain

$$\gamma p + 2 = \frac{Nq[(r-1)(p-2)+r]}{Nr - pq(r-1)}\,,$$

which implies, by (2.16),

$$s = \frac{qr(N+p) + N(p-2)(qr-q+r)}{Nr-pq(r-1)}\,,$$

which is the value given by (1.10). Passing to the limit as n tends to infinity in (2.15), we thus obtain an estimate for u in $L^s(Q)$.

We now have to check that $\gamma \ge 0$ and that θ, defined in (2.9), belongs to $[0,1]$. After easy calculations, we obtain that $\gamma \ge 0$ if and only if

$$p < \frac{N}{q} + \frac{p}{r} \le \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np - N + 2p}{2}\,,$$

while the condition $\theta \le 1$ holds is satisfied if and only if

$$\frac{N}{q} + \frac{p}{r} \le \frac{N}{r} + p\,.$$

The condition $\theta \ge 0$ is automatically satisfied if $\gamma \ge 0$. Thus, we obtain for r and q (both strictly greater than 1) the conditions given in the statement.

We now deal with the case $r = 1$, which corresponds to $q \geq 2$. In this case, we choose $\varphi = |T_n(u)|^{q-2}\, T_n(u)\, \chi_{(0,\tau)}$ as test function in (1.7). By means of the same technique used above, we arrive after straightforward passages to

$$|T_n(u)|^{\frac{q-2+p}{p}} \in L^\infty(0,T; L^{\frac{pq}{q-2+p}}(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))\,,$$

with a bound on the norms of $T_n(u)$ in these spaces which is uniform with respect to n. Using inequality (2.3), this implies (again after a passage to the limit as n tends to infinity) that u is bounded in $L^s(Q)$, with

$$s = \frac{(N+p)q + N(p-2)}{N}\,,$$

which is the value given by (1.10) in the case $r = 1$.

Finally, we have to treat the case $q = 1$, which corresponds to the case $p > N$. As before, we choose as test function $\varphi = |T_n(u)|^{\gamma p}\, T_n(u)\, \chi_{(0,\tau)}(t)$, and we obtain (setting $w = |T_n(u)|^{\gamma+1}$),

$$\|w\|^{\frac{\gamma p+2}{\gamma+1}}_{L^\infty(0,T;L^{\frac{\gamma p+2}{\gamma+1}}(\Omega))} + \|\nabla w\|^p_{L^p(Q;\boldsymbol{R}^N)} \leq c\left[\int_0^T \|w(t)\|^{\frac{(\gamma p+1)\,r'}{\gamma+1}}_{L^\infty(\Omega)}\,dt\right]^{\frac{1}{r'}}\,.$$

Observe that in this case $q' = +\infty$. Using the Gagliardo-Nirenberg inequality (2.1) on the right hand side, with $\rho = (\gamma p+2)/(\gamma+1)$ and $\sigma = +\infty$, we get (2.10), with θ given by

$$\theta = \frac{\frac{1}{\rho}}{\frac{1}{N} - \frac{1}{p} + \frac{1}{\rho}}\,.$$

Going on as in the first part of the proof, one obtains the bound on u in $L^s(Q)$, with s given by (1.10) with $q = 1$. The only condition we have to check is that γ, defined in (2.12), is nonnegative, since this will then imply that $\theta \in [0,1]$. After straightforward calculation, one can check that $\gamma \geq 0$ if and only if

$$\frac{Np+2p-2N}{Np+2p-3N} \leq r < \frac{p}{p-N}\,,$$

and these are exactly the bounds on r given by (2.4) for $q = 1$ (see Remark 2.3).

It remains to prove the estimate in $L^p(0,T;W_0^{1,p}(\Omega))$, and in order to do this, we take $T_1(u)$ as test function in (1.7) (observe that this corresponds to $\gamma = 0$). Throwing away positive terms we obtain, after using (1.4),

$$\alpha \int_{\{|u|\le 1\}} |\nabla u|^p \le \|f\|_{L^1(Q)} \le c\,\|f\|_{L^r(0,T;L^q(\Omega))},$$

since both r and q are not smaller than 1. On the other hand, if γ is as in (2.12) (if $r > 1$) or is such that $\gamma p + 2 = q$ (if $r = 1$), we have

$$\int_{\{|u|>1\}} |\nabla u|^p \le \int_Q |\nabla u|^p\,|u|^{\gamma p} \le c\,,$$

by the first part of the proof. Thus, we obtain

$$\int_Q |\nabla u|^p = \int_{\{|u|\le 1\}} |\nabla u|^p + \int_{\{|u|>1\}} |\nabla u|^p \le c\,,$$

and this gives us the $L^p(0,T;W_0^{1,p}(\Omega))$ estimate. ■

The following result is the analogue of Proposition 2.2 in the case of data in divergence form.

Proposition 2.4 *Let r_1 and q_1 be real numbers such that*

$$p-1 < \frac{N}{q_1} + \frac{p}{r_1} \le \frac{N}{r_1} + p - 1\,, \qquad r_1 \ge p'\,, \qquad q_1 > p'\,. \tag{2.17}$$

Let $F : \Omega \to \boldsymbol{R}^N$ be such that $|F|$ belongs to $L^{r_1}(0,T;L^{q_1}(\Omega))$. Let u be a solution in $L^p(0,T;W_0^{1,p}(\Omega))$ of

$$\begin{cases} u' + A(u) = -\mathrm{div}\,(F) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{2.18}$$

and let

$$s_1 = \frac{(N+2)(p-1)q_1 r_1 + N(r_1 - q_1)(p-2)}{r_1(N+q_1) - p q_1 (r_1 - 1)}\,. \tag{2.19}$$

Then the norm of u in $L^{s_1}(Q)$ is bounded by a constant which depends on α, p, N, q_1, r_1, Q and on the norm of $|F|$ in $L^{r_1}(0,T;L^{q_1}(\Omega))$.

Moreover, if $p \ne N$ we can choose $q_1 = p'$ in (2.17).

Remark 2.5 In the case of minimal regularity of the datum, that is, in the case where $r_1 = q_1 = p'$, the value given by (2.19) is $s_1 = p(N+2)/N$, which is the exponent of the embedding (2.3) for functions in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Proof of Proposition 2.4. The proof follows the same lines as Proposition 2.2.

As before, for $\gamma \geq 0$, and $\tau \in (0,T)$, we take $\varphi = |T_n(u)|^{\gamma p} T_n(u) \chi_{(0,\tau)}(t)$ as test function in (2.18). As in the proof of Proposition 2.2, we obtain (after straightforward passages),

$$\|T_n(u)\|^{\gamma p+2}_{L^\infty(0,T;L^{\gamma p+2}(\Omega))} + \int_Q |\nabla T_n(u)|^p \, |T_n(u)|^{\gamma p} \leq c \int_Q |F| \, |\nabla T_n(u)| \, |T_n(u)|^{\gamma p} .$$

Using the Young inequality with exponents p' and p in the right hand side, we have

$$\int_Q |F| \, |\nabla T_n(u)| \, |T_n(u)|^{\gamma p} \leq \varepsilon \int_Q |\nabla T_n(u)|^p \, |T_n(u)|^{\gamma p} + c_\varepsilon \int_Q |F|^{p'} \, |T_n(u)|^{\gamma p} ,$$

so that, choosing ε small enough,

$$\|T_n(u)\|^{\gamma p+2}_{L^\infty(0,T;L^{\gamma p+2}(\Omega))} + \int_Q |\nabla T_n(u)|^p \, |T_n(u)|^{\gamma p} \leq c \int_Q |F|^{p'} \, |T_n(u)|^{\gamma p} .$$

Starting from here, we can repeat exactly the same steps as in the proof of Proposition 2.2, obtaining the result given in the statement (again, one has to treat separately the "borderline" cases in which either $r_1 = p'$ or $q_1 = p'$).

■

Remark 2.6 What kind of estimates can be expected if the datum of the problem is of the form $f - \mathrm{div}\,(F)$, where f belongs to $L^r(0,T;L^q(\Omega))$, and $|F|$ belongs to $L^{r_1}(0,T;L^{q_1}(\Omega))$, with (r,q) and (r_1,q_1) as in Proposition 2.2 and Proposition 2.4, respectively? What is the regularity of the solutions with respect to Lebesgue spaces on Q?

If the problem is linear (for example, if $a(x,t,s,\xi) = \xi$, so that we get the Laplacian operator), then the solution of the problem with datum $f - \mathrm{div}\,(F)$ is the sum of the solutions of the problems with data f and $-\mathrm{div}\,F$, respectively, so that, in general, it has the regularity of the less regular of the two solutions. This fact is also true in the general case.

To see this, let us define $s(r,q)$ the value given by (1.10), and $s_1(r_1,q_1)$ the value given by (2.19); denote by E and E_1 the sets of pairs (r,q) and (r_1,q_1) which satisfy the hypotheses of Proposition 2.2 and Proposition 2.4 respectively. If $(r,q) \in E$ and $(r_1,q_1) \in E_1$ are such that $s(r,q) = s_1(r_1,q_1) = s$, then clearly we can "combine" the proofs of Proposition 2.2 and Proposition 2.4, choosing the same test function and thus obtaining a regularity result for u in $L^s(Q)$.

If (for example) $s(r,q) < s_1(r_1,q_1)$ then we can use the following property of the set E_1 (also valid for E), which is easily seen to hold true: given $(r_1,q_1) \in E_1$ and given $s \in [(N+2)p/N, s_1(r_1,q_1))$, there exists $(r_2,q_2) \in E_1$, with $r_2 \le r_1$ and $q_2 \le q_1$, such that $s = s_1(r_2,q_2)$. In other words, we can use the classical embeddings between Lebesgue spaces, and consider a function in $L^{r_1}(0,T;L^{q_1}(\Omega))$ as a function in the bigger space $L^{r_2}(0,T;L^{q_2}(\Omega))$. But then, as we said before, the regularity of u is $L^{s(r,q)}(Q)$, that is, u has the regularity of the less regular of the solutions with data f and $-\mathrm{div}\,(F)$.

Moreover it is clear that the solutions belong to $L^p(0,T;W_0^{1,p}(\Omega))$. We thus have proved the following result.

Proposition 2.7 *Let r and q be real numbers satisfying (2.4), and let r_1 and q_1 be real numbers satisfying (2.17).*

Let f be a function in $L^r(0,T;L^q(\Omega)) \cap L^{p'}(0,T;W^{-1,p'}(\Omega))$, and let $F : Q \to \boldsymbol{R}^N$ be such that $|F|$ belongs to $L^{r_1}(0,T;L^{q_1}(\Omega))$. Let u be a solution in $L^p(0,T;W_0^{1,p}(\Omega))$ of

$$\begin{cases} u' + A(u) = f - \mathrm{div}\,(F) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma, \end{cases}$$

and let

$$\overline{s} = \min\{s, s_1\},$$

where s is given by (1.10), and s_1 is given by (2.19).

Then the norms of u in $L^{\overline{s}}(Q)$ and in $L^p(0,T;W_0^{1,p}(\Omega))$ are bounded by a constant which depends on α, p, N, q, r, q_1, r_1, Q, and on the norms of f in $L^r(0,T;L^q(\Omega))$ and of $|F|$ in $L^{r_1}(0,T;L^{q_1}(\Omega))$.

Remark 2.8 If (r,q) satisfy (1.11) then the *a priori* estimate of Proposition 2.7 yields an existence result for problem (1.7) with datum $f - \operatorname{div}(F)$; this result can be obtained proceeding as in the proof of Theorem 1.3 (see next section).

3 Proof of the results

Proof of Theorem 1.1. The result follows directly from Proposition 2.2, since in this case $L^r(0,T;L^q(\Omega))$ is embedded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ (see Remark 1.2). ∎

If we are under the hypotheses of Theorem 1.3, then we need to prove the existence of solutions since the datum f is not in the dual space. As stated in the Introduction, we will proceed by approximation.

Proof of Theorem 1.3. Let $\{f_n\}$ be a sequence of functions in $L^\infty(Q)$ that converges to f strongly in $L^r(0,T;L^q(\Omega))$, with r and q as in the statement of the theorem. Let u_n be a solution of

$$\begin{cases} u_n' + A(u_n) = f_n & \text{in } Q, \\ u_n(x,0) = 0 & \text{in } \Omega, \\ u_n(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Observe that such a solution exists (since f_n belongs to the dual space) and belongs to the space $L^p(0,T;W_0^{1,p}(\Omega))$. By Proposition 2.2, the sequence $\{u_n\}$ is bounded both in $L^s(Q)$ (with s as in the statement) and in $L^p(0,T;W_0^{1,p}(\Omega))$. Thus, we can extract a subsequence, still denoted by $\{u_n\}$, which converges weakly to some function u both in $L^s(Q)$ and in $L^p(0,T;W_0^{1,p}(\Omega))$.

Moreover, since from the equation one obtains that $\{u_n'\}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$, using compactness arguments (see for example [11]), we obtain that u_n converges strongly to u in $L^1(Q)$; in particular, it converges to u almost everywhere in Q.

Furthermore, standard results (see again [11]) yield the compactness of $\{u_n\}$ in $C^0([0,T];W^{-1,1}(\Omega))$, so that the initial condition $u_n(0) = 0$ passes to the limit as n tends to ∞.

Since the operator is nonlinear with respect to the gradient, all the convergence results obtained so far are not enough in order to pass to the limit in the weak formulation of (3.1). To proceed, we need to apply the following result, which is proved in [3], Theorem 3.3, in a more general setting (see also [4] for an analogous result).

Theorem 3.1 *Let* $\{u_n\}$ *be a sequence of solutions of (3.1) with data* f_n *which are convergent in the weak-∗ topology of measures; suppose that* u_n *converges to some* u *weakly in* $L^\sigma(0,T;W_0^{1,\sigma}(\Omega))$ *for some* $\sigma > 1$*. Then, up to subsequences,*

$$\nabla u_n \to \nabla u \qquad \textit{almost everywhere in } Q.$$

Since our sequences $\{u_n\}$ and $\{f_n\}$ satisfy the hypotheses of the above theorem, we have the almost everywhere convergence of ∇u_n. This fact implies that u_n converges strongly to u in $L^\rho(0,T;W_0^{1,\rho}(\Omega))$ for every $\rho < p$, so that, by (1.5),

$$a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u) \qquad \text{strongly in } L^{\frac{\rho}{p-1}}(Q;\boldsymbol{R}^N),$$

for every $\rho < p$. Thus, writing the weak formulation of (3.1), that is

$$-\int_Q u_n\varphi' + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla\varphi = \int_Q f_n\,\varphi\,,$$

for every $\varphi \in C_0^\infty(Q)$, and using all the convergences proved so far, it is easy to see that u satisfies

$$-\int_Q u\,\varphi' + \int_Q a(x,t,u,\nabla u)\cdot\nabla\varphi = \int_Q f\,\varphi\,,$$

and so is a weak solution of (1.7). ■

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