Asymptotic Analysis 42 (2005) 123–132
IOS Press

Homogenization of forward-backward parabolic equations

M. Amar [a,*], A. Dall'Aglio [a] and F. Paronetto [b]

[a] *Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", Via A. Scarpa 16, 00161 Roma, Italy*
E-mails: {amar, aglio}@dmmm.uniroma1.it

[b] *Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, Via per Arnesano, 73100 Lecce, Italy*
E-mail: fabio.paronetto@unile.it

Abstract. We study the homogenization of the equation

$$R(\varepsilon^{-1}x)\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f,$$

where R is a periodic function which may vanish or change sign, with appropriate initial/final conditions. The main tool is a compactness result for sequences of functions which have bounded norms in the spaces associated to the problems.

1. Introduction, notations and preliminary results

In this paper, we consider the homogenization for the following problem:

$$\begin{cases} R(\varepsilon^{-1}x)\dfrac{\partial u_\varepsilon}{\partial t}(x,t) - \Delta u_\varepsilon(x,t) = f(x,t) & \text{in } \Omega \times (0,T) = \Omega_T, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega_{+,\varepsilon}, \\ u_\varepsilon(x,T) = \psi(x) & \text{in } \Omega_{-,\varepsilon}, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbf{R}^N$, $N \geqslant 1$, R, f, φ, ψ are the data of the problem, for which appropriate assumptions will be required in the following, while the sets $\Omega_{\pm,\varepsilon}$ depend on the sign of $R(\varepsilon^{-1}x)$ and will be specified below. The main feature of this problem is that the function R may vanish and change sign.

We recall that, for fixed ε, particular cases of equations like that in (1.1), arising in kinetic theory (see, for instance, [4]), have been already considered in [2,6].

*Corresponding author.

1.1. Notations

Let $\Omega \subseteq \mathbf{R}^N$, $N \geqslant 1$, be a given open set, whose boundary is indicated by $\partial\Omega$. We denote by $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, the standard Lebesgue spaces, and by $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$, having distributional derivatives in $L^2(\Omega)$, which vanish on $\partial\Omega$.

Let I be a real interval and X a Banach space. We denote by $L^p(I;X)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions $h: I \to X$ such that

$$\|h\|^p_{L^p(I;X)} = \int_I \|h(t)\|^p_X \, \mathrm{d}t < +\infty \quad \text{if } 1 \leqslant p < +\infty,$$

$$\|h\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t \in I} \|h(t)\|_X < +\infty \quad \text{if } p = +\infty.$$

Let $Y = (0,1)^N$ be the unit cell in $\mathbf{R}^N$. A function defined on $\mathbf{R}^N$ is said to be Y-periodic if it is periodic of period 1 with respect to each variable x_i, with $1 \leqslant i \leqslant n$.

Throughout this paper, Ω is an open bounded subset of $\mathbf{R}^N$ with Lipschitz boundary and T is a positive number; we set $\Omega_T = \Omega \times (0,T)$. Finally, the letter C denotes a strictly positive constant which may vary each time.

1.2. Assumptions and preliminary results

Let us assume that the data φ, ψ, f satisfy

$$\varphi, \psi \in L^2(\Omega), \qquad f \in L^2\big(0,T; H^{-1}(\Omega)\big).$$

Let $R: \mathbf{R}^N \to \mathbf{R}$ be a measurable bounded Y-periodic function which may vanish and change sign and define $Y_+ = \{y \in Y\colon\ R(y) > 0\}$, $Y_0 = \{y \in Y\colon\ R(y) = 0\}$, $Y_- = \{y \in Y\colon\ R(y) < 0\}$. About the regions Y_+ and Y_- we make the following assumption: we suppose the existence of two open sets A_+ and A_-, up to modify R on some set of null measure, such that

$$A_+, A_- \subset Y, \quad A_+ \cap A_- = \emptyset, \quad Y_+ \subseteq A_+,\ Y_- \subseteq A_-. \tag{1.2}$$

This avoids the situation in which, for example, $Y_0 = \emptyset$ and Y_+ is a Cantor type set with positive measure.

We also assume that the mean value of R is different from zero, that is,

$$\overline{R} = \int_Y R(y)\, \mathrm{d}y \neq 0.$$

This assumption is not used for the existence result (i.e., for fixed ε), but will be essential in the proof of the compactness result, Theorem 2.3 below.

For instance, one could consider the simple case where $N = 1$, $R(y) \equiv -1$ in $(0,\alpha)$, $R(y) \equiv 1$ in $(\alpha, 1)$, with $\alpha \in (0,1)$ and $\alpha \neq 1/2$.

For simplicity of notation we set $R_\varepsilon(x) = R(\varepsilon^{-1}x)$. We denote by $\Omega_{+,\varepsilon}$ (resp. $\Omega_{-,\varepsilon}$) the subset of Ω where $R_\varepsilon > 0$ (resp. $R_\varepsilon < 0$).

We point out that in (1.1) the initial datum is only prescribed in the region where $R_\varepsilon > 0$, i.e., where the equation is "forward parabolic", the final datum only where $R_\varepsilon < 0$, i.e., where the equation is "backward parabolic", while no datum is given in the region where $R_\varepsilon = 0$, i.e., where the equation is "elliptic" in the variable x, with t as a parameter.

The first step is to state an existence result for the solution of (1.1). To this purpose, let us define the space

$$\mathcal{W}_\varepsilon = \{u \in L^2(0,T;H_0^1(\Omega)) \mid (R_\varepsilon u)' \in L^2(0,T;H^{-1}(\Omega))\} \tag{1.3}$$

(where $(R_\varepsilon u)'$ is the distributional derivative of $R_\varepsilon u$ with respect to t), endowed with the natural norm

$$\|u\|_{\mathcal{W}_\varepsilon} = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|(R_\varepsilon u)'\|_{L^2(0,T;H^{-1}(\Omega))}. \tag{1.4}$$

Definition 1.1. For every $\varepsilon > 0$, we say that a function $u_\varepsilon \in \mathcal{W}_\varepsilon$ is a solution of (1.1) if the first equation in (1.1) is satisfied in $H^{-1}(\Omega)$, for almost every $t \in (0,T)$ and

$$u_\varepsilon(x,0) = \varphi(x) \quad \text{for a.e. } x \in \Omega_{+,\varepsilon}, \qquad u_\varepsilon(x,T) = \psi(x) \quad \text{for a.e. } x \in \Omega_{-,\varepsilon}.$$

Under the assumptions made until now, Theorem 3.8 in [8] implies the following existence result (see also [7]).

Theorem 1.2. *For every $\varepsilon > 0$, problem* (1.1) *admits a unique solution $u_\varepsilon \in \mathcal{W}_\varepsilon$. Moreover,*

$$\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leqslant C\left[\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|\varphi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\right], \tag{1.5}$$

where $C > 0$ does not depend on ε.

2. Main results

Our aim is to study the limit of the solutions u_ε of (1.1), when $\varepsilon \to 0^+$. The main result is the following

Theorem 2.1. *For every $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{W}_\varepsilon$ be the unique solution of problem* (1.1). *Assume that $\overline{R} > 0$ and let u_0 be the unique solution of*

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t}(x,t) - \Delta u_0(x,t) = f(x,t) & \text{in } \Omega_T, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_0(x,0) = \varphi(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Then, for every $\delta > 0$,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0, \quad \|u_\varepsilon - u_0\|_{L^2(0,T-\delta;H_0^1(\Omega))} \to 0 \quad \text{for } \varepsilon \to 0^+. \tag{2.2}$$

Remark 2.2. It follows from (2.2) and from the first equation in (1.1) that the sequence $(R_\varepsilon u_\varepsilon)'$ converges strongly to $(\overline{R}u_0)'$ in $L^2(0,T-\delta;H^{-1}(\Omega))$, for every $\delta > 0$.

The main tool in order to prove Theorem 2.1 is the following compactness theorem (see also [3] for the case where $R(x) \geqslant c > 0$).

Theorem 2.3. *Assume that $\overline{R} \neq 0$. Let $\{u_\varepsilon\}$ be a sequence of functions such that $\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leqslant C$. Then $\{u_\varepsilon\}$ is relatively compact in $L^2(\Omega_T)$.*

Remark 2.4. In the case where $\overline{R} = 0$, the above theorem is false. For example, consider $\Omega = (0,1)$, $u_\varepsilon(x,t) = \eta(x)\sin(t/\varepsilon)$ with $\eta \in \mathcal{C}^1_c(0,1)$ and $R_\varepsilon(x) = \sin(x/\varepsilon)$. The sequence $\{u_\varepsilon\}$ is bounded in $\mathcal{W}_\varepsilon$, hence $u_\varepsilon \rightharpoonup 0$ weakly in $L^2(0,T;H^1_0(\Omega))$, but it does not converge strongly in $L^2(\Omega_T)$.

Remark 2.5. Actually, with the same proof one shows that Theorem 2.3 holds for any sequence $\{R_\varepsilon\}$ such that

$$R_\varepsilon(x) \rightharpoonup \overline{R}(x) \quad *\text{-weakly in } L^\infty(\Omega), \text{ with } \overline{R}(x) \neq 0 \text{ for a.e. } x \in \Omega.$$

Remark 2.6. The homogenization in the "critical" case $\overline{R} = 0$ will be treated with a different technique in a forthcoming paper [1], where more general linear operators will be considered.

A similar homogenization result as in Theorem 2.1 holds true, if we assume that $\overline{R} < 0$. Indeed we can state the following

Theorem 2.7. *For every $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{W}_\varepsilon$ be the unique solution of problem* (1.1). *Assume that $\overline{R} < 0$ and let u_0 be the unique solution of*

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t}(x,t) - \Delta u_0(x,t) = f(x,t) & \text{in } \Omega_T, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_0(x,T) = \psi(x) & \text{in } \Omega. \end{cases} \tag{2.3}$$

Then, for every $\delta > 0$,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0, \quad \|u_\varepsilon - u_0\|_{L^2(\delta,T;H^1_0(\Omega))} \to 0 \quad \text{for } \varepsilon \to 0^+. \tag{2.4}$$

3. Proof of the results

Proof of Theorem 2.3. Taking a subsequence, we can assume that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(0,T;H^1_0(\Omega))$. The sequence $\{R_\varepsilon u_\varepsilon\}$ is bounded in $L^2(\Omega_T)$, while $\{(R_\varepsilon u_\varepsilon)'\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Therefore using the classical compactness result by Aubin (see, for instance, [9]), up to subsequences one has

$$R_\varepsilon u_\varepsilon \to \xi \quad \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big) \text{ and weakly in } L^2(\Omega_T). \tag{3.1}$$

We want to show that $\xi = \overline{R}u$. Let $\phi(x)$ be any function in $H^1_0(\Omega)$. Then $R_\varepsilon\phi \to \overline{R}\phi$ weakly in $L^2(\Omega)$ and strongly in $H^{-1}(\Omega)$. If $\zeta(t) \in L^2(0,T)$, then $R_\varepsilon\phi\zeta \to \overline{R}\phi\zeta$ strongly in $L^2(0,T;H^{-1}(\Omega))$. Indeed

$$\int_0^T \|R_\varepsilon(x)\phi(x)\zeta(t) - \overline{R}\phi(x)\zeta(t)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t = \|\zeta\|^2_{L^2(0,T)}\|R_\varepsilon\phi - \overline{R}\phi\|^2_{H^{-1}(\Omega)} \to 0.$$

Therefore, recalling that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega))$, one has

$$\int_{\Omega_T} u_\varepsilon R_\varepsilon \phi\zeta \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} u\overline{R}\phi\zeta \,\mathrm{d}x\,\mathrm{d}t. \tag{3.2}$$

On the other hand, by (3.1),

$$\int_{\Omega_T} R_\varepsilon u_\varepsilon \phi\zeta \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \xi\phi\zeta \,\mathrm{d}x\,\mathrm{d}t. \tag{3.3}$$

Since $\phi(x)$ and $\zeta(t)$ are arbitrary, (3.2) and (3.3) imply $\xi \equiv \overline{R}u$; i.e.,

$$R_\varepsilon u_\varepsilon \to \overline{R}u \quad \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big) \text{ and weakly in } L^2(\Omega_T). \tag{3.4}$$

By the following Lemma 3.1, for every $\eta > 0$ and for ε small enough, one has

$$\begin{aligned}\|u_\varepsilon - u\|_{L^2(\Omega_T)} &\leqslant \eta\|u_\varepsilon - u\|_{L^2(0,T;H_0^1(\Omega))} + c_\eta\|R_\varepsilon(u_\varepsilon - u)\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\leqslant c_1\eta + c_\eta\|R_\varepsilon u_\varepsilon - \overline{R}u\|_{L^2(0,T;H^{-1}(\Omega))} + c_\eta\|(R_\varepsilon - \overline{R})u\|_{L^2(0,T;H^{-1}(\Omega))}.\end{aligned}$$

By (3.4) the norm $\|R_\varepsilon u_\varepsilon - \overline{R}u\|_{L^2(0,T;H^{-1}(\Omega))}$ tends to zero. Moreover, $(R_\varepsilon - \overline{R})u \to 0$ strongly in $L^2(0,T;H^{-1}(\Omega))$. Indeed, it is enough to prove that

$$\int_{\Omega_T} (R_\varepsilon - \overline{R})u\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{3.5}$$

for every weakly convergent sequence $\{\phi_\varepsilon\} \subseteq L^2(0,T;H_0^1(\Omega))$. To this purpose, let

$$\Psi_\varepsilon(x) = \int_0^T u(x,t)\phi_\varepsilon(x,t)\,\mathrm{d}t;$$

clearly, $\|\Psi_\varepsilon\|_{W^{1,1}(\Omega)} \leqslant C$, where C depends on $\|\phi_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}$ (which is equibounded) and $\|u\|_{L^2(0,T;H_0^1(\Omega))}$. This implies that the sequence $\{\Psi_\varepsilon\}$ strongly converges in $L^1(\Omega)$, so that, recalling that $R_\varepsilon - \overline{R} \rightharpoonup 0$ $*$-weakly in $L^\infty(\Omega)$, we obtain

$$\int_{\Omega_T} (R_\varepsilon - \overline{R})u\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (R_\varepsilon - \overline{R})\left(\int_0^T u\phi_\varepsilon \,\mathrm{d}t\right)\mathrm{d}x = \int_\Omega (R_\varepsilon - \overline{R})\Psi_\varepsilon \,\mathrm{d}x \to 0,$$

i.e., (3.5) is proven. Therefore we have shown that

$$\limsup_{\varepsilon\to 0^+}\|u_\varepsilon - u\|_{L^2(\Omega_T)} \leqslant c_1\eta;$$

but η is arbitrary, hence $u_\varepsilon \to u$ strongly in $L^2(\Omega_T)$. $\square$

Lemma 3.1. *Let $\{R_\varepsilon\}$ be a sequence in $L^\infty(\Omega)$, which $*$-weakly converges to $\overline{R} \neq 0$. Then, for every $\eta > 0$, there exist c_η and $\varepsilon_\eta > 0$ such that, for every $\varepsilon \in (0, \varepsilon_\eta)$ and for every $u \in H_0^1(\Omega)$*

$$\|u\|_{L^2(\Omega)} \leqslant \eta \|u\|_{H_0^1(\Omega)} + c_\eta \|R_\varepsilon u\|_{H^{-1}(\Omega)}. \tag{3.6}$$

Proof. By contradiction, assume that (3.6) is not true: then, for a convenient subsequence of indexes (which we still denote with ε) there exists $\bar{\eta} > 0$ and a sequence $\{u_\varepsilon\} \subset H_0^1(\Omega)$, with $\varepsilon \to 0$, such that

$$\|u_\varepsilon\|_{L^2(\Omega)} > \bar{\eta} \|u_\varepsilon\|_{H_0^1(\Omega)} + k(\varepsilon) \|R_\varepsilon u_\varepsilon\|_{H^{-1}(\Omega)}, \tag{3.7}$$

with $k(\varepsilon) \to +\infty$. We set $v_\varepsilon = u_\varepsilon / \|u_\varepsilon\|_{H_0^1(\Omega)}$. Since $\|v_\varepsilon\|_{H_0^1(\Omega)} = 1$, we can extract a subsequence (still denoted by $\{v_\varepsilon\}$) such that $v_\varepsilon \to v$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, for some $v \in H_0^1(\Omega)$. Moreover, from (3.7) one obtains

$$1 \geqslant \|v_\varepsilon\|_{L^2(\Omega)} > \bar{\eta} + k(\varepsilon) \|R_\varepsilon v_\varepsilon\|_{H^{-1}(\Omega)}, \tag{3.8}$$

which implies $R_\varepsilon v_\varepsilon \to 0$ strongly in $H^{-1}(\Omega)$. On the other hand, since $R_\varepsilon \rightharpoonup \overline{R}$ $*$-weakly in $L^\infty(\Omega)$, it follows that $R_\varepsilon v_\varepsilon \rightharpoonup \overline{R} v$ weakly in $L^2(\Omega)$, so that $\overline{R} v = 0$. Therefore, using the assumption $\overline{R} \neq 0$, one can conclude that $v = 0$; i.e., $v_\varepsilon \to 0$ strongly in $L^2(\Omega)$. Finally (3.8) implies

$$0 \geqslant \bar{\eta} + \limsup_{\varepsilon \to 0^+} k(\varepsilon) \|R_\varepsilon v_\varepsilon\|_{H^{-1}(\Omega)} \geqslant \bar{\eta},$$

which is absurd. $\square$

Proof of Theorem 2.1. By (1.5) and Theorem 2.3, there exists a function $u \in L^2(0, T; H_0^1(\Omega))$ and a subsequence, still denoted by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega_T) \text{ and weakly in } L^2\big(0, T; H_0^1(\Omega)\big) \text{ for } \varepsilon \to 0^+. \tag{3.9}$$

We now define the error function $r_\varepsilon(x,t) = u_\varepsilon(x,t) - u_0(x,t)$, where u_0 is the solution of (2.1). Assume for the moment that $\partial u_0 / \partial t \in L^2(\Omega_T)$. Then r_ε satisfies the equation

$$\begin{cases} R_\varepsilon(x) \dfrac{\partial r_\varepsilon}{\partial t}(x,t) - \Delta r_\varepsilon(x,t) = \big[\overline{R} - R_\varepsilon(x)\big] \dfrac{\partial u_0}{\partial t}(x,t) & \text{in } \Omega_T, \\ r_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ r_\varepsilon(x,0) = 0 & \text{on } \Omega_{+,\varepsilon}, \\ r_\varepsilon(x,T) = \psi(x) - u_0(x,T) & \text{on } \Omega_{-,\varepsilon}. \end{cases} \tag{3.10}$$

Fix $0 < \delta < T$ and let $\eta_\delta : [0,T] \to \mathbf{R}$ be a nonincreasing $\mathcal{C}^1$-function such that $\eta_\delta(t) \equiv 1$ in $[0, T-\delta]$, $\eta_\delta(T) = 0$. Let us now multiply the equation in (3.10) by $r_\varepsilon \eta_\delta$ and integrate by parts over Ω_T, obtaining

$$\begin{aligned} &\frac{1}{2} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_\Omega R_\varepsilon r_\varepsilon^2 \eta_\delta \,\mathrm{d}x \right] \mathrm{d}t - \frac{1}{2} \int_{\Omega_T} R_\varepsilon r_\varepsilon^2 \eta_\delta' \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} [\overline{R} - R_\varepsilon] \frac{\partial u_0}{\partial t} r_\varepsilon \eta_\delta \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Since $\eta_\delta(T)=0$, the first integral is non-negative, therefore

$$\int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_{\Omega_T} R_\varepsilon r_\varepsilon^2 \eta_\delta' \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} [\overline{R}-R_\varepsilon]\frac{\partial u_0}{\partial t} r_\varepsilon \eta_\delta \,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the lim sup for $\varepsilon \to 0^+$ in the previous inequality and taking into account that $\eta_\delta' \leqslant 0$, that $R_\varepsilon \rightharpoonup \overline{R}$ $*$-weakly in $L^\infty(\Omega)$ and that, by (3.9), $r_\varepsilon \to u-u_0$ strongly in $L^2(\Omega_T)$, we obtain

$$0 \leqslant \limsup_{\varepsilon\to 0^+} \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\overline{R}\int_{\Omega_T} (u-u_0)^2 \eta_\delta' \,\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

This implies that $\nabla r_\varepsilon \eta_\delta^{1/2} \to 0$ strongly in $L^2(\Omega_T)$. In particular, it follows that, for every $\delta>0$, $\|u_\varepsilon - u_0\|_{L^2(0,T-\delta;H_0^1(\Omega))} \to 0$. Therefore, the limit u in (3.9) actually coincides with the function u_0 and hence the whole sequence $\{u_\varepsilon\}$, not just a subsequence, converges to u_0. If $\frac{\partial u_0}{\partial t} \notin L^2(\Omega_T)$, we can proceed by approximation as follows. Let us take $\{f_k\} \subseteq L^2(\Omega_T)$, $\{\varphi_k\}, \{\psi_k\} \subseteq H_0^1(\Omega)$ such that

$$\begin{cases} f_k \to f & \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big), \\ \varphi_k \to \varphi, \quad \psi_k \to \psi & \text{strongly in } L^2(\Omega). \end{cases}$$

Then, by the linearity of problems (1.1) and (2.1), and by the estimate (1.5), we obtain that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2(0,T-\delta;H_0^1(\Omega))} &\leqslant \|u_\varepsilon - u_\varepsilon^k\| + \|u_\varepsilon^k - u_0^k\| + \|u_0^k - u_0\| \\ &\leqslant C\big(\|f_k - f\|_{L^2(0,T;H^{-1}(\Omega))} + \|\varphi_k - \varphi\|_{L^2(\Omega)} + \|\psi_k - \psi\|_{L^2(\Omega)}\big) \\ &\quad + \|u_\varepsilon^k - u_0^k\|_{L^2(0,T-\delta;H_0^1(\Omega))}, \end{aligned} \tag{3.11}$$

where, for every $\varepsilon>0$ and every $k\in\mathbf{N}$, u_ε^k and u_0^k are the unique solutions of (1.1) and (2.1), respectively, with f,φ and ψ replaced by f_k,φ_k and ψ_k. Since $\partial u_0^k/\partial t \in L^2(\Omega_T)$ (see, for instance, [5]), we can conclude by letting first $\varepsilon\to 0^+$ and then $k\to+\infty$ in (3.11). □

The proof of Theorem 2.7 is analogous to the previous one, up to obvious modifications.

4. The semilinear case

The compactness Theorem 2.3 allows us to study, with a similar technique as in the proof of Theorem 2.1, the homogenization of equations with nonlinear reaction terms, as showed in the next theorem. As in the linear case, there is no loss of generality in supposing that $\overline{R}>0$. The adaptation for $\overline{R}<0$ is straightforward.

Theorem 4.1. *Assume that $R(y)\neq 0$ a.e. in $\mathbf{R}^N$, that $\overline{R}>0$, and let $f:\Omega_T\times\mathbf{R}\to\mathbf{R}$ be a Caratheodory function with linear growth*; *i.e.*,

(i) *for every $s\in\mathbf{R}$, $f(\cdot,\cdot,s)$ is a measurable function on Ω_T*;
(ii) *for a.e. $(x,t)\in\Omega_T$, $f(x,t,\cdot)$ is a continuous function on $\mathbf{R}$*;

(iii) *there exists a constant* $\gamma > 0$*, such that, for a.e.* $(x,t) \in \Omega_T$ *and for every* $s \in \mathbf{R}$,

$$|f(x,t,s)| \leqslant \gamma(1+|s|).$$

For every $\varepsilon > 0$*, let* $u_\varepsilon \in \mathcal{W}_\varepsilon$ *be a solution of the problem*

$$\begin{cases} R_\varepsilon(x)\dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f(x,t,u_\varepsilon) & \text{in } \Omega_T, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega_{+,\varepsilon}, \\ u_\varepsilon(x,T) = \psi(x) & \text{in } \Omega_{-,\varepsilon}. \end{cases} \tag{4.1}$$

Then there exists a subsequence, still denoted by $\{u_\varepsilon\}$*, and a function* $u_0 \in L^2(0,T;H^1_0(\Omega))$ *such that, for every* $\delta > 0$,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0, \quad \|u_\varepsilon - u_0\|_{L^2(0,T-\delta;H^1_0(\Omega))} \to 0 \quad \text{for } \varepsilon \to 0^+. \tag{4.2}$$

Moreover, u_0 *is a solution of the problem*

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t} - \Delta u_0 = f(x,t,u_0) & \text{in } \Omega_T, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_0(x,0) = \varphi(x) & \text{in } \Omega. \end{cases} \tag{4.3}$$

Remark 4.2. In the case where the solution of problem (4.3) is unique, the whole sequence $\{u_\varepsilon\}$, not only a subsequence, converges to the solution u_0 of (4.3). This happens, for instance, if f is nonincreasing with respect to the last variable or if there exists a constant $C > 0$, such that

$$|f(x,t,s_1) - f(x,t,s_2)| \leqslant C|s_1 - s_2| \quad \text{for a.e. } (x,t) \in \Omega_T,\ \forall s_1, s_2 \in \mathbf{R}.$$

The first step is to state an existence result for the solution of (4.1) (which is defined in analogy with Definition 1.1).

Theorem 4.3. *For every* $\varepsilon > 0$*, problem* (4.1) *admits at least a solution* $u_\varepsilon \in \mathcal{W}_\varepsilon$*. Moreover,*

$$\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leqslant C(\gamma, \|\varphi\|_{L^2(\Omega)}, \|\psi\|_{L^2(\Omega)}), \tag{4.4}$$

where $C > 0$ *does not depend on* ε.

Proof. Let us fix $\varepsilon > 0$ and define the map $S_\varepsilon : L^2(\Omega_T) \to L^2(\Omega_T)$ by $S_\varepsilon(v) = u_\varepsilon$, where u_ε is the unique solution of

$$\begin{cases} R_\varepsilon(x)\dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f(x,t,v) & \text{in } \Omega_T, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = \varphi(x) & \text{in } \Omega_{+,\varepsilon}, \\ u_\varepsilon(x,T) = \psi(x) & \text{in } \Omega_{-,\varepsilon}. \end{cases}$$

Clearly, by Theorem 1.2, S_ε is well defined and $S_\varepsilon(v) = u_\varepsilon \in \mathcal{W}_\varepsilon$. Moreover, using also (iii) of Theorem 4.1, we have

$$\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leqslant C\big(\|v\|_{L^2(\Omega_T)}, \gamma, \|\varphi\|_{L^2(\Omega)}, \|\psi\|_{L^2(\Omega)}\big);$$

hence, by Remark 2.5, S_ε is a compact operator from the Banach space $L^2(\Omega_T)$ into itself. This implies that it admits at least a fixed point; i.e., there exists at least a function $u_\varepsilon \in L^2(\Omega_T)$ such that $S_\varepsilon(u_\varepsilon) = u_\varepsilon$, that is, u_ε is a solution of (4.1).

Estimates (4.4) easily follows by multiplying the first equation in (4.1) by u_ε, integrating by parts and using Gronwall's inequality. □

Proof of Theorem 4.1. By (4.4) and Theorem 2.3, there exist a function $u \in L^2(0,T;H_0^1(\Omega))$ and a subsequence, still denoted by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega_T) \text{ and weakly in } L^2\big(0,T;H_0^1(\Omega)\big) \text{ for } \varepsilon \to 0^+. \tag{4.5}$$

Passing to the limit in the weak formulation of the first equation in (4.1) and taking into account the continuity of $f(x,t,\cdot)$, it follows that u satisfies, in the sense of distribution, the equation

$$\overline{R}\frac{\partial u}{\partial t} - \Delta u = f(x,t,u) \quad \text{in } \Omega_T.$$

The crucial point is now to state the correct initial/final conditions for the limit function u. To this purpose, define $u_0 \in L^2(0,T;H_0^1(\Omega))$ to be the unique solution of

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t} - \Delta u_0 = f(x,t,u) & \text{in } \Omega_T, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_0(x,0) = \varphi(x) & \text{in } \Omega. \end{cases} \tag{4.6}$$

We claim that u coincides with u_0, therefore u_0 is indeed a solution of (4.3).

Let us define the error function $r_\varepsilon(x,t) = u_\varepsilon(x,t) - u_0(x,t)$. Again, by approximating the initial datum with regular functions, one can assume that $\partial u_0/\partial t \in L^2(\Omega_T)$. Then r_ε satisfies the equation

$$\begin{cases} R_\varepsilon(x)\dfrac{\partial r_\varepsilon}{\partial t}(x,t) - \Delta r_\varepsilon(x,t) = f_\varepsilon(x,t) & \text{in } \Omega_T, \\ r_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ r_\varepsilon(x,0) = 0 & \text{in } \Omega_{+,\varepsilon}, \\ r_\varepsilon(x,T) = \psi(x) - u_0(x,T) & \text{in } \Omega_{-,\varepsilon}, \end{cases} \tag{4.7}$$

where

$$f_\varepsilon(x,t) = \big[\overline{R} - R_\varepsilon(x)\big]\frac{\partial u_0}{\partial t}(x,t) + f(x,t,u_\varepsilon) - f(x,t,u).$$

Fix $0 < \delta < T$ and let $\eta_\delta : [0,T] \to \mathbf{R}$ be a nonincreasing $\mathcal{C}^1$-function such that $\eta_\delta(t) \equiv 1$ in $[0, T-\delta]$, $\eta_\delta(T) = 0$. Let us now multiply the equation for r_ε by $r_\varepsilon\eta_\delta$ and integrate by parts over Ω_T, obtaining, as in the proof of Theorem 2.1,

$$\int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \, \mathrm{d}x \, \mathrm{d}t \leqslant \frac{1}{2} \int_{\Omega_T} R_\varepsilon r_\varepsilon^2 \eta_\delta' \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_T} [\overline{R} - R_\varepsilon] \frac{\partial u_0}{\partial t} r_\varepsilon \eta_\delta \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_T} \big[f(x,t,u_\varepsilon) - f(x,t,u)\big](u_\varepsilon - u_0)\eta_\delta \, \mathrm{d}x \, \mathrm{d}t.$$

Passing to the lim sup for $\varepsilon \to 0^+$ in the previous inequality and taking into account that $\eta_\delta' \leqslant 0$, that $R_\varepsilon \rightharpoonup \overline{R}$ $*$-weakly in $L^\infty(\Omega)$, that $r_\varepsilon \to u - u_0$ strongly in $L^2(\Omega_T)$ and that, by (ii) and (iii), $f(x,t,u_\varepsilon) \to f(x,t,u)$ strongly in $L^2(\Omega_T)$, we obtain

$$0 \leqslant \limsup_{\varepsilon \to 0^+} \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \, \mathrm{d}x \, \mathrm{d}t \leqslant \frac{1}{2} \overline{R} \int_{\Omega_T} (u - u_0)^2 \eta_\delta' \, \mathrm{d}x \, \mathrm{d}t \leqslant 0.$$

This implies that $\nabla r_\varepsilon \eta_\delta^{1/2} \to 0$ strongly in $L^2(\Omega_T)$. In particular, it follows that, for every $\delta > 0$, $\|u_\varepsilon - u_0\|_{L^2(0,T-\delta;H_0^1(\Omega))} \to 0$. Therefore, $u = u_0$ and the claim is proved. □

Acknowledgements

We wish to thank Prof. A. Pankov for suggesting this problem and Prof. A. Piatnitski for some useful discussion.

References

[1] M. Amar, A. Dall'Aglio and F. Paronetto, Homogenization of changing-type evolution equations, *J. Convex Analysis* **12**(1) (2005), 221–237.
[2] M.S. Baouendi and P. Grisvard, Sur une équation d'evolution changeant de type, *J. Funct. Anal.* **2** (1968), 352–367.
[3] S. Brahim-Otsmane, G. Francfort and F. Murat, Correctors for the homogenization of the wave and heat equations, *J. Math. Pures Appl.* **71** (1992), 197–231.
[4] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer, New York, 1988.
[5] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.
[6] C.D. Pagani and G. Talenti, On a forward-backward parabolic equation, *Ann. Mat. Pura Appl.* **90** (1971), 1–57.
[7] A A. Pankov and T.E. Pankova, Nonlinear evolution equations with non-invertible operator coefficient at the derivative, *Dokl. Akad. Nauk Ukrainy* **9** (1993), 18–20 (in Russian).
[8] F. Paronetto, Existence results for a class of evolution equations of mixed type, *J. Funct. Anal.* **212**(2) (2004), 324–356.
[9] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146**(4) (1987), 65–96.